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DEFORMATIONS OF TRANSVERSELY HOLOMORPHIC FLOWS ON SPHERES AND DEFORMATIONS OF HOPF MANIFOLDS

A. Haefliger

Abstract

In this note we consider a transversely holomorphic foliation $\mathscr{F}$ of dimension one on S^{2n-1} obtained by intersecting the orbits of a holomorphic flow on $\mathbb{C}^n$ having zero as a contracting fixed point. It is shown that any deformation of $\mathscr{F}$ (in the class of transversely holomorphic foliations) is still obtained by intersecting S^{2n-1} with the orbits of a deformation of the holomorphic flow.

We use an analogue of the theorem of Kodaira-Spencer on the existence of a versal deformation for transversely holomorphic foliation (see [6] or [7]) and the classification of germs of holomorphic contracting vector fields (Poincaré-Dulac theorem) as explained in the book of Arnold [1]. This book was the main inspiration for this paper.

In an appendix which can be read independently, we show that parallel considerations leads to a complete classification of Hopf manifolds. This completes results of C. Borcea [2].

1. Statement of the main theorem

1.1. λ-RESONANT VECTOR FIELDS. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a sequence of complex numbers with strictly negative real part. Following Arnold ([1], p. 178), an additive *) λ-resonant monomial vector field in $\mathbb{C}^n$ is a vector field of the form $az^m\partial/\partial z_s$, where $m = (m_1, \dots, m_n)$ is a multiindex of non negative integers m_i such that

$$\lambda_s = (m, \lambda).$$

Here $(m, \lambda) = \Sigma m_i\lambda_i$, $z^m = z^{m_1} \dots z^{m_n}$ and $a \in \mathbb{C}$. This condition implies that the m_i are not all zero.

DEFINITION: $\mathscr{G}_\lambda$ denotes the vector space of λ-resonant vector fields, i.e. vector fields which are sum of λ-resonant monomial vector fields. It is a finite dimensional vector space. It can also be characterized as the subspace of holomorphic vector fields on $\mathbb{C}^n$ commuting with the diago-

* In the appendix, we shall define multiplicative μ-resonant vector fields.

nal vector field $\Sigma\lambda_s\partial/\partial z_s$. Therefore $\mathscr{g}_\lambda$ is a Lie subalgebra of the Lie algebra of holomorphic vector fields on $\mathbf{C}^n$.

For generic λ, $\dim \mathscr{g}_\lambda = n$. For $n = 2$, $\dim \mathscr{g}_\lambda$ can be 2, 3 or 4. For $n \geqslant 3$, $\dim \mathscr{g}_\lambda$ is not bounded.

1.2. THE THEOREM OF POINCARÉ-DULAC. Let $\xi = \Sigma a^s_m z^m \partial/\partial z_s$ be a holomorphic vector field on a neighbourhood of 0 in $\mathbf{C}^n$, vanishing at 0 and such that the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix (a^s_i) of the linear part of ξ have strictly negative real parts (in other words the flow generated by ξ is contracting). We can order the λ_i's so that $\operatorname{Re}\lambda_1 \leqslant \dots \leqslant \operatorname{Re}\lambda_n < 0$.

The theorem of Poincaré-Dulac (Cf. Arnold [1], p. 183) asserts that one can find new coordinates such that ξ is under the form of a λ-resonant vector field, with linear part under Jordan form, so

$$\xi = \sum \lambda_s z_s \partial/\partial z_s + \sum_{(s,m)} a^s_m z^m \partial/\partial z_s \tag{1.2.1}$$

where the sum is over the sequences (s, m) such that

$$\lambda_s = \sum m_i \lambda_i, \quad m_i = 0 \qquad \text{for} \quad i \geqslant s.$$

Remark that after a diagonal change of coordinates, one can assume that the coefficients a^s_m are as small as we want. Indeed if h is the diagonal linear map with entries $\mu_1, \dots, \mu_n$ in the diagonal, then

$$(h_*\xi)(z) = \sum \lambda_s z_s \partial/\partial z_s + \sum_{(s,m)} \frac{\mu_s}{\mu^m} a^s_m z^m \partial/\partial z_s.$$

So we can choose $\mu_k = \epsilon^{-k}$, where ϵ is very small.

In other words, under the action of the group of linear automorphisms of $\mathbf{C}^n$, the orbit of the diagonal vector field $\Sigma\lambda_s z_s \partial/\partial z_s$ is in the adherence of the orbit of ξ.

The vector field ξ generates a global holomorphic flow $z(t)$ of the form

$$z_s(t) = e^{\lambda_s t}\Big(z_s(0) + \sum_{(s,m),r} b^s_{m,r} t^r z^m(0) \Big) \tag{1.2.2}$$

(the coefficients $b^s_{m,r}$ are determined step by step, starting with $s = n$).

The orbits of this flow in $\mathbf{C}^n$-$\{0\}$ are complex curves which are the leaves of a holomorphic foliation $\mathscr{F}_\xi$. Note that ξ vanishes only at 0. Those leaves are transversal to the unit sphere S^{2n-1} in $\mathbf{C}^n$, if the coefficients a^s_m are small enough. It follows that their intersection with S^{2n-1} are curves which are the leaves of a transversely holomorphic foliation $\mathscr{F}^0_\xi$ on S^{2n-1} induced by $\mathscr{F}_\xi$.

THEOREM: *Let S be a small enough neighbourhood of 0 in a vector subspace of $\mathscr{g}_\lambda$ (cf. 1.1) complementary to the vector subspace generated by $[\xi, \mathscr{g}_\lambda]$ and ξ.*

The family $\mathscr{F}^0_{\xi+s}$ of transversely holomorphic foliations on S^{2n-1} obtained by intersecting the orbits of the flows generated by $\xi + s$ is a versal deformation of $\mathscr{F}^0_\xi$ parametrized by $s \in S$ (in the sense of [7]).

Note that if ξ is the diagonal vector field $\Sigma \lambda_s z_s \partial/\partial z_s$, then the dimension of S is $\dim \mathscr{g}_\lambda - 1$. In any case, $\dim S \geqslant n - 1$.

Let $\theta^{\mathrm{tr}}_{F^0_\xi}$ be the sheaf of germs of transversely holomorphic vector fields for $\mathscr{F}^0_\xi$ (cf. Lemma 3.2), we obtain that

$$\dim H^0\left(S^{2n-1}, \theta^{\mathrm{tr}}_{F^0_\xi}\right) = \dim H^1\left(S^{2n-1}, \theta^{\mathrm{tr}}_{F^0_\xi}\right) = \dim S$$

and for $i > 1$

$$H^i\left(S^{2n-1}, \theta^{\mathrm{tr}}_{F^0_\xi}\right) = 0.$$

REMARK: An open neighbourhood of 0 in a vector subspace of $\mathscr{g}_\lambda$ complementary to $[\xi, \mathscr{g}^\lambda]$ parametrizes a versal deformation of the holomorphic vector field ξ (see Arnold [1], p. 302, and N. Brouchlinskaïa [2] where the existence of a versal deformation is proved).

2. Infinitesimal deformation defined by a deformation of a vector field

2.1. Let ξ be an everywhere non zero holomorphic vector field on a complex manifold X and let $\mathscr{F}$ be the holomorphic foliation whose leaves are the orbits of the flow generated by ξ.

We consider the following sheaves:

θ = sheaf of germs of holomorphic vector fields on X,
$\theta_{\mathscr{F}}$ = subsheaf of θ of vectors fields preserving $\mathscr{F}$,
θ^ξ = subsheaf of θ of vector fields commuting with ξ,
$\theta^{\mathrm{tr}}_{\mathscr{F}}$ = sheaf of germs of transversely holomorphic vector fields for $\mathscr{F}$ (quotient of $\theta_{\mathscr{F}}$ by vector fields tangent to the leaves of $\mathscr{F}$)
σ = sheaf of germs of holomorphic functions on X
$\sigma^{\mathrm{tr}}_{\mathscr{F}}$ = subsheaf of germs of holomorphic functions locally constant on the leaves of $\mathscr{F}$.

We have the following exact sequences of sheaves

$$0 \to \sigma^{\mathrm{tr}}_{\mathscr{F}} \to \sigma \xrightarrow{L_\xi} \sigma \to 0 \tag{2.1.1}$$

$$0 \to \theta^\xi \to \theta \xrightarrow{L_\xi} \theta \to 0 \tag{2.1.2}$$

$$0 \to \sigma^{\mathrm{tr}}_{\mathscr{F}} \to \theta^\xi \to \theta^{\mathrm{tr}}_{\mathscr{F}} \to 0 \tag{2.1.3}$$

where $L_\xi : \sigma \to \sigma$ is the derivative in the direction of ξ, $L_\xi : \theta \to \theta$ the Poisson bracket with ξ and the inclusion $\sigma_{\mathscr{F}}^{\mathrm{tr}} \to \theta^\xi$ the multiplication by ξ. The multiplication by ξ send the sequence (2.1.1) in the sequence (2.1.2).

To check exactness, choose local coordinates $(z, w_1, \ldots, w_{n-1})$ such that ξ is given by $\partial/\partial z$.

Let S be a germ of analytic space with a base point 0; its Zariski tangent space of 0 is denoted by $T_0 S$. Let ξ_s be a holomorphic family of everywhere non zero vector fields on X parametrized by S and such that $\xi_0 = \xi$. Denote by $\mathscr{F}_s$ the corresponding family of holomorphic foliations on X.

2.2. Proposition: *The Kodaira-Spencer map*

$$\rho : T_0 S \to H^1(X, \theta_{\mathscr{F}})$$

measuring the infinitesimal deformations of the family $\mathscr{F}_s$ *is given by*

$$\rho(\partial/\partial s) = -i \cdot \delta(\partial \xi_s / \partial s\,|_{s=0})$$

where $\delta : H^0(X, \theta) \to H^1(X, \theta^\xi)$ *is the connecting homomorphism associated to* (2.1.2) *and* $i : H^1(X, \theta^\xi) \to H^1(X, \theta_{\mathscr{F}})$ *is induced by the inclusion of* θ^ξ *in* $\theta_{\mathscr{F}}$.

Proof: Let $\{U_i\}_{i \in I}$ be an open covering of X such that there are families of holomorphic charts $\varphi_i^s : U_i \to \mathbb{C} \times \mathbb{C}^{n-1}$ so that $\varphi_{i*}^s(\xi_s) = \partial/\partial z$, where $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$.

Let g_{ij}^s be the change of charts defined by $\varphi_i^s = g_{ij}^s \varphi_j^s$ on $U_i \cap U_j$.

The element $\rho(\partial/\partial s)$ of $H^1(X, \theta_{\mathscr{F}})$ corresponding to the infinitesimal deformation $\partial/\partial s$ is represented by the 1-cocycle associating to $U_i \cap U_j$ the vector field

$$\theta_{ij} = \mathrm{d}/\mathrm{d}s\Big(\big(\varphi_i^0\big)^{-1}\varphi_i^s\big(\varphi_j^s\big)^{-1}\varphi_j^0\Big)_{s=0} = \Big(\big(\varphi_i^0\big)^{-1}\Big)_*\Big(\frac{\mathrm{d}}{\mathrm{d}s} g_{ij}^s\Big|_{s=0}\Big).$$

Let η_i be the holomorphic vector field on U_i defined by $\eta_i = \mathrm{d}/\mathrm{d}s((\varphi_i^s)^{-1}\varphi_i^0)_{s=0}$. On $U_i \cap U_j$, we have $\theta_{ij} = \eta_j - \eta_i$.

Moreover $L_\xi \eta_i = -\mathrm{d}/\mathrm{d}s \xi_s\,|_{s=0}$ because $\xi_s = ((\varphi_i^s)^{-1}\varphi_i^0)_* \xi$ hence $\mathrm{d}/\mathrm{d}s$ $\xi_s\,|_{s=0} = [\eta_i, \xi]$.

This shows that $\rho(\partial/\partial s) = -i \circ \delta(\mathrm{d}\xi_s/\mathrm{d}s\,|_{s=0})$.

2.3. Corollary: *If we consider* $\mathscr{F}_s$ *as a family of transversely holomorphic foliations, then the Kodaira-Spencer map*

$$\rho : T_0 S \to H^1(X, \theta_{\mathscr{F}}^{\mathrm{tr}})$$

is the composition of δ *and of the map* $p : H^1(X, \theta^\xi) \to H^1(X, \theta_{\mathscr{F}}^{\mathrm{tr}})$ *induced by the projection* $\theta^\xi \to \theta_{\mathscr{F}}^{\mathrm{tr}}$.

3. Proof of the theorem

3.1. We denote by $\mathscr{F}$ the holomorphic foliation on $W = \mathbb{C}^n - \{0\}$ whose leaves are the orbits of the flow generated by ξ and by $\mathscr{F}^0$ the transversely holomorphic foliation on S^{2n-1} induced from $\mathscr{F}$ by the inclusion $i: S^{2n-1} \to \mathbb{C}^n$. As before we denote by $\theta^{\mathrm{tr}}_{\mathscr{F}}$ (resp. $\theta^{\mathrm{tr}}_{\mathscr{F}^0}$) the sheaf of germs of transversely holomorphic vector fields for $\mathscr{F}$ (resp. $\mathscr{F}_0$); clearly $\theta^{\mathrm{tr}}_{\mathscr{F}^0} = i^*\theta^{\mathrm{tr}}_{\mathscr{F}}$.

By the analogue of the Kodaira-Spencer theorem proved in [6] or in [7] for transversely holomorphic foliations, it will be sufficient to prove that the Kodaira-Spencer map $\rho: T_0 S \to H^1(S^{2n-1}, \theta^{\mathrm{tr}}_{\mathscr{F}^0})$ is an isomorphism.

For the real parameter t, the orbits (1.2.2) of the flow generated by ξ are tangent to the leaves of $\mathscr{F}$, transveral to the sphere S^{2n-1}, tend to 0 for $t \to +\infty$ and to infinity in norm for $t \to -\infty$. So there is a projection $\pi: W \to S^{2n-1}$ mapping the point z on the point of intersection with S^{2n-1} of the orbit passing through z. The pull back by π of $\mathscr{F}^0$ is the transversely holomorphic foliation associated to $\mathscr{F}$. Hence $\sigma^{\mathrm{tr}}_{\mathscr{F}} = \pi^*\sigma^{\mathrm{tr}}_{\mathscr{F}^0}$ and

$$H^i\left(S^{2n-1}, \sigma^{\mathrm{tr}}_{\mathscr{F}^0}\right) \approx H^i\left(W, \sigma^{\mathrm{tr}}_{\mathscr{F}}\right) \tag{3.1.1}$$

because W retracts by deformation on S^{2n-1} along the orbits. For the same reasons and with the notations of 2, we have

$$H^i(S^{2n-1}, i^*\theta^{\xi}) \approx H^i(W, \theta^{\xi}). \tag{3.1.2}$$

For $n > 1$, any holomorphic vector field on W extends to a holomorphic vector field on $\mathbb{C}^n$, so any element of $H^0(W, \theta)$ is represented by a convergent series $\Sigma a^s_m z^m \partial/\partial z_s$ (here θ denotes the sheaf of germs of holomorphic vector fields on W; it is isomorphic to σ^n, where σ is the sheaf of germs of holomorphic functions on W).

Consider the cohomology long exact sequences associated to the short exact sequences (2.1.1) and (2.1.2):

$$0 \to H^0\left(W, \sigma^{\mathrm{tr}}_{\mathscr{F}}\right) \to H^0(W, \sigma) \xrightarrow{L_\xi} H^0(W, \sigma) \to H^1\left(W, \sigma^{\mathrm{tr}}_{\mathscr{F}}\right) \to \dots \tag{3.1.3}$$

$$0 \to H^0\left(W, \theta^{\xi}_{\mathscr{F}}\right) \to H^0(W, \theta) \xrightarrow{L_\xi} H^0(W, \theta) \to H^1\left(W, \theta^{\xi}_{\mathscr{F}}\right) \to \dots. \tag{3.1.4}$$

Let $\mathscr{g}^{\perp}_{\lambda}$ be the vector subspace of $H^0(W, \theta)$ of holomorphic vector fields which are sum of monomial vector fields $az^m \partial/\partial z_s$ which are not λ-resonant. It is clear that L_ξ maps $\mathscr{g}_\lambda$ in $\mathscr{g}_\lambda$ and $\mathscr{g}^{\perp}_{\lambda}$ in $\mathscr{g}^{\perp}_{\lambda}$, because the bracket of a λ-resonant monomial vector field with a monomial vector field which is not λ-resonant is not λ-resonant.

3.2. LEMMA:

(a) $H^i(W, \sigma) = H^i(W, \theta) = 0$ *for* $i \neq 0,\ n-1$. $H^{n-1}(W, \sigma)$ (*resp.* $H^{n-1}(W, \theta)$) *is isomorphic to the vector space of convergent series* $\Sigma a_m z^m$ (*resp.* $\Sigma a^s_m z^m \partial/\partial z_s$) *on* $(\mathbf{C}\text{-}\{0\})^n$ *where the sum is over the sequences* m *such that all* $m_i < 0$.

(b) *For* $i > 0$, *the maps* $L_\xi: H^i(W, \sigma) \to H^i(W, \sigma)$ *and* $L_\xi: H^i(W, \theta) \to H^i(W, \theta)$ *are isomorphisms.*

(c) *The kernel and cokernel of* $L_\xi: H^0(W, \sigma) \to H^0(W, \sigma)$ *are generated by the constant function* 1. $L_\xi: \mathscr{G}_\lambda^\perp \to \mathscr{G}_\lambda^\perp$ *is an isomorphism.*

PROOF OF (A): Consider the covering $\mathscr{U} = \{U_i\}$ of W by the Stein open sets $U_i = \{z \in W: z_i \neq 0\}$. By the theorem of Leray, $H^k(W, \sigma)$ is isomorphic to the Čech cohomology $H^k(\mathscr{U}, \sigma)$ computed using alternate cochains. Hence $H^k(W, \sigma) = 0$ for $k \geqslant n$. In dimension $n-1$, cochains are cocycles and are holomorphic functions on $\bigcap U_i = (\mathbf{C} - \{0\})^n$; their Laurent expansion are of the form $\Sigma a_m z^m$, where $m = (m_1, \ldots, m_n)$, $m_i \in \mathbf{Z}$. Modulo the coboundaries, each element of $H^{n-1}(W, \sigma)$ has a unique representative with all $m_i < 0$.

To prove that $H^k(W, \sigma) = 0$ for $0 < k < n-1$, one can consider W as the union of $\{z \in W: z_1, \ldots, z_{n-1}$ not all zero$\} = (\mathbf{C}^{n-1} - \{0\}) \times \mathbf{C}$ and $\{z \in W: z_n \neq 0\} = (\mathbf{C} - \{0\}) \times \mathbf{C}^{n-1}$, and write the Mayer-Vietoris cohomology exact sequence associated to this covering. Part a) of the lemma for σ follows by induction on n, using Künneth formula.

As $\theta = \sigma^n$, the similar result for $H^i(W, \theta)$ follows.

PROOF OF (B): If ξ is diagonal, namely if $\xi = \Sigma \lambda_s z_s \partial/\partial z_s$, then

$$L_\xi(z^m) = (m, \lambda) z^m \quad \text{and}$$

$$L(z^m \partial/\partial z_s) = \left[(m, \lambda) - \lambda_s\right] z^m \partial/\partial z_s.$$

If all the m_i are strictly negative, (m, λ) and $(m, \lambda) - \lambda_s$ have strictly positive real part, hence are non zero. It follows that the endomorphism L_ξ of $H^{n-1}(W, \sigma)$ and $H^{n-1}(W, \theta)$ are injective. Surjectivity is also easy because $|(m, \lambda)|^{-1}$ and $|(m, \lambda) - \lambda_s|^{-1}$ are smaller than 1 for $|m|$ big enough.

In general we can assume that ξ is under the form 1.2.1. With respect to the lexicographic order,

$$L_\xi(z^m) = (m, \lambda) z^m + \text{bigger monomials,} \quad \text{and}$$

$$L_\xi(z^m \partial/\partial z_s) = \left[(m, \lambda) - \lambda_s\right] z^m + \text{bigger monomial vector fields,}$$

if we decide that $z^m \partial/\partial z_s < z^n \partial/\partial z_t$ for $s > t$. It follows that L is injective. One should be able to prove directly that L is also surjective.

We give another argument using the upper semi-continuity of the dimension of the space of solutions of a differential elliptic operator on a compact manifold depending smoothly on a parameter (cf. Kodaira-Spencer III, [8] Th. 4, p. 48). When ξ is diagonal, we have checked that L_ξ is an isomorphism. Hence from the exact sequence 3.1.3, we have $H^i(W, \sigma_{\mathscr{F}}^{\mathrm{tr}}) = H^i(S^{2n-1}, \sigma_{\mathscr{F}_0}^{\mathrm{tr}}) = 0$ for $i \geqslant 2$. As this group is isomorphic to the space of solutions of an elliptic differential operator (cf. Kalka-Duchamp [5]), for all ξ close enough to a diagonal vector field (this is always the case by 1.2), we still have $H^i(S^{2n-1}, \sigma_{\mathscr{F}_0}^{\mathrm{tr}}) = 0$ for $i \geqslant 2$. Hence $L_\xi\colon H^i(W, \sigma) \to H^i(W, \sigma)$ is an isomorphism for $i > 0$. The similar argument works for σ replaced by θ.

Proof of (c): The elements of $H^0(W, \sigma)$ are convergent series $\Sigma a_m z^m$, with all $m_i \geqslant 0$. For ξ diagonal, it is easy to check that the kernel and cokernel of L_ξ are generated by 1. So from the exact sequence (3.1.3) and (3.1.1), we have $\dim H^0(S^{2n-1}, \sigma_{\mathscr{F}}^{\mathrm{tr}}) = \dim H^1(S^{2n-1}, \sigma_{\mathscr{F}}^{\mathrm{tr}}) = 1$. Hence $\Sigma(-1)^i \dim H^i(S^{2n-1}, \sigma_{\mathscr{F}_0}^{\mathrm{tr}}) = 0$. This number is the index of an elliptic complex (cf. Kalka-Duchamp [5]), so it is constant under deformation. Hence when ξ is not diagonal, we still have $\dim H^0(S^{2n-1}, \sigma_{\mathscr{F}_0}^{\mathrm{tr}}) = \dim H^1(S^{2n-1}, \sigma_{\mathscr{F}_0}^{\mathrm{tr}}) \leqslant 1$ (we also use semi-continuity as above). But the kernel of L_ξ contains 1, hence $\dim H^1(S^{2n-1}, \sigma_{\mathscr{F}_0}^{\mathrm{tr}}) = \dim H^1(W, \sigma_{\mathscr{F}}^{\mathrm{tr}}) = 1$. Therefore L_ξ surjects on the space of holomorphic functions vanishing at zero.

Similarly L_ξ restricted to $\mathscr{g}_\lambda^\perp$ is injective and, when ξ is diagonal, it is an isomorphism on $\mathscr{g}_\lambda^\perp$. As above, we see that $\dim H^0(W, \theta^\xi) = \dim H^1(W, \theta)$, for all ξ. But $H^0(W, \theta^\xi) = \operatorname{Ker} L_\xi = \operatorname{Ker}(L_\xi \mid \mathscr{g}_\lambda) = \mathscr{g}_\lambda / L_\xi(\mathscr{g}_\lambda)$, because $\mathscr{g}_\lambda$ is finite dimensional, and $H^1(W, \theta^\xi) = \operatorname{Coker} L_\xi$. Hence L_ξ maps $\mathscr{g}_\lambda^1$ surjectively on itself.

3.3. End of the proof of the theorem. Consider the commutative diagram

$$
\begin{tikzcd}
{} \arrow[r] & H^0(W,\sigma) \arrow[r] \arrow[d] & H^0(W,\sigma) \arrow[r, "\delta"] \arrow[d] & H^1(W,\sigma_{\mathscr{F}}^{\mathrm{tr}}) \arrow[r] \arrow[d] & H^1(W,\sigma) \arrow[r] \arrow[d] & H^1(W,\sigma) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^0(W,\theta) \arrow[r] & H^0(W,\theta) \arrow[r, "\delta"] \arrow[dr, "p\circ\delta"'] & H^1(W,\theta^\xi) \arrow[r] \arrow[d, "p"] & H^1(W,\theta) \arrow[r] & H^1(W,\theta) \arrow[r] & {} \\
 & & & H^1(W,\theta_{\mathscr{F}}^{\mathrm{tr}}) \arrow[d] & & & \\
 & & & H^2(W,\theta_{\mathscr{F}}^{\mathrm{tr}}) & & &
\end{tikzcd}
$$

where the first row is mapped in the second one by the multiplication by

ξ, and the vertical column is the cohomology exact sequence associated to (2.1.3).

By 2.3, we have to check that the restriction of $p \circ \delta$ to the subspace $T_0 S$ of $H^0(W, \theta)$ is an isomorphism on $H^1(W, \theta^{\mathrm{tr}}_{\mathscr{F}})$.

By the lemma, the map p as well as both maps δ are surjective. Also the restriction of δ to the vector subspace of $H^0(W, \theta)$ generated by $T_0 S$ and ξ is an isomorphism on $H^1(W, \theta^{\xi})$. But $\delta\xi$ generates the kernel of p; hence $p \circ \delta \,|\, T_0 S$ is an isomorphism.

Appendix: Versal deformation of Hopf manifolds

A.1. μ-RESONANT MAPS. Let $\mu = (\mu_1, \dots, \mu_n)$ be a sequence of non zero complex numbers such that $|\mu_i| < 1$. A (multiplicative) μ-resonant monomial (cf. Arnold [1], p. 185) is a polynomial map of $\mathbb{C}^n$ in $\mathbb{C}^n$ of the form $z \to a z^m e_s$ such that

$$\mu_s = \mu^m.$$

Here $m = (m_1, \dots, m_n)$ is a sequence of positive integers, $\mu^m = \mu_1^{m_1} \dots \mu_n^{m_n}$, $a \in \mathbb{C}$ and $e_1, \dots, e_n$ is the canonical basis of $\mathbb{C}^n$. We shall assume that

$$0 < |\mu_1| \leqslant |\mu_2| \leqslant \dots \leqslant |\mu_n| < 1.$$

A μ-resonant polynomial map $f: \mathbb{C}^n \to \mathbb{C}^n$ is a sum of μ-resonant monomials. Equivalently f is a polynomial map of $\mathbb{C}^n$ in $\mathbb{C}^n$ which commutes with the diagonal linear map $d_\mu : (z_1, \dots, z_n) \to (\mu_1 z_1, \dots, \mu_n z_n)$. Note that $f(0) = 0$.

The set of μ-resonant polynomial maps is a subalgebra R_μ of the algebra of polynomial maps of $\mathbb{C}^n$ in $\mathbb{C}^n$. It is finite dimensional. The elements of degree one in R_μ are represented by matrices with possibly non zero blocs along the diagonal with size equal to the number of times a μ_i is repeated.

Let G_μ be the group of invertible elements in R_μ. An element f in R_μ is invertible iff its linear part (or equivalently its derivative $f'(0)$ at 0) is invertible. Indeed, after conjugation with a linear automorphism in R_μ, one can assume that f is under lower triangular Jordan form. The i^{th}-coordinate is the sum of a non zero multiple of z_i and a polynomial containing only $z_1, \dots, z_{i-1}$.

G_μ is a connected complex Lie group, open in R_μ. The kernel of the projection of G_μ on the group G_μ^1 of linear automorphisms in R_μ is nilpotent.

The Lie algebra $\mathscr{g}_\mu$ of G_μ is the space of (multiplicatively) μ-resonant vector fields on $\mathbb{C}^n$, namely those vector fields which are linear combination of vector fields of the form $z^m \partial/\partial z_s$, where $\mu_s = \mu^m$. The μ-resonant vector fields can also be characterized as vector fields invariant by d_μ. As a vector space, $\mathcal{g}_\mu$ is isomorphic to R_μ.

A.2. DEFINITION OF HOPF MANIFOLDS. In this paragraph, we give several equivalent definitions for a Hopf manifold. We begin with the one which is apparently the most general.

A Hopf manifold of dimension $n > 1$ is a complex manifold W_f which is the quotient of $W = \mathbf{C}^n - \{0\}$ by an infinite cyclic group acting holomorphicaly, and properly discontinuously on W. It is proved by C. Borcea [2] that this group is generated by an element f such that $\sup_{|z|<a} |f^m(z)| \to 0$ when $m \to \infty$ for any a, and that W_f is compact. f extends to an automorphism of $\mathbf{C}^n$. We claim that the eigenvalues of the differential $f'(0)$ of f at 0 are of absolute value smaller than one. Indeed let v be an eigenvector corresponding to an eigenvalue μ of $f'(0)$, and let p be a linear projection of $\mathbf{C}^n$ on the one-dimensional subspace V generated by v. For m big enough, the restriction of f^m to V composed with p is a holomorphic map mapping the unit disk in V in a disk of smaller radius. The derivative at 0 of this map is μ^m, and by Schwarz lemma, $|\mu|$ must be smaller than one.

So we could have defined a Hopf manifold of dimension n as a compact complex manifold W_f which is the quotient of W by a properly discontinuous group generated by an automorphism f of $\mathbf{C}^n$ fixing 0 and such that the eigenvalues $\mu_1, \ldots, \mu_n$ of $f'(0)$ are inside the unit circle. According to the Poincaré-Dulac theorem (Cf. Arnold [1], p. 187), there is a holomorphic isomorphism h of a neighbourhood of 0 on a neighbourhood of 0 such that hfh^{-1} is the restriction of a polynomial map $\tilde{f}$ of $\mathbf{C}^n$ which is μ-resonant. We have seen in A.1 that $\tilde{f}$ is bijective and as each orbit of f and $\tilde{f}$ meets an arbitrarily small neighbourhood of 0 in $\mathbf{C}^n$, the map $\tilde{h}$ extends to a global automorphism h of $\mathbf{C}^n$ such that $\tilde{f} = \tilde{h}f\tilde{h}^{-1}$.

Eventually we can equivalently define a Hopf manifold as the quotient W_f of $W = \mathbf{C}^n - \{0\}$ by a polynomial automorphism f of $\mathbf{C}^n$ whose derivative $f'(0)$ at 0 has eigenvalues $\mu = (\mu_1, \ldots, \mu_n)$ inside the unit circle, and such that f is μ-resonant.

A.3. THEOREM: *Let f be as above. A versal deformation of the Hopf manifold W_f is obtained as follows. Let S be a small complex submanifold in G_μ passing through f and whose tangent space at f is complementary to the tangent space of the orbit of f under the action of G_μ by conjugation on itself. Then the family W_s, where $s \in S$, is a versal deformation of W_f parametrized by S.*

Let θ_f be the sheaf of germs of holomorphic vector fields on W_f. Then $H^i(W_f, \theta_f) = 0$ for $i > 1$ and $\dim H^0(W_f, \theta_f) = \dim H^1(W_f, \theta_f)$ is the dimension of the centralizer of f in G_μ (which is also the dimension of the kernel of the endomorphism $1 - f_*$ of $\mathcal{g}_\mu$).

For instance if f is diagonal, then $\dim H^1(W_f, \theta_f) = \dim \mathscr{g}_\mu$. In general $n \leqslant \dim H^1(W_f, \theta_f) \leqslant \dim \mathscr{g}_\mu$.

For $n \geqslant 3$, the dimension of $\mathscr{g}_\mu$ is not bounded. For instance, if $n = 3$ and μ_1, μ_2, μ_3 are real numbers such that $\mu_2 = \mu_3$ and $\mu_1 = \mu_2^p$, then $\dim \mathscr{g}_\mu = p + 6$.

A.4. Proof of the theorem: Consider the exact sequence (cf. [4], [2], [9]) analogous to (3.1.4):

$$0 \to H^0(W_f, \theta_f) \to H^0(W, \theta) \overset{1-f_*}{\to} H^0(W, \theta) \to H^1(W_f, \theta_f)$$

$$\to H^1(W, \theta) \overset{1-f_*}{\to} H^1(W, \theta) \tag{A.4.1.}$$

where θ is the sheaf of germs of holomorphic vector fields on W.

The main facts are:

(a) for $i > 0$, $1 - f_* : H^i(W, \theta) \to H^i(W, \theta)$ is an isomorphism.

(b) Denote by $\mathscr{g}_\mu^\perp$ the subspace of $H^0(W, \theta)$ of those vector field of the form $\Sigma a_m^s z^m \partial/\partial z_s$, where the monomials $a_m^s z^m \partial/\partial z_s$ are not μ-resonant. Then 1-f_* maps $\mathscr{g}_\mu$ in $\mathscr{g}_\mu$ and is an isomorphism of $\mathscr{g}_\mu^\perp$ on $\mathscr{g}_\mu^\perp$.

(c) Let f_s be a holomorphic family of automorphisms of $\mathbb{C}^n$ depending on a parameter t in a small neighbourhood of 0 in $\mathbb{C}$, such that $f = f_0$ and $f_t \in G_\mu$. For the corresponding family W_{f_t} of Hopf manifolds, the infinitesimal deformation $\rho(\partial/\partial t) \in H(W_f, \theta_f)$ is the image by δ of the vector field $\mathrm{d}/\mathrm{d}t(f_t f_0^{-1})\,|_{t=0}$ on W.

Those facts imply the theorem. By (a) and (b), the restriction of δ to the subspace $\mathscr{g}_\mu$ of $H^0(W, \theta)$ is a surjection on $H^1(W_f, \theta_f)$. The differential $f^{-1}{}_*$ of the right translation by f^{-1} in G_μ maps isomorphically the tangent space at f to the orbit of f on the subspace $(1 - f_*)\mathscr{g}_\mu$ of $\mathscr{g}_\mu$, and maps isomorphically T_fS on a complement in $\mathscr{g}_\mu$ of $(1 - f_*)\mathscr{g}_\mu$. By c), the Kodaira-Spencer map $\rho : T_fS \to H^1(W_f, \theta_f)$ is the composition of $f^{-1}{}_*$ with δ, hence is an isomorphism by the exactness of (A.4.1).

Also (a), (b) and the exactness of (A.4.1) imply that $H^i(W_f, \theta_f) = 0$ for $i > 1$ and that $H^0(W_f, \theta_f) = H^1(W_f, \theta_f)$. The Lie algebra of the centralizer of f is the kernel of the map $1 - f_* : \mathscr{g}_\mu \to \mathscr{g}_\mu$, so is canonically isomorphic to $H^0(W_f, \theta_f)$. Hence the connected component of the centralizer of f in G_μ (which acts naturally on W_f) is the connected component of the group of analytic automorphisms of W_f.

(c) is proved in Douady [4] and the proof of a) and b) is parallel to the proof of lemma 3.2 and is partly contained in [2] or [9]. We have seen in lemma 3.2 that $H^i(W, \theta) = 0$ for $i \neq 0$, $n - 1$ and that $H^{n-1}(W, \theta)$ is isomorphic to the convergent series in $(\mathbb{C}\text{-}\{0\})^n$ of the form $\Sigma a_m^s z^m \partial/\partial z_s$, with $m_i < 0$. It is easy to check directly that $(1 - f_*)\,|\,\mathscr{g}_\mu^\perp$ is injective, as well as $1-f_*: H^i(W, \theta) \to H^i(W, \theta)$ for $i>0$. Indeed we can assume that $0<|\mu_1| \leqslant \ldots \leqslant |\mu_n|<1$ and that the linear part of f is under upper triangular form. Then for the order defined in 3.2 (proof of b)), we have

$$(1-f_*) z^m \partial/\partial z_s = \left(1 - {}_{\mu_s}/_{\mu^m}\right) z^m \partial/\partial z_s + \text{bigger terms.}$$

Moreover $(1-\mu_s/_{\mu^{m_i}}) \neq 0$ if all m_i are negative or if $z^m \partial/\partial z_s \in \mathscr{g}_\mu^\perp$. The surjectivity of $(1-f_*): H^i(W, \theta) \to H^i(W, \theta)$ for $i>0$ is also obvious if f is diagonal.

From the preceding discussion and the exactness of A.3.1, it follows that $H^i(W_f, \theta_f)=0$ for $i>1$ in case f is diagonal. By upper semi-continuity, this is still true for f close enough to a diagonal map; this is always the case up to conjugation (cf. A.2). As in the proof of lemma 3.2, c), we see that $H^0(W_f, \theta_f)=H^1(W_f, \theta_f)$, because $\Sigma(-1)^i H^i(W_f, \theta_f)=0$ for f diagonal, hence also for f close to diagonal. The restriction of $1-f_*$ to $\mathscr{g}_\mu$ has kernel and cokernel in $\mathscr{g}_\mu$ of the same dimension. We have seen that the kernel of $1-f_*$ restricted to $\mathscr{g}_\mu^\perp$ is zero, so its cokernel in $\mathscr{g}_\mu^\perp$ must also be zero. This completes the proof of b).

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