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DECOMPOSITIONS OF MANIFOLDS INTO CODIMENSION ONE SUBMANIFOLDS

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1. Introduction

This paper investigates decompositions of boundaryless manifolds into closed, connected, codimension one submanifolds. It was inspired by recent work of V.T. Liem [22], with a related but distinct emphasis: typically the hypotheses in Liem's results require that the decomposition elements have the shape of a codimension one sphere. It can be construed as an outgrowth of the same spirit leading to the studies by D.S. Coram [11] and by Coram and P.F. Duvall [13] of decompositions of S^3 into simple closed curves. For example, in that spirit, because of the severe limitations on the decomposition elements within the source manifold M, one expects to discover a good deal of information about the decomposition space, M/G. We prove that M/G is a 1-manifold, possible with boundary.

A unifying theme is the question asking: which manifolds M admit such decompositions? The obvious examples that come to mind are the locally trivial fiber bundles over E^1 or S^1. Twisted line bundles over closed $(n-1)$-manifolds provide other examples; twisted I-bundles over two $(n-1)$-manifolds N_1 and N_2 can be glued together along their boundaries, when homeomorphic, to produce still others. L.S. Husch [20] has described yet another, exhibiting a decomposition G of an n-manifold M into closed $(n-1)$-manifolds, all embedded in M as bicollared subsets, such that the decomposition space is equivalent to S^1 but where the decomposition map π: $M \to M/G$ is not homotopic to a locally trivial fiber map; this manifold M is built from a non-trivial h-cobordism W having homeomorphic boundary components by identifying these two ends.

At first glance the decompositions under consideration may appear to provide partitions of M similar to those associated with foliations. (Recall that a k-dimensional foliation of an n-manifold M is a partition

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$\mathscr{F}$ of M into k-dimensional submanifolds such that each $x \in M$ has a neighborhood U equivalent to $E^n = E^k \times E^{n-k}$, where every level corresponding to $E^k \times \{\text{point}\}$ belongs to some element of $\mathscr{F}$.) Such appearances are rather ephemeral, for the differences are more pronounced than the similarities. First, upper semicontinuity necessitates that all the decomposition elements be compact; the more interesting foliations, for the most part, have non-compact leaves. Second, upper semicontinuity forces setwise convergence (in) to g_0 of a sequence of decomposition elements $\{g_i\}$ for which some sequence of points $x_i \in g_i$ converges to a point $x_0 \in g_0 \in G$; foliations have no comparable feature – in fact, individual leaves can be dense in M. Third, given a foliation $\mathscr{F}$ of M into k-manifolds, one knows that locally the leaves are aligned with the levels $E^k \times \{\text{point}\} \subset E^k \times E^{n-k}$ in some atlas of E^n-charts covering M; the elements of the decompositions studied here need not be locally flatly embedded in M, and even when they happen to be, they need not line up in any way parallel to other elements nearby.

The theorems here frequently supply structural data about the source manifold M involving the allied concepts of "standard structures" and 'standard decompositions". The *standard* (n-dimensional) *structures* are of two types: (i) twisted I-bundles W_T over (arbitrary) closed, connected $(n-1)$-manifolds, and (ii) near-product h-cobordisms W (the term "near-product" means that, if B_0 and B_1 denote the components of ∂W, then $W - B_i \approx B_j \times [0, 1)$ for $i \neq j$). Each type of standard structure W admits certain natural (non-unique) *standard decompositions* $\mathscr{G}$ into closed $(n-1)$-manifolds, each locally flatly embedded in W, such that $W/\mathscr{G}$ is topologically $[0, 1]$. An n-manifold M is said to have a *standard formation* $\{W_i\}$ provided it has a locally finite cover $\{W_i\}$, where each W_i is a standard structure and where $W_i \cap W_j \neq \emptyset$ $(i \neq j)$ implies $W_i \cap W_j$ is a boundary component of each. Given any standard formation $\{W_i\}$ for M, and given some standard decomposition $\mathscr{G}_i$ for each W_i, one obtains a *standard decomposition* $\mathscr{G}$ *for* M by setting $\mathscr{G} = \bigcup_i \mathscr{G}_i$.

The first Structure Theorem, Theorem 4.4, attests that if M admits a (upper semicontinuous) decomposition G into closed $(n-1)$-manifolds locally flatly embedded in M, then it also admits a standard formation $\{W_i\}$ and associated standard decomposition $\mathscr{G}$. Furthermore, in some sense $\mathscr{G}$ serves as a reasonable approximation to G.

However, when elements of G are allowed to be wildly embedded in M, then M can take on drastically different forms. For instance, Example 5.5 depicts a decomposition G of an n-manifold M $(n \geqslant 6)$ for which M/G is E^1 and the fundamental group of M at infinity is infinitely generated. In contrast to what occurs when decompositions elements are locally flat, this example has an infinite number of (pairwise) homotopically inequivalent elements.

Nevertheless, the decomposition G of Example 5.5 has the important property that each inclusion $g \to M$ induces homology isomorphisms.

This holds whenever M/G is E^1 (Lemma 6.2). It suggests the general structure theorem (Theorem 6.6): any manifold M admitting a decomposition into codimension one closed submanifolds has a formation $\{\tilde{W}_i\}$, where the $\tilde{W}_i$'s are compact manifolds with boundary that satisfy homological analogues of the homotopy restrictions imposed upon the objects of a standard formation.

As a consequence, we can specify (Corollary 7.4) the 3-manifolds M^3 that admit decompositions into closed 2-manifolds, by showing first that every such M^3 also has a standard formation and associated standard decomposition $\mathcal{G}$.

Finally, we investigate some special cases in which decomposition elements are known to possess some homotopy-theoretic compatibility, which is significant to view of the aforementioned Example 5.5. According to Corollary 8.4, if $\partial(M^n/G_A) = \phi$ and each $g \in G_A$ has Abelian fundamental group, then all the elements of G_A are homotopy equivalent; as a result, for $n \geqslant 5$, (whether or not $\partial(M^n/G_A) = \phi$) M^n has a standard decomposition $\mathcal{G}$ approximating G_A (Theorem 8.7). The same conclusion is valid if G is a decomposition of M^n ($n \geqslant 5$) such that each inclusion $g \to M$ induces an isomorphism of fundamental groups (Theorem 8.8).

2. Definitions, conventions, and preliminary results

By an *n-manifold*, we mean a separable metric space modelled on Euclidean n-space E^n. Accordingly, an n-manifold has no boundary. If we wish to consider the possibility of boundary, we shall say so explicitly, referring to an *n-manifold with boundary* Q (a separable metric space modelled on the n-cell I^n), and we denote its boundary as ∂Q.

One should always regard the manifolds, with or without boundary, discussed herein as connected spaces. In particular, the source manifold M (or, M^n) appearing densely throughout is presumed to be connected.

The symbol G is reserved to denote a decomposition of the source n-manifold M^n into closed (i.e., compact), connected, $(n-1)$-manifolds. Usually we presume G to be upper semicontinuous (abbreviated as usc), except in Section 3, where in some instances we assume only that G is a partition (but then prove it to be usc). We use the symbol π (and, sometimes, p) to denote the natural decomposition map of M to the decomposition space, M/G. Of course, G is usc iff π is a closed mapping.

By an (n-dimensional) *h-cobordism* W we mean a compact n-manifold W having two boundary components, B_0 and B_1, such that each inclusion $B_i \to W$ ($i = 0, 1$) is a homotopy equivalence. (Note that W is not required to be simply connected.) Such a W is called a *near-product* h-cobordism provided $W - B_0 \approx B_1 \times (0, 1]$ and $W - B_1 \approx B_0 \times [0, 1)$. (The symbol $\approx$ means "is homeomorphic to.") Each near-product

h-cobordism W has a standard decomposition $\mathcal{G}_W$ associated with any homeomorphism ψ of $B_i \times [0, 1)$ onto $W - B_j$ $(i, j \in \{0, 1\}, i \neq j)$; namely

$$\mathcal{G}_W = \{B_j\} \cup \{\psi(B_i \times \{t\}) \mid t \in [0,1)\}.$$

It should be clear that $W/\mathcal{G}_W \approx [0, 1]$.

The true purpose of the "near-product" phraseology is apparent only for $n = 3, 4$. For $n \geqslant 5$ the following holds:

PROPOSITION 2.1: *For $n \geqslant 5$, each n-dimensional h-cobordism W is a near-product.*

This was established by E.H. Connell [10].

For $n = 3$ this language helps circumvent potential ambiguities that would stem from failure of the 3-dimensional Poincaré Conjecture.

PROPOSITION 2.2. *If W is a 3-dimensional near-product h-cobordism, with boundary components B_0 and B_1, then $W \approx B_0 \times I$.*

PROOF: Let ψ: $B_1 \times (0, 1] \to \mathrm{W} - \mathrm{B}_0$ be a homeomorphism. Construct an embedding λ of $B_0 \times [0, 1]$ in W giving a collar on B_0 there. Then $\psi^{-1}(\lambda(B_0 \times \{1/2\}))$ is locally flat and incompressible (cf. [18, pp. 42–43]) in $B_1 \times (0, 1]$; according to [18, Appendix], the region bounded by $\psi^{-1}(\lambda(B_0 \times \{1/2\})]$ and $B_1 \times \{1\}$ is homeomorphic to $B_0 \times [1/2, 1]$. The result follows.

Let N denote a closed $(n-1)$-manifold, on which is defined a $2-1$ covering map p: $N \to X$. A *twisted I-bundle over X* is a space W_T equivalent to $(N \times I)/R$, where R represents the (usc) partition of $N \times I$ whose only nondegenerate sets are those of the form

$$\{\langle z_1, 1\rangle, \langle z_2, 1\rangle \in N \times I \mid p(z_1) = p(z_2)\}.$$

Then W_T has a *standard decomposition* $\mathcal{G}_W$ into the $(n-1)$-manifolds corresponding to the images of $N \times \{t\}$, $t \in I$. When $t < 1$, these are homeomorphic to N, but the image of $N \times \{1\}$ is homeomorphic to X. As before, $W_T/\mathcal{G}_W \approx [0, 1]$.

Repeating a definition given previously, we say that $\{W_i\}$ is a *standard formation for an n-manifold M* provided $\{W_i\}$ is a locally finite cover of M, each W_i is either a near-product h-cobordism or a twisted I-bundle over some $(n-1)$-manifold, and $W_i \cap W_j \neq \emptyset$ $(i \neq j)$ implies $W_i \cap W_j$ is a boundary component of each. Associated with any standard formation $\{W_i\}$ for M is a *standard decomposition* $\mathcal{G}$ of M given by $\mathcal{G} = \bigcup_i \mathcal{G}_i$, where $\mathcal{G}_i$ represents some standard decompostion of W_i.

Here is an obvious result:

PROPOSITION 2.3: *If $\{W_i\}$ is a standard formation for M such that each W_i is homeomorphic to $N_i \times [0, 1]$, then M is a locally trivial fiber bundle over S^1 or E^1 (according to whether or not M is compact) with fibers homeomorphic to N_i.*

Recall the example of Husch [20, p. 912] demonstrating the necessity, for our purposes, of considering near-product *h*-cobordisms that are not products.

3. Analysis of the decomposition space.

The aim in this section is to prove that M/G is a 1-manifold with boundary. Boundary must be tolerated because $\pi(g) \in \partial(M/G)$ iff g is 1-sided in M (equivalently, g separates no connected neighborhood of itself).

In addition, it is shown that, for compact manifolds M, the objects of study can be regarded as arbitrary partitions G of M into closed, connected codimension one manifolds; every such G is usc.

First, a technical result.

LEMMA 3.1: *Each closed, connected (n-1)-manifold g in M has a connected neighborhood V_g such that every closed $(n-1)$-manifold in $V_g - g$ separates V_g.*

PROOF: Select a connected neighborhood V_g of g that deformation retracts to g in M.

Suppose to the contrary that some closed $(n-1)$-manifold N in $V_g - g$ fails to separate V_g. Then an arc A in V_g meeting N at just one point and piercing N at that point can be completed to a simple closed curve J in V_g, with $J \cap N = A \cap N$. By construction of V_g, J is homotopic in M to a loop J' in g. This yields: $J \cap N = \{\text{point}\}$ while $J' \cap N = \phi$, violating the invariance of (mod 2) intersection number.

COROLLARY 3.2: *If G is a usc decomposition of M into closed, connected $(n-1)$-manifolds, then the set Q of all $g^* \in G$ that are 1-sided in M is locally finite.*

Next, the central result of this section.

THEOREM 3.3: *If G is a usc decomposition of M into closed, connected $(n-1)$-manifolds, then M/G is a 1-manifold with boundary.*

PROOF: Focus first on some $g_0 \in G$ having a connected neighborhood U_0 separated by g_0. Apply Lemma 3.1 to obtain a G-saturated connected neighborhood V_0, with $g_0 \subset V_0 \subset U_0$, such that every $g \in G$ in V_0 separates V_0. Then $\pi(V_0)$ is a non-compact, locally compact, connected, locally connected, separable metric space, each point of which separates $\pi(V_0)$ into two components. This implies $\pi(V_0) \approx E^1$.

Let Q denote the set of all $g^* \in G$ having no connected neighborhood separated by g^*. By Corollary 3.2, Q is locally finite.

In case $Q = \phi$, clearly $\pi(M) = M/G$ must be a 1-manifold.

In case $Q \neq \phi$, the set $M^Q = M - \cup\{g^* \mid g^* \in Q\}$ is a G-saturated, connected, open manifold, and $\pi(M^Q)$ must be topologically (0, 1), the only boundaryless non-compact 1-manifold. For distinct elements g^*, g^{**} of Q, $\pi(g^*)$ and $\pi(g^{**})$ compactify distinct ends of $\pi(M^Q)$. Hence, Q consists of either two or one elements, and M/G is homeomorphic to [0, 1] or [0, 1), according to whether M is compact or not.

THEOREM 3.4: *Suppose M is a closed n-manifold and G is a partition of M into closed, connected $(n-1)$-manifolds. Then G is usc.*

PROOF: There exists a finite subset F of G such that every $g \in G - F$ separates $\tilde{M} = M - \cup\{g \mid g \in F\}$. To see this, suppose otherwise. Choose successively $g_1, g_2, \ldots$ from G such that $M - (g_1 \cup g_2 \cup \cdots \cup g_k)$ is connected. The impossibility of such choices becomes clear for k larger than the rank (over Z_2) of $H_1(M; Z_2)$. For $i \in \{1, 2, \ldots, k\}$ one can construct a simple closed curve J_i in M that meets and pierces g_i at precisely one point and that intersects no other g_j selected. By the homological invariance of (mod 2) intersection number, no J_i is null-homologous (over Z_2) in M. The size of k ensures that some curve, say J_1, is homologous in M to some union of the other J_i's $(i > 1)$, another violation of intersection number invariance.

Fix $g_0 \in G - F$ and a neighborhood U_0 of g_0 in M. Determine a smaller neighborhood V_0 of g_0 with $g_0 \subset V_0 \subset Cl\, V_0 \subset U_0 \cap \tilde{M}$ and $Cl\, V_0$ compact. Let $\tilde{M}^+$ represent the closure (in $\tilde{M}$) of one of the components of $\tilde{M} - g_0$. Define G^+ as the set of all $g \in G$, $g \neq g_0$, in $\tilde{M}^+$. For $g \in G^+$, let X_g denote the closure of the component of $\tilde{M}^+ - g$ containing g_0. The collection $\{X_g \mid g \in G^+\}$ is totally ordered and its intersection is g_0. The compactness of $Fr\, V_0$ guarantees the existence of some $g^+ \in G^+$ for which $X_{g^+} \subset V_0$. If g^- is the analogous element of G in $M - \tilde{M}^+$, then every $g \in G$ in the component of $M - (g^+ \cup g^-)$ containing g_0 satisfies: $g \subset V_0 \subset U_0$, revealing that G satisfies the usual definition of upper semicontinuity at $g_0 \in G - F$.

One can argue, in similar fashion, that G is upper semicontinuous at each $g \in F$. The only extra technical matter to account for is that some $g \in F$ may separate no connected neighborhood of itself.

This argument establishes the stronger result below.

COROLLARY 3.5: *Suppose M is an n-manifold without boundary such that $H_1(M; Z_2)$ is finite, and suppose G is a partition of M into closed, connected $(n-1)$-manifolds. Then G is usc.*

COROLLARY 3.6: *If M is a closed, connected n-manifold such that $H_1(M; Z)$ is finite and if G is a partition of M into closed connected $(n-1)$-manifolds, then there exist exactly two elements of G that are 1-sided in M.*

PROOF: By Theorem 3.4, G is usc; thus, by Theorem 3.3, M/G is either $[0, 1]$ or S^1. It cannot be S^1 because then one could produce a loop J in M such that $\pi | J: J \to S^1$ has degree 1, by finding J that pierces some $g \in G$ exactly once, indicating that $\pi_*: H_1(M; Z) \to H_1(M/G; Z)$ is an epimorphism (monotonicity implies that automatically).

COROLLARY 3.7: *If M is a closed n-manifold such that $H_1(M; Z_2) = 0$, then M admits no partition into closed, connected $(n-1)$-manifolds.*

PROOF: By the Universal Coefficient Theorem, $H_1(M; Z)$ is finite. Moreover, since $H_1(M; Z_2) = 0$, each closed $(n-1)$-manifold in M separates. This contradicts the preceding Corollary.

We close by stating, without proof, improvements to the main results that can be established by similar arguments.

THEOREM 3.3′: *If K is a usc decomposition of an n-manifold M into compacta having the shapes of closed, connected, $(n-1)$-manifolds, then M/K is a 1-manifold with boundary.*

COROLLARY 3.5′: *If M is an n-manifold such that $H_1(M; Z_2)$ is finite and K is a partition of M into compacta having the shapes of closed, connected, $(n-1)$-manifolds, then K is usc.*

4. Decompositions with locally flat submanifolds

LEMMA 4.1: *Suppose $M/G \approx E^1$ and each $g \in G$ is locally flat in M. Then for each $g_0 \in G$, M is homeomorphic to $g_0 \times (-1, 1)$, with $g_0 \subset M$ corresponding to $g_0 \times \{0\}$.*

PROOF: Without loss of generality, $M/G = E^1$ and $\pi(g_0) = 0$. It suffices to prove that $\pi^{-1}([0, \infty)) = M^+$ is homeomorphic to $g_0 \times [0, 1)$.

Since g_0 is locally flat in M and separates M, it is bicollared [5]. Fix an open collar $C \approx g_0 \times [0, 1)$ on g_0 in M^+. Let X denote the set of all points t in $[0, \infty)$ for which there exists a homeomorphism h of M^+ onto itself having compact support, fixing (pointwise) some neighborhood of g_0, with the property that $h(C) \supset \pi^{-1}[0, t]$. Certainly X, which contains a neighborhood of 0, is nonempty and connected. It has no upper bound, because if x_0 were the least upper bound, one could stretch the collar C out very near $\pi^{-1}(x_0)$ and use the bicollar on $\pi^{-1}(x_0)$ to expand it still farther to encompass $\pi^{-1}[0, x_0]$. Hence, $X = [0, \infty)$.

By the techniques developed by M. Brown in [6], M^+ is homeomorphic to $g_0 \times [0, 1)$.

The proof of Lemma 4.1 also establishes:

COROLLARY 4.2: *Suppose each $g \in G$ is locally flat in M, $M/G \approx E^1$, and $A \subset M/G$ is an arc. Then $\pi^{-1}A$ is a near-product h-cobordism.*

Nothing about Corollary 4.2 should be construed as suggesting any compatibility between the layers $\pi^{-1}a$, $a \in A$, and the levels in a standard decomposition $\mathcal{G}$ resulting from the near-product features of $\pi^{-1}A$.

COROLLARY 4.3: *Suppose each $g \in G$ is locally flat in M and $\partial(M/G) = \emptyset$. Then all pairs $\{g_1, g_2\}$ of elements in G are homotopy equivalent.*

PROOF: In M/G choose an open set $U \approx E^1$ and an arc $A \subset U$ with $\partial A = \{\pi(g_1), \pi(g_2)\}$. Then apply Corollary 4.2.

At this point we are prepared to give the first Structure Theorem.

THEOREM 4.4: *Suppose M admits a usc decomposition G into closed, connected $(n-1)$-manifolds, each of which is locally flat in M. Then M has a standard formation $\{W_i\}$ and a standard decomposition $\mathcal{G}$, with decomposition map p: $M \to M/\mathcal{G}$. Moreover, for each continuous ε: $M/G \to (0, \infty)$, $\mathcal{G}$ can be obtained so that there exists a homeomorphism λ of $M/\mathcal{G}$ onto M/G satisfying*

$$\operatorname{dist}(\lambda p(x), \pi(x)) < \epsilon\pi(x), \quad \textit{for all } x \in M.$$

PROOF: In case $\partial(M/G) = \emptyset$, express M/G as the union of a collection of arcs $\{A_i\}$, where $A_i \cap A_j \neq \emptyset$ $(i \neq j)$ implies $A_i \cap A_j$ is an endpoint of each, and where the arcs are small enough that diam $A_i < \varepsilon(a_i)$ for all $a_i \in A_i$. Corollary 4.2 certifies that each $\pi^{-1}A_i$ is a near-product h-cobordism. Hence, $\{\pi^{-1}A_i\}$ is a standard formation on M, as required. The homeomorphism λ almost constructs itself; one can readily define it so $\lambda p(\pi^{-1}A_i) = A_i$ for all i.

Somewhat more interesting is the case $M/G = [0, 1)$. Set $g_0 = \pi^{-1}(0)$. Since g_0 does not locally separate M, it has a nice twisted I-bundle neighborhood B in M. There exists a map μ: $M \to M$ with compact support whose nondegenerate preimages are the fiber arcs corresponding to the I-"factor" in B. Let G' denote the decomposition of M induced by $\pi\mu$: $M \to [0, 1)$; that is,

$$G' = \{\mu^{-1}\pi^{-1}(t) \mid t \in [0,1)\}.$$

Note that $\mu^{-1}\pi^{-1}(0) = B$, and that $\mu^{-1}\pi^{-1}(t) \approx \pi^{-1}(t)$ for $t > 0$. Modify G' to form another decomposition G^* consisting of the natural levels, from a standard decomposition of B, together with the elements of $G' - \{B\}$. Clearly G^* is a decomposition of M with locally flat, closed, connected $(n-1)$-manifolds. Furthermore, if π^*: $M \to M/G^*$ denotes the new decomposition map, we can equate M/G^* with $[0, 1)$ and presume that π^* is sufficiently close to π. Now $M = B \cup \mathrm{Cl}(M - B)$ and $\mathrm{Cl}(M - B) = (\pi^*)^{-1}[t_0, 1)$, where $t_0 = \pi^*(\partial B) > 0$. Cover M/G^* by small arcs $\{A_0, A_1, \ldots\}$, as before, with $A_0 = [0, t_0]$. Then $\{(\pi^*)^{-1}A_i\}$ is a standard formation on M, as required.

Finally, the case $M/G \approx [0, 1]$ is just a double-barreled variation to the preceding one, obtained by modifying G in the same fashion over neighborhoods of both points in $\partial(M/G)$.

COROLLARY 4.5: *Suppose M admits a usc decomposition G into closed, connected, locally flat $(n-1)$-manifolds. Then*

(*i*) *if $M/G \approx [0, 1)$, M is homeomorphic to the interior of a twisted I-bundle over $\pi^{-1}(0)$;*
(*ii*) *if $M/G \approx [0, 1]$, M has a standard formation $\{W_1, W_2, W_3\}$ consisting of two twisted I-bundles and a near-product h-cobordism;*
(*iii*) *if $M/G \approx (0, 1)$, then $M \approx g_0 \times (-1, 1)$, $g_0 \in G$; and*
(*iv*) *if $M/G \approx S^1$, then M can be obtained from some near-product h-cobordism W by homeomorphically identifying the two components of ∂W.*

Since every 1-sphere in a 2-manifold is locally flat there, we have another proof of a result given by Liem [22, Corollary 1′].

COROLLARY 4.6: *Suppose G is a usc decomposition of a 2-manifold M^2 without boundary into 1-spheres. then*

(*i*) *if $M^2/G \approx [0, 1)$, M^2 is an open Mobius band;*
(*ii*) *if $M^2/G \approx [0, 1]$, M^2 is a Klein bottle;*
(*iii*) *if $M^2/G \approx E^1$, M^2 is an open annulus; and*
(*iv*) *if $M^2/G \approx S^1$, M^2 is a torus $(S^1 \times S^1)$ or a Klein bottle.*

Theorem 4.4 also brings us to the subject of approximate fibrations, introduced and analyzed by Coram and Duvall. See [12] for the definition.

COROLLARY 4.7: *Suppose each $g \in G$ is locally flat in M. Then π: $M \to M/G$ is an approximate fibration if and only if $\partial(M/G) = \emptyset$.*

PROOF: When $\partial(M/G) = \emptyset$, π is an approximate fibration because it can be approximated, in a reasonable sense, by the approximate fibrations p: $M \to M/\mathcal{G}$ associated with standard decompositions $\mathcal{G}$. When $\partial(M/G)$

$= \emptyset$, π fails to be an approximate fibration because the inclusion of g_0 into a twisted I-bundle B over g_0 is not homotopic in B to a map whose image misses g_0.

5. Examples

This section is devoted to displaying some of the bizarre possibilities for the decompositions under consideration. First, we give an explicit reminder that wildness can occur by incorporating a classical, wild 2-sphere in E^3 as a decomposition element in a decomposition of E^3-origin into 2-spheres, and then we produce more complex examples revealing unexpected changes, due to wildness, among the homotopy types of decomposition elements.

EXAMPLE 1: A decomposition G_1 of $M^3 = E^3$-origin into 2-spheres, one of which is wildly embedded in M^3.

Let S denote the 2-sphere in E^3 described by Fox and Artin [17, Example 3.2], which bounds a 3-cell C there. Coordinatize E^3 so that the origin is placed interior to C. One can fill up Int $S \cap M^3$ with 2-spheres that cobound with S 3-dimensional annuli in M^3; by construction, Ext $S \approx S^2 \times (1, \infty)$, and Ext S can be filled up with 2-spheres as well, but the regions bounded by S and any of the latter are not manifolds with boundary.

Controls on the Fox-Artin construction can be imposed by noting the following: there exists a decomposition of $S^2 \times [-1, 1]$ into 2-spheres, where both $S^2 \times \{\pm 1\}$ are members and some other member is (wildly) embedded in $S^2 \times (-1, 1) \subset E^3$ like the Fox-Artin sphere. With this, the Fox-Artin wildness can be inserted repeatedly and densely, to obtain:

EXAMPLE 2: A decomposition G_2 of $M^3 = E^3$-origin such that the image of those $g \in G_2$ wildly embedded in M^3 is dense in M^3/G_2.

By taking products with S^{n-3}, one can produce similar decompositions of n-manifolds.

Although examples like the preceding illustrate that wildness must be accounted for, they do not exhibit the full range of pathology that can be present. Among the decompositions encountered thus far, as long as the decomposition space has empty boundary, all the elements have been homotopy equivalent. The example below shows that this is not always the case.

Fix $n \geqslant 6$. Let F^{n-2} denote a nonsimply-connected homology $(n-2)$-cell (explicitly, a compact $(n-2)$-manifold having trivial integral homology), and let X denote a spine of F^{n-2} (that is, the complement in F^{n-2} of an open collar on F^{n-2}). Form $V = F^{n-2} \times [-1, 1]$. Let N^{n-1} be the homology $(n-1)$-sphere obtained by doubling V along its boundary, with $N^{n-1} = V_+ \cup V_-$ representing this double. Set $M' =$

$N^{n-1} \times E^1$. Name a Cantor set C in $(-1, 1)$. Let K denote the decomposition of M' into singletons and the sets $X \times \{c\} \times \{0\}$, $c \in C$, in $V_+ \times \{0\} \subset N^{n-1} \times \{0\}$. Finally, define M as the decomposition space M'/K, with decomposition map p: $M' \to M'/K = M$.

According to the Main Lemma of [16], M is an n-manifold. Obviously, it admits a decomposition G_3, where

$$G_3 = \{ p(N^{n-1} \times \{t\}) \mid t \in E^1 \}$$

For $t \neq 0$, $p(N^{n-1} \times \{t\})$ is naturally homeomorphic to the nonsimply-connected homology sphere N^{n-1}, while $p(N^{n-1} \times \{0\})$ is homeomorphic to S^{n-1} (cf. the proof of Proposition 1 in [16]). Thus, we have:

Example 3: For $n \geqslant 6$ there exists a decomposition G_3 of an n-manifold M into codimension 1 closed submanifolds such that $M/G_3 \approx E^1$ yet not all pairs of elements from G_3 are homotopy equivalent.

Example 4: For $n \geqslant 6$ there exists a decomposition G_4 of an n-manifold M into codimension 1 closed submanifolds such that $M/G_4 \approx [0, 1)$ and each $g \in G_4$ that separates M is locally flat, but M is not a twisted line bundle over any $(n-1)$-manifold.

For Example 4, start with a twisted line bundle M^* over some closed $(n-1)$-manifold S^*. Let S' denote the connected sum of S^* and N^{n-1}, the homology sphere of Example 3, and let M' denote the line bundle produced by piecing together relevant parts of M^* and $N \times E^1$. Let $\mathcal{G}$ denote a standard kind of decomposition on M' (where M' is considered as the interior of a twisted I-bundle over S').

Name a decomposition K_4 of M' into singletons and the sets $X \times \{c\}$ $(c \in C)$ in that copy of $V = F^{n-2} \times [-1, 1]$ contained in the trivial section S' of M'. Define M as M'/K_4 and p: $M' \to M$ as the decomposition map. As before, M is an n-manifold, and it admits a decomposition $G_4 = \{ p(g) \mid g \in \mathcal{G} \}$. Then the fundamental group of M is isomorphic to $\pi_1(S^*)$, while $\pi_1(M - p(S'))$ is isomorphic to the free product of $\pi_1(M^* - S^*)$ and two copies of $\pi_1(N^{n-1}) \cong \pi_1(F)$. Certainly S^* can be chosen with sufficiently simple $\pi_1(S^*)$ that it has no subgroup of index 2 isomorphic to $\pi_1(M^* - S^*) * \pi_1(F) * \pi_1(F)$ ($\pi_1(S^*) \cong Z_2$, for example). This prevents M from being an E^1-bundle over any manifold X.

Finally, to indicate how the pathology can proliferate, we give:

Example 5: For $n \geqslant 6$ there exists a decomposition G_5 of an n-manifold M into codimension 1, closed submanifolds such that $M/G_5 \approx E^1$ and the fundamental group of M at ∞ is infinitely generated.

Let D denote the complement of a standardly embedded $(n-1)$-cell in N^{n-1}, the homology sphere of Examples 3 and 4, arranged so D

contains one of the homology cells $V = F \times [-1, 1]$ of which N^{n-1} is the double. In S^{n-1} pick out a null sequence of pairwise disjoint $(n-1)$-cells $B_0, B_1, B_2, \ldots$ converging to some point and replace each B_i with a copy D_i of D, thereby forming an object Y (a non-ANR, topologized so as to be compact and metric). Let $Z = Y \times E^1$. Form a decomposition K_5 of Z into singletons and

$$\{(X \times \{c\})_i \times \{t\} \mid c \in C,\ t \leqslant i,\ i \in \{0, 1, 2, \ldots\}\},$$

where $(X \times \{c\})_i$ denotes the compactum in D_i corresponding to $X \times \{c\} \subset V \subset D$.

It can be proved that the decomposition space $M = Z/K_5$ is an n-manifold (by showing it is generalized n-manifold that satisfies the Disjoint Disk Property). The central reason behind its being a generalized manifold is that, for $t \leqslant i$, $p(D_i \times \{t\})$ is a contractible $(n-1)$-manifold bounded by an $(n-2)$-sphere (cf. [16]), forcing it to be an $(n-1)$-cell. As usual, p: $Z \to M$ denotes the decomposition map; as should now be expected, the decomposition G_5 is defined to be $\{p(Y \times \{t\}\}$.

The manifold M deformation retracts to $p(Y \times \{0\})$, which is homeomorphic to S^{n-1}. Moreover, for each integer $i \geqslant 1$, $p(Y \times [i, \infty))$ deformation retracts to $p(Y \times \{i\})$, the fundamental group of which has rank i times the rank of $[\pi_1(D) = \pi_1(F)]$. Explicitly, for $t \in (i, i+1]$, $p(Y \times \{t\})$ is topologically the connected sum of $i+1$ copies of N^{n-1}.

6. Decompositions into arbitrary submanifolds.

In view of the pathology manifested in Section 5, we now attempt to gain some understanding of the manifolds M that admit decompositions G into closed, codimension 1 submanifolds, possible wildly embedded. The goal is to establish a structure theorem that is the homology analogue to Theorem 4.4. In particular, we want to represent M as a union of some quasi-standard objects $\{\tilde{W}_i\}$, and the initial concern is to find objects $\tilde{W}_i$ that are n-manifolds with boundary.

Lemma 6.1: *The set $\mathcal{D}$ of all $x \in M/G$ such that $\pi^{-1}(x)$ is bicollared in M contains a dense G_δ subset of M/G.*

Proof: Without loss of generality, we assume $M/G = E^1$. According to [9], there exists (up to homeomorphism) just a countable collection $\{N_i \mid i = 1, 2, \ldots\}$ of distinct, closed, connected $(n-1)$-manifolds. Partition E^1 into subsets T_i $(i = 1, 2, \ldots)$ by the rule: $x \in T_i$ iff $\pi^{-1}(x)$ is homeomorphic to N_i. For each $x \in T_i$ name a specific homeomorphism λ_x of N_i onto $\pi^{-1}(x)$. Topologize the various sets $\{\lambda_x \mid x \in T_i\}$ by means of the sup-norm metric in M.

As suggested by Bryant [7, p. 478] or in the unpublished work of Bing [3], one can prove the following claim:

For each integer j there exists a countable subset C_j of T_j such that to each $x \in T_j - C_j$ there correspond two sequences $\{s(i)\}$ and $\{u(i)\}$ of real numbers in T_j such that $s(i) < x < u(i)$ for all i and each of the associated sequences $\{\lambda_{s(i)}\}$ and $\{\lambda_{u(i)}\}$ converges to λ_x (in the sup-norm metric).

Because then $\pi^{-1}(x)$ can be homeomorphically approximated in each component of $M - \pi^{-1}(x)$, a classical argument (cf. [4, Theorem 9]) shows that $M - \pi^{-1}(x)$ is 1-*LC* at each point of $\pi^{-1}(x)$ (for $x \in T_j - C_j$). As a result, $\pi^{-1}(x)$ is bicollared in M (this comes from [4, Theorem 6] in case $n = 3$, from [8] or [14] in case $n \geqslant 5$, and from [25, Theorem 2.7.1] in case $n = 4$.

Now it is transparent that the Lemma holds, since $M/G - \mathscr{D}$ is contained in the countable set $\cup C_j$.

Our intention is to find quasi-standard objects $\{\tilde{W}_i\}$ in M such that $\tilde{W}_i$ is homologically like an object in a standard formation $\{W_i\}$. The crux of that matter is given in the next lemma.

LEMMA 6.2: *Suppose $c \in E^1 = M/G$. Then, for any coefficient module Γ, the inclusion-induced i_*: $H_*(\pi^{-1}(c); \Gamma) \to H_*(M; \Gamma)$ is an isomorphism.*

PROOF: For simplicity we assume $c = 0$ and we suppress any further mention of Γ.

First, we show that i_* is $1-1$. Define

$$S = \{s \in [0, \infty) \mid H_*(\pi^{-1}(0)) \to H_*(\pi^{-1}[-s, s]) \text{ is } 1-1\}.$$

Certainly $S \neq \{0\}$, for $\pi^{-1}(0)$ is a deformation retract in M of some neighborhood of itself. Moreover, if $s_0 \in S$ and $0 \leqslant s' < s_0$, then obviously $s' \in S$. Consequently, S must be one of: $[0, \infty)$, $[0, d]$, or $[0, d)$ (for some $d > 0$). However, $S = [0, d]$ is impossible by the same reasoning indicating $S = \{0\}$ is impossible. In addition, because the inclusion-induced $H_*(\pi^{-1}(-d, d)) \to H_*(\pi^{-1}[-d, d])$ is an isomorphism, $S = [0, d)$ is also impossible. Therefore, $S = [0, \infty)$. The claim that $H_*(\pi^{-1}(0)) \to H_*(M)$ is $1-1$ follows.

Next we show that i_* is onto. We compare images by setting

$$S' = \{s \in [0, \infty) \mid \operatorname{im}[H_*(\pi^{-1}[-s, s]) \to H_*(M)] = \operatorname{im} i_*\},$$

and we argue, as above, that $S' = [0, \infty)$, which yields $\operatorname{im} i_* = H_*(M)$. Details are left to the reader.

COROLLARY 6.3: *Suppose $M/G = E^1$, $[a, b] \subset E^1$ and $c \in [a, b]$. Then, for every coefficient module Γ, the inclusion-induced α_*: $H_*(\pi^{-1}(c); \Gamma) \to H_*(\pi^{-1}[a, b]; \Gamma)$ is an isomorphism.*

PROOF: This can be viewed as a corollary to the proof of Lemma 6.2. Alternately, one can select an interval $(a', b') \supset [a, b]$ such that $\pi^{-1}(a', b')$ deformation retracts in M to $\pi^{-1}[a, b]$, with ρ denoting the ultimate retraction. The diagram

$$\begin{array}{ccc} & \overset{\cong}{\nearrow} & H_*(\pi^{-1}(a', b'); \Gamma) \\ & & \downarrow \rho_* \text{ (onto)} \\ H_*(\pi^{-1}(c); \Gamma) & \overset{\alpha_*}{\rightarrow} & H_*(\pi^{-1}[a, b]; \Gamma) \\ & & \downarrow \\ & \underset{\cong}{\searrow} & H_*(M; \Gamma) \end{array}$$

displays that α_* is an isomorphism.

COROLLARY 6.4: *Suppose* $M/G = [0, 1)$ *and* $0 < c \leqslant b$. *Then for every coefficient module* Γ, $H_*(\pi^{-1}(c); \Gamma) \rightarrow H_*(\pi^{-1}(0, b]; \Gamma)$ *is an isomorphism.*

The *quasi-standard structures* are of two types. Each is a compact n-manifold with boundary $\tilde{W}$ endowed with some (quasi-standard) usc decomposition $\tilde{\mathscr{G}}$ into closed, connected $(n-1)$-manifolds, where each component B of $\partial\tilde{W}$ belongs to $\mathscr{G}$. Moreover, $\partial\tilde{W}$ can have either 1 or 2 components. In case it has 2, then each $H_*(\tilde{g}; \Gamma) \rightarrow H_*(\tilde{W}; \Gamma)$ is required to be an isomorphism ($\tilde{g} \in \tilde{\mathscr{G}}$); in case it has just 1, then there is required to be one exceptional element $\tilde{g}_e \in \tilde{\mathscr{G}}$ such that $H_*(\tilde{g}; \Gamma) \rightarrow H_*(\tilde{W} - \tilde{g}_e; \Gamma)$ is an isomorphism for all $\tilde{g} \neq \tilde{g}_e \in \tilde{\mathscr{G}}$ (thus, $\tilde{g}_e$ represents the unique 1-sided element of $\tilde{\mathscr{G}}$ in Int $\tilde{W}$). We say that an n-manifold M has a *quasi-standard formation* $\{\tilde{W}_i\}$ provided it has a locally finite cover $\{\tilde{W}_i\}$ by quasi-standard structures (of dimension equal to that of M) such that $\tilde{W}_i \cap \tilde{W}_j \neq \emptyset$ implies $\tilde{W}_i \cap \tilde{W}_j$ is a boundary component of each. When M has a quasi-standard formation $\{\tilde{W}_i\}$, it is equipped thereby with an *associated quasi-standard* decomposition $\tilde{\mathscr{G}} = \cup \tilde{\mathscr{G}}_i$, where $\mathscr{G}_i$ denotes any such decomposition on $\tilde{W}_k$.

Implicit in this discussion is the trust that if $\tilde{W}$ (quasi-standard) has just one boundary component, it behaves homologically like a twisted I-bundle. To support that trust, the reader should check the (straightforward) result below.

LEMMA 6.5: *Let* $\tilde{\mathscr{G}}$ *be a quasi-standard decomposition of* $\tilde{W}$, *where* $\tilde{W}$ *has one boundary component, and let* $\tilde{g}_e$ *be the exceptional element of* $\tilde{\mathscr{G}}$. *Then* $H_*(\tilde{g}_e, \Gamma) \rightarrow H_*(W; \Gamma)$ *is an isomorphism but no* $H_1(\tilde{g}; Z_2) \rightarrow H_1(\tilde{W}; Z_2)$ *is an epimorphism* ($\tilde{g} \in \tilde{\mathscr{G}} - \{g_e\}$).

The second assertion can be proved with another linking number argument.

THEOREM 6.6: *Suppose M^n admits a usc decomposition G into closed, connected $(n-1)$-manifolds. Then M^n has a quasi-standard formation $\{\tilde{W}_i\}$. Moreover, given a continuous function ε: $M^n/G \to (0, \infty)$, the quasi-standard formation $\{\tilde{W}_i\}$ can be chosen so that, for any quasi-standard decomposition $\tilde{\mathscr{G}}$ associated with $\{\tilde{W}_i\}$, there exists a homeomorphism λ of $M^n/\tilde{\mathscr{G}}$ onto M^n/G satisfying*

$$\operatorname{dist}(\pi(x), \lambda p(x)) < \varepsilon\pi(x), \quad \textit{for each } x \in M^n.$$

PROOF: Choose a collection $\{A_i\}$ of (at least two) arcs filling up M^n/G so that any two intersect in, at most, a single boundary point of each, with the additional property that diam $A_i < \varepsilon(a_i)$ for all $a_i \in A_i$. Apply Lemma 6.1 to modify this collection slightly so that each $\pi^{-1}(A_i)$ is an n-manifold-with-boundary.
By Corollary 6.3 or 6.4, the natural decomposition on each $\pi^{-1}A_i$, obtained as a restriction of G, is itself quasi-standard. Thus, $\{\pi^{-1}A_i\}$ is a quasi-standard formation on M^n, and G represents one associated quasi-standard decomposition. In case $\tilde{\mathscr{G}}$ is any other, one can produce the required homeomorphism λ as in the proof of Theorem 4.4.

In the argument above, the quasi-standard formation $\{\pi^{-1}A_i\}$ imposes a quasi-standard decomposition $\mathscr{G}$ identical to the given decomposition G. That is not to say that $\mathscr{G} = G$ is always preferred. In the next result we show that $(n \neq 4)$ quasi-standard twisted I-bundles can be exchanged for genuine twisted I-bundles.

PROPOSITION 6.7: *Let $\tilde{W}$ denote a quasi-standard n-manifold with one boundary component B, where $n \neq 4$. Then there exist* (1) *a twisted I-bundle W_T and* (2) *a quasi-standard $\tilde{W}_2$ having* 2 *boundary components such that $\tilde{W} = W_T \cup \tilde{W}_2$, $W_T \cap \tilde{W}_2 = \partial W_T \subset \partial \tilde{W}_2$, and $\partial \tilde{W}_2 - \partial W_T = \partial \tilde{W}$.*

PROOF: By hypothesis, W has an associated quasi-standard decomposition $\tilde{\mathscr{G}}$ with exceptional element $\tilde{g}_e$. To get this argument underway, one should substantiate the claim that the subgroup of $\pi_1(\tilde{W})$ generated by loops in $\tilde{W} - \tilde{g}_e$ has index 2 in $\pi_1(\tilde{W})$. Let θ: $\tilde{V} \to \tilde{W}$ denote the corresponding $2-1$ covering map. Then $\tilde{V}$ has 2 boundary components; in fact, $\theta^{-1}(\tilde{g})$ has 2 components for all $\tilde{g} \in \tilde{\mathscr{G}} - \{\tilde{g}_e\}$ but $\theta^{-1}(\tilde{g}_e)$ is a connected manifold N_e. Furthermore, N_e separates $\tilde{V}$ into two components having compact closures $\tilde{V}_+$ and $\tilde{V}_-$, and every $\theta^{-1}(\tilde{g})$ meets both $\tilde{V}_+$ and $\tilde{V}_-$.

Since $n \neq 4$, we can apply the crumpled cube reembedding theorem (cf. [19] or [23] in case $n = 3$, [15] in case $n \geqslant 5$) to obtain disjoint embeddings $\psi_\pm$ of $\tilde{V}_\pm$ in $\tilde{V}$ such that $\psi_\pm | \tilde{V}_\pm \cap \partial\tilde{V} =$ identity and the closure of $\tilde{V} - [\psi_+(\tilde{V}_+) \cup \psi_-(\tilde{V}_-)]$ is homeomorphic to $N_e \times [-1, 1]$

(with $z \times 1$ corresponding to $\psi_+(z)$ for all $z \in N_e$). This correspondence does not respect the given embedding of N_e in $\tilde{V}$, which in all likelihood is wild.

Form a new manifold with boundary $\tilde{U}$ from $(N_e \times [0, 1]) \cup \psi_+(V_+)$ by identifying those points $z_1 \times 0$, $z_2 \times 0$ ($z_1, z_2 \in N_e$) for which $\theta(z_1) = \theta(z_2)$. Note that $\tilde{U}$ can be expressed as the union of a twisted I-bundle $\tilde{U}_T$, the image of $N_e \times [0, 1/2]$ in $\tilde{U}$, and a quasi-standard $\tilde{U}_2$ with 2 boundary components, where $\tilde{U}_2 = \tilde{U} - \text{Int } \tilde{U}_T$. The required decomposition of $\tilde{U}_2$ consists of the various $\psi_+(\theta^{-1}(\tilde{g}) \cap \tilde{V}_+)$ and the levels $N_e \times \{s\}$, $s \in [1/2, 1]$. (Remark: although the image of $N_e \times [0, 1]$ in $\tilde{U}$ gives what may seem to be a more natural twisted I-bundle, the closure of its complement has the undesirable feature of not necessarily being a manifold with boundary.)

Having found structures of the required kind in $\tilde{U}$, we complete the proof by showing $\tilde{U}$ and $\tilde{W}$ to be homeomorphic. Consider the decomposition K of $\tilde{U}$ into singletons and all the image arcs obtained from $\{z_1, z_2\} \times [0, 1] \subset \tilde{V}$, where $\theta(z_1) = \theta(z_2)$ and $z_1 \neq z_2 \in N_e$. The decomposition space $\tilde{U}/K$ is naturally equivalent to $\theta(\tilde{V}_+) = \tilde{W}$; in other words, the decomposition map essentially provides a cell-like map of $\tilde{U}$ onto $\tilde{W}$ (all of whose nondegenerate preimages are found in Int $\tilde{U}$). The Armentrout-Siebenmann Cell-Like Approximation Theorem [2, Theorem 1] [26, Theorem A] attests that $\tilde{U}$ and $\tilde{W}$ are homeomorphic.

ADDENDUM TO THEOREM 6.6: *In case $n \neq 4$, M^n has such a quasi-standard formation $\{\tilde{W}_i\}$ in which each $\tilde{W}_i$ with connected boundary is a twisted I-bundle, and the associated quasi-standard decomposition for such $\tilde{W}_i$ is standard.*

7. Decompositions of 3-manifolds

In case $n = 3$ and $M/G \approx E^1$, Lemma 6.2 has the important consequence that all $g \in G$ are homeomorphic. Consequently, none of the pathology appearing in Examples 5.3 through 5.5 can be present when $n = 3$. What impact is produced by the remaining possible wildness? We show that the impact is negligible, by proving that the 3-manifolds admitting arbitrary decompositions G into 2-manifolds also admit (approximating) standard decompositions $\mathcal{G}$, into locally flat 2-manifolds.

LEMMA 7.1: *Suppose G is a usc decomposition of M^3 into closed, connected 2-manifolds such that $M^3/G = E^1$, A is an arc in M/G such that $\pi^{-1}A$ is a 3-manifold with boundary and is locally flat in M, and $a \in \partial A$. Then $\pi^{-1}A \approx \pi^{-1}(a) \times [0, 1]$.*

PROOF: According to Theorem 2 of [24], each $t \in A$ has a neighborhood U_t in M/G where $\pi^{-1}U_t$ lies interior to 3-manifold Q_t locally flatly embedded in M, Q_t is obtained from a space $\approx \pi^{-1}(t)\times[-1, 1]$ by adding (in M) a finite number of pairwise disjoint 1-handles, and the obvious homeomorphism $\pi^{-1}(t)\to\pi^{-1}(t)\times\{0\}$ is homotopic in M to the inclusion. It is important to observe that $\pi^{-1}(t)\times[-1, 1]\subset M$ separates M (since $\pi^{-1}(t)$ does); thus, each 1-handle attached in forming Q_t meets either $\pi^{-1}(t)\times\{1\}$ or $\pi^{-1}(t)\times\{-1\}$, but not both.

Subdivide A into a finite chain of arcs $\{A_j\}$ such that $A_j\cap A_{j+1}$ is an endpoint of each, $\{\pi^{-1}A_j\}$ refines $\{\pi^{-1}U_t \mid t\in A\}$, and each $\pi^{-1}A_j$ is a (locally flat) 3-manifold with boundary (cf. Lemma 6.1). The proof will be completed by showing that each $\pi^{-1}A_j$ is topologically $\pi^{-1}(a)\times[0, 1]$, where $a\in\partial A_j$.

Let $\partial A_j=\{a, a'\}$ and $U_t\subset M/G$ the promised neighborhood containing A_j. The handle attaching properties of Q_t make it possible to embed Q_t in $\pi^{-1}(t)\times[-2, 2]=Q_t'$, with the portion $\pi^{-1}(t)\times[-1, 1]\subset Q_t$ included naturally in Q_t'.

We argue that both $\pi^{-1}(a)$ and $\pi^{-1}(a')$ are incompressible in Q_t'. If not, Dehn's Lemma and standard 3-manifold arguments give a 2-cell D, locally flat in Q_t', with $\partial D\subset\pi^{-1}(a)$, say, and $D\cap\pi^{-1}(a)=\partial D$. Since $\pi^{-1}(t)$ separates the two ends of M, it must separate the two components of ∂Q_t in Q_t, which implies it does the same for the two components of $\partial Q_t'$ in Q_t'. After performing a surgery on $\pi^{-1}(t)$ along D, we find that the resulting (possibly disconnected) 2-manifold F still separates the components of $\partial Q_t'$ in Q_t. Then we have

$$\chi(F)>\chi(\pi^{-1}(a))=\chi(\pi^{-1}(t)),$$

where χ denotes the Euler characteristic, and where the equality stems from Lemma 6.2. By Proposition 3.2 of [21], this is impossible.

Hence, by [18, pp. 91–92] or by [21, Proposition 3.1] $\pi^{-1}(a)$ and $\pi^{-1}(a')$ can be rearranged in Q_t' to be parallel to the boundary. This indicates that the region W in Q_t' bounded by them is topologically $\pi^{-1}(a)\times[0, 1]$. Clearly, $W\subset Q_t\subset M$.

THEOREM 7.2: *If the 3-manifold M^3 admits a usc decomposition G into closed, connected 2-manifolds, then it admits a standard decomposition $\mathcal{G}$ of the same sort. Moreover, given any continuous ε: $M^3/G\to(0, \infty)$, there exists a homeomorphism λ: $M^3/\mathcal{G}\to M^3/G$ such that*

$$\operatorname{dist}(\pi(x), \lambda p(x))<\varepsilon\pi(x) \quad \text{for all } x\in M^3$$

(*where p: $M^3\to M^3/\mathcal{G}$ denotes the decomposition map*).

PROOF: By the Addendum to Theorem 6.6, M^3 has a quasi-standard formation $\{\tilde{W}_j\}$ consisting of twisted I-bundles and quasi-standard objects $\tilde{W}_j$ having 2 boundary components. Lemma 7.1 establishes that each of the latter is a standard product.

Corollary 7.3: *If G is a usc decomposition of M^3 into closed, connected 2-manifolds and $\partial(M^3/G)=\emptyset$, then π: $M^3 \to M^3/G$ is an approximate fibration (see also Corollary 4.7).*

From Theorem 7.2 we obtain a strengthening of Corollary 4.5 for $n=3$, somewhat comparable to what Corollary 4.6 does in case $n=2$.

Corollary 7.4: Suppose M^3 admits a usc decomposition into closed, connected 2-manifolds. Then:

(i) if $M^3/G \approx [0, 1)$, M^3 is a twisted line bundle over some $g_0 \in G$;
(ii) If $M^3/G \approx [0, 1]$, M^3 is obtained by homeomorphically identifying the boundaries of twisted I-bundles over some pair $\{g_0, g_1\} \subset G$;
(iii) if $M^3/G \approx (0, 1)$, $M^3 \approx g_0 \times (-1, 1)$ for each $g_0 \in G$;
(iv) if $M^3/G \approx S^1$, M^3 is a locally trivial fiber bundle over S^1, with fiber any $g_0 \in G$.

8. Decompositions into homotopy equivalent submanifolds

Having considered decompositions into rather arbitrary codimension one submanifolds, here we specialize to study certain decompositions into homotopy equivalent submanifolds. The central result reveals that if $\pi_1(g) \to \pi_1(M^n)$ is an isomorphism for all $g \in G$, then each $g \to M^n$ is a homotopy equivalence and (for $n \geqslant 5$) $M^n \approx g \times (-1, 1)$. This has straightforward applications (primarily for $\partial(M^n/G)=\emptyset$) in case each $g \in G$ has Abelian fundamental group.

Due to the frequent reference to homotopy groups, we shall attempt to minimize the potential for notational confusion in this section by denoting the decomposition map $M \to M/G$ as p instead of π.

Proposition 8.1: *Suppose G is a usc decomposition of M into closed, connected $(n-1)$-manifolds such that, for each $g \in G$, the inclusion-induced i_*: $\pi_1(g) \to \pi_1(M)$ is an isomorphism. Then $M/G \approx E^1$ and i: $g \to M$ is a homotopy equivalence.*

Proof: Let θ: $\tilde{M} \to M$ denote the universal covering. The collection $\tilde{G} = \{\theta^{-1}(g) \mid g \in G\}$ forms a partition of $\tilde{M}$ into closed subsets that are $(n-1)$-manifolds. Since $\pi_1(g) \to \pi_1(M)$ is an isomorphism, each $\tilde{g} = \theta^{-1}(g)$ is connected and $\theta \mid \tilde{g}$: $\tilde{g} \to g$ is the universal covering. Although $\tilde{G}$ is not necessarily usc, due to the likely noncompactness of $\tilde{g}$, $p\theta$: $\tilde{M} \to M/G$ is equivalent to the quotient map $\tilde{M} \to \tilde{M}/\tilde{G}$.

Fix some $\tilde{g} \in \tilde{G}$. Lifting properties of θ gives rise to a $\tilde{G}$-saturated neighborhood $\tilde{U}$ of $\tilde{g}$ that deformation retracts to $\tilde{g}$ in $\tilde{M}$ (under a deformation fixing all points of $\tilde{g}$). Thus, $\tilde{g}$, being simply connected, separates such a $\tilde{U}$, which implies that $\theta(\tilde{g})$ separates $\theta(\tilde{U})$. As a result, $\partial(M/G)=\emptyset$. It is impossible for M/G to be S^1, for $p_*: \pi_1(M)\to\pi_1(S^1)$ is an epimorphism but $(p\,|\,g)_*: \pi_1(g)\to\pi_1(S^1)$ is always trivial. Consequently, $M/G\approx E^1$.

The crucial claim is that the inclusion-induced $H_*(\tilde{g};\ Z)\to H_*(\tilde{M};\ Z)$ is an isomorphism. Armed with the observation that $\tilde{U}$ deforms to $\tilde{g}$, we establish the claim by simply repeating the argument given for Lemma 6.2.

According to the Whitehead Theorem [28, p. 399], $\pi_k(\tilde{g})\to\pi_k(\tilde{M})$ is an isomorphism for all $k\geqslant 2$. Since θ induces isomorphisms of higher homotopy groups $i_*: \pi_k(g)\to\pi_k(M)$ is an isomorphism for all $k\geqslant 2$ (as well as for $k=1$). Hence, i: $g\to M$ is a homotopy equivalence.

Corollary 8.2: *Under the hypothesis of Proposition* 8.1, *all pairs of elements of G are homotopy equivalent.*

Proposition 8.3: *If G is a usc decomposition of M^n into closed, connected $(n-1)$-manifolds, each with Abelian fundamental group, and if $M/G\approx E^1$, then i: $g\to M$ is a homotopy equivalence, for all $g\in G$.*

Proof: Each $g\in G$ has a (connected) neighborhood U_g that deforms to g in M, implying that the images of $[\pi_1(g,\ x)\to\pi_1(M,\ x)]$ and $[\pi_1(U_g,\ x)\to\pi_1(M,\ x)]$ coincide. Let ρ: $\pi_1(M)\to H_1(M;\ Z)$ denote the Abelianization. It follows from the hypothesis and Lemma 6.2 that $\rho\,|\,i_*\pi_1(g,\ x)$ is an isomorphism, so the same is true of $\rho\,|$image $\pi_1(U_g,\ x)$, independent of the choice of $g\in G$.

Express M/G as the union of a chain of arcs $\{A_j\,|\,j\in\mathbb{Z}\}$, in the usual way, chosen small enough that $\{p^{-1}A_j\}$ refines $\{U_g\,|\,g\in G\}$. Piecing the various $p^{-1}A_j$ together, we can invoke the Siefert-van Kampen Theorem repeatedly to prove that, on the image of

$$\pi_1\Big(\cup\big\{p^{-1}A_j\,|\,-k\leqslant j\leqslant k\big\}\Big)\to\pi_1(M),$$

ρ is an isomorphism. Therefore, $\pi_1(M)\cong H_1(M;\ Z)$. Furthermore, each $\pi_1(g)\to\pi_1(M)$ must be an isomorphism, since each group is its own abelization, and Lemma 6.2 applies once again. Now Proposition 8.1 certifies that $g\to M$ is a homotopy equivalence.

Corollary 8.4: *If G is a usc decomposition of M^n into closed, connected $(n$-$1)$-manifolds, each with Abelian fundamental group, and if $\partial(M/G)=\emptyset$, then all pairs of elements of G are homotopy equivalent.*

Corollary 8.5: *If G is a usc decomposition of M^n into simply-connected, closed $(n-1)$-manifolds, then all pairs of elements of G are homotopy equivalent.*

LEMMA 8.6: *Suppose $g \to M$ is a homotopy equivalence, for each $g \in G$; $[a, b] \subset E^1 = M/G$; and $c \in [a, b]$. Then $p^{-1}(c) \to p^{-1}[a, b]$ is a homotopy equivalence. Furthermore, $p^{-1}[a, b]$ is an h-cobordism whenever it is an n-manifold.*

PROOF: Since M retracts to $p^{-1}[a, b]$ under some map r, the homotopy h_t: $M \to M$ between Id and a retraction $M \to p^{-1}(c)$ leads to a homotopy $r\, h_t | p^{-1}[a, b]$ of $p^{-1}[a, b]$ to itself revealing the required equivalence.

THEOREM 8.7: *Suppose M^n admits a usc decomposition G into closed, connected $(n-1)$-manifolds, $n \geqslant 5$, each with Abelian fundamental group. Then M^n has a standard formation $\{W_i\}$. Moreover, the associated standard decomposition $\mathscr{G}$ of M^n can be obtained as an approximation to G, in the sense of Theorem 4.4.*

PROOF: By the Addendum to Theorem 6.6, M^n admits a quasi-standard formation $\{\tilde{W}_j\}$ into twisted I-bundles and quasi-standard objects $\tilde{W}_j$ with two boundary components. It follows from the construction there that each $\tilde{g}$ in the associated quasi-standard decomposition for $\tilde{W}_j$ has Abelian fundamental group. Proposition 8.3 and Lemma 8.6 reveal that such a $\tilde{W}_j$ is an h-cobordism; by [10], $\tilde{W}_j$ is a near-product h-cobordism. This can be done so that the associated standard $\mathscr{G}$ approximates G.

THEOREM 8.8: *Suppose M^n admits a usc decomposition into closed, connected $(n-1)$-manifolds, $n \geqslant 5$, such that i_*: $\pi_1(g) \to \pi_1(M^n)$ is an isomorphism for each $g \in G$. Then, for every $g_0 \in G$, $M^n \approx g_0 \times (-1, 1)$.*

PROOF: By Proposition 8.1, $M^n/G \approx E^1$.

First, assume g_0 has a bicollared embedding. Then M/G can be expressed as the union of a countable collection $\{A_j\}$ of arcs, in the usual way, where each $p^{-1}A_j$ is a compact manifold with boundary (cf. Lemma 6.1) and where $g_0 \subset \partial p^{-1}A_0$. From Lemma 8.6 and [10], each $p^{-1}A_j$ is a near-product h-cobordism, so M^n has a standard decomposition $\mathscr{G}$ associated with this standard formation $\{p^{-1}A_j\}$, and $g_0 \in \mathscr{G}$. Corollary 4.5 indicates that $M^n \approx g_0 \times (-1, 1)$.

In general, however, g_0 may not be bicollared. Select some $g_1 \in G$ that does have a bicollar (recall Lemma 6.1 again). Use the Locally Flat Approximation Theorem of Ancel and Cannon [1, p. 63] to find a bicollared embedding λ of g_0 in $M^n - g_1$ that is homotopic in $M^n - g_1$ to the inclusion. Set W equal to the compact manifold bounded by $\lambda(g_0)$ and g_1. Argue, as before, that W is an h-cobordism, which necessarily is a near-product. Therefore, Int W is homeomorphic to both $g_0 \times (-1, 1)$

and $g_1 \times (-1, 1)$. From what was shown in the preceding paragraph,

$$M^n \approx g_1 \times (-1, 1) \approx g_0 \times (-1, 1).$$

From Proposition 8.3, we obtain:

COROLLARY 8.9: *If M^n $(n \geqslant 5)$ admits a usc decomposition G into closed, connected $(n-1)$-manifolds, each with Abelian fundamental group, and if $M^n/G \approx E^1$, then, for each $g_0 \in G$, $M^n \approx g_0 \times (-1, 1)$.*

The argument given for Theorem 8.8 actually establishes:

THEOREM 8.10: *Under the hypothesis of Theorem 8.8, for all pairs $\{g_1, g_2\} \subset G$, there exists a near-product h-cobordism W, locally flatly embedded in M^n, with ∂W homeomorphic to $g_1 \cup g_2$.*

THEOREM 8.11. *If M^n admits a usc decomposition into simply connected, closed $(n-1)$-manifolds, $n \geqslant 5$, then M^n has a standard formation $\{W_i\}$ consisting solely of simply connected (near-product) h-cobordisms. Thus, for each $g_0 \in G$,*
(1) M^n noncompact implies M^n is homeomorphic to $g_0 \times (-1, 1)$, and
(2) M^n compact implies M^n is homeomorphic to a locally trivial fiber bundle over S^1 with fiber g_0.

PROOF: The only part of this result not subsumed by earlier results is statement (2). In that situation, we know $M^n/G \approx S^1$.

Assume g_0 is bicollared in M^n. By the same procedures used in proving Theorem 4.4, we can modify G slightly on one side (locally) of g_0 so as to express M/G as the union of two arcs A and B with
(i) $p(g_0) \in \partial A = A \cap B$,
(ii) $p^{-1}(B)$ homeomorphic to $g_0 \times [-1, 0]$ and
(iii) $p^{-1}A$ a near-product h-cobordism.

The simply connected case of L.C. Siebenmann's Topological s-cobordism [26] Theorem implies that $p^{-1}A$ is also homeomorphic to $g_0 \times I$, for $n \geqslant 6$; F. Quinn [25] has announced the same result for $n = 5$ (simply-connected case).

In case g_0 is not bicollared, choose two $g_1, g_2 \in G$ that are bicollared. Find a bicollared embedding λ of g_0 in $M - (g_1 \cup g_2)$ that is homotopic there to the inclusion [1]. Then the closure W of that component of $M - g_2$ bounded by $\lambda(g_0)$ and g_1 is a (near-product) h-cobordism. Similarly, the closure of $M - W$ is a simply-connected h-cobordism. Consequently, M admits a decomposition G^* consisting of $\lambda(g_0)$, g_1 and other manifolds homeomorphic to g_0. Since $\lambda(g_0) \in G^*$ is bicollared, the result follows from the previous case.

PROPOSITION 8.12: *If G is a usc decomposition of the n-manifold M^n, $n \geqslant 5$, into simply connected, closed $(n-1)$-manifolds, then all pairs of elements of G are homeomorphic.*

This follows because, by Theorem 8.9, there is a simply connected h-cobordism between embedded copies of any two elements of G. It must be a product [25] [26].

Using the full strength of the Topological *s*-cobordism Theorem [26], we can improve some of the results of this section as follows.

PROPOSITION 8.13: *Suppose G is a usc decomposition of the n-manifold M^n, $n \geqslant 6$, into closed, connected $(n-1)$-manifolds such that each $\pi_1(g) \to \pi_1(M^n)$ is an isomorphism and the Whitehead group of $\pi_1(M^n)$ is trivial. Then all pairs of elements of G are homeomorphic.*

THEOREM 8.14: *Suppose M^n is a closed n-manifold, $n \geqslant 6$, that admits a usc decomposition G into closed, connected $(n-1)$-manifolds, each with Abelian fundamental group having trivial Whitehead group, and suppose $M/G \approx S^1$. Then, for each $g_0 \in G$, M^n is a locally trivial fiber bundle over S^1 with fiber homeomorphic to g_0.*

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