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GEOMETRIC INVARIANT THEORY FOR GENERAL ALGEBRAIC GROUPS

Amassa Fauntleroy

The solutions of many of the moduli problems which occur in algebraic geometry involve the construction of orbit spaces for the action of an algebraic group on a suitable variety. In most cases the groups involved are reductive. Mumford in [10] worked out the theory of reductive group actions, particularly those arising from linear representations in the affine cone over a projective space, to a point sufficient for applications to moduli problems. With a view toward similar applications of more general algebraic groups acting on quasi-projective varieties, this paper studies the problem of constructing quotients of varieties under the action of arbitrary connected linear groups.

Our approach is to first treat the case of unipotent groups and then to reduce to the cases in which Mumford's geometric invariant theory applies. Section 1 of this paper discusses the case of unipotent groups acting on quasi-affine varieties. We give necessary conditions for the existence of a quotient and give a complete description of proper actions with a quotient.

In section 2 we give a local criterion for the action of a connected unipotent group G on a quasi-affine variety X to be properly stable (c.f. 2.1) under the assumption that the ring of global sections $\Gamma(X, O_X)$ is factorial. We define stable points $X^s(G)$ of X and in (2.2) we give an inductive procedure for constructing the quotient of $X^s(G)$ by G when the action on X is proper. Finally, in sections 3 and 4 we apply these results to the problem of constructing quotients by general connected linear groups.

Most of the results of this paper are valid in any characteristic. However, we have assumed that the ground field has characteristic zero in order to avoid too many p-pathologies. In this paper all ground fields-generally called k - are algebraically closed. The word 'scheme' here means reduced, irreducible algebraic scheme over k. A variety is a separated scheme. Points of a scheme are assumed closed unless otherwise stated. All algebraic groups are affine and we identify $\Gamma(X, O_X)$ with the subring of $k(X)$ consisting of everywhere defined rational functions on the scheme X.

0. Generalities on group actions

This section gives a brief summary of the results on actions of algebraic groups on varieties which will be used in the following sections. They are given here essentially for convenience of reference.

0.1. Let G be an algebraic group acting rationally on a scheme X. A pair (Y, q) consisting of a scheme Y and a morphism $q: X \to Y$ is a geometric quotient of X by G denoted X/G if the following conditions hold:

(i) q is open and surjective

(ii) $q_*(O_X)^G = O_Y$

(iii) q is an orbit map; i.e., the fiber of closed points are orbits.

The action of G on X is said to be locally trivial if each point $x \in X$ is contained in a G-stable open subset U of X which is equivariantly isomorphic to $G \times S$ for some scheme S.

THEOREM 0.1 (*Generic Quotient Theorem* [12]): *Let G act on an algebraic scheme X. Then there exists a G-stable open subset U of X such that $Y = U/G$ exists and Y is a quasi-projective variety.*

In general the determination of the open set described in 0.1 is a non-trivial task. However, for reductive algebraic groups somewhat more can be said. Let G be reductive and V a finite dimensional rational G-module. Let $P(V)$ denote the associated projective space consisting of lines through the origin in V and let R be the ring of polynomial functions on V (with respect to some basis). A point $v \in P(V)$ is called semi-stable if there exists an invariant nonconstant homogeneous element $f \in R$ with $f(v) \neq 0$. A point $v \in P(V)$ is stable if it is semi-stable and the orbit $G \cdot v$ is closed.

THEOREM 0.2 (*Mumford*; [10: 1.10]): *Let G and V be as above and let X be the set of stable points of $P(V)$. Then X is open and $Y = X/G$ exists.*

The only other result of a general nature aside from Mumford's theorem is a result due to Seshadri which we now describe. If G is a connected algebraic group acting on a scheme X, then the action is said to be proper if the map $G \times X \to X \times X$ given by $(g, x) \to (gx, x)$ is proper.

THEOREM 0.3 (*Seshardi* [16]): *Let G be a connected algebraic group acting on a variety X such that for each point x in X the isotropy subgroup of G at x is finite. Then there exists a morphism $p: Z \to X$ such that*

(i) *Z is a normal variety, G operates on Z and p is a finite surjective G-morphism*

(ii) *G operates freely on Z, the geometric quotient $W = Z/G$ exists and the quotient map $q: Z \to W$ is a locally trivial principal fibre space with structure group G.*

(iii) *If the action of G on X is proper then the action of G on Z is proper and W is separated.*

(iv) *$k(Z)$ is a finite normal extension of $k(X)$ and the canonical action of $\mathrm{Aut}(k(Z)/k(X))$ on Z commutes with the action of G.*

In the situation described in 0.3 we will call (Z, W, p) a *Seshardi cover* of X.

When X is a variety on which the connected algebraic group G operates then a quotient map can be characterized in a slightly different way. If $q: X \to Y$ is a surjective orbit map and Y is a normal variety, then Y is the geometric quotient of X mod G. This is the content of [1; 6.6]. The fact that q is open follows from Chevalley's result [1; AG 0, 10.3]. This condition is often the easiest to check.

Recall that a scheme Q is called a *categorical quotient* of X by G if there is a morphism $q: X \to Q$ such that whenever $f: X \to S$ is a morphism constant on G orbits, then there is a unique morphism $h: Q \to S$ with $f = h \circ q$. Clearly, a geometric quotient is a categorical quotient. If G is reductive and acts linearly on $\mathbb{P}^n$, that is via a linear action on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$ then $x \in \mathbb{P}^n$ is semi-stable provided there is a G-invariant section $s \in H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(m))$ for some $m > 0$ with $s(x) \neq 0$. The set of semi-stable points U of $\mathbb{P}^n$ is open and a projective categorical quotient of U by G exists. [11; Theorem 3.21.]

There is a similar result for unipotent groups. Let H be a connected unipotent group acting on a normal quasi-affine variety V. We assume that the stability group of each point of V is finite. Let $B = \Gamma(V, O_V)$ and $A = B^H$. There is a canonical morphism $c: V \to \mathrm{Spec}\, A$. A point $v \in V$ will be called *semi-stable* if $\dim c^{-1}(c(v)) = \dim H$. Let V^{ss} denote the set of semi-stable points of V.

THEOREM 0.4 [4; *Proposition* 6]: *The set V^{ss} is open and H-stable in V. There exists a quasi-affine variety Q and a morphism $q: V^{\mathrm{ss}} \to Q$ constant on H-orbits satisfying the following*:

Given any variety W and a morphism $f: V^{\mathrm{ss}} \to W$ constant on H-orbits, there is a unique map $h: Q \to W$ such that $f = h \circ q$.

If further $B = \Gamma(V, O_V)$ is factorial and U is any open set in V, stable under the action of H, such that $Y = U/H$ exists and $U \to Y$ is affine, then $U \subset V^{\mathrm{ss}}$. If Y is separated then the natural map $Y \to Q$ is an open immersion. This result is generalized in 3.4 of the present paper to arbitrary connected groups acting on quasi-projective varieties.

Finally we want to make explicit mention of a result due originally to Rosenlicht concerning invariant rational functions [15]. If H is unipotent and acts on a quasi-affine variety X then any invariant rational function is a quotient of global invariant functions. Unfortunately if f is such a rational function and f is defined at a point $x \in X$ there may be no way of expressing f as a/b with a, b global invariants and $b(x) \neq 0$. This observation escaped us in [5] but was pointed out to the author by Mumford. It explains the need for our factoriality assumptions in sections 2 and 3 of this paper. *

1. Necessary and sufficient conditions for the existence of quotients

Let X be a normal variety on which the connected unipotent group G operates via the morphism $\sigma: G \times X \to X$ and let $\phi := \sigma \times 1 : G \times X \to X \times X$. We will assume throughout this section that X is quasi-affine and ϕ quasi-finite. We put $B = \Gamma(X, O_X)$ and denote by A the subring of G-invariant functions of B.

DEFINITION 1.1: The variety $X \times X$ can be considered in a natural way as an open subscheme of Spec$(B \otimes B)$. The G-diagonal of X, denoted $G \vee \Delta(X)$ is the closed subset defined by the common zeroes of the functions $a \otimes 1 - 1 \otimes a$, $a \in A$.

Note that the image of Φ is always contained in $G \vee \Delta(X)$.

LEMMA 1.2: *Let G, X and Φ be as above. Assume that a geometric quotient $Y = X/G$ exists and is affine. Then* $\operatorname{Im} \Phi = G \vee \Delta(X)$.

PROOF: Since Y is affine we have $\operatorname{Im} \Phi = (q \times q)^{-1}(\Delta(Y))$ is closed in $X \times X$ where $q: X \to Y$ is the quotient map. On the other hand $\Delta(Y)$ is defined by the ideal in $\Gamma(Y, O_Y)$ generated by $\{a \otimes 1 - 1 \otimes a : a \in \Gamma(Y, O_Y) = A\}$ so $(q \times q)^{-1}(\Delta(Y)) = G \vee \Delta(X)$. □

DEFINITION 1.3: Let G and X be as above. We say that the action σ is *stable* if $\operatorname{Im} \Phi = G \vee \Delta(X)$. We call a point $x \in X$ *stable* if there exists a G-stable open neighborhood U containing x such that the action of G on U is stable. We note by $X^s(G)$ the set of stable points of X.

THEOREM 1.4: *Let G and X be as above. The set $X^s(G)$ is open and G-stable in X. Moreover a geometric quotient $Y = X^s(G)/G$ exists as an algebraic scheme.*

PROOF: The definition of $X^s(G)$ implies that this set is open and that $G \cdot X^s(G) = X^s(G)$. Since the existence of a geometric quotient is local on $X^s(G)$ it suffices to show that if G acts stably on an open set $V \subset X^s(G)$ then V/G exists and is quasi-affine.

* In this regard see [10: p. 154].

Let $A = \Gamma(V, O_V)^G$ and consider the canonical morphism $q: V \to \operatorname{Spec} A$. It is evidently constant on G orbits and separable. Suppose x and y are in the same fiber of q (above a closed point). Then $a(x) = a(y)$ for all $a \in A$. But this means $(x, y) \in G \vee \Delta(V) = \operatorname{Im} \Phi$ so that x and y are in the same G-orbit. Hence q is an orbit map We next show that A can be replaced if necessary by a finitely generated subring R of A. Indeed V is locally noetherian and quasi-compact and at each point of $G \vee \Delta(V)$ the functions $\{a \otimes 1 - 1 \otimes a : a \in A\}$ generate the ideal of $G \vee \Delta(V)$. Using quasi-compactness we can find a finite family of functions $\{a_\alpha : a_\alpha \in A, \alpha \in I\}$ such $\{a_\alpha \otimes 1 - 1 \otimes a_\alpha : \alpha \in I\}$ defines $G \vee \Delta(V)$ at each point. Let R be the smallest finitely generated k-subalgebra of A which is normal, contains all the a_α and has the same quotient field as A. Then $q' : V \to \operatorname{Spec} R$ will define a separable open orbit map from V to $Y = q(V)$ with $Y \subset \operatorname{Spec} R$ open hence quasi-affine. By [1; 6.6] Y is the geometric quotient of V by G. This proves the theorem. □

REMARK: Even though the quotients above are constructed using rings of invariants, they can in fact be non separated schemes (cf. [6; Example 2]). Note that we have not assumed – indeed we cannot – that rings of invariants are actually affine rings; i.e. finitely generated k-algebras. Indeed, it appears that the "14th Problem" is not the issue here!

COROLLARY 1.5: *Let G and X be as in the theorem. If X/G exists then $X = X^s(G)$.*

PROOF: Let $Y = X/G$ and $\{Y_\alpha\}$ an affine open cover of Y. Apply Lemma 1.2 to the covering $\{U_\alpha = q^{-1}(Y_\alpha)\}$ of X. □

Recall that the action of G on X is separated if the image of Φ is closed in $X \times X$.

COROLLARY 1.6: *If σ is separated then $Y = X^s(G)/G$ is a variety.*

PROOF: Since $X^s(G)$ is G-stable and open, the image of Φ restricted to $G \times X^s(G)$ is closed in $X^s(G) \times X^s(G)$. By [10; p. 13] this implies that Y is separated. □

DEFINITION 1.7: The action of G on X is called *properly stable* if $X = X^s(G)$ and Φ is proper.

THEOREM 1.8: *Let X be a normal quasi-affine variety on which the connected unipotent group G acts. Assume that the action of G on X is properly stable and let $Y = X/G$. Then the quotient map $q: X \to Y$ is affine and X is locally trivial. Conversely, if $Y = X/G$ exists, is separated, and X is locally trivial then the action of G is properly stable.*

PROOF: Let $Z \to X$ be a Seshadri cover of X (see Theorem 0.3) Then $W = Z/G$ is separated and hence the natural map $\bar{p}: W \to Y$ is finite. If $Y_0 \subset Y$ is an open affine, then $W_0 = \bar{p}^{-1}(Y_0)$ is affine and hence $Z_0 = q^{-1}(W_0) \simeq G \times W_0$ is also affine.

Consider the commutative square

$$\begin{array}{ccc} Z_0 \simeq G \times W_0 & \to & X_0 = q^{-1}(Y_0) \\ \downarrow & & \downarrow \\ W_0 & \to & Y_0 \end{array}$$

It is immediate that $Z_0 = p^{-1}(X_0)$ so $X_0 = p(Z_0)$ is affine. Thus q is an affine map. Further the separable degree of p is $|\Gamma|$ where $\Gamma = \mathrm{Aut}(k(Z)/k(X))$. Let $T_0 = p(0 \times W_0)$. Then the natural map $G \times T_0 \to X_0$ is surjective and since p factors as $G \times W_0 \to G \times T_0 \to X_0$ and the separable degree of the first map in $|\Gamma|$, it follows that $G \times T_0 \to X_0$ is bijective. Now p is proper and $G \times W_0 \to G \times T_0$ is proper. It follows from [2; 5.4.3] that $G \times T_0 \to X_0$ is proper hence finite so that X_0 is trivial as a G-space.

Assume conversely that X is locally trivial and $Y = X/G$ is separated. Let $\{Y_\alpha\}$ be an affine open cover of Y and $\{X_\alpha = q^{-1}(Y_\alpha) \simeq G \times Y_\alpha\}$ the corresponding affine open cover of X. Since the property of properness of a morphism is local on the range it suffices to show that $\Phi: \Phi^{-1}(X_\alpha \times X_\beta) \to X_\alpha \times X_\beta$ is proper for each pair of indices α, β.

Now $\Phi^{-1}(X_\alpha \times X_\beta) = G \times (X_\alpha \cap X_\beta)$ and we may factor Φ as follows:

$$G \times (X_\alpha \cap X_\beta) \simeq G \times \big(G \times (Y_\alpha \cap Y_\beta)\big)$$

$$\xrightarrow{\psi} G \times G \times Y_\alpha \times Y_\beta \simeq X_\alpha \times X_\beta$$

where the morphism ψ is $\tau \times \Delta_Y$ with Δ_Y the diagonal morphism of Y and $\tau(g, h) = (gh, h)$ an isomorphism. Since Y is separated Δ_Y and hence ψ is a closed immersion, thus proper. This proves the theorem. □

REMARK: In principle one should be able to strengthen the converse to the case where X is locally trivial in the finite radicial topology. Theorem 1.8 is similar to Propositions 0.8 and 0.9 of [10].

COROLLARY 1.9: *Let X be a normal variety on which the connected algebraic group H operates. Assume $Y = X/H$ exists, is separated and that $q: X \to Y$ is a locally trivial principal H-bundle. Then the action of H on X is proper.*

PROOF: The proof of the converse of 1.8 is independent of any particular property of the group G. □

It may happen that properties of the variety X force the geometric quotient $Y = X/G$ to be well behaved. We give an important special case of this phenomena next.

DEFINITION 1.10: A variety X is called quasi-factorial if it is quasi-affine and $\Gamma(X, O_X)$ is a unique factorization domain. Any open subset of a quasi-factorial variety is quasi-factorial. In particular, any open subset of affine space is quasi-factorial.

PROPOSITION 1.11: *Let G be a connected unipotent group acting on the quasi-factorial variety X. Assume that the action of G on X is properly stable. Then $Y = X/G$ is quasi-factorial.*

PROOF: By [Theorem 0.4] there exists a quasi-affine variety Q and a morphism $\varphi: X \to Q$ making Q the categorical quotient of X by G. But Y being a geometric quotient is also a categorical quotient hence $Y \simeq Q$. Since $\Gamma(Y, Q_Y) = \Gamma(X, O_X)^G$ is factorial by [9] Y is quasi-factorial. □

REMARK: It is easily seen that this is a result about unipotent groups. If X is the cone over $\mathbb{P}^n$ for example, then X is certainly quasi-factorial and $X \to \mathbb{P}^n$ is a principal G_m bundle, but $\mathbb{P}^n$ is evidently not quasi-affine.

2. Local criteria for properly stable actions

Let X be a quasi-affine variety on which the connected unipotent group G operates. We have defined stability in the last section as a local property on X. However, the notion of properly stable is a global property of the morphism $\Phi: G \times X \to X \times X$. In this section we will show that if X is quasi-factorial then properly stable is actually a local property of the action of G on X (Theorem 2.4). We also investigate the connection between the set of properly stable points $X^{\mathrm{ps}}(G)$ and the set $X^{\mathrm{ps}}(N)$ where N is a normal subgroup of G. Throughout this section X denotes a quasi-factorial variety and G a unipotent algebraic group. We assume the base field k has characteristic zero and put $B = \Gamma(X, O_X)$.

THEOREM 2.1: *Suppose the connected unipotent group G acts properly on the quasi-factorial variety X. Then the following hold:*

(1) *There exists an algebraic space Y and a morphism $q: X \to Y$ making Y the geometric quotient of X by G in the category of algebraic spaces.*

(2) *If $A = B^G$ the square*

$$\begin{array}{ccc} X & \xrightarrow{c} & \operatorname{Spec} B \\ q\downarrow & & \downarrow \\ Y & \xrightarrow{c} & \operatorname{Spec} A \end{array}$$

commutes where the horizontal maps are the canonical ones.

(3) *Let* $Y_0 = \{y \in Y : |c^{-1}(c(y))| < \infty\}$. *Then* Y_0 *is open in* Y *and* c/Y_0 *is an open immersion.*

(4) *If* Y_0 *is as in* (3) *then* $q^{-1}(Y_0) = X^s(G)$.

Further, if $k = \mathbb{C}$ *and* X *is nonsingular, then* Y *is a complex manifold.*

PROOF: Let $(Z, W\ \rho)$ be a Seshadri cover of X (note that since X is quasi-affine and Φ is proper, the stability groups are finite). Now G acts properly on Z by [Theorem 0.3] so W is a variety. By [8; p. 183] $Y = W/\Gamma$ exists as an algebraic space. This quotient is clearly the geometric quotient of X by G. This proves (1). The assertion (2) is immediate.To see (3) consider the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{c} & \operatorname{Spec} R \\ \pi \downarrow & & \downarrow \\ Y = W/\Gamma & \xrightarrow{c} & \operatorname{Spec} A \end{array}$$

where $R = \Gamma(W, O_W)$ and the horizontal maps are again the canonical ones. If $y_0 \in Y_0$ and $w \in \pi^{-1}(y_0)$ we also have $|c^{-1}(c(w))| < \infty$. But $c : W \to \operatorname{Spec} R$ is birational so by Zariski's Main Theorem, c is an open immersion in a neighborhood of w. Thus if $W_0 = \pi^{-1}(Y_0)$ then $c : W_0 \to \operatorname{Spec} R$ is an open immersion so W_0 is quasi-affine. Clearly W_0 is Γ-stable so $Y_0 = \pi(W_0)$ is open in Y. As $Y_0 = W_0/\Gamma$, Y_0 is also quasi-affine. Now from this it follows that $X_0 = q^{-1}(Y_0)$ has a quotient (viz Y_0) in the category of algebraic varieties and by [Theorem 0.4] $Y_0 \to \operatorname{Spec} A$ is an open immersion.

To see (4) note that $X_0 = q^{-1}(Y_0) \subset X^s(G)$ and that G acts properly on any open G-stable subset. By Proposition 1.11 $X^s(G)/G$ exists and is quasi-affine. Using this fact and the definition of the Seshadri cover (Z, W, p) it is easily verified that

(a) if $W_0 = \{w \in W : |c^{-1}(c(w))| < \infty\}$ then $\pi(W_0) = Y_0$.

(b) if $Z_0 = p^{-1}(X^s(G))$ then $Z_0 = q^{-1}(W_0)$.

It then follows that $q(X^s(G)) = (q \circ p)(Z_0) = (\pi \circ q)(Z_0) = \pi(W_0) = Y_0$ Thus $X^s(G) \subset q^{-1}(Y_0)$ and the desired equality follows.

Finally, if $k = \mathbb{C}$ and X is smooth then for each $x \in X$ we can find a smooth subvariety T_x transversal to the orbit of x under G. By [10; Appendix 1] this implies Y is a smooth algebraic space over $\mathbb{C}$, i.e. a complex manifold. □

Let H be a closed subgroup of G. Note that if G acts properly on X then so does H. Indeed, $\Phi_H : H \times X \to X \times X$ is the composition $H \times X \to G \times X \to X \times X$ and the first map being a closed immersion (hence proper) implies the composition Φ_H is proper provided Φ is proper.

THEOREM 2.2: *Let* G *act properly on* X *and let* N *be a closed normal subgroup of* G. *Let* $X_0 = X^s(N)$ *and* $Y_0 = X_0/N$. *Then* G/N *acts canoni-*

cally on Y_0 and this action is again proper. Further if $q: X_0 \to Y_0$ is the quotient map then $X^s(G) = q^{-1}(Y_0^s(G/N)$.

PROOF: Let $B = \Gamma(X, O_X)$ and let Y be the quotient of X by N in the category of algebraic spaces given in 2.1. Then G/N acts on Y and it is clear that $c: Y \to \operatorname{Spec} B^N$ is a G/N morphism. It follows from the defining properly of Y_0 that G/N acts on Y_0. Now the natural map $\operatorname{Spec} B \to \operatorname{Spec} B^N$ is also a G-morphism where G acts on B^N via the natural map $G \to G/N$. Hence $X_0 = q^{-1}(Y_0)$ is G-stable and G acts properly on X_0.

Note that if $x = h \cdot x'$, $h \in N$, then $g \cdot x$ and $g \cdot x'$ are in the same N orbit: for $g \cdot x = g(h \cdot x') = (gh) \cdot x' = (h''g) \cdot x' = h''(g \cdot x')$, $h'' \in N$. Now consider the diagram

$$\begin{array}{ccc} G \times X_0 & \overset{\Phi}{\to} & X_0 \times X_0 \\ \downarrow & & \downarrow \\ G/N \times Y_0 & \overset{\Phi_0}{\to} & Y_0 \times Y_0. \end{array}$$

Here $\Phi_0 = \sigma \times 1_{Y_0}$ as usual. The vertical maps are quotient maps for the action of $N \times N$. If we let $N \times N$ act on $G \times X_0$ by $(h_1, h_2)(g, x) = (h_1 g h_2^{-1}, h_2 x)$ then Φ is $N \times N$ equivariant and $G/N \times Y_0$ is still the quotient of $G \times Y_0$ by $N \times N$ since N is normal in G. Since $(h_1 g \cdot x, h_2 \cdot x)$ and $(g \cdot x, x)$ have the same image in $Y_0 \times Y_0$ for all $h_1, h_2 \in N$, $g \in G$, $x \in X_0$ the square commutes. By [3; 1.3] Φ_0 is proper which proves the first assertion of the theorem.

Now since G acts properly on X, the map $X^s \to X^s/G$ is a locally trivial principal G-bundle by Theorem 1.8. It follows that X^s/N exists so $X^s \subset X_0$ and is N-stable. Then $q(X^s)$ is open in Y_0 and G/N acts properly stably on it by the first part of the theorem. Clearly X^s/G is the quotient $q(X^s)/(G/N)$ so $X^s \subset q^{-1}(Y_0^s(G/N))$. Conversely, since $Y_0^s(G/N)/(G/N)$ exists, this quotient must be a geometric quotient of $q^{-1}(Y_0^s(G/N))$ by G. Thus $q^{-1}(Y^s(G/N)) \subset X^s(G)$ and the desired equality follows. □

COROLLARY 2.3: *Let G act properly on X. Then a point x in X is unstable if there exists a 1-dimensional subgroup H of G such that x is unstable for H.*

PROOF: We argue by induction on $\dim G$. If $\dim G = 1$ there is nothing to prove. Let H be a proper one dimensional subgroup of G with x not in $X^s(H)$. I claim there is a normal subgroup N of G with $H \subseteq N$ and $\dim N < \dim G$. Granting this claim we have, since $\dim N < \dim G$, $x \notin X^s(N)$. But by the theorem $X^s(G) = q^{-1}(Y_0^s(G/N)) \subset X^s(N)$ so $x \notin X^s(G)$.

To establish the claim recall that if $H_1 = \mathrm{Norm}_G(H)$ then dim $H_1 <$ dim H (cf [13; p. 140]). Thus defining $H_{i+1} = \mathrm{Norm}_G(H_i)$ we obtain a normal series $H < H_1 < H_2 < \ldots < H_p = G$. Put $N = H_{p-1}$ if $p > 1$ otherwise $N = H$. □

The next result is an extension of a result in [5] to arbitrary unipotent groups. In the context of the present development it establishes that the properness of an action is local on X.

THEOREM 2.4: *Let X be a quasi-factorial variety on which the connected unipotent group G operates. If the action of G on X is locally trivial, then $\Phi: G \times X \to X \times X$ is proper. In particular $Y = X/G$ exists and is quasi-factorial.*

PROOF: We argue by induction on dim G. If $G = G_a$ this is just Theorem 7 of [5]. Assume then that dim $G > 1$ and let N be a 1-dimensional connected central subgroup of G. Now if X_0 is an open G-stable subset of X with $X_0 \simeq G \times Y_0$ then $X_0 \simeq N \times ((G/N) \times Y_0)$ by [13; p. 150] so X_0 is also trivial as an N-stable open subset of X. Since X can be covered by such sets it follows that N acts locally trivially on X and hence the action is properly stable. Thus by 1.11 X/N is quasi-factorial again. The action of G/N is clearly locally trivial on X/N and it is proper by the induction hypothesis. Thus $Y = X/G = (X/N)/(G/N)$ is quasi-factorial and $X \to Y$ is a locally trivial principal G-bundle with Y separated so by Corollary 1.9 the action of G on X is proper. □

The above results say that an action of G on a quasi-factorial variety X is properly stable if and only if it is locally trivial. Theorem 2.2 suggest we look for the set of properly stable points by looking for properly stable points $X^{\mathrm{ps}}(N)$ of a proper normal subgroup N and then finding the properly stable points of $Y(N) = X^{\mathrm{ps}}(N)/N$. Since we may always choose N to be one-dimensional i.e. $N \simeq G_a$ it seems worthwhile to give a description of properly stable points in this case. If G_a acts on the quasi-factorial variety X (with finite stability groups) then for each $f \in B = \Gamma(X, O_X)$ we can write

$$f(tx) = f(x) + f_1(x)t + \ldots + f_n(x)t^n$$

$$x \in X,\ t \in G_a,\ f_i \in B.$$

If $\sigma_* : B \to B \otimes k[G_a] = B[T]$ is the co-action then we have

$$\sigma_* f = \sum_{i=0}^{n} f_i T^i \; f_0 = f.$$

In this case we call n the G_a-order of f and f_n the *weight form* of f.

PROPOSITION 2.5. *Let G_a act on the quasi-factorial variety X. Then a point x in X lies in $X^{ps}(G_a)$ if and only there exists a function $f \in \Gamma(X, O_X)$ of G_a-order n for some n such that*
(i) *$f(x)=0$ and*
(ii) *if f_n is the weight form of f then $f_n(x) \neq 0$.*
In this case there exists such a function of G_a-order 1.

PROOF: If $x \in U \subset X$ with U G_a-stable affine and isomorphic to $G_a \times Y$ then $U \simeq \operatorname{Spec} B[h^{-1}]$ for some $h \in B^{G_a}$ since X is quasi-factorial. Then the coordinate function T on G_a gives rise via this isomorphism to a rational function $a = f/h^m$ with $a(t \cdot x) = a(x) + t$ for each $x \in V$, $t \in G_a$. It follows that $\sigma_* f = f + Th^m$. This proves the only if part of the statement provided we adjust f by a suitable constant.

Conversely if (i) and (ii) hold write $\sigma_* f = \sum_{i=0}^{n} f_i T^i$. A straightforward computation shows that f_n is invariant and that $\sigma_* f_{n-1} = f_{n-1} + nTf_n$. If we put $b = n \cdot f_n$ and $a = f_{n-1}$ then $\sigma_* a = a + bT$. It follows (cf. Lemma 5 of [5]) that X_b is trivial and the Proposition follows easily from this fact. □

COROLLARY 2.6. *Let X be a quasi-factorial variety on which the connected unipotent group G acts. Assume that the stability group of each point of X is finite. Then the set of properly stable points $X^{ps}(G)$ of X is non empty.*

PROOF: We argue by induction on $n = \dim G$. If $n = 1$ then $G = G_a$ and we choose any non constant non invariant function f in $B = \Gamma(X, O_X)$ and apply the proposition to f_{n-1}, f_n. Then X_{f_n} is trivial so the action of G is properly stable.

In the general case let N be a one dimensional central subgroup of G. Then $X^{ps}(N) \neq \phi$ and by Theorem 2.2 (proof) $X^{ps}(N)$ is G-stable. Thus G/N acts on $Y = X^{ps}(N)/N$. By induction $Y^{ps}(G/N) \neq \phi$ and $X_0 = q^{-1}(Y^{ps}(G/N))$ is open non empty in X and G acts properly stably on it. □

3. Semi-stability and stability for general algebraic groups

Let G be a connected linear algebraic group with unipotent radical H. Let V be a normal quasi-projective variety on which G acts and suppose $L \in \operatorname{Pic}(V)$. A *G-linearization of L* is a G-equivariant morphism α from V into a projective space $\mathbb{P}^d$ on which G acts such that $\alpha^* O_{\mathbb{P}^d}(1) = L$.

Let V be a fixed projective, normal, G-variety and $\alpha: V \to \mathbb{P}^d$ a G-equivariant immersion into a projective space on which G operates linearly; i.e., via a linear action on the cone $\mathbb{A}^{d+1}$ over $\mathbb{P}^d$. Let $L = \alpha^* O_{\mathbb{P}^d}(1)$. Let $C(V)$ be the affine cone over V minus the origin and $C_\nu(V)$ the normalization of $C(V)$. Then since G acts linearly on $\mathbb{P}^d$ it acts on $C(V)$ and hence by [16; Sec. 6] on $C_\nu(V)$. We say that the action of G on V is *regular* if for almost all $v \in V$, $\dim G \cdot v = \dim G$. If the action of G on V is regular then so is the action on $C(V)$ and $C_\nu(V)$. We assume from now on that the action of G on V is regular. Let R be the coordinate ring of $C_\nu(V)$ and let $A = R^H$.

Recall that a point $x \in C_\nu(V)$ is called *semi-stable* with respect to H if $\dim c^{-1}(c(x)) = \dim H$ where $c: C_\nu(V) \to \operatorname{Spec} A$ is the canonical map. We denote by $C_\nu^{ss}(V)(H)$ the set of *semi-stable points* of $C_\nu(V)$. By [0.4] $C_\nu^{ss}(V)(H)$ is open H-stable and there is a quasi-affine variety Y and a morphism $q: C_\nu^{ss}(V)(H) \to Y$ constant on H-orbits satisfying a universal mapping property. Moreover, Y is an open subvariety of $\operatorname{Spec} A$ so for each $y \in Y$ there is a non constant function f in A with $f(y) \neq 0$. We call a point $v \in V$ *semi-stable* for H if there exists $x \in C_\nu^{ss}(V)(H)$ lying over v. We denote by $V^{ss}(H)$ the set of semi-stable points for H in V. Evidently $V^{ss}(H)$ is open an H-stable in V.

Let X be a G-variety. A pair (Y, q) where Y is a variety with trivial G-action and $q: X \to Y$ a G-equivariant morphism will be called an *s-categorical quotient* if whenever $f: X \to T$ is a morphism from X to a variety T constant on G-orbits there is a unique morphism $\bar{f}: Y \to T$ such that $f = \bar{f} \circ q$.

THEOREM 3.1: *Let (V, L) be a G-linearized projective variety and H the unipotent radical of G. Then $V^{ss}(H)$ is G-stable and an s-categorical quotient W of $V^{ss}(H)$ by H exists and is quasi-projective. Moreover, G/H acts regularly on $\overline{W}$, $\overline{W}$ carries a natural G/H linearization and $q: V^{ss}(H) \to W$ is a G-morphism.*

PROOF: Let $X = C_\nu(V)$ be the normalization of $C(V)$, $B = \Gamma(X, O_X)$ and $A = B^H$. Then G/H acts on A and A is a graded subring of the naturally graded ring B. Let $c: X \to \operatorname{Spec} A$ be the canonical map. Then $x \in X^{ss}(H)$ if and only if $\dim c^{-1}(cx) = \dim H$. But c is evidently G-equivariant. Thus $\dim c^{-1}c(g \cdot x) = \dim c^{-1}(g \cdot cx) = \dim g(c^{-1}c(x)) = \dim c^{-1}(cx))$ and hence $x \in X^{ss}(H)$ implies $gx \in X^{ss}(H)$. Since the natural map $X \to V$ is G-equivariant, $V^{ss}(H)$ is also G-stable.

Now by Theorem 0.4 there is an s-categorical quotient Y of $X^{ss}(H)$ by H and $Y \to \operatorname{Spec} A$ is an open immersion. Since Y is the image of $X^{ss}(H)$, Y is G/H-stable in $\operatorname{Spec} A$. Let $A_0 \subset A$ be a finitely generated graded k-subalgebra of A satisfying

(i) A_0 is integrally closed and G/H-stable.

(ii) $Y \to \operatorname{Spec} A_0$ is an open immersion.

Such an A_0 exists because G (via G/H) acts rationally on A and Y is locally of finite type and quasi-compact. Note that for each $y \in Y$ there is a homogeneous element $a \in A$ of positive degree such that $a(y) \neq 0$. Now put $P_0 = \operatorname{Proj} A_0$. Then G/H acts on P_0 and $O_{P_0}(m)$ gives a G/H-linearization if $m = \text{l.c.m}\,(d_1, d_2, \ldots, d_s)$ if A_0 is generated over k by its elements of degree $d_1, \ldots, d_s$.

Now $Y \subset \operatorname{Spec} A_0 - (0)$ so the image W of Y in P_0 is open. Evidently, W is G/H-stable and $\overline{W} = P_0$ has a G/H-linearization. The action of G/H is regular on $\overline{W}$ if and only if it is regular on Y. But G acts regularly on X so also on $X^{ss}(H)$ and it follows easily that G/H acts regularly on Y.

It remains only to show that W is an s-categorical quotient of $V^{ss}(H)$ by H. By our construction $\operatorname{Spec} A_0 - (0) \to P_0$ is a principal G_m-bundle and Y is G_m-stable. Thus $Y \to W$ is the geometric quotient of Y by G_m so in particular it is a categorical quotient. Since $A_0 \subset B$ we have a natural rational mapping $V = \operatorname{Proj} B \to P_0$. This mapping is regular on $V^{ss}(H)$ because $A_0 \subset B$ and if $v \in V^{ss}(H)$ then there exist $a \in A_0$ homogeneous of positive degree with $a_0(v) \neq 0$. The image of the restriction q of this mapping is evidently W. We have q constant on H-orbits and G-equivariant. Let T be a variety and $g: V^{ss}(H) \to T$ a morphism constant on H-orbits. Then we get a map $\bar{g}: X^{ss}(H) \to T$ constant on H-orbits and hence a unique map $\tilde{\beta}: Y \to T$ with $\tilde{\beta}\tilde{q} = \tilde{g}$. But the map $\tilde{\beta}$ is evidently constant on G_m-orbits in Y so we get a unique map $\beta: W \to T$ such that $\beta\pi = \tilde{\beta}$, $\pi: Y \to W$ the quotient map. Since

$$\begin{array}{ccc} X^{ss}(H) & \xrightarrow{\tilde{q}} & Y \\ \downarrow \pi_X & & \downarrow \pi_Y \\ V^{ss}(H) & \xrightarrow{q} & W \end{array}$$

commutes we have $\tilde{\beta}\tilde{q} = \beta\pi_Y\tilde{q} = \beta q \pi_X = \tilde{g} = g\pi_X$. Since $V^{ss}(H)$ is the quotient of $X^{ss}(H)$ by an action of G_m, $\beta q = g$ as desired. □

DEFINITION 3.2: Let V be a normal projective variety on which G operates and L in $\operatorname{Pic}(V)$ a G-linearized ample invertible sheaf. A point v of V is *semi-stable* with respect to this linearization if there is an invariant section a of L^n for some n such that $a(v) \neq 0$ and $v \in V_a^{ss}(H)$. A point v in V is *stable* if there is a G-invariant section a of L^m for some m such that $v \in V_a^{ps}(H)$ and the action of G on V_a is closed. We denote by $V^{ss}(L)$ (resp. $V^s(L)$) the set of semi-stable (resp. stable) points.

THEOREM 3.3: *Let V be a normal projective variety on which the connected algebraic group G operates. Let L in* Pic(V) *be a G-linearized very ample invertible sheaf. Then*

(i) *An s-categorical quotient of $V^{ss}(L)$ by G exists and is quasi-projective.*

(ii) *If $C_v(V)$ is quasi-factorial, then a quasi-projective geometric quotient of $V^s(L)$ by G exists.*

PROOF: Let $q: V^{ss}(H) \to W$ be the s-categorical quotient of $V^{ss}(H)$ by $H = \mathrm{Rad}_u G$. Let $P_0 = \overline{W}$, A_0 be as in the proof of 3.1. By enlarging A_0 if necessary we may assume that there is a finite set of homogeneous elements of A_0^G say $a_1, \dots, a_r$ representing invariant sections of various powers of L such that $V^{ss}(L) \subset \cup_{i=1}^r V_{a_i}$. Note that $V^{ss}(L)$ is open and G-stable and that $g(V^{ss}(L)$ is open and G (i.e. G/H) stable in W. It follows from 3.1 and the definitions that $O_{P_0}(m)$ is $G' = G/H$ linearized and that $q(V^{ss}(L)) \subset P_0^{ss}(O_{P_0}(m))$. If $\overline{Q}$ is the categorical quotient of $P_0^{ss}(O_{P_0}(m))$ by G', then the image Q of $q(V^{ss}(L))$ in $\overline{Q}$ is the categorical quotient of $q(V^{ss}(L))$ by G'. It follows that the composition

$$V^{ss}(L) \to q(V^{ss}(L)) \to Q$$

makes Q the s-categorical quotient of $V^{ss}(L)$ by G. This gives (i).

Now let π: $C_v(V) \to V$ be the natural projection. From the definition of $V^s(L)$ it follows that $X = \pi^{-1}(V^s(L))$ is contained in $C_v(V)^{ps}(H)$. Denote by $V^{ps}(H)$ the image of $C_v(V)^{ps}(H)$ by π and let $W' = q(V^{ps}(H)) \subset W$. Then we have

LEMMA: *$V^{ps}(H)$ is G-stable and the restriction of $q: V^{ss}(H) \to W$ to $V^{ps}(H)$ makes W' the geometric quotient of $V^{ps}(H)$ by H.*

Granting the lemma we proceed with the proof of ii). First, $W' \supset q(V^s(L))$ and both these sets are open and G/H-stable in P_0. Let $L_0 = O_{P_0}(m)$ be the G/H-linearized sheaf constructed in the proof of 3.1. We may assume as above that $q(V^s(L)) \subset \cup_{i=1}^t (P_0)_{b_i}$, b_i G/H-invariant sections of suitable powers of L_0 with the action of G/H on $(P_0)_{b_i}$ closed, $i = 1, \dots, t$. By [10: 2.2] $q(V^s(L)) \subset P_0^s(L_0)$. Let (φ, Y_1) be the geometric quotient of $P_0^s(L_0)$ by G/H and Y the image of $q(V^s(L))$ under φ. It follows that Y is the geometric quotient of $V^s(L)$ by G.

It remains only to prove the lemma. Let $X = C_v(V)$ and put $B = \Gamma(X, O_X)$. By Corollary 2.6 $X^{ps}(H) \neq \phi$. Let $U \subset X$ be an H-stable open subset of X with U/H affine. Since $A = B^H$ is factorial, $U/H \simeq$ Spec $A[a^{-1}]$ for suitable $a \in A$. Let $g \in G$ be fixed. Then $HgU = gHU = gU$ so gU is stable. The group G acts on A via the homomorphisms $\tau_g : A \to A$ given by $\tau_g(a)(x) = a(g^{-1}x)$. We claim $gU/H \simeq$ Spec $\tau_g A[(\tau_g b)^{-1}]$. Using the commutative square

$$\begin{array}{ccc} U & \xrightarrow{\simeq} & gU \\ \downarrow & & \downarrow \\ \operatorname{Spec} A[a^{-1}] & \xrightarrow{\simeq} & \operatorname{Spec} \tau_g A[(\tau_g a)^{-1}] \end{array}$$

it is easy to see that g_g is a separable morphism constant on orbits. Now let f_α generate $A[a^{-1}]$ over k. Then $f_\alpha(x) = f_\alpha(y)$ for x, $y \in U$ and all α

if and only if $x = h \cdot y$ for some $h \in H$, i.e. the fibers of q are H-orbits. But $(\tau_g f_\alpha)(g \cdot x) = f_\alpha(g^{-1} \cdot g \cdot x) = f_\alpha(x)$ so $\tau_g(f_\alpha)(g \cdot x) = \tau_g(f_\alpha)(g \cdot y)$ for all α if and only if $x = h \cdot y$ for some $h \in H$ if and only if $gx = ghy = h'gy (H \triangleleft G!)$ i.e. the fibers of q_g are also H-orbits. By [1; 6.6] q_g is a quotient map. This $gU \subset X^s(H)$. Applying this result to an arbitrary element of G implies $X^s(H)$ is G-stable.

Now the map $\pi : X \to V$ is G-equivariant and so $\pi(X^{ps}(H)) = V^{ps}(H)$ is open and G-stable. Note that $X^{ps}(H) \subset X^{ss}(H)$ and we have a commutative diagram

$$\begin{array}{ccc} X^{ss}(H) & \to & Y \\ \pi \downarrow & & \downarrow \bar{\pi} \\ V^{ss}(H) & \to & W \end{array}$$

where Y is the s-categorical quotient of $X^{ss}(H)$. The image, say Y', of $X^{ps}(H)$ in Y is the geometric quotient of $X^{ps}(H)$ by H. Thus $W' = \bar{\pi}(Y')$ is the geometric quotient of $V^{ps}(H)$ by H. It is evidently G/H-stable in $\overline{W}$. This proves the lemma and completes the proof of the theorem.

REMARK: Let M be an n-dimensional G-module and $P = P(M)$ the projective space of lines through the origin in M. If $r < n$ let $G(r, M)$ be the Grassmann space of r-planes in M. Then P and $G(r, M)$ carry natural G-linearized very ample invertible sheaves and the hypothesis in (ii) of the theorem are satisfied in both cases [15].

In [10] a criterion for stability is given in terms of 1-parameter subgroups. We give now an analog in this more general setting of the key result used to establish this criterion. Recall that a 1-parameter subgroup of G is a homomorphism $\lambda : G_m \to G$. We put $G(\lambda) = \text{Image}(\lambda) \cdot H$. The group $G(\lambda)$ is a closed connected subgroup of G of semi-simple rank 1. If $\pi : G \to G/H$ is the natural projection then π induces a 1-parameter subgroup $\bar{\lambda} = \pi \circ \lambda$ of G/H. Let $\Lambda(G)$ (resp. $\Lambda(G/H)$) denote the set of 1-parameter subgroups of G (resp. G/H).

LEMMA 3.4. *π induces a bijection $\pi^* : \Lambda(G) \to \Lambda(G/H)$.*

PROOF: Fix a maximal torus T of G and let $P = T \cdot H$. Every $G(\lambda)$ is conjugate to a subgroup of P. If $x \in \Lambda(G/H)$ then $G(x) = [\pi^{-1}(\text{Im}(x))]^0$ is conjugate to a subgroup of P. Say $G(x)^{g(x)} \in P$. Then if $T(x)$ is a maximal torus of $G(x)^{g(x)}$ then $T(x) \subset T$ so $T(x)$ is given by a 1-parameter subgroup $\bar{\lambda}_0(x) : G_m \to T$. Define λ by $\lambda(\alpha) = (\bar{\lambda}_0(x)(\alpha))^{g^{-1}(x)}$. Then $\pi \circ \lambda = x$. Since H is unipotent $\pi \circ \lambda = \pi \circ \lambda_1$ if and only if $\lambda = \lambda_1$. Indeed, $\pi(\lambda(\alpha)) = \pi(\lambda_1(\alpha))$ if and only if $\lambda(\alpha)\lambda_1(\alpha)^{-1} \in H$. This means $\lambda(\alpha)\lambda_1(\alpha)^{-1} = e$ so $\lambda = \lambda_1$. □

Let (V, L) be as above and B the ring of regular functions on $C_\nu(V)$. We shall say that the action of G on V is *k-noetherian* if B^H is a finitely generated k-algebra. In this case B^G is also finitely generated as a k-algebra.

PROPOSITION 3.5: *Let V be a normal projective variety on which the connected algebraic group G operates. Let L in* Pic(V) *be a very ample G-linearized invertible sheaf and assume that the action of G on (V, L) is k-noetherian. Then*

1. *A point v in V is semi-stable if and only if it is semi-stable for $G(\lambda)$ for all λ in $\Lambda(G)$.*
2. *If $C_\nu(V)$ is quasi-factorial, then v in V is stable if and only if it is stable for $G(\lambda)$ for all $\lambda \in \Lambda^{(G)}$.*

PROOF: Let $P_0 = \text{Proj } A$ where $A = B^H$, $B = \Gamma(C_\nu(V), O_{C_\nu(V)})$. The s-categorical quotient (q, W) of $V^{ss}(H)$ is canonically identified with an open subset of P_0. Let $L_0 = O_{P_0}(m)$ be the natural G-linearized invertible sheaf on P_0. Then $V^{ss}(L) = V^{ss}(H) \cap q^{-1}(P_0^{ss}(L_0))$ and $V^s(L) = V^{ps}(H) \cap q^{-1}(P_0^s(L_0))$. But $P_0^{ss}(L_0)$ and $P_0^s(L_0)$ are determined by $\Lambda(G/H)$ by [10: 3.2]. The proposition now follows in a straight forward manner from lemma 3.4. We leave the details to the reader.

A somewhat more useful criterion is the following result.

PROPOSITION 3.6: *Let G, V and L be as above. Assume that G is a closed subgroup of a reductive group M which acts on V and that the G-linearization of (V, L) is the restriction of an M-linearization of (V, L). Suppose $C_\nu(V)$ is quasi-factorial. Then if $v \in V^{ss}(H)$ (resp. $V^s(H)$) is semi-stable (respectively stable) with respect to M, it is also semi-stable (respectively stable) with respect to G.*

PROOF: Since every M-invariant section of L is also G-invariant the assertion with regard to semi-stability is clear. Suppose s is an M-invariant section of L^r some r and the action of M on V_s is closed. Since G is closed in M, the action of G on V_s is also closed. The proposition is then an immediate consequence of the definition of stability. □

4. Applications and examples

In this section we discuss several examples of actions of nonreductive groups to which the foregoing theory can be applied. The last two examples are based on work of Mori [19] and Catanese [18]. We begin by recalling Mumford's 1-parameter subgroup criteria.

Let G be a reductive algebraic group acting linearly on a vector space V. This gives an action of G on the associated projective space $P(V)$ which is linear with respect to $O(1)$. Recall that a point $x \in P(V)$ is stable if (i) there is a homogeneous polynomial $f \in S(V^*)^G$ such that $f(x) \neq 0$ and (ii) the action of G on the affine open G-stable subset $P(V)_f$ is closed. A point is semi-stable if and only if the first condition holds. The point $x \in P(V)$ is said to be a properly stable point if it is stable and the dimension of the stability group of x in G is zero. Let $\lambda: G_m \to G$ be a 1-parameter subgroup of G. With respect to a suitable basis of V, the action of $\lambda(G_m)$ can be diagonalized. Thus $\lambda(\alpha) = \operatorname{diag}(\alpha^{r_0}, \dots, \alpha^{r_m})$. If $x \in P(V)$ and $v \in V$ lies over x then we can write $v = v_0 + v_1 + \dots + v_m$ v_i in the r_i-eigenspace of λ. Then define

$$\mu(x, \lambda) = \max\{-r_i | i \text{ such that } v_i \neq 0\}.$$

THEOREM [10; 2.1]: *A point $x \in P(V)$ is stable (res. semi-stable) if and only if $\mu(x, \lambda) > 0$ (resp. $\mu(x, \lambda) \geqslant 0$) for all 1-parameter subgroups λ.*

We also have [10; 2.2] for $x \in P(V)$ and any $v(x) \in V$ lying over x:

(i) x is semi-stable if and only (0) is not in the closure of the orbit of $v(x)$.

(ii) x is properly stable if and only if the map $\psi_{v(x)}: G \to V$ given by $g \to g \cdot v(x)$ is proper.

(iii) If $P(V)^s$ (resp. $P(V)^{ss}$) denotes the set of properly stable points (resp. semi-stable points) of $P(V)$ then a geometric (resp. categorical) quotient of $P(V)^s$ (resp. $P(V)^{ss}$) by G exists and is quasi-projective (resp. projective).

When $G \subset GL(V)$ contains the scalar matrices Z then the above criterion cannot be applied directly. However, we can modify it slightly as follows. Let $G \subset GL(V)$ be a reductive subgroup containing the center Z of $GL(V)$. Let $G' = G \cap SL(V)$. If X_0^s denotes the set of properly stable points of G' in $P(V)$ then X_0^s is G-stable and X_0^s/G exists and is quasi-projective since Z acts trivially on X_0^s and $X_0^s/G = X_0^s/G'$.

4.2. First examples

Let $P \subset SL(2, \mathbb{C})$ be the Borel subgroup of upper triangular matrices in $SL(2, \mathbb{C})$. Then P acts on $S^2(\mathbb{C}^2)$ in a natural way. The group P is the semi-direct product of $P_u \cong G_a$ and G_m and we can choose affine coordinates for the vector space $V = S^2(\mathbb{C}^2)$ say x, y and z so that the action of P is given by

$$(\alpha a)x = \alpha^2(x + ay + a^2 z)$$

$$(\alpha a)y = y + 2az$$

$$(\alpha a)z = \alpha^{-2} z$$

where $\alpha \in G_m$, $a \in G_a$, $\alpha a \in P$. The P_u invariant functions are generated over $\mathbb{C}$ by z and $y^2 - xz$ and V_z is the set of properly stable points of P_u.

Let $f: V \to \mathbb{A}^2$ be the map induced by the ring inclusion $\mathbb{C}[u, v] \to \mathbb{C}[x, y, z]$, $f_*(u) = y^2 - xz$, $f_*(v) = z$. Then f is P equivariant and $P/P_u \simeq G_m$ acts on $\mathbb{A}^2$ by $(u, v) \to {}^{\lambda}(u, \lambda^{-2}v)$. The open set $\mathbb{A}^2_v$ is G_m-stable and in fact $\mathbb{A}^2_v/G_m$ exists and is affine. Thus V_z/P exists and is affine.

On the other hand if we consider the natural action of P on $W = S^3(\mathbb{C}^2)$ then $\Gamma(W, O_W)^{P_u} = \mathbb{C}[a, b, c, d]$ with one relation $a^2d = b^3 - c^2$ and P/P_u acts by

$$(a, b, c, d) \xrightarrow{\lambda} (\lambda^3 a, \lambda^2 b, \lambda^3 c, d).$$

The set of properly stable points for P_u is $W_a \cup W_b$. Again we can map W to $\mathbb{A}^4$ by $\mathbb{C}[u_1, u_2, u_3, u_4] \to \Gamma(W, O_W)$ $u_1 \to a$, $u_2 \to b$, $u_3 \to c$, $u_4 \to d$. Let $Y_i = \mathbb{A}^4_{u_i}$ $i = 1, 2$. Then Y_i/G_m exists and is affine for $i = 1, 2$. Thus a quotient Q as algebraic scheme exists. On $Y = Y_1 \cup Y_2$ the points $(1, 0, 0, 1)$ and $(0, 1, 0, 1)$ have distinct orbits. But the closures of both these orbits in $\mathbb{A}^4$ contains $(0, 0, 0, 1)$. Thus Q is certainly not quasi-affine. However, $T = \text{Proj } \mathbb{C}[u_1, u_2, u_3, u_4]$ is quasi-projective if we declare the grading to be of type $(3, 2, 3, 0)$. Then T is the geometric quotient of $\mathbb{A}^4 - (0)$ mod G_m and since $u_1^2u_4 - u_{32} + u_{23}$ is homogeneous with respect to this grading, its image in T is a geometric quotient by G_m. But Y is open and G_m-stable in the variety $u_1^2u_4 - u_{32} + u_{23} = 0$ thus, it has a quotient by G_m. The usefulness of weighted projective spaces will reappear in a later example. Note here that the use of invariants of G_m in $\mathbb{C}[u_1, u_2, u_3, u_4]$ (u_4 is the generator of the ring of invariants) does not lead to a quotient.

4.3 Morphisms of projective spaces

Let $\Phi(d) \subset \text{Hom}_{\mathbb{C}}(\mathbb{P}^n, \mathbb{P}^m)$ be the set of all morphisms $\varphi: \mathbb{P}^n \to \mathbb{P}^m$ such that $\varphi^* O_{\mathbb{P}^m}(1) = O_{\mathbb{P}^n}(d)$. It is well known that $\Phi(d)$ is a quasi-projective variety. Let $\Psi(d)$ be the cone over $\Phi(d)$ identified with $m + 1$ sections in $H^0(\mathbb{P}^n, O(d))$. Then $\Psi(d)$ is an open subset of the affine space of dimension $N = (m + 1) \dim H^0(\mathbb{P}^n, O(d))$. For $\psi \in \Psi(d)$ we let $s_j(\psi)$ be the j-th component section of $\psi = (s_0(\psi), \ldots, s_m(\psi))$. Each $s_j(\psi)$ can be naturally viewed as a polynomial of degree d in homogeneous coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n$. Denote by $\Psi(d, j)$ the set of all $\psi \in \Psi(d)$ such that x_0^d occurs in $s_j(\psi)$ with non zero coefficient. The fact that ψ induces a morphism $\mathbb{P}^n \to \mathbb{P}^m$ ensures us that $\Psi(d) = \bigcup_{j=0}^{m+1} \Psi(d, j)$. It is easily seen that $\Psi(d, j)$ is open in $\Psi(d)$.

Let $v_\infty = [1, 0, \ldots, 0]$ in $\mathbb{P}^n$ and P the stability group of v_∞ in $G = PGL(n)$. If $H = \text{Rad}_u P$ then $H \simeq G_a^n$ and if $h = (h_1, \ldots, h_n) \in H$ then h acts on $\mathbb{P}^n$ by

$$h \cdot [x_0, \ldots, x_n] = \left[x_0 + \sum_{i=1}^{n} h_i v_i, v_1, v_2, \ldots, v_n \right].$$

The group G (and so also P) acts on $\Phi(d)$ by $(g\varphi)(v)=\varphi(g^{-1}v)$ and this action lifts to an action of $SL(n+1, \mathbb{C})$ on $\Psi(d)$.

LEMMA: *The action of H on $\Phi(d)$ is properly stable. In particular $\Phi(d)/H$ exists and is quasi-projective.*

PROOF: Let H' be the preimage of H under the natural isogeny $SL(n+1)\to PGL(n)$. The action of $SL(n+1)$ on $\Psi(d)$ exhibits a natural linearization of the action of $SL(n+1)$ on $\Phi(d)$. Let $\varphi_0\in\Phi(d)$ and suppose ψ_0 lies over it in $\Psi(d)$. We can assume that $\psi_0\in\Psi(d, i)$. Define $W_i(\psi_0)$ to be the closed subset of $\Psi(d, i)$ defined by the condition:

$$\frac{\text{coefficient of } x_j^d \text{ in } s_i(\psi)}{\text{coefficient of } x_0^d \text{ in } s_i(\psi)}=\frac{\text{coefficient of } x_j^d \text{ in } s_i(\psi_0)}{\text{coefficient of } x_0^d \textit{ in } s_i(\psi_0)}.$$

Let $\beta: H\times W_i(\psi_0)\to\Psi(d, i)$ be the morphism $\beta(h, \psi)=h\cdot\psi$. We will show that β is bijective onto its image. Indeed, if $\beta(h_1, \psi_1)=\beta(h_2, \psi_2)$ then $h_1\psi_1=h_2\psi_2$ so $\psi_2=h\psi_1$ where $h=h_2^{-1}h_1$. Let $s_i(\psi_1)=\lambda_0x_0^d+\ldots+\lambda_nx_n^d+M_1$ and $s_i(\psi_2)=\mu_0x_0^d+\ldots+\mu_nx_n^d+M_2$ with M_1, M_2 forms of degree d in $x_0,\ldots, x_n$ not containing any monomials of the form αx_j^d, $\alpha\in\mathbb{C}$. Then we have

$$s_i(h\psi_1)=\lambda_0x_0^d+(\lambda_1+\lambda_0h_1^d)x_1^d+\ldots+(\lambda_n+\lambda_0h_n^d)x_n^d+M.$$

Since $h\cdot\psi_1=\psi_2$, evaluating both functions at v_∞ gives $\lambda_0=\mu_0$. Since ψ_1, $\psi_2\in W_i(\psi_0)$ we have $\lambda_j=\mu_j$ $j=1,\ldots,n$. But $h\cdot\psi_1=\psi_2$ implies $\lambda_j+\lambda_0h_j^d=\mu_j$ and so $\lambda_0h_j^d=0$ all j. But $\lambda_0\neq 0$ because $\psi\in\Psi(d, i)$ thus $h_j^d=0$ all j; i.e., $h=e$ in H. Since $W_i(\psi_0)$ is easily seen to be normal (indeed it is the intersection of $\Psi(d, i)$ with a linear subvariety) we have by Zariski's Main Theorem (the characteristic is zero) that β is an open immersion. It follows that $\Psi(d, i)$ is locally trivial. But then $\Psi(d)$ is also locally trivial and being open in an affine space is also quasi-factorial. By Theorem 2.4 the action of H on $\Psi(d)$ is properly stable so by definition 3.2 the action of H on $\Phi(d)$ is properly stable.

We next show that $\Phi(d)/P$ exists and is quasi-projective. Let $\pi: SL(n+1, \mathbb{C})\to PGL(n, \mathbb{C})$ be the canonical isogeny and put $Q=\pi^{-1}(P)^0$. Then Rad_uQ is naturally isomorphic to H. It is easy to verify that $Q\simeq GL(n, \mathbb{C})\cdot H$.

LEMMA: *Every point of $\Phi(d)$ is properly stable for the action of $SL(n+1, \mathbb{C})$.*

PROOF: Let λ be a 1-parameter subgroup of $SL(n+1, \mathbb{C})$. Replacing λ by a suitable conjugate if necessary we may assume

$$\lambda(t)=\text{diag}(t^{r_0}, t^{r_1},\ldots, t^{r_n})\quad t\in G_m$$

(cf. [10: Chapt. 3]). Let $\psi \in \Psi(d)$ lie over $\varphi \in \Phi(d)$. Fix j, $0 \leqslant j \leqslant n$. For at least one i between 0 and m we have

$$s_i(\psi) = a_0 x_0^d + \ldots + a_n x_n^d + M \quad \text{with} \quad a_j \neq 0.$$

For $t \in G_m$, we then have

$$\lambda(t) s_i(\psi) = a_0 t^{dr_0} x^d + \ldots + a_n t^{dr_n} x_n + \lambda(t) \cdot M.$$

Now the basis of $H^0(\mathbb{P}^n, O(d)) \oplus \ldots \oplus H^0(\mathbb{P}^n, O(d))$ $(m+1$ copies) given by monomials of degree d in each factor diagonalizes the action of λ. For the corresponding coordinates we see that in the i-th factor the coordinate of $s_i(\psi)$ corresponding to x_j^d gives the character dr_j. Since $\lambda(t) \in SL(n+1, \mathbb{C})$, det $\lambda(t) = 1$ so $\sum_{i=0}^n r_i = 0$. It follows that at least one r_j is positive and one is negative. From this it is clear that

$$\mu(\lambda) = \max\{-dr_j\}$$

is positive so by the numerical criterion [10 : 3.1] φ is a stable point for $SL(n+1, \mathbb{C})$. Since the stability groups are 0-dimensional, φ is actually a properly stable point and the lemma is established.

COROLLARY: *Every point of $\Phi(d)$ is properly stable for the action of P.*

PROOF: By Proposition 3.6 every point is properly stable for the action of Q and it follows immediately that every point is also properly stable for the action of P.

The example of Mori

Let d, n, m be as above. Let $X \subset \mathbb{P}^m$ be an n-dimensional subariety, $\varphi_0 : \mathbb{P}^n \to X$ a morphism such that $\varphi_0^* O_{\mathbb{P}^m}(1) = O_{\mathbb{P}^n}(d)$ (here $\varphi_0 : \mathbb{P}^n \to \mathbb{P}^m$ by composing with the inclusion $X \subset \mathbb{P}^m$). Let $j : \{v_\infty\} \to X$ be the restriction of φ_0 to $\{v_\infty\}$ and put $\Phi(d, j, X) = \{\varphi \in \Phi(d) : \varphi(\mathbb{P}^n) \subset X, \varphi|v_\infty = j\}$. *I* claim P acts on $\Phi(d, j, X)$ and in particular on the component containing φ_0. Further, a quasi-projective quotient of this component by P exists.

Note first that since P fixes v_∞, $(g\varphi)(v_\infty) = \varphi(v_\infty)$ for all φ so if $\varphi|_{v_\infty} = j$ then $g \cdot \varphi|_{v_\infty} = j$ all $g \in P$. Next let $e : \Phi(d) \times \mathbb{P}^n \to \mathbb{P}^m$ be the evaluation morphism

$$e(\varphi, x) = \varphi(x).$$

If P acts diagonally on the product $\Phi(d) \times \mathbb{P}^n$ then e becomes a P-equivariant morphism where P acts trivially on $\mathbb{P}^m$. Hence $e^{-1}(X)$ is closed in $\Phi(d)\times\mathbb{P}^m$ and is P-stable. Let $Z=e^{-1}(X)$ and $p_1: Z\to\Phi(d)$ the restriction of the projection onto the first factor. Note that $[\varphi_0]\times\mathbb{P}^n\subset Z$. Let $E_n=\{z\in Z: \dim p_1^{-1}(p_1(z))\geqslant n\}$. Then E_n is closed in Z (cf. [1; A.G., 10.3]). Since p_1 is proper $p_1(E_n)$ is closed in $\Phi(d)$. But $z\in E_n$ if and only if $p_1^{-1}(p_1(z))=p_1(z)\times\mathbb{P}^n$ and this holds if and only if $p_1(z)(\mathbb{P}^n)\subset X$. Now let $Z(\varphi_0)$ be the component of E_n containing $[\varphi_0]\times\mathbb{P}^n$. Then

(1) $p_1(E_n)=\Phi(d, X)$ and

(2) $p_1(Z(\varphi_0))=\Phi(d, X, j)$ are closed

P-stable subsets of $\Phi(d, X)$. Thus both have quasi-projective geometric quotients since $\Phi(d)$ does. This example is based on a construction of Mori in his proof of Hartshorne's conjecture [19].

4.3. Moduli of surface with $K^2=p_g=1$

The last example is taken from a paper of F. Catanese [18]. Let S be a smooth minimal surface with $p_g=K^2=1$ where p_g is the geometric genus and K the canonical divisor. Then the canonical ring $R(S)=\oplus_{m=0}^{\infty}H^0(S, O(mK))$ gives rise to a scheme $P=\operatorname{Proj} R(S)$ which is a weighted complete intersection of type (6,6) in $Q=\operatorname{Proj}\mathbb{C}[x_0, y_1, y_2, z_3, z_4]$ with $R=\mathbb{C}[x_0, y_1, y_2, z_3, z_4]$ a graded ring having deg $x_0=1$, deg $y_i=2$, deg $z_j=3$ ($i=1, 2,\ j=3, 4$). Here type (6,6) refers to the degree of the two homogeneous polynomials F and G in R which define P.

Catanese shows that the polynomials F and G may be written in the form $F=z_3^2+x_0z_4\alpha(y)+A(y)$ and $G=z_4^2+x_0z_4\beta(y)+B(y)$ where $\alpha(y)$ and $\beta(y)$ are linear in $y_0=x_0^2$, y_1 and y_2 and $A(y)$, $B(y)$ are cubic forms in y_0, y_1 and y_2. For a general choice of α, β, A and B the complete intersection $F=G=0$ in Q is smooth and hence isomorphic to a minimal surface with $p_g=K^2=1$.

Let G_1 be the subgroup of Aut Q determined by the substitutions

$$x_0\to d_0x_0 \qquad d_0\in\mathbb{C}^*$$

$$y_i\to d_{i1}y_1+d_{i2}y_2+d_{i0}v_0 \qquad i=1,2,\quad (d_{ij})\in GL(2,\mathbb{C})$$

$$z_j\to c_{jj}z_j \qquad j=3,4,\quad c_{jj}\in\mathbb{C}^*$$

and the involution i which permutes z_3 and z_4. Then the connected component H of the corresponding algebraic group $\tilde{H}$ is isomorphic to a semi-direct product

$$H\simeq G_m\cdot[P\times T_1]$$

where P is the group of matrices

$$P = \left\{ \begin{bmatrix} 1 & d_{10} & d_{20} \\ 0 & d_{11} & d_{12} \\ 0 & d_{21} & d_{12} \end{bmatrix} \right\}$$

T_1 is given as

$$T_1 = \left\{ \begin{bmatrix} c_{33} & 0 \\ 0 & c_{44} \end{bmatrix} \right\}$$

and the semi-direct product structure is given by

$$\lambda p \lambda^{-1} = \begin{bmatrix} 1 & \lambda^2 d_{10} & \lambda^2 d_{20} \\ 0 & d_{11} & d_{12} \\ 0 & d_{21} & d_{22} \end{bmatrix} \qquad p \in P$$

$$\lambda t \lambda^{-1} = t, \quad t \in T_1.$$

THEOREM [18; 1.9]: *If X and X' are defined by two pairs of canonical equations (F, G) and (F', G') then they are isomorphic if and only if (F, G) and (F', G') are in the same orbit under the action of the group $\tilde{H}$.*

We want to consider the problem of constructing the moduli space for these surfaces defined by canonical forms F and G. A straightforward computation shows that if $h = d_0 \cdot p \cdot t$ with $d_0 \in G_m$, $p \in P$, $t \in T_1$ then

$$h \cdot F = z_3^2 + x_0 z_4 \left[c_{33}^{-2} c_{44} d_0 \cdot ((d_0 p) \cdot \alpha(y)) \right] + c_{33}^{-2} \cdot (d_0 p) \cdot A(y)$$

and

$$h \cdot G = z_4^2 + x_0 z_3 \left[c_{44}^{-2} c_{33} d_0 ((d_0 p) \cdot \beta(y)) \right] + c_{44}^{-2} \cdot ((d_0 p) \cdot B(y)).$$

The unipotent radical of $\tilde{H} = H_u = P_u \simeq G_a^2$. We look at the action of this group next.

Let $\alpha(y) = a_0 y_0 + a_1 y_1 + a_2 y_2$, $\beta(y) = b_0 y_0 + b_1 y_1 + b_2 y_2$. Then if $p_u = (d_{10}, d_{20}) \in H_u$ we have

$$p_u \cdot \alpha(y) = (a_0 + d_{10} a_1 + d_{20} a_2) y_0 + a_1 y_1 + a_2 y_2$$

$$p_u \cdot \beta(y) = (b_0 + d_{10} b_1 + d_{20} b_2) y_0 + b_1 y_1 + b_2 y_2.$$

If

$$D(\alpha, \beta) = \det\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$$

then we can find unique d_{10}, d_{20} with $p_u(a_1y_1 + a_2y_2) = \alpha(y)$ and $p_u(b_1y_1 + b_2y_2) = \beta(y)$. If $D(\alpha, \beta) = 0$ then the stability group of $\alpha(y)$, $\beta(y)$ is positive dimensional. Next write

$$A(y) = \sum_{0 \leqslant i \leqslant j \leqslant k} a_{ijk} y_i y_j y_k$$

and

$$B(y) = \sum_{0 \leqslant i \leqslant j \leqslant k} b_{ijk} y_i y_j y_k.$$

Then a straightforward computation shows the following relations:

$$a_{011} \to a_{011} + 3d_{10}a_{111} + d_{20}a_{112}$$

$$a_{012} \to a_{012} + 2d_{10}a_{112} + 2d_{20}a_{122}$$

$$a_{022} \to a_{022} + d_{10}a_{122} + 3d_{20}a_{222}.$$

$$a_{111} \to a_{111}$$

$$a_{112} \to a_{112}$$

$$a_{122} \to a_{122}$$

$$a_{222} \to a_{222}.$$

The same relations hold with a_{ijk} replaced by b_{ijk} throughout. Using the same argument as above we can conclude the following.

PROPOSITION: *Let W be the affine space of dimension 26 parametrizing the canonical forms F and G. Then a point $w \in W$ is properly stable for the action of H_u if the rank of the following matrix is two*:

$$M(F, G) = \begin{pmatrix} a_1 & b_1 & 3a_{111} & 2a_{112} & a_{122} & 3b_{111} & 2b_{112} & b_{122} \\ a_2 & b_2 & a_{112} & 2a_{122} & 3a_{222} & b_{112} & 2b_{122} & 3b_{222} \end{pmatrix}$$

An interesting connection exists between this matrix and the following result of Catanese.

THEOREM [10: 3.1]: *Let S be a smooth weighted complete intersection of type* (6,6) *defined by the vanishing of the canonical equations F and G.*

Then the local universal deformation space has a smooth base of dimension 18 *and the differential of the local period mapping* μ *is injective if and only if the determinant of the following (generically invertible) matrix* $\Delta(\alpha, \beta, A, B)$ *is non zero.*

$$\Delta(\alpha, \beta, A, B) = \begin{bmatrix} a_1 & 0 & 0 & 3a_{111} & 0 & a_{112} \\ 0 & a_2 & 0 & a_{122} & 0 & 3a_{222} \\ a_2 & a_1 & 0 & 2a_{112} & 0 & 2a_{122} \\ c_1 & 0 & 3c_{111} & 0 & c_{112} & 0 \\ 0 & c_2 & c_{122} & 0 & 3c_{222} & 0 \\ c_2 & c_1 & 2c_{112} & 0 & 2c_{112} & 0 \end{bmatrix} \quad \square$$

If the determinant is expanded along the last column then it can be readily seen that det $\Delta(\alpha, \beta, A, B) \neq 0$ implies $M(F, G)$ has rank 2. Thus if W_Δ is the open subset of W where det $\Delta(\alpha, \beta, A, B) \neq 0$ then W_Δ is H_u stable (because the entries of $\Delta(\alpha, \beta, A, B)$ are invariants) and a quasi-factorial quotient of W_Δ by H_u exists. In fact W_Δ/H_u is rational since $W_\Delta \to W_\Delta/H_u$ is a locally trivial principal H_u-bundle.

Now H/H_u is reductive and standard techniques can be applied to this group acting on W_Δ/H_u to construct (at least generically) a quotient.

REMARK: Catanese appeals to a theorem of Geiseker to show that a moduli space exists. The construction of moduli spaces of smooth varieties sitting as complete intersections in weighted projective spaces can be handled slightly differently however. This topic will be discussed by us elsewhere.

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