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STABLY FREE, PROJECTIVE RIGHT IDEALS

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A finitely generated module P over a ring R is called *stably free* if $P \oplus R^{(m)} \simeq R^{(n)}$ for some integers m and n. Of course, P is necessarily projective. If P is stably free but not free then we call it a *non-trivial stably free* module. If R is a commutative ring then it has long been known that ideals of R cannot be non-trivial stably free modules (see, for example, [4, Theorem 4.11, p 29]). However, if R is not commutative then it is also well known that non-trivial stably free right ideals can exist. For example, if R is a Weyl algebra, or the group ring of a poly (infinite cyclic) group or a polynomial extension in at least two variables over a division ring, then such right ideals do exist - see [13], [1] and [6] respectively. One of the aims of this paper is to provide other examples of this phenomenon. In particular, we prove:

THEOREM 2.6: *Enveloping algebras of non-abelian, finite dimensional Lie algebras always possess non-trivial, stably free right ideals.*

Thus stably free right ideals are a frequent occurrence over non-commutative rings. Unfortunately, the proofs in [1], [6] and [13] and the author's first proof of Theorem 2.6 are basically computational and so don't explain *why* these right ideals are so common. The second aim of this paper is to try to understand this question, which we do by abstracting as much of the argument as possible. In so doing, we are also able to provide a reasonably unified way of producing stably free right ideals over each of the four rings mentioned above.

Let us describe the idea behind our approach. The method that always seems to work is to take two regular elements, a and b, in one's ring A such that "by virtue of the fact that a and b do not commute", $A = aA + bA$. Then $K = aA \cap bA$ is obviously a right ideal satisfying $K \oplus A \simeq A \oplus A$. Furthermore, the noncommutativity of a and b usually enables one to show that K is not cyclic; i.e., K is a non-trivial, stably free right ideal. We need to formalise this approach, and the appropriate setting is that of an Ore extension $S = R[x; \sigma, \delta]$ or skew Laurent extension $S = R[x, x^{-1}; \sigma]$ (see Section 1 for the definitions). For the remainder of this introduction we will concentrate on Ore extensions, since the method for skew Laurent extensions is similar and produces no new results. Now there are two obvious situations when $S = R[x; \sigma, \delta]$ cannot have non-trivial, stably free right ideals; viz when

(0.1) S is commutative, or

(0.2) R is a division ring, as S is now a principal ideal domain.

One way of excluding these two cases is to demand that there exists a non-unit $r \in R$ and some $s \in R$ such that r and $x + s$ do not commute. The trick is to go a bit further and demand

there exists a non-unit $r \in R$ *and some* $s \in R$ *such that* $S = rS + (x + s)S$. $(*)$

It is an easy exercise to show that $(*)$ will never hold if r and $x + s$ commute, and so $(*)$ can be regarded as a formal way of producing two elements of S that "by reason of their non-commutativity" generate the ring as a right ideal. Furthermore, this condition is sufficient for our purposes.

THEOREM 1.2: *Let* R *be a Noetherian domain and suppose that* $S = R[x; \sigma, \delta]$ *satisfies* $(*)$. *Then* $K = rS \cap (x + s)S$ *is a non-trivial, stably free right ideal of* S.

This theorem and its analogue for skew Laurent extensions can then be used to obtain the desired module over each of the four rings mentioned at the beginning of the introduction. This follows from the fact that the given ring contains an Ore extension or skew Laurent extension. (Actually, there is one exceptional case that has to be dealt with separately; if U is the enveloping algebra of a non-split semisimple Lie algebra, then U may not contain an Ore extension, and so a slightly different method has to be used.)

Unfortunately, this method is also not sufficient to deal with all Ore extensions. For example, if $(R, \mathfrak{m})$ is a local commutative domain and $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$, then $(*)$ can never hold for $S = R[x; 1, \delta]$. Indeed, if $R = k[y]_{(y)}$ and $\delta(y) = y$, then every projective module over $S = R[x; 1, \delta]$ is free. (This ring may also be viewed as a localisation of the enveloping algebra of the 2-dimensional, solvable Lie algebra.) The final section of this paper, therefore, considers Ore extensions of local rings and proves the following.

COROLLARY 4.6: *Let* $(R, \mathfrak{m})$ *be a commutative, Noetherian, regular local domain and* δ *a non-zero derivation of* R. *Then every stable free right ideal of* S *is free if and only if* (i) $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$ *and* (ii) $K \dim R = 1$.

The method used here is similar to the earlier one, except one now considers elements r, s and t of R that satisfy $rS + (xt + s)S = S$.

Unfortunately, there seems to be no easy general result along the lines of Theorem 1.2 that will cover this case.

1. Ore and skew Laurent extensions

In this section we will prove Theorem 1.2 of the introduction and give some of its easier corollaries. The applications to group rings and enveloping algebras will be given in Section 2.

We begin with the appropriate notation. Let R be a domain, σ and automorphism of R and δ a σ-derivation; that is, $\delta(ab) = a\delta(b) + \delta(a)b^{\sigma}$ for a and b in R. Then the *Ore extension* $S = R[x; \sigma, \delta]$ is the ring that additively is isomorphic to the polynomial extension of R in one variable, but multiplication is defined by

$$rx = xr^{\sigma} + \delta(r).$$

We will always use σ for an automorphism and δ for a σ-derivation and so we may, without confusion, write $R[x; \sigma]$ for $R[x; \sigma, 0]$ and $R[x; \delta]$ for $R[x; 1, \delta]$. Note that by inverting x in the ring $R[x; \sigma]$ one obtains the *skew Laurent extension* $S = R[x, x^{-1}; \sigma]$. In order to save repetitions we will, for example, use the phase "let $S = R[x; \sigma, \delta]$" to mean "let R be a domain with an automorphism σ and a σ-derivation δ and let $S = R[x; \sigma, \delta]$". Any element $a \in S = R[x; \sigma, \delta]$ can be uniquely written as $a = \sum_0^n x^i a_i$ for some $a_i \in R$ with $a_n \neq 0$. In this case, n is the *degree* of a, written $\deg a = n$. The element a is called *monic* if $a_n = 1$. Similarly, if $a = \sum_m^n x^i a_i \in S = R[x, x^{-1}; \sigma]$ with $a_m \neq 0$ and $a_n \neq 0$, then $\deg(a) = n - m$. Again, a is *monic* if $a_n = 1$. The reason for the two distinct definitions is the following easy lemma, the proof of which is left to the reader.

LEMMA 1.1: *If a and b are non-zero elements of either $S = R[x; \sigma, \delta]$ or $S = R[x, x^{-1}; \sigma]$, then $\deg(ab) = \deg(a) + \deg(b)$. In particular, units of S must have degree zero.*

As remarked in the introduction, when we consider an Ore extension we will need the condition:

If $S = R[x; \sigma, \delta]$, then there exists a non-unit $r \in R$ and some $s \in R$ such that $S = rS + (x+s)S$. $(*)$

Suppose that one now considers the skew Laurent extension $S = R[x, x^{-1}; \sigma]$. Then $(*)$ is automatically satisfied if $s = 0$ or $s = r$ and so it must be inadequate for our purposes. (Consider the case (0.1) of the introduction.) In this case we will use the slightly stronger hypothesis:

If $S = R[x, x^{-1}; \sigma]$, *then there exists a non-unit* $r \in R$ *and some*

$$s \in R \text{ such that } S = rS + (x+s)S \text{ but } sr^{\sigma} \notin rR. \tag{$\substack{*\\*}$}$$

We can now prove the main result of this section.

THEOREM 1.2: *Let* R *be a Noetherian domain and suppose that either* (i) $S = R[x; \sigma, \delta]$ *is an Ore extension of* R *that satisfies* $(*)$ *or* (ii) $S = R[x, x^{-1}; \sigma]$ *is a skew Laurent extension of* R *that satisfies* ($\substack{*\\*}$). *Set* $K = \{g \in S : rg \in (x+s)S\}$. *Then* K *is a non-trivial, stably free right ideal of* S, *satisfying* $K \oplus S \simeq S \oplus S$.

REMARK: Since r is a regular element of S, $K \simeq rS \cap (x+s)S$ and so we could equally well work with the latter module.

PROOF: With the exception of the last few lines, the proof is identical in the two cases. Note that S is a domain. There exists a short exact sequence with the obvious homomorphisms

$$0 \to K \to rS \oplus (x+s)S \to rS + (x+s)S \to 0,$$

which ensures that $K \oplus S \simeq S \oplus S$. This shows that K has all the desired properties, with the exception of proving that K is not free. This is equivalent to showing that K is not cyclic.

We next find some elements in K. Since R is a Noetherian ring, the set of monic elements of S forms an Ore set [8]. So there exists a monic element, say $f \in S$, such that $rf \in (x+s)S$. Thus $f \in K$. Secondly,

$$rx = xr^{\sigma} + \delta(r) = (x+s)r^{\sigma} + (\delta(r) - sr^{\sigma})$$

(where $\delta(r) = 0$ when $S = R[x, x^{-1}; \sigma]$). Now R is an Ore domain, so there exist r_1 and r_2 in R, with $r_1 \neq 0$, such that $(\delta(r) - sr^{\sigma})r_1 = rr_2$. Thus

$$r(xr_1 - r_2) = (x+s)r^{\sigma}r_1 \in (x+s)S.$$

Thus $g = xr_1 - r_2 \in K$.

The existence of these two elements in K is enough to ensure that K is not cyclic. For, suppose that $K = kS$. Since $xr_1 - r_2 \in K$, Lemma 1.1 shows that $\deg k \leqslant 1$. Multiplying k by a unit if necessary, we may therefore suppose that $k = x\lambda + \mu$ for some $\lambda, \mu \in R$. Since $rk \in (x+s)S$, Lemma 1.1 shows that $\deg rk \geqslant 1$ and hence $\deg k \geqslant 1$. Thus $\lambda \neq 0$. Indeed, λ must be a unit. For, $f \in K$; say $f = kd$ with $f = x^{\mathrm{u}} + \sum_s^{\mathrm{u}-1} x^t f_t$ and $d = \sum_t^{\mathrm{u}-1} x^t d_t$. Then, comparing coefficients of x^{u} gives $1 = \lambda^{\sigma^{\mathrm{u}-1}} d_{\mathrm{u}-1}$ which forces λ to be a unit. Multiplying by λ^{-1}, we may suppose that $k = x + \mu$. As $k \in K$ this implies that

$$rk = r(x+\mu) = xr^{\sigma} + \delta(r) + r\mu = (x+s)t \tag{1.3}$$

for some $t \in S$. Thus $t = r^{\sigma}$ and $sr^{\sigma} = r\mu + \delta(r)$.

We claim that this contradicts the hypotheses $(*)$ and $(\ast\!\ast)$. When $S = R[x, x^{-1}; \sigma]$ is the skew Laurent extension, this is exactly what is excluded (remember that $\delta = 0$ in this case). Now consider $S = R[x; \sigma, \delta]$. Then by $(*)$,

$$1 = (x + s)a + rb \tag{1.4}$$

for some a and b in S. We may write $b = (x + \mu)c + d$ where $c \in S$ but $d \in R$. Combining equations (1.3) and 1.4) gives

$$1 = (x + s)(a + tc) + rd.$$

By comparing degrees, this forces $a + tc = 0$ and $1 = rd$. Finally, this contradicts the fact that r is not a unit and completes the proof.

The elements f and g of K that were found in the above proof are clearly not unique and do not necessarily generate K. However, in each of the special cases that concern us there are obvious 'smallest' elements of these two forms and these will generate K. In Section 3, explicit generators will be given for the module K for each of these cases. We end this section with several easy applications of Theorem 1.2.

COROLLARY 1.5. [6] *Let D be a division ring that is not commutative and $S = D[x_1, \dots, x_n]$ a polynomial extension of D in $n \geqslant 2$ commuting indeterminates. Then S has a non trivial, stably free right ideal.*

PROOF: Pick elements $a, b \in D$ such that $[a, b] = ab - ba = c \neq 0$. Then $[x_n + a, x_{n-1} + b] = c$ is a unit of S. The corollary therefore follows from the theorem by taking

$$R = D[x_1, \dots, x_{n-1}] \subset S, \quad r = x_{n-1} + b,$$

$$x = x_n \quad \text{and} \quad s = a.$$

The next corollary is a useful special case of Theorem 1.2 which will in fact be used for all future applications of that Theorem to Ore extensions.

COROLLARY 1.6: *Let R be a Noetherian domain and $S = R[x; \sigma, \delta]$. Suppose that there exists a non-unit $r \in R$ such that $\Sigma \delta^i(r) R = R$. Then $K = rS \cap xS$ is a non-trivial, stably free right ideal of S.*

REMARK: One can even obtain Corollary 1.5 from Corollary 1.6. For, in the notation of Corollary 1.5, set $y = x_n + a$ and $\delta = [?, a]$. Then $S \simeq R[y; 1, \delta]$ and $\delta(r) = c$ is a unit when $r = x_{n-1} + b$.

PROOF: Note that $\delta^{i+1}(r) = \delta^i(r)x - x(\delta^i(r))^\sigma$ for any integer i. Since

$\Sigma\delta^i(r)R = R$, this implies that $rS + xS = S$. Thus the Corollary follows from Theorem 1.2 and the remark thereafter.

COROLLARY 1.7 [13]: *Let $S = A_n(D)$ be the n^{th} Weyl algebra over a Noetherian domain D. Then S has a non-trivial, stably free right ideal.*

PROOF: If $E = A_{n-1}(D)$ then S is defined to be $S = E[y][x; -\partial/\partial y]$. Now apply Corollary 1.6 with $R = E[y]$ and $r = y$.

If D is a field of characteristic zero then $A_n(D)$ is known to be a simple ring. Corollary 1.7 can be extended to a large number of similar simple rings as follows.

COROLLARY 1.8: *Let δ be a derivation (respectively σ an automorphism) of a commutative, Noetherian domain R and suppose that $S = R[x; \delta]$ (respectively $S = R[x, x^{-1}; \sigma]$) is a simple ring. Then:*

(*i*) *If R is not a field then S has a non-trivial, stably free right ideal. Indeed, $K = rS \cap (x+1)S$ is a non-trivial, stably free right ideal of S whenever r is a non-zero non-unit from R.*

(*ii*) *S is a principal ideal domain if and only if R is a field.*

PROOF: If R is a field then S is a principal ideal domain by the obvious analogue of Euclid's algorithm. Thus we may suppose that R is not a field, in which case it suffices to prove (i). The fact that S is a simple extension of a Noetherian domain implies that δ (respectively σ) leaves no ideal of R invariant. Pick a non-zero, non-unit $r \in R$ and consider $S = R[x, x^{-1}; \sigma]$. Then $J = \Sigma_0^\infty r^{\sigma^i}R$ is an ideal of R left invariant by σ. Thus $J = R$. The equations

$$r^{\sigma^i}(x+1) - (x+1)r^{\sigma^{i+1}} = r^{\sigma^i} - r^{\sigma^{i+1}} \qquad \text{for} \quad i \geqslant 0$$

ensure that $rS + (x+1)S \supseteq JS = S$. Also, $r^\sigma \notin rR$ as rR cannot be σ-invariant. Thus ($\ast$) is satisfied and the Corollary follows from the Theorem. A similar argument works for the ring $S = R[x; \delta]$.

2. Enveloping algebras and group rings

In this section we show how to apply Theorem 1.2 to obtain non-trivial, stably free right ideals over enveloping algebras of finite dimensional Lie algebras and group rings of poly (infinite cyclic) groups. In both cases, the proof is in three parts. Prove the result for certain special rings; show that the general enveloping algebra or group ring contains one of these special rings; and finally, pull the result up to this general case.

LEMMA 2.1. *Let $\mathfrak{h}$ be a finite dimensional, abelian-by-(one dimensional) Lie algebra over a division ring D. If $\mathfrak{h}$ is not abelian, then $\mathrm{U}(\mathfrak{h})$ has a non-trivial, stably free right ideal.*

PROOF: By hypothesis, there exists an ideal $\mathfrak{r}$ of $\mathfrak{h}$ such that $\mathfrak{r}$ is abelian and $\mathfrak{h}/\mathfrak{r}$ is one-dimensional. Write $R = \mathrm{U}(\mathfrak{r}) \subset S = \mathrm{U}(\mathfrak{h})$ and pick $x \in \mathfrak{h} \setminus \mathfrak{r}$. Then $S \simeq R[x; \delta]$ where δ is the non-zero derivation $[?, x]$. Pick $y_0 \in \mathfrak{r}$ such that $[y_0, x] \neq 0$. Define, inductively, $y_i = [y_{i-1}, x]$ for $i \geqslant 0$. As $\mathfrak{h}$ is finite dimensional there exists an integer n such that $\{y_0, \dots, y_n\}$ are linearly independent but $y_{n+1} \in \Sigma D y_i$.

If $y_{n+1} = 0$, set $r = y_n y_{n-1} + 1$. Then $\delta(r) = [r, x] = y_n^2$. As $\mathrm{U}(\mathfrak{r})$ is commutative, $1 = \delta(r) y_{n-1}^2 - r(y_n y_{n-1} - 1) \in rR + \delta(r)R$. Corollary 1.6 can now be used to show that $K = rS \cap xS$ is a non-trivial, stably free right ideal.

If $y_{n+1} \neq 0$ then $y_{n+1} = \Sigma_0^n \lambda_i y_i$ for some $\lambda_i \in D$, not all of which are zero. By replacing y_0 by an appropriate y_i, we may suppose that $\lambda_0 \neq 0$. In this case take $r = y_0 + 1$ and write $J = \Sigma \delta^i(r) R$. Then the elements $y_1 = [r, x]$, $y_i = [y_{i-1}, x]$ for $2 \leqslant i \leqslant n$ and $\Sigma \lambda_i y_i = [y_n, x]$ are all contained in J. Since λ_0 is a unit, this implies that $y_0 \in J$. Thus $J = R$ and, again, the result follows from Corollary 1.6.

LEMMA 2.2: *Let $\mathfrak{g}$ be a non-abelian, finite dimensional Lie algebra over a division ring D. Suppose that either $\mathfrak{g}$ is solvable or that D is an algebraically closed field. Then $\mathfrak{g}$ contains an abelian-by-(one dimensional) sub-Lie algebra $\mathfrak{h}$ that is not abelian.*

REMARK: If $\mathfrak{g}$ is a non-solvable Lie algebra over a non-algebraically closed field then the Lemma may fail for $\mathfrak{g}$, and so it is not clear how to find an Ore extension inside $\mathrm{U}(\mathfrak{g})$. This happens, for example, when $\mathfrak{g} = SO_3(R)$. This case will be dealt with later.

PROOF: If $\mathfrak{g}$ is solvable then we may take $\mathfrak{h}$ to be any sub-Lie algebra of $\mathfrak{g}$ such that $\mathfrak{h}$ is minimal with respect to being non-abelian. For, $[\mathfrak{h}, \mathfrak{h}]$ is abelian. Furthermore, either $[\mathfrak{h}, \mathfrak{h}]$ is not central, in which case $\mathfrak{r} = [\mathfrak{h}, \mathfrak{h}]$ is the desired ideal of codimension one, or $[\mathfrak{h}, \mathfrak{h}]$ is central, in which case $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is 2-dimensional and we can take $\mathfrak{r} = [\mathfrak{h}, \mathfrak{h}] + Dy$ for any $y \in \mathfrak{h} \setminus [\mathfrak{h}, \mathfrak{h}]$.

Suppose, alternatively, that D is an algebraically closed field and $\mathfrak{g}$ is not nilpotent. In this case $\mathfrak{g}$ contains a copy of the 2-dimensional, solvable Lie algebra [3, Ex. 4, p. 54] (this follows easily from the fact that $\mathfrak{g}$ now has a Cartan subalgebra and a root space decomposition).

We next need to show how to pull non-free projective modules from $\mathrm{U}(\mathfrak{h})$ up to $\mathrm{U}(\mathfrak{g})$. This follows from the following result.

PROPOSITION 2.3: *Let $A \subset B$ be domains such that B is faithfully flat as a left A-module and satisfies*:

$$\textit{If } a \textit{ and } b \textit{ are non-zero elements of } B \textit{ such that } ab \in A, \textit{ then } a = a_1 c$$
$$\textit{and } b = c^{-1} b_1 \textit{ for some unit } c \in B \textit{ and elements } a_1, b_1 \in A. \quad (2.4)$$

Let P be a projective right ideal of A that is not cyclic. Then $PB \simeq P \otimes_A B$ is a projective right ideal of B that is not cyclic. If P is stably free then so is PB.

PROOF: The final assertion is a triviality. Since ${}_AB$ is flat, $PB \simeq P \otimes_A B$ and so is projective. Suppose that $PB = \alpha B$ for some $\alpha \in B$. Pick $p \neq 0 \in P$. Since $P \subset PB$, there exists $\lambda \in B$ such that $p = \alpha\lambda$. By (2.4), $\alpha = \alpha_1 c$ where $\alpha_1 \in A$ and c is a unit. Thus, replacing α by α_1, we may assume that $\alpha \in A$.

Since B is faithfully flat over A, given right ideals $I \subsetneqq J$ of A, then $IB \subsetneqq JB$. In particular $PB \cap A = P$ and $\alpha \in P$. But now $P = \alpha B \cap A = \alpha A$; giving the required contradiction.

LEMMA 2.5: *Let $\mathfrak{h}$ be a sub-Lie algebra of a finite dimensional Lie algebra $\mathfrak{g}$ over a field k. Then* $\mathrm{U}(\mathfrak{h}) \subset \mathrm{U}(\mathfrak{g})$ *satisfy the hypotheses of Proposition* 2.3.

PROOF: It is an easy exercise using the Poincaré-Birkhoff-Witt Theorem to show that $\mathrm{U}(\mathfrak{h})$ and $\mathfrak{U}(\mathfrak{g})$ are domains such that $\mathrm{U}(\mathfrak{g})$ is a free, and hence faithfully flat, $\mathrm{U}(\mathfrak{h})$-module. The same graded ring argument, but with [2, Behauptung 3.6] replacing the PBW, can be used to show that (2.4) will hold.

The last four results can now be combined to prove all but one special case of the following theorem.

THEOREM 2.6: *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k. Then the following are equivalent*:

(*i*) *$\mathfrak{g}$ is abelian,*

(*ii*) *every stably free right ideal of* $\mathrm{U}(\mathfrak{g})$ *is free,*

(*iii*) *every projective right ideal of* $\mathrm{U}(\mathfrak{g})$ *is free.*

REMARK: In the special case of the 2-dimensional, solvable Lie algebra, this result has also been proved independently by T.J. Hodges [unpublished].

PROOF: By [7, Theorem 3.2] every finitely generated projective $\mathrm{U}(\mathfrak{g})$-module is stably free, and so (ii) and (iii) are equivalent. If $\mathfrak{g}$ is abelian then stably free right ideals are free by [4, Theorem 4.11, p 29]. So we may suppose that $\mathfrak{g}$ is not abelian. If $\mathfrak{g}$ contains a non-abelian, solvable Lie algebra, then it follows from the first four results of this section that $\mathrm{U}(\mathfrak{g})$ does have a non-trivial, stably free right ideal.

This leaves the case when the only solvable subalgebras of $\mathfrak{g}$ are abelian. The problem with this case is, of course, that $\mathrm{U}(\mathfrak{g})$ presumably does not contain an Ore extension and so Theorem 1.2 cannot be applied. However, the proof is rather similar to that of Theorem 1.2 and so some of the details may be left to the reader. By Proposition 2.3 and

Lemma 2.5, we may replace $\mathfrak{g}$ by any sub-Lie algebra, minimal with respect to bring non-abelian. We may therefore assume that $\mathfrak{g}$ is not solvable, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and that $\mathfrak{g}$ is generated as a Lie algebra by any two elements that do not commute. In particular, pick $x, y \in \mathfrak{g}$ such that $[x, y] \neq 0$. Then

$$[x, y+1] = [x, y] = z,$$

and $z \notin kx + ky$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, this equation ensures that y is contained in the Lie ideal of $\mathrm{U}(\mathfrak{g})$ generated by x and $y+1$. Thus, writing $S = \mathrm{U}(\mathfrak{g})$, we can see that $S = xS + (y+1)S$ and $K = \{f\colon xf \in (y+1)S\}$ is a stably free right ideal of S.

As in the proof of Theorem 1.2, we next choose some distinguished elements from K. Since $k[y]^*$ is an Ore set in S [2, Satz 3.3], $xf \in (y+1)S$ for some $f \neq 0 \in k[y]$. Similarly, $zg = xh$ for some $g \neq 0 \in k[x]$. Thus

$$x\{(y+1)g - h\} = (y+1)xg \tag{2.7}$$

and $(y+1)g - h \in K$. Now suppose that $K = aS$. Since $f \in K \cap k[y]$, Proposition 2.3 implies that $a \in k[y]$. Now g and h have the same total degree and so the leading term of $(y+1)g - h$ is just λyx^m for some $\lambda \in k$ and integer m. Since $(y+1)g - h \in aS$, this is only possible if a has degree $\leqslant 1$. Clearly a is not a unit, so we may suppose that $a = y + \mu$ for some $\mu \in k$. But $a \in K$ and so $x(y+\mu) = (y+1)t$, for some $t \in S$. This contradicts the fact that $z \notin kx + ky$ and completes the proof.

We now repeat the above process for group rings of poly (infinite cyclic) groups, thereby giving another proof of Artamonov's Theorem [1]. There are similarities between the two proofs and, in particular, equation (2.9) is due to him. However, our more general approach does provide a less technical proof.

Lemma 2.8: *Let G be a non-abelian group with a normal subgroup H such that H is free abelian of finite rank and G/H is infinite cyclic. Then, given any Noetherian domain k, the group ring $S = kG$ has a non-trivial, stably free right ideal.*

Proof: Write $H = \langle y_1, \ldots, y_n \rangle$ and $G = H\langle x \rangle$. Then $S = kG \simeq R[x, x^{-1}; \sigma]$, where $R = kH$ and σ is defined by $r^\sigma = x^{-1}rx$ for $r \in R$. By reordering the y_i's we may suppose that $y_1^\sigma \neq y_1$. Now, for any $r \in R$,

$$rS + (x + r^{\sigma^{-1}})S \ni (x + r^{\sigma^{-1}})(x - r)x^{-2} + rr^{\sigma^{-1}}x^{-2} = 1. \tag{2.9}$$

Thus, in order to apply Theorem 1.2 and complete the proof we need only find a non-unit $r \in R$ such that

$$r^{\sigma^{-1}} r^\sigma \notin rR. \tag{2.10}$$

If $y_1^\sigma \neq y_1^{-1}$ then, since $(1-y_1)R$ is a prime ideal of R, it is easy to see that (2.10) holds with $r = 1 - y_1$. If $y_1^\sigma = y_1^{-1}$ then (2.10) holds for $r = 1 + y_1 + y_1^3$.

LEMMA 2.11: *Let A be a domain and suppose that either $B = A[x;\ \sigma, \delta]$ or $B = A[x, x^{-1};\ \sigma]$. Then $A \subset B$ satisfy the hypothesis of Proposition* 2.3.

PROOF: This follow by an easy induction on degree.

THEOREM 2.12 [11]: *Let G be a non-abelian, poly (infinite cyclic) group and k a Noetherian domain. Then $S = kG$ has a non-trivial, stably free right ideal.*

PROOF: By hypothesis, there exists a subnormal chain

$$G_0 = (1) \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$$

such that each G_i/G_{i+1} is infinite cyclic. Let j be the minimal i such that G_i is not abelian. By Lemma 2.8, kG_j has a non-trivial, stably free right ideal. The Theorem now follows, by induction, from Lemma 2.11 and Proposition 2.3.

By making a suitable, rather technical generalisation of Theorem 1.2 one can show that Theorem 2.12 holds for the group ring AG where A is any ring and that Theorem 2.6 holds for $B \otimes_k U(\mathfrak{g})$ where B is any ring containing the field k. However, we feel that the present results are sufficiently general.

In [5] Lewin proves that $\mathbb{Q}G$ has a non-trivial, stably free right ideal whenever G is a torsion-free, polycyclic-by-finite group that is not nilpotent (of course, by combining Artamonov's result with Lewin's one can drop the nilpotence condition). It is not clear how to generalise Theorem 1.2 in order to incorporate Lewin's result.

Proposition 2.3 shows that non-free, projective right ideals remain non-free under various extensions. This is slightly surprising since the corresponding result for projective modules of rank greater than one is false. This is illustrated by the following example from [11].

EXAMPLE 2.13: Let $R = \mathbb{R}[x_1, \ldots, x_{2n+1}]/(1 - \Sigma x_i^2)$ be the real $2n$-sphere and define a homomorphism θ: $R^{(2n+1)} \to R$ by $\theta(a_1, \ldots, a_{2n+1}) = \Sigma x_i a_i$. Then $P = \ker \theta$ is a nontrivial, stably free module of rank $2n$ [12, p. 269]. Let $S = R[y][x;\ \partial/\partial y]$ be the first Weyl algebra over R. Then $P \otimes_R S$ is a free S-module.

REMARK: Clearly, given any *commutative* polynomial extension $T = R[y_1, \ldots, y_n]$ of R, then $P \otimes_R T$ remains non-free.

PROOF: Write $d = (x_1, \ldots, x_{2n+1}) \in R^{(2n+1)}$. Then $R^{(2n+1)} = dR \oplus Q$ with

$Q \simeq P$. Similarly, $S^{(2n+1)} = dS \oplus QS$ with $QS \simeq P \otimes_R S$. Given a and b in S write

$$I_{ab} = \Sigma_1^{2n-2} S x_i + S(x_{2n-1} + a x_{2n+1}) + S(x_{2n} + b x_{2n+1}).$$

In order to prove that QS is free, it suffices to find $a, b \in S$ such that $I_{ab} = S$ [4, Corollary 4.9, p. 24 and Proposition 5.3, p. 35]. However,

$$x_{2n+1}^2 = [x_{2n-1} + y x_{2n+1}, x_{2n} + x x_{2n+1}].$$

It follows that $I_{yx} = S$ and $P \otimes S$ is free.

3. Generators of projective modules

Theorem 1.2, by its generality, does not provide explicit generators for the projective module that it produces. However, for each of the rings to which that result has been applied, one can easily write down the required two generators, as we do in this section. In each case $K = \{f: rf \in (x+s)S\}$ is the projective module described by the appropriate corollary.

(i) If $S = A_n(D)$ is as in Corollary 1.7, then $K = x^2 S + (xy+1)S$.

(ii) If $S = D[x_1, \ldots, x_n]$ is as in Corollary 1.5, then $K = \{(x_n + b)(x_{n-1} + a) - c\}S + (x_n + b)c^{-1}(x_n + b)S$.

(iii) Let $\mathfrak{h} = kx + ky + kz$, where $[y, x] = z$, be the 3-dimensional nilpotent Lie algebra over a field k and set $S = \mathrm{U}(\mathfrak{h})$; as in Lemma 2.1. Then $K = x^2 S + (x(yz+1) - z^2)S$.

(iv) Let $\mathfrak{h} = \mathrm{k}x + \mathrm{k}y$, where $[y, x] = y$, be the 2-dimensional solvable Lie algebra over k and set $S = \mathrm{U}(\mathfrak{h})$, as in Lemma 2.1. Then

$$K = (x^2 - x)S + \{(x-1)(y+1) + 1\}S.$$

At least when k is algebraically closed, Lemma 2.2 shows that the Lie algebras of (iii) and (iv) are the only ones that one need consider. The general case is left to the reader.

(v) Let $S = kG$ as in Lemma 2.8. Then

$$K = (x^2 - r^{\sigma^{-1}} r^{\sigma^2})S + (xr + r^{\sigma} r^{\sigma^{-1}})S.$$

In each case the assertion is proved as follows. It is easy to check that the given two elements belong to K. Notice that one of them, say f, is monic of degree 2 while the other, say g, has degree one. Now suppose that $h \in K$. As $f \in K$ we may assume that $\deg h \leqslant 1$. If $h \notin gS$ then it is easily checked that $hR + gR$ contains a non-zero element from the subring R. But, as we saw in the proof of Theorem 1.2, this is not possible.

4. Ore extensions of local rings

One might be tempted to suppose that Theorem 1.2 can be applied to any Ore extension (apart from the two trivial exceptions (0.1) and (0.2)). Unfortunately, this is not the case. The easiest example is when $(R, \mathfrak{m})$ is a local, commutative, Noetherian domain, δ satisfies $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$ and $S = R[x; \delta]$. For, $\mathfrak{m}S$ is now an ideal of S and $S/\mathfrak{m}S$ is isomorphic to an Ore extension of the field $R/\mathfrak{m}$. Thus given a non-unit $r \in R$ and any $s \in R$,

$$rS + (x+s)S \subseteq \mathfrak{m}S + (x+s)S \neq S.$$

In other words, $(*)$ cannot hold in this case and so Theorem 1.2 cannot be applied. In this section we therefore study projective right ideals over Ore extensions of local rings. The first result shows that the rings in (0.1) and (0.2) are not the only Ore extensions for which stably free right ideals are free.

PROPOSITION 4.1: *Let $(R, \mathfrak{m})$ be a local, commutative, principal ideal domain and δ a derivation of R such that $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$. Then every projective right ideal of $S = R[x; \delta]$ is cyclic.*

REMARK: The proof of this result can be generalised to show that every projective S-module is free.

PROOF: If R is a field, then the result is obvious. Thus we may suppose that $K \dim R = 1$. Suppose that P is a projective, non-free, right ideal of S. The aim of the proof is to show that one can assume that $\mathfrak{m} \subset P$, after which the contradiction will be easy to establish.

Write $\mathfrak{m} = yR$ for some $y \in R$ and F for the field of fractions of R. Then the localisation $T = S_{\{y^n\}} \simeq F[x; \delta]$ is a principal ideal domain. Thus $PT \simeq T$. By [10, Lemma 5.3] this means we can, by replacing P by an isomorphic right ideal, assume that $y^n \in P$ for some n. We may further suppose that $y^{n-1} \notin P$. Since P is not cyclic, $P \neq y^nS$ and it follows that $P \cap y^{n-1}S \neq y^nS$. For, $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$ and so $y^sS = Sy^s$ for each integer s. So, pick $p \in (P \cap y^{r-1}S) \backslash y^rS$ for some $r \leqslant n$. Then $py^{n-r} \in (P \cap y^{n-1}S) \backslash y^nS$.

Given a module M over a ring A, write $hd_A M$ for its homological dimension. Then

$$hd_S(P + y^{n-1}S) < \mathrm{gl.dim}\, S \leqslant 1 + \mathrm{gl.dim}\, R = 2;$$

by [9, Corollary 1.8]. Thus $hd_S(P + y^{n-1}S) \leqslant 1$. *However, there exists a short exact sequence*

$$0 \to P \cap y^{n-1}S \to P \oplus y^{n-1}S \to P + y^{n-1}S \to 0.$$

Thus $P \cap y^{n+1}S$ is projective. Since $P \cap y^{n-1}S \supsetneqq y^nS$ but $y^{n-1} \notin P$, $P \cap y^{n-1}S$ cannot be cyclic. We may, therefore, replace P by $y^{1-n}(P \cap y^{n-1}S)$ and assume that P is a non-cyclic, projective right ideal containing y.

Let $\alpha \in P \backslash yS$. Since R/yR is a field, α may be assumed to be monic. In particular, S/P is a non-zero, finitely generated R-module which, since $y \in P$, is also torsion as an R-module. Thus, by [9, Corollary 1.7],

$$hd_S(S/P) = 1 + hd_R(S/P) \geqslant 2.$$

This contradicts the fact that P is projective and completes the proof.

We now turn to local rings of Krull dimension greater than one, for which the following lemma is needed. The height of an ideal I will be denoted by htI.

LEMMA 4.2: *Let* $(R, \mathfrak{m})$ *be a local, commutative, Noetherian domain with* $K \dim R \geqslant 2$. *Suppose that* δ *is a non-zero derivation of* R *such that* $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$. *Then there exists* $r \in \mathfrak{m}$ *such that* $ht(rR + \delta(r)R) \geqslant 2$.

PROOF: The proof breaks into two cases according to whether $\mathbb{Q} \subseteq R$ or not. Suppose first that $\mathbb{Q} \not\subseteq R$. Then either char $R = p > 0$ or $\mathbb{Z} \subset R$ with $\mathbb{Z} \cap \mathfrak{m} \neq 0$. In either case, there exists $a \in \mathfrak{m}$ with $\delta(a) = 0$. There also exists $b \in \mathfrak{m}$ such that $\delta(b) \neq 0$ but $ht(aR + bR) \geqslant 2$. For, pick any $c \in \mathfrak{m}$ such that $ht(aR + cR) \geqslant 2$. If $\delta(c) \neq 0$ set $b = c$. If $\delta(c) = 0$ then take $b = c + ad$ where d is any element of R for which $\delta(d) \neq 0$.

For any integer n, $\delta(b + a^n) = \delta(b)$. So let $Q_1, \ldots, Q_m$ be prime ideals minimal over $\delta(b)$. In order to complete the proof we need to find n such that $r = b + a^n \notin \cup Q_i$. If no such n exists, then there exist integers $u < v$ and i such that both $b + a^u$ and $b + a^v$ belong to Q_i. Thus $a^u - a^v \in Q_i$. Since $1 - a^{v-u}$ is a unit this forces $a^u \in Q_i$ and $b \in Q_i$. By the choice of a and b this means that $htQ_i > 1$; contradicting the principal ideal theorem.

Now suppose that $\mathbb{Q} \subseteq R$. Pick $a \in \mathfrak{m}$ with $\delta(a) \neq 0$ and some $b \in \mathfrak{m}$ such that $ht(aR + bR) \geqslant 2$. We now use the idea behind the first part of the proof, using the elements $a^{2^n} + b^{2^n}$. Suppose that P is a height one prime ideal containing both $a^{2^n} + b^{2^n}$ and $\delta(a^{2^n} + b^{2^n})$. Then

$$P \ni \delta(a^{2^n} + b^{2^n})b - 2^n(a^{2^n} + b^{2^n})\delta(b) = 2^n a^{2^n - 1}(\delta(a)b - a\delta(b)).$$

Since $\mathbb{Q} \subseteq R$ and $a \notin P$ this forces $\delta(a)b - a\delta(b) \in P$. Thus, as before, either there exists n such that $r = a^{2^n} + b^{2^n}$ satisfies the conclusion of the lemma, or there exists a height one prime ideal P which contains $a^{2^u} + b^{2^u}$ and $a^{2^v} + b^{2^v}$ for some $v > u \geqslant 1$. But 2^{v-u} is even. Thus, in the latter case,

$$(a^{2^v} - b^{2^v}) \in (a^{2^u} + b^{2^u})R \subseteq P.$$

Thus P contains $2a^{2^v}$ and $2b^{2^v}$; that is, $aR + bR \subseteq P$. This contradicts the choice of a and b and completes the proof.

THEOREM 4.3: *Let $(R, \mathfrak{m})$ be a local, commutative, Noetherian domain with $K\dim' R \geqslant 2$. Suppose that δ is a non-zero derivation of R satisfying $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$. Then $S = R[x; \delta]$ has non-trivial, stably free right ideals.*

PROOF: Given $a \in R$, we will write a' for $\delta(a)$. Pick an element $r \in \mathfrak{m}$ by Lemma 4.2, set $\alpha = r'x + 1 + r''$ and consider $I = rS + \alpha S$. Then $(r')^2 = [r, \alpha] \in I$. Thus

$$r' = \alpha r' - (r')^2 x \in I.$$

In particular, $1 + r'' \in I$. However, $r'' \in \delta^2(\mathfrak{m}) \subseteq \mathfrak{m}$ and so $1 + r''$ is a unit. Thus $I = S$.

As usual, this quickly implies that $K = \{f\colon rf \in \alpha S\}$ is a stably free right ideal of S. It remains to show that K is not cyclic and we start by finding some elements of K. First,

$$\alpha r^2 = r\alpha r - (r')^2 r = r\{\alpha r - (r')^2\}.$$

Thus $\alpha r - (r')^2 \in K$. Next, set $u = 1 + r''$. Then, either by inverting the proof that $I = S$, or directly,

$$1 = \alpha\{u^{-1} - r'xu^{-1} - rx^2u^{-1}\} + r\alpha x^2 u^{-1}.$$

Thus $r\alpha x^2 u^{-1}\alpha \in \alpha S$ and $\alpha x^2 u^{-1}\alpha \in K$. Expanding these two elements from K we see that

$$\alpha r - (r')^2 = xrr' + r(1 + 2r'') - (r')^2 \tag{4.4}$$

and

$$\alpha x^2 u^{-1}\alpha = x^4 (r')^2 u^{-1} + \text{lower order terms}. \tag{4.5}$$

Suppose that K is cyclic; say $K = sS$. By (4.4) $\deg s \leqslant 1$; say $s = x\lambda + \mu$. Since $\alpha S \cap R = 0$, clearly $\lambda \neq 0$. By (4.4) and (4.5) we see that $rr' \in \lambda R$ and $(r')^2 \in \lambda R$. Thus $\lambda R \supseteq r'T$ where $T = rR + r'R$ is, by hypothesis, an ideal of height at least two. Now

$$xrr' + r(1 + 2r'') - (r')^2 = (x\lambda + \mu)t$$

for somt $t \in S$. By comparing degrees, $t \in R$. Thus $\lambda t = rr'$ and $\mu t = r(1 + 2r'') - (r')^2$. However

$$\lambda t T = rr'T \subseteq r\lambda R,$$

and so $tT \subseteq rR$. Therefore $\{r(1+2r'') - (r')^2\}T = \mu tT \subseteq \mu rR \subseteq rR$. Thus $(r')^2 T \subseteq rR$. Finally, this says that

$$T^3 \subseteq rR + (r')^2 T \subseteq rR.$$

Since $htT^3 \geqslant 2$, this contradicts the principal ideal theorem and completes the proof.

COROLLARY 4.6: *Let $(R, \mathfrak{m})$ be a commutative, Noetherian, regular, local domain and δ a non-zero derivation of R. Then every stably free right ideal of $S = R[x; \delta]$ is free if and only if (i) $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$ and (ii) K dim $R \leqslant 1$.*

PROOF: If $\delta(\mathfrak{m}) \not\subseteq \mathfrak{m}$ then pick $r \in \mathfrak{m}$ such that $\delta(r) \notin \mathfrak{m}$ and apply Corollary 1.6. If $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$ but K dim $R \geqslant 2$, apply Theorem 4.3. Finally, if K dim $R \leqslant 1$, then R is a principal ideal domain and the result follows from Proposition 4.1.

It would be interesting to know whether Corollary 4.6 can be extended to an arbitrary (regular) commutative domain R. A tempting conjecture is:

(4.7) If δ is a non-zero derivation of R, then every stably free right ideal of $S = R[x; \delta]$ is free if and only if δ leaves every prime ideal of R invariant.

If R satisfies the final condition of (4.7) then Corollary 4.6 implies that K dim $R = 1$. However, R need not be semilocal. For example, let $R_1 = k[x_1, \ldots]$, $\mathscr{C} = R_1 \setminus \cup x_i R_1$ and $R = (R_1)_{\mathscr{C}}$. Take δ to be the derivation defined by $\delta(x_i) = x_i$ for each i. It is easy to see that R is a principal ideal domain, each of whose infinitely many prime ideals is left invariant by δ. The proof of Proposition 4.1 can be used to show that every projective right ideal of $S = R[x; \delta]$ is free.

Of course, (4.7) would be a triviality if one could answer the following question in the affirmative. Suppose that S is a ring such that every stable free right ideal over $S_{\mathscr{C}}$ is free? Unfortunately we suspect that this result is not true, although this is only based on analogy with an example of Swan (see [4, pp. 147–148]).

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