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MONODROMY OF FUNCTIONS DEFINED ON ISOLATED SINGULARITIES OF COMPLETE INTERSECTIONS

Alexandru Dimca

A basic tool in the study of an analytic function germ f: $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin (or of the corresponding hypersurface germ $Y = f^{-1}(0)$) is the wellknown local monodromy group ([4], [8], [12]).

This widely studied monodromy group can be defined in two equivalent ways:

(i) Using a morsification of the function f.

(ii) Using a line in the base space B of a versal deformation for Y, in general position with respect to the discriminant hypersurfaces $\Delta \subset B$.

In this paper we extend the construction (i) above to function germs f: $(X, 0) \to (\mathbb{C}, 0)$ defined on a complete intersection $(X, 0) \subset (\mathbb{C}^{n+p}, 0)$ with an isolated singular point at the origin and such that $X_0 = f^{-1}(0)$ is also a complete intersection with an isolated singularity at 0 (here $n = \dim X > 0$).

In this way we obtain an action of a fundamental group $\pi = \pi_1(\text{disc} \setminus \{s \text{ points}\})$ on the exact sequence of the pair $(\tilde{X}, \tilde{X}_0)$ in homology (with Z-coefficients):

$$0 \to H_n(\tilde{X}) \to H_n(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} H_{n-1}(\tilde{X}_0) \to 0 \qquad (*)$$

where $\tilde{X}$, $\tilde{X}_0$ are the Milnor fibers of X and X_0 ([5]) chosen such that $\tilde{X}_0 \subset \tilde{X}$ and $s = \mu(X) + \mu(X_0)$ is the sum of their Milnor numbers.

More precisely, the action of π on $H_n(\tilde{X})$ is trivial, while the actions on the other two homology groups can be described in terms of Picard-Lefschetz formulas with respect to thimbles $\Delta_k \in H_n(\tilde{X}, \tilde{X}_0)$ and corresponding vanishing cycles $\delta_k = \partial\Delta_k \in H_{n-1}(\tilde{X}_0)$.

The π-exact sequence $(*)$ is proved to be a contact invariant of the function f i.e. it depends only on the isomorphism class (in a natural sense) of the pair of complete intersections (X, X_0). This fact, as well as the independence of the sequence $(*)$ on the choice of the morsification for f is obtained by a simple application of the Thom-Mather Second Isotopy Lemma.

To give some explicit examples, we compute next the π-sequence $(*)$ for all the $\mathscr{R}$-simple functions f defined on an isolated hypersurface singularity X of dimension $n > 1$, as listed in [1].

Note that the π-sequence $(*)$ gives us in particular two monodromy groups

$$G_0(f) = \operatorname{Im}\{\pi \to \operatorname{Aut} H_{n-1}(\tilde{X}_0)\}$$

$$G(f) = \operatorname{Im}\{\pi \to \operatorname{Aut} H_n(\tilde{X}, \tilde{X}_0)\}.$$

We prove that $G_0(f)$ is precisely the monodromy group of the complete intersection X_0 defined as in (ii). In fact the morsification process used above gives rise to a line in the base space B of a (suitable chosen) versal deformation of X_0, whose direction depends on the function f and is *not* generic with respect to the discriminant $\Delta \subset B$.

That is why we need a slightly modified version of a result of Hamm-Lê on the fundamental group $\pi_1(B \setminus \Delta)$ (see Lemma 3.5).

Then we show that the other monodromy group $G(f)$ is a semidirect product of $G_0(f)$ with a free abelian group $\mathbb{Z}^\alpha$ and we also give some estimates for the rank α.

Finally we remark that constructions similar to some of ours (i.e. morsifications and connections with versal deformations) have been used many a time before (e.g. by Iomdin [7] and Lê [10]) but always with different aims in view, as far as we known.

We would like to express our deep gratitude to Professor V.I. Arnold for a very stimulating discussion.

§1. Morsifications and monodromy map of pairs

Let X: $g_1 = \ldots = g_p = 0$ be an analytic complete intersection in a neighbourhood of the origin of $\mathbb{C}^{n+p}$, with an isolated singular point at 0. ($n \geqslant 1$, $p \geqslant 0$). Consider also an analytic function germ

$$f\colon (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}, 0)$$

such that $X_0 = f^{-1}(0) \cap X$ is again a complete intersection with an isolated singularity at 0.

For $\epsilon \gg \delta > 0$ chosen sufficiently small, it is known that the Milnor fiber of X

$$X_r = \{x \in B_\epsilon;\, g(x) = r\}$$

is a compact C^∞-manifold with boundary for any $r \in \mathbb{C}^p$ sufficiently general with $0 < |r| \leqslant \delta$, where

$$B_\epsilon = \{x \in \mathbb{C}^{n+p};\, |x| \leqslant \epsilon\}. \qquad [5]$$

The space X_r (denoted in the introduction by $\tilde{X}$) has the homotopy type of a bouquet of n-spheres, the number of which is by definition the Milnor number $\mu(X)$ of the complete intersection X.

For r small enough, it is easy to see that $f' = f|\text{Int } X_r$ has only a finite number of critical points $a_1, \ldots, a_k$ and moreover $a_i \to 0$ when $r \to 0$ for any $i = 1, \ldots, k$.

Let us denote by $\mu(f', a_i)$ the Milnor number of the function f' at the critical point a_i.

One has the following property, in analogy with a result of Lê ([10], (3.6.4)).

PROPOSITION 1.1:

$$\sum_{i=1,k} \mu(f', a_i) = \mu(X) + \mu(X_0).$$

PROOF: Let D_δ denote the open disc $\{z \in \mathbb{C};\ |z| < \delta\}$. For ϵ, δ and r suitable chosen, the inclusion

$$E = X_r \cap f^{-1}(D_\delta) \hookrightarrow X_r \tag{1.2}$$

is a homotopy equivalence (see for instance [10] (3.5)) and moreover the restriction

$$f|_{\partial E}\colon \partial E \to D_\delta \qquad \text{where} \quad \partial E = \partial X_r \cap f^{-1}(D_\delta) \tag{1.3}$$

is a submersion.

Let $b \in D_\delta$ be a regular value of $\tilde{f} = f|E$ and let $c_i = f(a_i) \in D_\delta$ be the (not necessarily distinct) critical values of $\tilde{f}$.

Then $F = \tilde{f}^{-1}(b)$ is the Milnor fiber of the complete intersection X_0 and the exact sequence of the pair (E, F) shows that $H_n(E, F)$ is a free abelian group of rank $s = \mu(X) + \mu(X_0)$. ($\mathbb{Z}$-coefficients for homology are used throughout in this paper).

We compute now this group in a different way, following ([9], §5). Choose small disjoint closed discs D_i centered at the critical values c_i and fix some points $b_i \in \partial D_i$.

For each i, take a C^∞-embedded interval l_i from b to b_i such that $l = \cup l_i$ can be contracted within itself to b and D_δ can be contracted to $C = \cup D_i \cup l$.

Since $\tilde{f}$ induces a (proper) locally trivial fibration

$$E \setminus \tilde{f}^{-1}\{c_i\}_i \to D_\delta \setminus \{c_i\}_i$$

these retractions can be lifted to the corresponding subsets of E and we get the following isomorphisms

$$H_n(E, F) \xleftarrow{\sim} H_n(\tilde{f}^{-1}(C), F) \xrightarrow{\sim} H_n(\tilde{f}^{-1}(C), \tilde{f}^{-1}(l)).$$

By excision, the last group is equal to

$$\bigoplus_i H_n(\tilde{f}^{-1}(D_i), \tilde{f}^{-1}(b_i))$$

Assume that $a_{i1}, \ldots, a_{im}$ are the critical points of $\tilde{f}$ in the fiber over c_i. Let B_j be the intersection of a small closed ball centered at a_{ij} with $\tilde{f}^{-1}(D_i)$ and denote with F_i the fiber $\tilde{f}^{-1}(b_i)$.

It follows that

$$H_n(\tilde{f}^{-1}(D_i), F_i) \simeq H_n\left(\bigcup_{j=1}^{m} B_j \cup F_i, F_i\right) \simeq \bigoplus_{j=1}^{m} H_n(B_j, B_j \cap F_i).$$

Moreover

$$H_n(B_j, B_j \cap F_i) \underset{\sim}{\overset{\partial}{\to}} H_{n-1}(B_j \cap F_i)$$

is a free abelian group of rank $\mu(f', a_{ij})$ by the definition of the Milnor numbers of f', if the discs D_i and the balls B_j are chosen small enough. □

We consider now the problem of the existence of morsifications of the function $f'\colon X_r \to \mathbb{C}$, i.e. small deformations of f' having only nondegenerate critical points with distinct critical values.

If P denotes the vector space of polynomials in $x_1, \ldots, x_{n+p}$ of degree $\leqslant 3$, it is easy to show by standard transversality arguments that there is a Zariski open subset $U \subset P$ such that the function

$$f_q = (f + q) | X_r$$

is a Morse function for any $q \in U$.

Moreover, if we have chosen already $\epsilon \gg \delta > 0$ such that (1.2) and (1.3) hold true for any generic $r \in \mathbb{C}^p$ with $|r| \leqslant \delta$, then there is an $\eta > 0$ such that $|q| < \eta$ implies similar properties for f_q.

Suppose now we have two polynomials q_0, $q_1 \in U$ such that $|q_i| < \eta$. We can find a C^∞-path q_t in U such that $q_t = q_0$ for $0 \leqslant t \leqslant a$, $q_t = q_1$ for $1 - a \leqslant t \leqslant 1$ and $|q_t| < \eta$ for any $t \in [0, 1]$, where $a \in (0, 1/3)$.

Consider the spaces

$$\tilde{D} = D_\delta \times (0, 1) \quad \text{and} \quad \tilde{E} = \{(x, t) \in X_r \times (0, 1); f_{q_t}(x) \in D_\delta\}$$

and the proper map

$$\varphi\colon \tilde{E} \to \tilde{D}, \quad \varphi(x, t) = \left(f_{q_t}(x), t\right).$$

If $a_i(t)$ (resp. $c_i(t)$) denote the critical points (resp. critical values) of f_{q_t} for $i = 1, \dots, s = \mu(X) + \mu(X_0)$, then we can stratify the map φ as follows ([2], Chap. I). The strata in $\tilde{D}$ are given by

$$\tilde{D}_1 = \{(c_i(t), t);\ t \in (0, 1), i = 1, \dots, s\} \quad \text{and} \quad \tilde{D}_3 = \tilde{D} \setminus \tilde{D}_1.$$

The strata in $\tilde{E}$ are given by

$$\tilde{E}_1 = \{(a_i(t), t);\ t \in (0, 1), i = 1, \dots, s\}$$

$$\tilde{E}_{2n-2} = \left\{(x, t);\ t \in (0, 1), x \in \left(f_{q_t}\right)^{-1}(c_i(t)) \cap \partial X_r, i = 1, \dots, s\right\}$$

$$\tilde{E}_{2n-1} = \left\{(x, t);\ t \in (0, 1), x \in \left(f_{q_t}\right)^{-1}(c_i(t)) \cap \operatorname{Int} X_r, i = 1, \dots, s\right\}$$

$$\tilde{E}_{2n} = \left(\partial X_r \times (0, 1)\right) \cap \left(\tilde{E} \setminus \tilde{E}_{2n-2}\right)$$

$$\tilde{E}_{2n+1} = \tilde{E} \setminus \text{the union of the other strata } \tilde{E}_k \text{ defined above.}$$

The lower index gives the real dimension of the stratum. (These definitions work for $n \geqslant 2$. The simpler case $n = 1$ is left to the reader.)

The Whitney-Thom regularity conditions are obviously satisfied for any pair of strata.

By Thom-Mather Second Isotopy Lemma ([2], II, (5.8)) we obtain a commutative diagram

$$\begin{array}{ccc} \varphi^{-1}(D_\delta \times \alpha) & \xrightarrow{H} & \varphi^{-1}(D_\delta \times (1-\alpha)) \\ {\scriptstyle f_{q_0}}\big\downarrow & & \big\downarrow{\scriptstyle f_{q_1}} \\ D_\delta \times \alpha & \xrightarrow{h} & D_\delta \times (1-\alpha) \end{array}$$

where $\alpha \in (0, a)$ and H, h are homomorphisms compatible with the induced stratifications.

In particular we get the following result.

LEMMA 1.4: *The topological type of the map of pairs*

$$f_q\colon \left(f_q^{-1}(D_\delta), f_q^{-1}(C)\right) \to (D_\delta, C)$$

where C is the set of critical values of the function f_q is independent of the polynomial $q \in U$, $|q| < \eta$.

It is also clear the independence of the topological type of the map above of the choice of (suitable) ϵ, δ and r. Moreover, if we change the function f to a function $f_1 = f + k$, where k is a function in the ideal $(g_1, \dots, g_p)$ of the complete intersection X, note that the distance $\|f_1 - f\|_{X_r}$ can be made as small as we want by taking r small enough.

Using a stratification argument as above it follows that the topological type of the map of pairs in (1.4) depends only on the restriction $f|X$ i.e. on a function in $m_X = m/(g_1, \dots, g_p)$, where $m \subset \mathcal{O}_{n+p}$ is the maximal ideal.

(We shall consider throughout in this paper only functions $f \in m_X$ such that $X_0 = f^{-1}(0)$ is a complete intersection with an isolated singularity at 0).

The discussion below will also imply independence from the defining equations $g_i = 0$ of X, and hence we can give the following.

DEFINITION 1.5: The topological type of the map of pairs in (1.4) will be called the *monodromy map of pairs* of the function $f \in m_X$ and will be denoted simply by

$$f^*: (E^*, E_c^*) \to (D, C).$$

This topological object is constant in μ-constant families in the following precise sense (compare to [12], §9).

Let $(X_t, 0) \subset (\mathbb{C}^{n+p}, 0)$ be a smooth family of complete intersections with isolated singular points at the origin such that $\dim X_t = n$ and $\mu(X_t) = \text{const.}$ for $t \in [0, 1]$. Assume that $f_t \in m_{X_t}$ is a smooth family of function germs such that $\mu(f_t^{-1}(0)) = \text{const.}$

Using the construction of morsifications and stratification arguments as above, one can then show that the monodromy map of pairs of the function f_t is independent of t.

A special case of this situation is the following.

DEFINITION 1.6 [1]: We say that two function germs $f_1, f_2 \in m_X$ defined on the complete intersection $(X, 0)$ are *$\mathcal{K}$-(contact)-equivalent* if there is an automorphism u of the local $\mathbb{C}$-algebra $\mathcal{O}_X$ such that $(u(f_1)) = (f_2)$, where (a) means the ideal generated by a in $\mathcal{O}_X$.

Since the complete intersections X and $X_{0i} = f_i^{-1}(0)$ $i = 1, 2$ have isolated singularities at the origin, the question of $\mathcal{K}$-equivalence of f_1 and f_2 can be settled in a jet space $J^k(n+p, p+1)$, via the action of a *connected* algebraic group $G_{\mathcal{K}}^k$ (the particular case when X is a hypersurface is treated in detail in [1]).

It follows that (X, f_1) and (X, f_2) can be connected by a μ-constant family (X_t, f_t) as above and we get thus the following.

COROLLARY 1.7: *If two function germs $f_1, f_2 \in m_X$ are $\mathcal{K}$-equivalent then their associated monodromy maps f_1^* and f_2^* are the same.*

§2. Monodromy exact sequence. Examples

Let $f^*: (E^*, E_c^*) \to (D, C)$ be the monodromy map of pairs of a function $f \in m_X$ as in §1.

If $b \in D \setminus C$ and $F = (f^*)^{-1}(b)$, then the locally trivial fibration $E^* \setminus E_c^* \to D \setminus C$ defines in the usual way an action of the fundamental group $\pi = \pi_1(D \setminus C)$ on the middle homology group $H_{n-1}(F)$ of the fiber.

Moreover, for any homotopy class $w \in \pi$ there is a well defined homomorphism

$$\tau_w: H_{n-1}(F) \to H_n(E^*, F)$$

called *the extension along the path w*. For a detailed construction and the main properties of τ_w we send to ([9], (6.4)).

We can define an action of the fundamental group π on the homology group $H_n(E^*, F)$ by the formula

$$w \cdot x = x + (-1)^{n-1} \tau_w(\partial x) \tag{2.1}$$

where ∂ is the connecting homomorphism in the exact sequence of the pair (E^*, F)

$$0 \to H_n(E^*) \xrightarrow{i} H_n(E^*, F) \xrightarrow{\partial} H_{n-1}(F) \to 0. \tag{2.2}$$

If we consider the trivial action of π on $H_n(E^*)$, then this exact sequence is a *π-exact sequence*, i.e. the homomorphisms i and ∂ are π-equivariant.

Let $\tilde{X}$ (say equal to X_r in §1) and $\tilde{X}_0$ (say equal to $X_r \cap f^{-1}(b)$) denote the associated Milnor fibers of the complete intersections X and X_0.

The corresponding exact sequence

$$0 \to H_n(\tilde{X}) \to H_n(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} H_{n-1}(\tilde{X}_0) \to 0 \tag{2.3}$$

is isomorphic to the exact sequence (2.2) and via this isomorphism we can transfer the π-actions on the homology groups in (2.3).

DEFINITION 2.4: The π-exact sequence (2.3) constructed as above is called *the monodromy exact sequence* of the function f.

EXAMPLE 2.5: If the complete intersection X is smooth, then the sequence (2.3) becomes

$$0 \to 0 \to H_n(\tilde{X}, \tilde{X}_0) \underset{\sim}{\overset{\partial}{\to}} H_{n-1}(\tilde{X}_0) \to 0$$

and hence it contains the same information as the action of π on $H_{n-1}(\tilde{X}_0)$ i.e. the classical monodromy action for the hypersurface X_0. □

Put again $s = \mu(X) + \mu(X_0) = rkH_n(\tilde{X}, \tilde{X}_0)$ and let $C = \{c_1, \dots, c_s\}$. We denote by $w_k \in \pi$ the elementary path encircling c_k ([9] (6.1)) and chose the order of these paths such that

$$w_s \cdot \dots \cdot w_1 = w_0$$

where w_0 is the class of the path $w_0(t) = b \cdot e^{2\pi it}$, $0 \leqslant t \leqslant 1$ (we assume here $|b| > |c_k|$ for any $k = 1, \dots, s$).

We recall from the proof of (1.1) the isomorphisms

$$H_n(\tilde{X}, \tilde{X}_0) \simeq H_n(E^*, F) \simeq \bigoplus_{k=1}^{s} H_n\big((f^*)^{-1}(D_k), (f^*)^{-1}(b_k)\big)$$

Since f^* is a morsification, each of the last homology groups is free abelian of rank one.

We shall denote by $\Delta_1, \dots, \Delta_s$ the corresponding generators of the group $H_n(\tilde{X}, \tilde{X}_0)$, which are precisely the *thimbles* of Lefschetz ([9] (6.2)).

With these notations, the π-actions in the exact sequence (2.3) can be described in terms of Picard-Lefschetz formulas.

LEMMA 2.6:

$$\textit{For} \quad x \in H_n(\tilde{X}, \tilde{X}_0)\colon w_k \cdot x = x + (-1)^{n(n+1)/2}(\partial x, \partial\Delta_k)\Delta_k$$

$$\textit{For} \quad x \in H_{n-1}(\tilde{X}_0)\colon w_k \cdot x = x + (-1)^{n(n+1)/2}(x, \partial\Delta_k)\partial\Delta_k$$

where (,) *denotes the intersection form on* $H_{n-1}(\tilde{X}_0)$ *and* $k = 1, \dots, s$.

PROOF. The second formula is the usual Picard-Lefschetz formula (see for instance ([8], §5)). The first one follows from (2.1) and the formula for τ_w given in ([9], (6.7.1)). □

It follows that in order to determine the monodromy exact sequence it is enough to fix a basis $\{\delta_k\}$ of the group $H_{n-1}(\tilde{X}_0)$ and to compute with respect to it the *vanishing cycles* $\partial\Delta_i$ and the intersection form.

As examples of this method, we give the description of the monodromy exact sequences of the $\mathscr{R}$-simple functions defined on an isolated hypersurface singularity X with $\dim X > 1$ which were classified in ([1], §3).

In all these cases X_0 is an isolated hypersurface singularity of type A_k for some k and we can chose a distinguished basis of vanishing cycles $\{\delta_i\}$ for $H_{n-1}(\tilde{X}_0)$ corresponding to a Dynkin diagram of type A_k ([4], (2.4)).

Moreover, using the stabilization of singularities (i.e. addition of a sum of squares to the given equation of X_0 as described in [4] (2.3)), we can assume $n = 1$ when we compute $\partial\Delta_i$.

The results are given below, without these tedious computations.

Proposition 2.7: *For the simple function of type B_m ($m \geqslant 2$) given by X: $x_1^m + x_2^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_1$ there is a basis of thimbles $\Delta_1, \ldots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a vanishing cycle δ which generates $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_k = \delta$ for any $k = 1, \ldots, m$.*

Proposition 2.8: *For the simple function of type C_{m+1} ($m \geqslant 1$) given by X: $x_1x_2 + x_3^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_1 + x_2^m$ there is a basis of thimbles $\Delta_0, \ldots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles $\delta_1, \ldots, \delta_m$ of $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_0 = \delta_1 + \ldots + \delta_m$ and $\partial\Delta_k = \delta_k$ for any $k = 1, \ldots, m$. (Note that $C_2 \equiv B_2$).*

Proposition 2.9: *For the simple function of type F_4 given by X: $x_1^3 + x_2^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_2$ there is a basis of thimbles $\Delta_1, \ldots, \Delta_4$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles δ_1, δ_2 of $H_{n-1}(\tilde{X}_0)$ such that*

$$\partial\Delta_1 = \delta_1, \quad \partial\Delta_3 = \delta_2, \quad \partial\Delta_2 = \partial\Delta_4 = \delta_1 + \delta_2.$$

Remark 2.10: It will follow from the results in the next section, that for $n \equiv 3 \pmod 4$ the monodromy group $G_0(f)$ (defined in the introduction) is a symmetric group for any $\mathscr{R}$-simple function f. More precisely

$$G_0(B_m) = S_2, \quad G_0(C_m) = S_m, \quad G_0(F_4) = S_3.$$

On the other hand, in these cases the monodromy groups $G(f)$ are all infinite (see 3.7 ii).

Therefore one cannot establish a simple connection between these monodromy groups and the Weyl groups associated to the root systems of type B_m, C_m and F_4.

Remark 2.11: It is easy to see that the action of the path w_0 on $H_{n-1}(\tilde{X}_0)$ is precisely the dual of the monodromy operator in cohomology h^* introduced in [5].

3. The monodromy groups $G_0(f)$ and $G(f)$

Let $(X_0, 0) \subset (Y, 0) \xrightarrow{F} (B, 0)$ be a versal deformation of the complete intersection X_0, with a smooth base space B and let us denote by $\Delta \subset B$ the discriminant hypersurface of F [3].

For a base point $b \in B \setminus \Delta$, the fundamental group $\pi_1(B \setminus \Delta, b)$ acts on the homology of the smooth fiber $F^{-1}(b) \sim \tilde{X}_0$ and we obtain in this way the *monodromy group of* X_0

$$G(X_0) = \operatorname{Im}\{\pi_1(B \setminus \Delta, b) \to \operatorname{Aut} H_{n-1}(\tilde{X}_0)\}.$$

This group is independent of the choice of the versal deformation F and of the base point b (provided we take B to be a small enough open ball in some $\mathbb{C}^N$).

Suppose we fix a morsification $f_q \colon X_r \to \mathbb{C}$ of the given function f as in (1.4). Then there is a versal deformation F of X_0 as above and a line l in the base space B such that after a natural identification $l \simeq \mathbb{C}$ we have a commutative diagram

$$\begin{array}{ccc} f_q^{-1}(D_\delta) & \simeq & F^{-1}(D_\delta) \\ & {\scriptstyle f_q} \searrow \quad \swarrow {\scriptstyle F} & \\ & D_\delta & \end{array} \tag{3.1}$$

To obtain such a versal deformation F it is enough to take a system of generators of the $\mathbb{C}$-vector space $\mathcal{O}_{X_0}^{p+1}/\partial G/\partial x_1 \cdot \mathcal{O}_{X0} + \ldots + \partial G/\partial x_{n+p} \cdot \mathcal{O}_{X_0}$ (where $\partial G/\partial x_i = (\partial g_1/\partial x_i, \ldots, \partial g_p/\partial x_i, \partial f/\partial x_i)$) including the constant vectors $e_1, \ldots, e_{p+1}$ and the vector $(0, \ldots, 0, q)$.

The set C of critical values of f_q corresponds via (3.1) to the intersection $l \cap \Delta$ and since f_q is a Morse function it follows that all the points $c_k \in l \cap \Delta$ are simple points on Δ and that the intersection $l \cap \Delta$ is transverse (situation denoted in the sequel by $l \pitchfork \Delta$). ([3], 1.3.i).

The number s of intersection points in $l \cap \Delta$ is equal to the intersection multiplicity $(\Delta, l_0)_0$, where l_0 is the line through $0 \in B$ with the same direction as l [10].

EXAMPLE 3.2: For the simple function of type B_m introduced in (2.7) one can take $F \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$

$$F(x) = \left(x_1^m + x_2^2 + \ldots + x_{n+1}^2, x_1\right)$$

Then the discriminant Δ is given by the equation $y_1 = y_2^m$ and the morsification $f_0 = x_1\colon X_r \to \mathbb{C}$ corresponds to the line $l\colon y_1 = r$. Hence in this case $s = m$, though Δ is smooth at 0. It follows that the direction l_0: $y_1 = 0$ is not generic with respect to the discriminant, as mentioned in the introduction. □

The main result of this section is the following.

PROPOSITION 3.3:

$$G_0(f) = G(X_0).$$

PROOF: Suppose that B is an open neighbourhood of 0 in $\mathbb{C}^N$ for some $N \geqslant 2$ and let $h = 0$ be the equation of the discriminant hypersurface Δ in B.

We denote here by B_ρ the closed ball of radius ρ centered at 0 in $\mathbb{C}^N$ and by d_a the line determined by a direction $d \in P(\mathbb{C}^N)$ and a point $a \in B$.

The results of Hamm-Lê [6] prove the existence of a Zariski open set $U \subset P(\mathbb{C}^N)$ such that for any $d \in U$ there is a $\rho_0 = \rho(d) > 0$ with the property that for any ρ with $0 < \rho \leqslant \rho_0$ there is a $\theta_\rho > 0$ such that the homomorphism

$$\pi_1\big((B_\rho \backslash \Delta) \cap d_a, b\big) \to \pi_1(B_\rho \backslash \Delta, b) \tag{3.4}$$

induced by the inclusion is an epimorphism for any point a with $0 < |a| \leqslant \theta_\rho$ and $b \in (B_\rho \backslash \Delta) \cap d_a$.

We cannot apply this result to the line l in our construction above, since l is not in general position with respect to the discriminant Δ (3.2).

That is why we need the following.

LEMMA 3.5: *Suppose that the direction $d \in P(\mathbb{C}^N)$ is chosen such that $d_0 \not\subset \Delta$. Then there is ρ, $\delta > 0$ such that* (3.4) *is an epimorphism for any point a with $|a| \leqslant \delta$ and $d_a \pitchfork \Delta$.*

PROOF: Let $\rho > 0$ be chosen such that

(i) $B_\rho \cap d_0 \cap \Delta = \{0\}$.

(ii) Inside the ball B_ρ we have a conical topological structure for Δ, i.e.

$$\big(B_\rho, \Delta \cap B_\rho\big) \simeq C\big(S_\rho, K\big)$$

where $S_\rho = \partial B_\rho$, $K = \Delta \cap S_\rho$ as in [11] (2.10).

There is a connected open neighbourhood V of d in $P(\mathbb{C}^N)$ such that $d' \in \overline{V}$ implies $d'_0 \cap K = \emptyset$.

We choose $\delta > 0$ small enough, such that $d'_a \cap K = \emptyset$ for any $d' \in \overline{V}$ and any point a with $|a| \leqslant \delta$.

Take now a point a with $|a| \leqslant \delta$ and $d_a \pitchfork \Delta$. Using a linear parametrization γ: $(\mathbb{C}, 0) \to (d_a, a)$, we define the function $\varphi = h\gamma$.

Then φ is defined on a neighbourhood of $0 \in \mathbb{C}$ which contains the disc $D = d_a \cap B_\rho$ (if ρ and δ are chosen small enough) and $\varphi^{-1}(0) = \{x_1, \ldots, x_s\}$ where the roots x_i are all in D and have multiplicity one.

We choose now a direction $d' \in V \cap U$ such that

$$(d_0', \Delta)_0 = m(\Delta)$$

where $m(\Delta)$ is the multiplicity of the discriminant Δ at the origin. An explicit formula for $m(\Delta)$ can be found in [3], [10] and it follows that $m(\Delta) \geqslant \mu(X_0)$ with equality iff X_0 is a hypersurface singularity.

Note that a path connecting d with d' within V gives rise to a homotopy φ_t: $D \to \mathbb{C}$, $0 \leqslant t \leqslant 1$ of $\varphi = \varphi_0$ with φ_1, the function defined as above with respect to d_a'.

Since the direction d' is in U, there is a $\rho' > 0$ and a $\theta' > 0$ such that, for any a' with $0 < |a'| \leqslant \theta'$, the corresponding homomorphism (3.4) is an epimorphism.

Choose a path $a(t)$ $1 \leqslant t \leqslant 2$ in B_δ such that $a(1) = a$, $a(2) = a'$ with $0 < |a'| \leqslant \theta'$ and $d'_{a(t)} \pitchfork \Delta$ for any t. This gives rise as above to a homotopy φ_t: $D \to \mathbb{C}$ $1 \leqslant t \leqslant 2$. Since all the functions φ_t have only simple roots $x_k(t)$ in Int D, we obtain in this way s paths $x_1(t), \ldots, x_s(t)$ for $0 \leqslant t \leqslant 2$.

We choose the order on the paths such that $x_1(2), \ldots, x_m(2)$ are precisely the end points within the disc $B_{\rho'} \cap d'_{a'} \subset D$, where $m = m(\Delta)$ (Note the identification $D \simeq d'_{a(t)} \cap B_\rho$ for any t).

Consider the following commutative diagram.

$$\begin{array}{ccc}
\pi_1((B_\rho \setminus \Delta) \cap d_a, b) & \xrightarrow{\quad i_\# \quad} & \pi_1(B_\rho \setminus \Delta, b) \\
\tilde{\varphi} \downarrow & & \downarrow c_* \\
\pi_1((B_\rho \setminus \Delta) \cap d'_{a'}, b') & \longrightarrow & \pi_1(B_\rho \setminus \Delta, b') \\
i_\# \uparrow & & \uparrow i_\# \\
\pi_1((B_{\rho'} \setminus \Delta) \cap d'_{a'}, b') & \overset{i_\#}{\to} \pi_1(B_{\rho'} \setminus \Delta, b') &
\end{array}$$

The isomorphism c_* is induced by a path in $B_\rho \setminus \Delta$ from b to b' and $\tilde{\varphi}$ is obtain via the homotopy φ_t.

If we denote by w_k (resp. w_k') the elementary path in $D \setminus \{x_1(t), \ldots, x_s(t)\}$ encircling the point $x_k(t)$ for $t = 0$ (resp. $t = 2$), then the left hand side of the diagram corresponds to

$$F(w_1', \ldots, w_m') \overset{i_\#}{\hookrightarrow} F(w_1', \ldots, w_s') \underset{\sim}{\overset{\tilde{\varphi}}{\to}} F(w_1, \ldots, w_s)$$

where $F(a_1,\dots,a_p)$ denotes the free group generated by $a_1,\dots,a_p$.
This ends the proof of (3.5) and hence of (3.3). □

COROLLARY 3.6: *Suppose X_0 is a hypersurface singularity and let $m = m(\Delta) = \mu(X_0)$. Then in the monodromy exact sequence* (2.3) *of the function f (up to a change of indexes) the vanishing cycles $\delta_k = \partial\Delta_k$ $(k = 1,\dots,m)$ form a basis of $H_{n-1}(\tilde{X}_0)$ and the Picard-Lefschetz transformations associated to the elementary paths w_k $(k = 1,\dots,m)$ generate the group $G_0(f)$.*

PROOF: The proof of (3.5) implies that (up to a change of indexes) the images of $w_1,\dots,w_m$ generate the group $G_0(f) = G(X_0)$.

The monodromy group $G(X_0)$ acts transitively on the set of vanishing cycles in $H_{n-1}(\tilde{X}_0)$ [4], (2.58).

Hence for any such cycle δ there is an element $g \in G_0(f)$ such that $\delta = \pm g \cdot \delta_1$.

Since g is a product of Picard-Lefschetz transformations associated to $w_1,\dots,w_m$, it follows that

$$\delta \in \mathbb{Z}\langle \delta_1,\dots,\delta_m \rangle$$

i.e. $\delta_1,\dots,\delta_m$ form a basis of $H_{n-1}(\tilde{X}_0)$. □

Finally we give some information about the other monodromy group of f, namely $G(f)$.

PROPOSITION 3.7:

(*i*) *There is an exact sequence of groups*

$$0 \to \mathbb{Z}^\alpha \to G(f) \to G_0(f) \to 1$$

for some $\alpha \in \mathbb{N}$ with $0 \leqslant \alpha \leqslant \mu(X) \cdot \mu(X_0)$.

(*ii*) *Suppose that X_0 is a hypersurface singularity and the intersection form on $H_{n-1}(\tilde{X}_0)$ is nondegenerate.*

Then $\alpha \geqslant \mu(X)$.

If moreover the action of $G_0(f)$ on $H_{n-1}(\tilde{X}_0) \otimes \mathbb{C}$ is irreducible, then $\alpha = \mu(X) \cdot \mu(X_0)$.

PROOF: Put $m = \mu(X_0)$, $m' = \mu(X)$ and $s = m + m'$. Assume that $\{\Delta_i\}$ is a basis of $H_n(\tilde{X}, \tilde{X}_0)$ (made of thimbles only in the proof of (ii)!) such that $\delta_k = \partial\Delta_k$ for $k = 1,\dots,m$ form a basis for $H_{n-1}(\tilde{X}_0)$.

Then for any $k > m$ there is a combination

$$v_k = \Delta_k + \sum_{i=1}^{m} a_{ki}\Delta_i \quad \text{such that} \quad \partial v_k = 0.$$

In the basis $v_{m+1},\ldots,v_s$, $\Delta_1,\ldots,\Delta_m$ the action of w_k on $H_n(\tilde{X}, \tilde{X}_0)$ is given by a matrix

$$T_k = \begin{pmatrix} 1 & A_k \\ 0 & B_k \end{pmatrix}$$

We define an epimorphism ρ: $G(f) \to G_0(f)$ by associating to an $s \times s$ matrix as above the $m \times m$ matrix in the lower right corner. We get thus an exact sequence

$$1 \to \ker \rho \to G(f) \overset{\rho}{\to} G_0(f) \to 1$$

where $\ker \rho$ is a subgroup in the (abelian!) multiplicative group of all the matrices

$$M = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$$

It follows that $\ker \rho \subset \mathbb{Z}^{m \cdot m'}$ and this gives us (i). To prove (ii) we assume the basis δ_k chosen as in (3.6). Note that the matrix A_k defined above is zero for $k \leqslant m$ and has a single nonzero row (that corresponding to the vector v_k) for $m < k \leqslant s$ if the intersection form is nondegenerate. This proves the first part of (ii).

Moreover, note that if

$$\left(\begin{array}{c|c} 1 & \begin{matrix} 0 \\ \ldots u \ldots \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right) \in \ker \rho$$

for some row vector $u \neq 0$, then the same is true for the vector $u \cdot B$ for any $B \in G_0(f)$.

If the action of $G_0(f)$ on the homology group $H_{n-1}(\tilde{X}_0; \mathbb{C})$ is irreducible, then it follows that

$$\dim \mathbb{C}\langle u \cdot B;\ B \in G_0(f)\rangle = m$$

Hence $\ker \rho$ contains in this case $m \cdot m'$ $\mathbb{C}$-linearly independent vectors and this implies the result in the second part of (ii). □

REMARKS 3.8:

a. The condition about the intersection form in (3.7.ii) is necessary. For instance, if f is a simple function of type B_k and n is even, it follows from (2.7) that $G_0(f) = G(f) = 0$.

On the other hand, note that both assumptions in (3.7.ii) hold when X_0 is one of Arnold simple hypersurface singularities A_n, D_n, E_6, E_7 or E_8 and $n \equiv 3 \pmod 4$ ([12], §8).

b. In general the subgroup $\ker \rho \subset \mathbb{Z}^{mm'}$ is not the whole group, even when they have the same rank.

For instance, for a function of type B_k and n odd, $\ker \rho = 2 \cdot \mathbb{Z}^{k-1} \subset \mathbb{Z}^{k-1}$.

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