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FIRST ORDER INTERPOLATION INEQUALITIES WITH WEIGHTS

L. Caffarelli, * R. Kohn ** and L. Nirenberg ***

Dedicated to the memory of Aldo Andreotti

In Lemma 7.1 of [2] we proved certain interpolation inequalities. These are analogous to the standard interpolation inequalities between functions and their first derivatives in various L^p norms on $\mathbb{R}^n$ (see [3], [8]), but with each term weighted by a power of $|x|$. Instances of these inequalities have been studied previously [1,6,7], but the general case seems to have not yet been treated; we present it here, in the belief that such inequalities may prove useful in other contexts. Lin [5] has generalized these results to include derivatives of any order.

For simplicity, we state our theorem for $u \in C_0^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support. Its extension to a more general class of functions is standard. In what follows p, q, r; α, β, σ; and a are fixed real numbers (called parameters) satisfying

$$p, q \geqslant 1, \quad r > 0, \quad 0 \leqslant a \leqslant 1 \tag{1.1}$$

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0, \tag{1.2}$$

where

$$\gamma = a\sigma + (1 - a)\beta. \tag{1.3}$$

THEOREM: *There exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(R^n)$*

$$\| |x|^\gamma u \|_{L^r} \leqslant C \| |x|^\alpha |Du| \|_{L^p}^{a} \| |x|^\beta u \|_{L^q}^{1-a} \tag{1.4}$$

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if and only if the following relations hold:

$$\frac{1}{r}+\frac{\gamma}{n}=a\left(\frac{1}{p}+\frac{\alpha-1}{n}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{n}\right) \tag{1.5}$$

(*this is dimensional balance*),

$$0 \leqslant \alpha-\sigma \quad \text{if} \quad a>0,$$

and

$$\alpha-\sigma \leqslant 1 \quad \textit{if} \quad a>0 \quad \textit{and} \quad \frac{1}{p}+\frac{\alpha-1}{n}=\frac{1}{r}+\frac{\gamma}{n}. \tag{1.7}$$

Furthermore, on any compact set in parameter space in which (1.1), (1.2), (1.5) *and* $0 \leqslant \alpha-\sigma \leqslant 1$ *hold, the constant* C *is bounded.*

We emphasize the curious fact that one needs the condition $\alpha-\sigma \leqslant 1$ only in case $a>0$ and $1/p+(\alpha-1)/n=1/r+\gamma/n$.

The proof is rather long, but elementary. We first verify necessity; then we verify the case $n=1$, $\sigma=\alpha-1$, using among other tools a weighted Hardy-type inequality proved by Bradley [1]. The case $n \geqslant 1$, $0 \leqslant \alpha-\sigma \leqslant 1$ is treated next; then finally the case $\alpha-\sigma>1$, $1/p+(\alpha-1)/n \neq 1/r+\gamma/n$. Since when $a=0$ there is nothing to prove, we shall always assume $a>0$.

Throughout, C denotes a constant, depending on the parameters, whose value may change from line to line. Although we will not estimate the constants explicitly, it will be clear from the arguments that the last assertion of the theorem holds.

I. Necessity

Note first that the inequalities (1.1) are necessary in order for the norms in (1.4) to be finite. If (1.4) holds for $u(x)$ then it holds also for $u(\lambda x)$, $\lambda>0$. Inserting this in (1.4) we obtain (1.5). This is merely dimensional analysis; if we think of u as dimensionless then the dimension of $\||x|^\gamma u\|_{L^r}$ is $\gamma+n/r$, that of $\||x|^\alpha|Du|\|_{L^p}$ is $\alpha-1+n/p$, etc.

Next, for some fixed function $v \in C_0^\infty(|x|<1)$, $v \not\equiv 0$, let $u(x)=v(x-x_0)$ with $|x_0|=R$ large. Inserting this in (1.4) we see that

$$R^\gamma \leqslant C R^{a\alpha+(1-a)\beta}$$

so that

$$a\sigma+(1-a)\beta \leqslant a\alpha+(1-a)\beta.$$

Hence $\sigma \leqslant \alpha$. Next we prove (1.7). Suppose

$$\frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}$$

$$= \frac{1}{q} + \frac{\beta}{n} \quad \text{by (1.5) if } a < 1. \tag{1.8}$$

We insert in (1.4) the function

$$u(x) = \begin{cases} 0 & \text{for } |x| \geqslant 1 \\ |x|^{-\gamma - n/r} \log \dfrac{1}{|x|} & \text{for } \epsilon \leqslant |x| \leqslant 1 \\ \epsilon^{-\gamma - n/r} \log \dfrac{1}{\epsilon} & \text{for } |x| \leqslant \epsilon. \end{cases}$$

This function is not in C^∞ but it is clear that (1.4) must also hold for it. Straightforward calculation shows that, if ρ, θ are polar coordinates, $\theta \in S^{n-1}$,

$$\int_{R^n} |x|^{\gamma r} |u|^r \geqslant C \int_{\epsilon < |x| < 1} \frac{1}{\rho} \log^r \frac{1}{\rho} \, \mathrm{d}\rho$$

$$\geqslant C \log^{1+r} \frac{1}{\epsilon}$$

Consequently (1.4) implies

$$\frac{1}{r} + 1 \leqslant a\left(\frac{1}{p} + 1\right) + (1 - a)\left(\frac{1}{q} + 1\right)$$

$$\frac{1}{r} \leqslant \frac{a}{p} + \frac{1 - a}{q}.$$

But according to (1.3) and (1.5)

$$\frac{1}{r} = \frac{a}{p} + \frac{1 - a}{q} + \frac{a}{n}(\alpha - 1 - \sigma) \tag{1.9}$$

Hence $\alpha - 1 - \sigma \leqslant 0$, i.e. (1.7) holds. Necessity is proved.

II. Preliminaries

We present some inequalities which will be useful in what follows. Several of these are special cases of (1.4).

(A) *If* $1 \leqslant p \leqslant r$, $\delta \in \mathbb{R}$, *and* $\alpha = \delta + 1/r + (p-1)/p$ *then for* $u \in C_0^\infty(\mathbb{R})$

$$\big\| |x|^\delta u \big|_{L^r} \leqslant C \big\| |x|^\alpha Du \big|_{L^p} \tag{2.1}$$

in case either

$$\text{(i) } \delta + \frac{1}{r} > 0$$

$$\text{(ii) } \delta + \frac{1}{r} < 0 \quad \text{and} \quad u(0) = 0.$$

The constant C in (2.1) *stays bounded as p, r, and δ range over any compact subset of* $\{1 \leqslant p \leqslant r,\ \delta r \neq -1\}$.

One easily deduces these facts from the weighted Hardy-type inequalities in [1]. For $r = p$ this is Theorem 330 in [4]. □

(B) *Assume* (1.1)–(1.3) *and* (1.5) *hold; for any* $\rho > 0$, *let*

$R_\rho = \{\rho < |x| \leqslant 2\rho\}$. If $u \in C_0^\infty(\mathbb{R}^n)$ and

$$\delta = \gamma + \frac{n}{r} - n \tag{2.2}$$

then

$$\int_{R_\rho} |x|^{\gamma r} |u|^r$$
$$\leqslant C\left(\int_{R_\rho} |x|^{\alpha p} |Du|^p\right)^{ar/p} \left(\int_{R_\rho} |x|^{\beta q} |u|^q\right)^{(1-a)r/q} + C\left(\int_{R_\rho} |x|^\delta |u|\right)^r,$$

with C independent of ρ. *If* $\int_{R_\rho} u = 0$ *then the latter term in* (2.3) *may be omitted.*

It suffices to consider $\rho = 1$, since the general case follows by scaling. Writing $R_1 = R$, we consider first the case that

$$\frac{1}{m} = \frac{a}{p} + \frac{1-a}{q} - \frac{a}{n} > 0. \tag{2.4}$$

Using a standard interpolation inequality ([3], [8]), and writing $\bar{u} = (\text{meas } R)^{-1} \int_R u$,

$$\int_R |u - \bar{u}|^m \leqslant C\left(\int_R |Du|^p\right)^{am/p} \left(\int_R |u - \bar{u}|^q\right)^{(1-a)m/q}. \tag{2.5}$$

Since $\alpha - \sigma \geqslant 0$, $r \leqslant m$; applying Hölder's inequality to (2.5) we find

$$\int_R |u-\bar{u}|^r \leqslant C\left(\int_R |u-\bar{u}|^m\right)^{r/m}$$

$$\leqslant C\left(\int_R |Du|^p\right)^{ar/p}\left(\int_R |u-\bar{u}|^q\right)^{(1-a)r/q} \tag{2.6}$$

If, on the other hand, (2.4) fails, then $a/p + (1-a)/q \leqslant a/n$. It follows that

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{n}; \tag{2.7}$$

and if (2.7) holds then

$$\int_R |u-\bar{u}|^r \leqslant C\left(\int_R |Du|^p\right)^{br/p}\left(\int_R |u-\bar{u}|^q\right)^{(1-b)r/q} \tag{2.8}$$

where

$$b = 0 \qquad \text{if} \quad r \leqslant q$$

$$b\left(\frac{1}{q} + \frac{1}{n} - \frac{1}{p}\right) = \frac{1}{q} - \frac{1}{r} \qquad \text{if} \quad r \geqslant q,$$

and in particular $b \leqslant a$. By Sobolev's inequality and (2.7),

$$\left(\int_R |u-\bar{u}|^q\right)^{1/q} \leqslant C\left(\int_R |Du|^p\right)^{1/p}. \tag{2.9}$$

combining (2.8) and (2.9) yields (2.6) once again. Rescaling and multiplying by $\rho^{\gamma r}$, we conclude that if $\int_{R_\rho} u = 0$ then

$$\int_{R_\rho} |x|^{\gamma r}|u|^r \leqslant C\left(\int_{R_\rho} |x|^{\alpha r}|Du|\right)^{ar/p}\left(\int_{R_\rho} |x|^{\beta q}|u|^q\right)^{(1-a)r/q}.$$

If $\bar{u} \neq 0$, we note that

$$\left(\int_R |u-\bar{u}|^q\right)^{1/q} \leqslant C\left(\int_R |u|^q\right)^{1/q}.$$

Therefore, using (2.6),

$$\left(\int_R |u|^r\right)^{1/r} \leqslant \left(\int_R |u-\bar{u}|^r\right)^{1/r} + \left(\int_R |\bar{u}|^r\right)^{1/r}$$

$$\leqslant C\left(\int_R |Du|^p\right)^{a/p}\left(\int_R |u|^q\right)^{(1-a)/q} + C\int_R |u|.$$

This proves (2.3) in case $\rho = 1$, and the general case follows once again by scaling. □

(C) *Suppose* (1.1)–(1.3) *and* (1.5) *hold, and* $\sigma = \alpha - 1$; *and suppose further that*

$$a > (1 + q - q/p)^{-1}. \tag{2.12}$$

Then (1.4) *holds, with constant* C *uniform so long as* $\gamma r + n$ *stays bounded away from zero.*

Since $\sigma = \alpha - 1$ implies $1/r = a/p + (1-a)/q$, the condition (2.12) is equivalent to

$$a > 1/r. \tag{2.13}$$

We prove (1.4) in this context using radial integration by parts:

$$\int |x|^{\gamma r}|u|^r \leqslant C\int |x|^{\gamma r+1}|u|^{r-1}|Du|$$

$$\leqslant C\int (|x|^{\alpha}|Du|)(|x|^{\beta}|u|)^{a^{-1}-1}(|x|^{\epsilon}|u|^{r-a^{-1}})$$

where $\epsilon = \gamma r + 1 - \alpha + \beta - \beta/a = \gamma(r - a^{-1})$. By (2.12), $a^{-1} - 1 \leqslant q$, so

$$\int |x|^{\gamma r}|u|^r \leqslant C\||x|^{\alpha}|Du|\|_{L^p}\||x|^{\beta}u\|_{L^q}^{a^{-1}-1}\||x|^{\epsilon}|u|^{r-a^{-1}}\|_{L^k} \tag{2.14}$$

with k chosen so that

$$\frac{1}{p} + \frac{1}{q}(a^{-1} - 1) + \frac{1}{k} = 1.$$

One checks that

$$(r - a^{-1})k = r \quad \text{and} \quad \epsilon k = \gamma r;$$

using this in (2.14) yields

$$\||x|^\gamma u|_{L^r}^{a^{-1}} \leqslant C\||x|^\alpha Du|_{L^p}\||x|^\beta u|_{L^q}^{a^{-1}-1},$$

from which (1.4) follows. □

(D) *If* $t, r \geqslant 1$; $\gamma + n/r$, $\epsilon + n/t$, $\beta + n/q > 0$; *and* $0 \leqslant b \leqslant 1$ *then*

$$\||x|^\gamma u|_{L^r} \leqslant \||x|^\epsilon u|_{L^t}^{b}\||x|^\beta u|_{L^q}^{1-b} \tag{2.15}$$

for $u \in C_0^\infty(\mathbb{R}^n)$ *provided that*

$$\frac{1}{r} = \frac{b}{t} + \frac{1-b}{q} \tag{2.16}$$

and

$$\gamma = b\epsilon + (1-b)\beta. \tag{2.17}$$

This is an easy consequence of Hölder's inequality. □

For notational convenience we set

$$\||x|^\alpha Du|_{L^p} = A, \quad \||x|^\beta u|_{L^q} = B \tag{2.18}$$

so that our goal is to show

$$\||x|^\gamma u|_{L^r} \leqslant CA^aB^{1-a}.$$

Throughout the paper, $\zeta(x)$ will represent a fixed C_0^∞ function on $\mathbb{R}^n$ with the properties

$$0 \leqslant \zeta \leqslant 1; \quad \zeta \equiv 1 \text{ if } |x| < \tfrac{1}{2}, \quad \zeta \equiv 0 \text{ if } |x| > 1. \tag{2.19}$$

III. Sufficiency when $n = 1$, $\sigma = \alpha - 1$

When possible, we shall use (IIA) to verify the case $a = 1$ and (IID) to interpolate between $a = 0$ and $a = 1$. Substantial complications arise, however, because (IIA) does not apply when $1/p + \alpha - 1 = 0$ (this corresponds to the case $\delta + 1/r = 0$ in (2.1)). Note that $\sigma = \alpha - 1$ implies

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q}, \quad \gamma = a(\alpha - 1) + (1-a)\beta. \tag{3.1}$$

(A) The case $\gamma + 1/r = 1/p + \alpha - 1$, $0 \leqslant a \leqslant 1$

By (IID),

$$\||x|^{\gamma}u\|_{L^r} \leqslant \||x|^{\alpha-1}u\|_{L^p}^{a}\||x|^{\beta}u\|_{L^q}^{1-a}, \tag{3.2}$$

while by (IIA),

$$\||x|^{\alpha-1}u\|_{L^p} \leqslant C\||x|^{\alpha}Du\|_{L^p}. \tag{3.3}$$

Combining (3.2) and (3.3) yields (1.4). □

The remainder of this section addresses the case $\gamma + 1/r \neq 1/p + \alpha - 1$. In that event one may rescale u so that $A = B = 1$; we henceforth assume such a normalization, so that our goal becomes to show

$$\||x|^{\gamma}u\|_{L^r} \leqslant C.$$

(B) The case $1/p + \alpha - 1 > 0$ and bounded away from zero

The argument used for part A applies here, too. Note however that as $1/p + \alpha - 1 \to 0$, the constant in (3.3) tends to ∞. □

The case $1/p + \alpha - 1 \approx 0$ will be handled in (C)–(E); part (F) will treat the case $1/p + \alpha - 1 < 0$ and bounded away from zero.

We choose a real number ν, depending on the parameters, such that $0 < \nu < \frac{1}{2}$ and

$$2\nu \leqslant \gamma + \frac{1}{r}, \quad 2\nu \leqslant \beta + \frac{1}{q} \leqslant (2\nu)^{-1}. \tag{3.4}$$

(C) The case $-\nu^3 \leqslant 1/p + \alpha - 1 \leqslant \nu$ and $1/p < 1 - \nu$

Note that a and $(1 + q - q/p)^{-1}$ are bounded away from 1 in this case:

$$a \leqslant 1 - 2\nu^2 \tag{3.5}$$

$$(1 + q - q/p)^{-1} < \frac{1}{1+\nu}. \tag{3.6}$$

Let $\mu = (1 + 2\nu^2)^{-1}$, and set $a_0 = a/\mu$, so that $a < a_0 \leqslant 1 - 4\nu^4$. By (IID),

$$\||x|^{\gamma}u\|_{L^r} \leqslant C\||x|^{\epsilon}u\|_{L^t}^{a_0}\||x|^{\beta}u\|_{L^q}^{1-a_0} \tag{3.7}$$

where ϵ and t are determined by

$$\frac{1}{t} = \frac{\mu}{p} + \frac{1-\mu}{q}$$

$$\epsilon = \mu(\alpha - 1) + (1-\mu)\beta.$$

Moreover we see that

$$\frac{1}{t} + \epsilon = \mu\left(\frac{1}{p} + \alpha - 1\right) + (1-\mu)\left(\beta + \frac{1}{q}\right) \geqslant 3\nu^3\mu$$

is bounded away from zero. Since $\nu < \frac{1}{2}$, (3.6) implies $\mu(1 + q - q/p) > 1$; hence by (IIC)

$$\| |x|^\epsilon u \|_{L^t} \leqslant CA^\mu B^{1-\mu}; \tag{3.8}$$

substitution of (3.8) into (3.7) yields (1.4). □

(D) The case $-\nu^3 \leqslant 1/p + \alpha - 1 \leqslant \nu, \quad 1 - \nu \leqslant 1/p \leqslant 1, a \geqslant \nu$

We set $\delta = \gamma + 1/r - 1$, and note that under the above hypotheses

$$\alpha \leqslant 2\nu \leqslant \delta + 1 \tag{3.9}$$

and

$$\delta - \beta \leqslant \frac{1}{q} - 1 - \nu^2 \tag{3.10}$$

We assert that

$$\left(\int |x|^{\gamma r}|u|^r\right)^{1/r} \leqslant CA^a B^{1-a} + C\int |x|^\delta |u|. \tag{3.11}$$

Indeed, if $R_k = \{2^k < |x| \leqslant 2^{k+1}\}$ for any integer k, then (IIB) yields

$$\int_{R_k} |x|^{\gamma r}|u|^r$$
$$\leqslant C\left(\int_{R_k} |x|^{\alpha p}|Du|^p\right)^{ar/p}\left(\int_{R_k} |x|^{\beta q}|u|^q\right)^{(1-a)r/q} + C\left(\int_{R_k} |x|^\delta |u|\right)^r. \tag{3.12}$$

We add (3.12) for $k \in \mathbb{Z}$, using the inequalities

$$\sum x_k^c y_k^d \leqslant \left(\sum x_k\right)^c \left(\sum y_k\right)^d, \qquad c + d \geqslant 1 \tag{3.13}$$

$$\sum x_k^c \leqslant \left(\sum x_k\right)^c, \qquad c \geqslant 1, \tag{3.14}$$

valid for x_k, y_k, c, $d \geqslant 0$. Since $ar/p + ((1-a)r)/q = 1$ and $r \geqslant 1$ by (3.1), these inequalities apply and yield (3.11).

Thus we need only show that $\int |x|^\delta |u| \leqslant C$. With ζ as in (2.19) we write

$$\int |x|^\delta |u| = \int |x|^\delta \zeta |u| + \int |x|^\delta (1-\zeta)|u| \tag{3.15}$$

and estimate the two terms separately. Since δ is bounded away from -1, we may use radial integration by parts in the first term:

$$\int |x|^\delta \zeta |u| \leqslant C \int |x|^{\delta+1} \zeta |Du| + C \int |x|^{\delta+1} |D\zeta| |u|$$

$$\leqslant C \int_{|x|<1} |x|^{\delta+1} |Du| + C \int_{\frac{1}{2}<|x|<1} |x|^{\delta+1} |u|.$$

$$\leqslant C \int_{|x|<1} |x|^{\delta+1} |Du| + CB$$

If $p = 1$ then

$$\int_{|x|<1} |x|^{\delta+1} |Du| \leqslant CA \tag{3.16}$$

since $\delta + 1 \geqslant \alpha$ by (3.4). If $p > 1$ then

$$\int_{|x|<1} |x|^{\delta+1} |Du| \leqslant A \cdot \left(\int_{|x|<1} |x|^{(\delta+1-\alpha)p'} \right)^{1/p'}$$

where $p' = p/(p-1)$. Since $\delta + 1 - \alpha \geqslant 0$, the integral converges; hence (3.16) holds also for $p > 1$. Thus

$$\int |x|^\delta \zeta |u| \leqslant C. \tag{3.17}$$

We argue similarly for the second term in (3.15), but without integrating by parts:

$$\int |x|^{\delta}(1-\zeta)|u| \leqslant \int_{|x|>\frac{1}{2}} |x|^{\delta}|u| \leqslant B\left(\int_{|x|>\frac{1}{2}} |x|^{(\delta-\beta)q'}\right)^{1/q'}$$

assuming $q > 1$, and setting $q' = q/(q-1)$. The last integral converges, by (3.10), so

$$\int |x|^{\delta}(1-\zeta)|u| \leqslant C. \tag{3.18}$$

If $q = 1$ we see from (3.10) that $\delta < \beta$, so

$$\int_{|x|>\frac{1}{2}} |x|^{\delta}|u| \leqslant C\int_{|x|>\frac{1}{2}} |x|^{\beta}|u| \leqslant CB,$$

which yields (3.18) also for $q = 1$. We have shown

$$\int |x|^{\delta}|u| \leqslant C,$$

with constant C uniform for fixed ν. By (3.11), the desired result (1.4) follows. □

(E) The case $-\nu^3 \leqslant 1/p + \alpha - 1 \leqslant \nu, \quad 0 \leqslant a < \nu$

We argue much as in part (C). Let ϵ and t satisfy

$$\frac{1}{t} = \frac{\mu}{p} + \frac{1-\mu}{q}, \quad \epsilon = \mu(\alpha-1) + (1-\mu)\beta$$

with $\mu = \frac{1}{2}$, we recall from (3.7) (with $\mu = \frac{1}{2}$) that

$$\||x|^{\gamma}u\|_{L^r} \leqslant C\||x|^{\epsilon}u\|_{L^t}^{2a}\||x|^{\beta}u\|_{L^q}^{1-2a}. \tag{3.19}$$

Since $\epsilon + 1/t \geqslant \frac{1}{2}(2\nu - \nu^3) \geqslant \frac{1}{2}\nu$, we have from cases (C) and (D) that

$$\||x|^{\epsilon}u\|_{L^t} \leqslant CA^{1/2}B^{1/2}. \tag{3.20}$$

Combining (3.19) and (3.20) yields (1.4). □

(F) The case $1/p + \alpha - 1 < -\nu^3$

Let $\tilde{u}(x) = u(x) - u(0)\zeta(x)$, with ζ as in (2.19). Arguing as in parts (A) and (B), we obtain

$$\begin{aligned}\left||x|^\gamma \tilde{u}\right|_{L^r} &\leqslant C\left||x|^\alpha D\tilde{u}\right|_{L^p}^a \left||x|^\beta \tilde{u}\right|_{L^q}^{1-a} \\ &\leqslant C\left(\left||x|^\alpha Du\right|_{L^p} + |u(0)|\right)^a \left(\left||x|^\beta u\right|_{L^q} + |u(0)|\right)^{1-a} \\ &\leqslant C(1 + |u(0)|).\end{aligned}$$

Thus to prove (1.4) we need only show that $|u(0)| \leqslant C$.

Now

$$u(0) = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}(u\zeta)$$

so that

$$\begin{aligned}|u(0)| &\leqslant C\int_0^1 |Du| + C\int_{1/2}^1 |u| \\ &\leqslant C\int_0^1 |Du| + C.\end{aligned}$$

If $p > 1$, then

$$\int_0^1 |Du| \leqslant \left||x|^\alpha Du\right|_{L^p} \cdot \left(\int_0^1 |x|^{-\alpha p'}\right)^{1/p'}$$

and the integral on the right converges, because $-\alpha p' \geqslant -1 + \nu^3 p'$; hence

$$\int_{|x|<1} |Du| \leqslant CA \leqslant C. \tag{3.22}$$

If $p = 1$, we still conclude (3.22), since in that case $\alpha < 0$. Thus we have shown

$$|u(0)| \leqslant C. \tag{3.23}$$

□

The proof of (1.4) for $n = 1$, $\sigma = \alpha - 1$ is now complete.

IV. Sufficiency when $n \geqslant 1$, $\alpha \geqslant \sigma \geqslant \alpha - 1$

Note that in this case

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q} + \frac{a(\alpha - \sigma - 1)}{n} \leqslant \frac{a}{p} + \frac{1-a}{q}. \tag{4.1}$$

(A) Radial functions

We consider $u(x) = f(|x|)$, where f is smooth on $[0, \infty)$ and vanishes for $|x|$ large. For integers k, let $R_k = \{2^k < |x| \leqslant 2^{k+1}\}$; by (IIB) we have

$$\int_{R_k} |x|^{\gamma r}|u|^r \tag{4.2}$$

$$\leqslant C\left(\int_{R_k} |x|^{\alpha p}|Du|^p\right)^{ar/p}\left(\int_{R_k} |x|^{\beta q}|u|^q\right)^{(1-a)r/q} + C\left(\int_{R_k} |x|^{\delta}|u|\right)^r$$

with $\delta = \gamma + n/r - n$. Let s be defined by

$$\frac{1}{s} = \frac{a}{p} + \frac{1-a}{q} \tag{4.3}$$

so that $1/r \leqslant 1/s \leqslant 1$. By Hölder's inequality,

$$\left(\int_{R_k} |x|^{\delta}|u|\right)^r \leqslant C\left(\int_{R_k} |x|^{\mu s}|u|^s\right)^{r/s}, \tag{4.4}$$

with

$$\mu = n\left(\frac{1}{r} - \frac{1}{s} + \frac{\gamma}{n}\right). \tag{4.5}$$

We add (4.2) for all k, using (4.4) and the inequalities (3.13), to obtain

$$\left(\int |x|^{\gamma r}|u|^r\right)^{1/r} \leqslant CA^aB^{1-a} + C\left(\int |x|^{\mu s}|u|^s\right)^{1/s}. \tag{4.6}$$

Now,

$$\left(\int_{\mathbb{R}^n} |x|^{\mu s}|u|^s\right)^{1/s} \leqslant C\left(\int_0^{\infty} \rho^{\bar{\mu}s}|f|^s \mathrm{d}\rho\right)^{1/s}, \qquad \bar{\mu} = \gamma + \frac{n}{r} - \frac{1}{s} \tag{4.7}$$

while

$$\left(\int_{\mathbb{R}^n} |x|^{\alpha p} |Du|^p\right)^{1/p} \geqslant C\left(\int_0^\infty \rho^{\bar{\alpha} p} |Df|^p\right)^{1/p}, \qquad \bar{\alpha} = \alpha + \frac{n-1}{p} \tag{4.8}$$

and

$$\left(\int_{\mathbb{R}^n} |x|^{\beta q} |u|^q\right)^{1/q} \geqslant C\left(\int_0^\infty \rho^{\bar{\beta} q} |f|^q\right)^{1/q}, \qquad \bar{\beta} = \beta + \frac{n-1}{q}. \tag{4.9}$$

Since

$$\frac{1}{s} + \bar{\mu} = a\left(\frac{1}{p} + \bar{\alpha} - 1\right) + (1-a)\left(\frac{1}{q} + \bar{\beta}\right),$$

we conclude from Section III that

$$\| |x|^{\bar{\mu}} f \|_{L^s} \leqslant C \| |x|^{\bar{\alpha}} Df \|_{L^p}^{a} \| |x|^{\bar{\beta}} f \|_{L^q}^{1-a}. \tag{4.10}$$

(Strictly speaking, one must first extend f to a function on $(-\infty, \infty)$, and then apply the results of Section III. Alternatively, one may simply note that all the proofs in Section III remain valid for functions on $[0, \infty)$.) Combining (4.6)–(4.10) yields $\| |x|^{\gamma} u \|_{L^r} \leqslant CA^a B^{1-a}$. □

(B) Non-radial functions

For any $u \in C_0(\mathbb{R})^n)$, let U: $(0, \infty) \to \mathbb{R}^n$ denote its spherical mean function

$$U(\rho) = ⨍_{|x|=\rho} u \tag{4.11}$$

and let u^* be the associated radial function on $\mathbb{R}^n$

$$u^*(x) = U(|x|). \tag{4.12}$$

We have

$$|DU(\rho)| \leqslant ⨍_{|x|=\rho} |Du|, \qquad |U(\rho)| \leqslant ⨍_{|x|=\rho} |u|$$

so that

$$\| |x|^{\alpha} Du^* \|_{L^p} \leqslant A, \qquad \| |x|^{\beta} u^* \|_{L^q} \leqslant B; \tag{4.13}$$

also, of course, $u - u^*$ has mean zero on each sphere $|x| = \rho$.

Let $R_k = \{2^k < |x| \leqslant 2^{k+1}\}$ for integers k; by (IIB) we have

$$\int_{R_k} |x|^{\gamma r}|u - u^*|^r$$

$$\leqslant C\left(\int_{R_k} |x|^{\alpha p}|Du - Du^*|^p\right)^{ar/p}\left(\int_{R_k} |x|^{\beta q}|u - u^*|^q\right)^{(1-a)r/q} \tag{4.14}$$

for each k. We add the inequalities (4.14), using (3.13) and (4.1), to conclude

$$\| |x|^\gamma (u - u^*)|_{L^r} \leqslant C \| |x|^\alpha D(u - u^*)|_{L^p}^{a} \| |x|^\beta (u - u^*)|_{L^q}^{1-a}, \tag{4.15}$$

whence using (4.13) and (IVA),

$$\| |x|^\gamma u|_{L^r} \leqslant CA^a B^{1-a} + \| |x|^\gamma u^*|_{L^r} \leqslant CA^a B^{1-a}. \qquad \square$$

(V) Sufficiency in case $1/p + (\alpha - 1)/n \neq 1/r + \gamma/n$ and $\sigma < \alpha - 1$

Notice that in this case $a < 1$ necessarily. We may assume $A = B = 1$, since this normalization may be achieved by scaling. Since (1.4) has been proved for $\sigma = \alpha$ and for $\sigma = \alpha - 1$, we know that

$$\| |x|^\delta u|_{L^s} \leqslant C, \qquad \| |x|^\epsilon u|_{L^t} \leqslant C \tag{5.1}$$

provided that δ, s, ϵ, and t are related by

$$\begin{aligned} &\delta = b\alpha + (1 - b)\beta \\ &\frac{1}{s} = \frac{b}{p} + \frac{1 - b}{q} - \frac{b}{n} \\ &\epsilon = d(\alpha - 1) + (1 - d)\beta \\ &\frac{1}{t} = \frac{d}{p} + \frac{1 - d}{q} \end{aligned} \tag{5.2}$$

for some choices of b and d, $0 \leqslant b, d \leqslant 1$, and provided that

$$\frac{\delta}{n} + \frac{1}{s} > 0, \qquad \frac{\epsilon}{n} + \frac{1}{t} > 0. \tag{5.3}$$

Under certain conditions upon b and d we shall see that (5.1) implies a bound for $\||x|^{\gamma}u|_{L^r}$. For ζ as in (2.19), we estimate

$$\left(\int |x|^{\gamma r}\zeta |u|^r\right)^{1/r} \leqslant \||x|^{\epsilon}u|_{L^t}\left(\int_{|x|<1} |x|^{(\gamma-\epsilon)tr/(t-r)}\right)^{1/r-1/t} \tag{5.4}$$

and

$$\left(\int |x|^{\gamma r}(1-\zeta)|u|^r\right)^{1/r} \leqslant \||x|^{\delta}u|_{L^s}\left(\int_{|x|>\frac{1}{2}} |x|^{(\gamma-\delta)sr/(s-r)}\right)^{1/r-1/s} \tag{5.5}$$

by Hölder's inequality, provided that

$$\frac{1}{t} \leqslant \frac{1}{r} \quad \text{and} \quad \frac{1}{s} \leqslant \frac{1}{r}. \tag{5.6}$$

The integrals on the right in (5.4) and (5.5) converge if

$$\frac{1}{t} + \frac{\epsilon}{n} < \frac{1}{r} + \frac{\gamma}{n} < \frac{1}{s} + \frac{\delta}{n}. \tag{5.7}$$

One computes that

$$\frac{1}{t} + \frac{\epsilon}{n} = d\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-d)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

$$\frac{1}{s} + \frac{\delta}{n} = b\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-b)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

so that (5.7) holds whenever

$$b < a < d \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - 1}{n} < \frac{1}{q} + \frac{\beta}{n}$$

$$d < a < b \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - 1}{n} > \frac{1}{q} + \frac{\beta}{n},$$

and (5.3) holds too if $|d-a|$ and $|b-a|$ are sufficiently small. One computes furthermore that

$$\frac{1}{r} - \frac{1}{s} = (a-b)\left(\frac{1}{p} - \frac{1}{q} - \frac{1}{n}\right) + \frac{a}{n}(\alpha - \sigma)$$

$$\frac{1}{r} - \frac{1}{t} = (a-d)\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{a}{n}(\alpha - \sigma - 1);$$

since $a > 0$ and $\sigma < \alpha - 1$,

$$0 < \frac{a}{n}(\alpha - \sigma - 1) < \frac{a}{n}(\alpha - \sigma);$$

therefore if $|b - a|$ and $|a - d|$ are small enough (5.6) will hold as well. For such choices of b and d, we use (5.1), (5.4), and (5.5) to conclude

$$\||x|^{\gamma} u|_{L^r} \leqslant C. \qquad \square$$

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