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UNITARY REPRESENTATIONS WITH NON-ZERO COHOMOLOGY

David A. Vogan, Jr. * and Gregg J. Zuckerman

1. Introduction

Let G be a real connected semisimple Lie group with finite center. Suppose $(\pi, \mathscr{H}_\pi)$ is an irreducible unitary representation of G, and (ρ, F) is an irreducible finite dimensional representation of G. Then one can consider the *continuous cohomology of G with coefficients in* $\pi \otimes \rho$,

$$H^*_{\mathrm{ct}}(G, \mathscr{H}_\pi \otimes F)$$

(see [2] or [5]). The zero cohomology group $H^0_{\mathrm{ct}}(G, \mathscr{H}_\pi \otimes F)$ consists of the G-invariant vectors in $\mathscr{H}_\pi \otimes F$, and the higher groups are the derived functors of H^0_{ct} in an appropriate category. One of the main reasons for the interest in this cohomology is its connection with the theory of automorphic forms. The simplest aspect of this connection can be described as follows. Let $K \subseteq G$ be a maximal compact subgroup, and let $\Gamma \subseteq G$ be a discrete subgroup. Assume that $\Gamma \backslash G$ is compact, and that Γ acts freely on G/K. (Such subgroups Γ always exist.) Then

$$X = \Gamma \backslash G/K$$

is a compact manifold. The action of G by right translation on the Hilbert space $L^2(\Gamma \backslash G)$ decomposes into a Hilbert space direct sum of irreducible unitary representations of G, each occurring with finite multiplicity:

$$L^2(\Gamma \backslash G) \cong \bigoplus_{\pi \in \hat{G}} m_\pi \mathscr{H}_\pi,$$

with m_π a non-negative integer. Matsushima's formula ([2], page 223) is

$$H^*_{\mathrm{top}}(X, \mathbb{C}) \cong \bigoplus_{\pi \in \hat{G}} m_\pi H^*_{\mathrm{ct}}(G, \mathscr{H}_\pi).$$

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The cohomology groups on the left are the ordinary topological ones for the manifold X. The numbers m_π are essentially dimensions of spaces of automorphic forms for Γ. In order to apply this formula (and refinements or generalizations of it) to compute the m_π, we need to understand $H^*_{\mathrm{ct}}(G, \mathscr{H}_\pi)$. The problem we consider (and more or less solve) is therefore this: describe, in as much detail as possible, the irreducible unitary representations $(\pi, \mathscr{H}_\pi)$ such that

$$H^*_{\mathrm{ct}}(G, \mathscr{H}_\pi \otimes F) \neq 0$$

for some finite dimensional representation F of G.

To understand the solution to this problem, we have to know what it means to describe a unitary representation. (The reader with any background in this area should now skip to Theorem 1.4.) The best description is usually a realization: we make G act on a vector bundle over a homogeneous space (say), and consider the representation on an appropriate Hilbert space of sections of the bundle. Unfortunately, the representations with cohomology rarely have such realizations. (This is perhaps a failure more of technology than of vision. Rawnsley, Schmid, and Wolf in [21] have suggested a possible realization which is a very sophisticated version of "sections of a bundle" but they have been able to produce it only in special cases.) Instead, we consider some invariants which any unitary representation has, and specify what they are in our representations. Two invariants are needed: the eigenvalue of the Casimir operator of the representation, and the restriction of the representation to K.

To describe the Casimir operator, we need to get a representation of the Lie algebra $\mathfrak{g}_0$ of G out of a unitary representation $(\pi, \mathscr{H}_\pi)$. There is a dense subspace $\mathscr{H}_\pi^\infty \subseteq \mathscr{H}_\pi$, the *smooth vectors* of $\mathscr{H}_\pi$. If $x \in \mathfrak{g}_0$ and $v \in \mathscr{H}_\pi^\infty$, then we define

$$\pi(X)v = \lim_{t \to 0} \frac{1}{t}(\pi(\exp(tX))v - v);$$

the limit exists, and belongs to $\mathscr{H}_\pi^\infty$. Therefore

$$\pi(X): \mathscr{H}_\pi^\infty \to \mathscr{H}_\pi^\infty \qquad (X \in \mathfrak{g}_0).$$

The Casimir operator is a certain element $\pi(\Omega)$ in the algebra generated by the operators $\pi(X)$; we will describe it more explicitly in a moment. It commutes with all of the operators $\pi(X)$. Since $(\pi, \mathscr{H}_\pi)$ is irreducible, this suggests that $\pi(\Omega)$ should be a scalar operator. (Schur's lemma does not immediately apply, since $\mathscr{H}_\pi$ is infinite dimensional, and $\pi(\Omega)$ is defined only on a dense subspace.) Nevertheless, I.E. Segal has shown that

$$\pi(\Omega) = c_\pi \cdot \mathrm{Id} \tag{1.1 (a)}$$

The (real) constant c_π is our first general invariant of π. To define $\pi(\Omega)$, recall the Killing form

$$\langle X, Y\rangle = \operatorname{tr} \operatorname{ad}(X) \operatorname{ad}(Y)$$

on $\mathfrak{g}_0$. It is non-degenerate since $\mathfrak{g}_0$ is semisimple. Fix a basis $\{X_i\}$ of $\mathfrak{g}$, and let $\{X^j\}$ be the dual basis:

$$\langle X_i, X^j\rangle = \delta_{ij}.$$

Then

$$\pi(\Omega) = \sum_i \tfrac{1}{2}\big(\pi(X_i)\pi(X^i) + \pi(X^i)\pi(X_i)\big). \tag{1.1 (b)}$$

It is easy to check that this is independent of the choice of basis.

If π is realized in a space of functions on G/K (which is a Riemannian manifold in a natural way), then c_π may be interpreted as the eigenvalue of the Laplace-Beltrami operator on that space of functions. Now Hodge theory suggests that it is the zero eigenspace of the Laplacian which should be related to cohomology; and this turns out to be the case.

PROPOSITION 1.2 ([2], *Proposition II.3.1*): *Suppose $(\pi, \mathscr{H}_\pi)$ is an irreducible unitary representation of G, and (ρ, F) is an irreducible finite dimensional representation of G. Write c_π, c_ρ for the respective eigenvalues of the Casimir operator* (*see* (1.1)). *Then $H^*_{\mathrm{ct}}(G, \mathscr{H}_\pi \otimes F) \neq 0$ only if $c_\pi = c_\rho$. In particular, $H^*_{\mathrm{ct}}(G, \mathscr{H}_\pi) \neq 0$ only if $c_\pi = 0$.*

We therefore know the value of this first invariant in a representation with non-zero cohomology.

Next, recall the maximal compact subgroup K of G. Any unitary representation $(\gamma, \mathscr{H}_\gamma)$ of K decomposes as a Hilbert space direct sum of copies of the various irreducible representations of K, which are finite dimensional. If we write $\hat{K}$ for the set of irreducible representations of K, we can write symbolically

$$\gamma = \sum_{\delta \in \hat{K}} m(\delta, \gamma) \cdot \delta.$$

Here $m(\delta, \gamma)$ is a cardinal number, the *multiplicity of* δ *in* γ. This means that (if δ acts on $\mathscr{H}_\delta$),

$$\mathscr{H}_\gamma \cong \bigoplus_{\delta \in \hat{K}} (\mathscr{H}_\delta)^{m(\delta,\gamma)}$$

(a Hilbert space direct sum), with the isomorphism respecting the actions of K. In particular, if $(\pi, \mathscr{H}_\pi)$ is an irreducible unitary representation of G, then

$$\pi|_K = \sum_{\delta \in \hat{K}} m(\delta, \pi) \cdot \delta;$$

here we have written $m(\delta, \pi)$ for the multiplicity of δ in the restriction of π to K. A theorem of Harish-Chandra says that all the cardinal numbers $m(\delta, \pi)$ are finite; that is, they are non-negative integers. The second general invariant of π which we have in mind is the set of integers $m(\delta, \pi)$. In practice, one usually uses much weaker information. One might know only (for some fixed π) that a particular $m(\delta_0, \pi)$ is non-zero. If π is realized in a space of functions on the symmetric space G/K, for example, then $m(\text{trivial}, \pi)$ must be non-zero: that is, π must contain the trivial representation of K.

Matsushima's formula shows that the representations having non-zero cohomology are connected with the cohomology of locally symmetric spaces; so analogy with the DeRham theorem suggests that such representations should be realized on the space of sections of the form bundle on G/K. This is essentially correct; and one concludes that they must contain certain particular representations of K.

PROPOSITION 1.3 ([2], *Proposition II.3.1*): *Suppose $(\pi, \mathscr{H}_\pi)$ is an irreducible unitary representation of G, and (ρ, F) is a finite dimensional representation of G. Write $\mathfrak{p}$ for the complexified tangent space of G/K at the origin. Suppose $H^j_{\mathrm{ct}}(G, \mathscr{H}_\pi \otimes F) \neq 0$. Then there is a $\delta \in \hat{K}$ such that δ occurs in both π and*

$$\mathrm{Hom}_C(F, \Lambda^j \mathfrak{p}).$$

Propositions 1.2 and 1.3 show how to get some information about our two general invariants from knowing that the cohomology of π is non-zero. Our results imply that this very weak information actually determines the representation.

THEOREM 1.4: *Suppose (ρ, F) is an irreducible finite dimensional representation of G, and $\delta \in \hat{K}$ occurs in*

$$\mathrm{Hom}_C(F, \Lambda^j \mathfrak{p}).$$

Then there is at most one irreducible unitary representation $(\pi, \mathscr{H}_\pi)$ of G with the following properties:

(a) $c_\pi = c_\rho$ (*notation* (1.1))

(b) δ *occurs in* π.

Assume that $(\pi, \mathscr{H}_\pi)$ *exists. Then the following things may be computed explicitly from* δ:

(1) $H^*_{\mathrm{ct}}(G, \mathscr{H}_\pi \otimes F)$ *(together with its Hodge structure, if* G/K *is Hermitian symmetric)*

(2) *the position of* π *in the Langlands classification of irreducible representations of* G

(3) *the character of* π *on a fundamental Cartan subgroup*

(4) *the multiplicity of any representation of* K *in* π.

This summarizes Propositions 6.1, 6.4, and 6.19, and Theorems 5.5 and 6.16. The theorem can be rephrased as follows. Suppose π is unitary and $\pi \otimes F$ has non-zero cohomology. If we can determine a single K type of π which lies in $\mathrm{Hom}_C(F, \Lambda^j\mathfrak{p})$, then we can compute all of the other things mentioned.

For simplicity of exposition, we treat the case of untwisted coefficients ($F = \mathbb{C}$) first. The first step is to exhibit a certain collection of representations of G (Theorem 2.5). These were first constructed by Parthasarathy in [12], but we need a characterization of them different from the one he gives. Next, we compute the cohomology of these representations (Theorem 3.3). Finally, we show (confirming a conjecture of Zuckerman) that any irreducible unitary representation of G having non-zero continuous cohomology belongs to our collection (Theorem 4.1). The method is due to Parthasarathy [13].

Our results rely heavily on work of Kumaresan in [10] (the cases $\lambda = 0$ of Propositions 5.7 and 5.16). For some applications, such as vanishing theorems, they do not improve on [10]. (We have completed the explicit calculation of Kumaresan's vanishing theorem in Section 8.) When G is complex, our results deduce to those of Enright [4], with essentially the same proof. When G/K is Hermitian symmetric and $\mathscr{H}_\pi$ is a highest weight representation, our results are those of Parthasarthy [13]. When G is $SL(n, \mathbb{R})$, sharper results than ours have been given by Speh in [15], [16].

There is one annoying gap in our results: we do not know how to prove that all of the representations which we construct are in fact unitary; so we have not actually classified the unitary representations with non-zero cohomology. This is unimportant for the obvious applications to automorphic forms; but it is of great importance in the study of the unitary dual of G. Some partial results are given in Section 6 (Propositions 6.3 and 6.5).

Finally, our results have some bearing on the theory of Dirac operators on locally symmetric spaces. This is discussed in Section 7.

This paper has deliberately been written at two levels. The results should be of interest outside of representation theory, so they have been formulated in as elementary a way as possible. The proofs are not really very deep, but they are a little convoluted; and it is difficult to imagine that they will appear natural or enlightening to a non-expert. They are accordingly addressed to a much smaller audience. From Section 5 on, in fact, many proofs are omitted or sketched on the grounds that this audience could easily supply the details.

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2. The representations $\mathrm{A}_{\mathfrak{q}}$

Recall that G is a connected real semisimple Lie group with finite center. Write $\mathfrak{g}_0$ for the Lie algebra of G, and $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ for its complexification. Analogous notation is used for other groups. Let $K \subseteq G$ be a maximal compact subgroup, θ the Cartan involution, and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

the corresponding Cartan decomposition. We write $\langle\ ,\ \rangle$ for the Killing form on $\mathfrak{g}_0$ and its various natural complexifications, restrictions, and so on.

Let $(\pi, \mathscr{H})$ be an irreducible unitary representation of G on a Hilbert space $\mathscr{H}$, and $\mathscr{H}^\infty$ the subspace of smooth vectors. Then $\mathscr{H}^\infty$ is a dense subspace of $\mathscr{H}$ invariant under G, and there is a natural action of $\mathfrak{g}$ on $\mathscr{H}^\infty$ (see the introduction). Define

$$\mathscr{H}^K = \{ v \in \mathscr{H}^\infty | \dim\langle \pi(K) v\rangle < \infty \};$$

here $\langle \pi(K)v\rangle$ denotes the linear span of all the vectors of the form $\pi(k)v$, with $k \in K$.

Proposition 2.1 (*Harish-Chandra* [6]): *$\mathscr{H}^K$ is stable under the actions of K and $\mathfrak{g}$ on $\mathscr{H}^\infty$. As a $\mathfrak{g}$ module, $\mathscr{H}^K$ is irreducible, and determines π up to unitary equivalence.*

We call $\mathscr{H}^K$ the *Harish-Chandra module of* π. If $x \in \mathfrak{g}$ and $v \in \mathscr{H}^K$, we will use the module notation $x \cdot v$ instead of $\pi(x)v$.

What we will actually describe are modules $A_{\mathfrak{q}}$ for $\mathfrak{g}$. The main theorem will assert that if $(\pi, \mathscr{H})$ has non-zero continuous cohomology, then $\mathscr{H}^K$ is isomorphic to some $A_{\mathfrak{q}}$ as a $\mathfrak{g}$ module. Because of the last assertion of Proposition 2.1, $A_{\mathfrak{q}}$ then determines π. (The problem discussed in the introduction is that, given $A_{\mathfrak{q}}$, we do not know how to find a unitary representation $(\pi, \mathscr{H})$ with $\mathscr{H}^K \cong A_{\mathfrak{q}}$.) The first problem is to describe the parameter $\mathfrak{q}$.

Fix an element $x \in \mathrm{i}\,\mathfrak{k}_0$; here $\mathrm{i} = \sqrt{-1}$. Since K is compact, the linear transformation $\mathrm{ad}(x)$ of $\mathfrak{g}$ is diagonalizable, with real eigenvalues; and complex conjugation interchanges the positive and negative eigenspaces. Define

$$\mathfrak{q} = \text{sum of non-negative eigenspaces of } \operatorname{ad}(x)$$

$$\mathfrak{u} = \text{sum of positive eigenspaces of } \operatorname{ad}(x)$$

$$\mathfrak{l} = \text{sum of zero eigenspaces of } \operatorname{ad}(x) = \text{centralizer of } x. \tag{2.2}$$

Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$, and

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$$

is a Levi decomposition. Furthermore, $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{q} \cap \mathfrak{g}_0$. Since $\theta x = x$, $\mathfrak{q}$, $\mathfrak{l}$, and $\mathfrak{u}$ are all invariant under θ, so

$$\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p},$$

and so on. In particular, $\mathfrak{q} \cap \mathfrak{k}$ is a parabolic subalgebra of $\mathfrak{k}$, with Levi decomposition

$$\mathfrak{q} \cap \mathfrak{k} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{u} \cap \mathfrak{k}.$$

We call the subalgebras $\mathfrak{q}$ obtained in this way *θ-stable parabolic subalgebras of* $\mathfrak{g}$. (Since not every parabolic subalgebra preserved by θ is of this form, the terminology is unfortunate.)

With notation as in the preceding paragraph, choose a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$ containing ix (as is possible). Then $\mathfrak{t}$ is automatically contained in $\mathfrak{l} \in \mathfrak{k}$. Let $\mathfrak{f} \subseteq \mathfrak{q}$ be any subspace stable under $\operatorname{ad}(\mathfrak{t})$. Then there are a subset $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathfrak{t}^*$ (the dual of $\mathfrak{t}$), and subspaces $\mathfrak{f}_{\alpha_i}$ of $\mathfrak{f}$, such that if $y \in \mathfrak{t}$ and $v \in \mathfrak{f}_{\alpha_i}$, then

$$\operatorname{ad}(x) v = \alpha_i(x) v.$$

Write

$$\Delta(\mathfrak{f}, \mathfrak{t}) = \Delta(\mathfrak{f}) = \{\alpha_1, \ldots, \alpha_r\}, \tag{2.3}$$

the *weights* or *roots* of $\mathfrak{t}$ in $\mathfrak{f}$. Often we assume that $\Delta(\mathfrak{f})$ is a set with multiplicities, with α_i having multiplicity $\dim \mathfrak{f}_{\alpha_i}$. Then if

$$\rho(\mathfrak{f}) = \rho(\Delta(\mathfrak{f})) = \tfrac{1}{2} \sum_{\alpha_i \in \Delta(\mathfrak{f})} \alpha_i \in \mathfrak{t}^*,$$

we have

$$\rho(\mathfrak{f})(y) = \tfrac{1}{2} \operatorname{tr}(\operatorname{ad}(y)|_{\mathfrak{f}}) \qquad (y \in \mathfrak{t}).$$

Fix a system $\Delta^+(\mathfrak{l} \cap \mathfrak{k})$ of positive roots in the root system $\Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$. (Of course $\Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$ as defined above includes zero, and so it is not really a root system; but we will overlook such abuses of terminology.) Then

$$\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{l} \cap \mathfrak{k}) \cup \Delta(\mathfrak{u} \cap \mathfrak{k})$$

is a positive root system for $\mathfrak{t}$ in $\mathfrak{k}$.

Recall that (given $\Delta^+(\mathfrak{k})$) the irreducible representations of the compact group K may be parametrized by their highest weights, which are elements of $\mathfrak{t}^*$. Define

$$\mu = \mu(\mathfrak{q}) = \text{representation of } K \text{ with highest weight } 2\rho(\mathfrak{u} \cap \mathfrak{p}). \tag{2.4}$$

Since not every element of $\mathfrak{t}^*$ is the highest weight of a representation, the existence of μ is not quite obvious. We will postpone this problem until Section 3, where μ will be exhibited as a subrepresentation of the exterior algebra $\Lambda^*\mathfrak{p}$. We can now describe the representation $A_{\mathfrak{q}}$.

THEOREM 2.5: *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ (see* (2.2)). *Then there is a unique irreducible module $A_{\mathfrak{q}}$ for $\mathfrak{g}$ with the following properties*:

(a) *The restriction of $A_{\mathfrak{q}}$ to $\mathfrak{k}$ contains the irreducible representation $\mu(\mathfrak{q})$ (see* (2.4)).
(b) *The center of the universal enveloping algebra of $\mathfrak{g}$ acts in $A_{\mathfrak{q}}$ by the same scalars as in the trivial representation of $\mathfrak{g}$.*
(c) *If the representation of $\mathfrak{k}$ of highest weight δ occurs in $A_{\mathfrak{q}}$ restricted to $\mathfrak{k}$, then δ must be of the form*

$$\delta = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta,$$

(*notation* (2.3)), *with n_β a non-negative integer.*

If $\mathfrak{l} \subseteq \mathfrak{k}$ (and in fact only then), $A_{\mathfrak{q}}$ is a discrete series representation. More generally, $A_{\mathfrak{q}}$ is a fundamental series representation if and only if $[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{k}$. If $[\mathfrak{l}, \mathfrak{l}] \not\subseteq \mathfrak{k}$, then $A_{\mathfrak{q}}$ is not tempered; that is, it does not appear in Harish-Chandra's Plancherel formula for G. If $\mathfrak{u} \cap \mathfrak{p} = 0$ (for example, if $\mathfrak{q} = \mathfrak{g}$), then $A_{\mathfrak{q}}$ is the trivial representation of $\mathfrak{g}$. The other $A_{\mathfrak{q}}$ are less familiar representations. For $SU(n, 1)$ and $SO(n, 1)$, they are all the representations having the same infinitesimal character as the trivial representation; they occur at the endpoints of certain complementary series. In Section 6, a simpler characterization of $A_{\mathfrak{q}}$ is given (Proposition 6.1, with $\lambda = 0$), as well as a formula for its global character on the fundamental Cartan subgroup (Proposition 6.4) and other properties.

The existence of $A_\mathfrak{q}$ is proved by Parthasarathy in [12]. Alternatively, the results of §6.3 of [19] show that the cohomologically induced representation $\mathscr{R}_\mathfrak{q}^S(\mathbb{C})$ (with $S = \dim \mathfrak{u} \cap \mathfrak{k}$) satisfies (a) – (c); so $A_\mathfrak{q}$ may be taken to be an appropriate irreducible subquotient of it. (Actually, it is not hard to show that $\mathscr{R}_\mathfrak{q}^S(\mathbb{C})$ is irreducible, and so coincides with $A_\mathfrak{q}$. This construction of $A_\mathfrak{q}$ is due to Zuckerman.) Still another construction is that of Speh-Vogan [17]: by their results, $X(\mathfrak{q}, \mathbb{C}, \mu(\mathfrak{q}))$ (notation [17], p. 247) satisfies (a), (b) and a weakening of (c): $\Delta(\mathfrak{u} \cap \mathfrak{p})$ is replaced by $\Delta(\mathfrak{u})$. (This weaker version of (c) suffices for our applications in this paper.)

The uniqueness of $A_\mathfrak{q}$ is more difficult, particularly because we need to establish it under a much weaker condition than Theorem 2.5(c). Choose a system $\Delta^+(\mathfrak{l})$ of positive roots for $\mathfrak{t}$ in $\mathfrak{l}$, containing the positive system $\Delta^+(\mathfrak{l} \cap \mathfrak{k})$ chosen above. Then

$$\Delta^+(\mathfrak{q}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u})$$

in a system of positive roots for $\mathfrak{t}$ in $\mathfrak{g}$. (It should be admitted that $\mathfrak{t}$ is not a Cartan subalgebra of $\mathfrak{l}$ or $\mathfrak{g}$ in general; but $\Delta(\mathfrak{l})/\{0\}$ and $\Delta(\mathfrak{g})/\{0\}$ can be shown to be root systems, so the discussion of "positive systems" is justified.) Set

$$W_K = W(\mathfrak{k}, \mathfrak{t}) \text{ (the Weyl group of } \mathfrak{t} \textit{ in } \mathfrak{k}\text{)},$$

$$W_K^1 = \{\sigma \in W_K | \sigma(\Delta^+(\mathfrak{k})) \supseteq \Delta^+(\mathfrak{l} \cap \mathfrak{k})\}.$$

We will be interested in representations of $\mathfrak{k}$ having a highest weight of the form

$$\delta = \sigma^{-1}[2\rho(\mathfrak{u} \cap \mathfrak{p}) + \rho(\Delta^+(\mathfrak{k})) - 2\rho(B_0)] - \rho(\Delta^+(\mathfrak{k})), \tag{2.6}$$

with $\sigma \in W_K^1$, and B_0 a subset of $\Delta(\mathfrak{u} \cap \mathfrak{p})$. We assume that either $\sigma \neq 1$, or $B_0 \neq \emptyset$.

LEMMA 2.7: *No weight $\delta \in \mathfrak{t}^*$ can satisfy both* (2.5) (*c*) *and* (2.6).

PROOF: Suppose (2.6) holds. We put $B = B_0 \cup \Delta^+(\mathfrak{l} \cap \mathfrak{p})$; then

$$2\rho(\Delta^+(\mathfrak{p})) = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p}))$$

$$2\rho(B) = 2\rho(B_0) + 2\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p})).$$

Therefore (2.6) may be rewritten as

$$\delta = [\sigma^{-1}\rho(\Delta^+(\mathfrak{k})) - \rho(\Delta^+(\mathfrak{k}))] + \sigma^{-1}[\rho(\Delta^+(\mathfrak{p})) - 2\rho(B)]$$
$$+ \sigma^{-1}\rho(\Delta^+(\mathfrak{p})).$$

As is well known (e.g., [20], 2.5.2.4), the first term is a sum of roots in

$-\Delta(\mathfrak{u} \cap \mathfrak{k})$. By [19], Lemma 5.4.5, each of the next two terms is of the form

$$\rho(\Delta^+(\mathfrak{p})) - 2\rho(C), \qquad C \subseteq \Delta^+(\mathfrak{p}).$$

Therefore

$$\delta = 2\rho(\Delta^+(\mathfrak{p})) - 2\rho(C_1) - 2\rho(C_2) - 2\rho(C_3)$$

$$\delta = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p}) - 2\rho(C_1) - 2\rho(C_2) - 2\rho(C_3).$$

Here $C_1 \subseteq \Delta(\mathfrak{u} \cap \mathfrak{k})$; C_2 and C_3 are contained in $\Delta^+(\mathfrak{p})$. Suppose now that δ also satisfies (2.5) (c). By inspection, this implies that all n_β are zero, and C_1 is empty. Thus $\delta = 2\rho(\mathfrak{u} \cap \mathfrak{p})$. Since C_1 is empty,

$$\sigma^{-1}\rho(\Delta^+(\mathfrak{k})) - \rho(\Delta^+(\mathfrak{k})) = -2\rho(C_1) = 0;$$

so $\sigma = 1$. Therefore (2.6) becomes

$$2\rho(\mathfrak{u} \cap \mathfrak{p}) = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \rho(\Delta^+(\mathfrak{k})) - 2\rho(B_0) - \rho(\Delta^+(\mathfrak{k})).$$

This forces B_0 to be empty, contrary to the hypotheses in (2.6).

□

Another proof of Lemma 2.7 can be given based on Lemma 4.7. Here is the result we actually need.

PROPOSITION 2.8: *Theorem* 2.5 *remains true with* (*c*) *replaced by* (*c*)′: *No representation of* $\mathfrak{k}$ *whose highest weight is of the form* (2.6) *occurs in* $A_{\mathfrak{q}}$.

PROOF: *Existence*. By Lemma 2.7, condition (*c*) implies condition (*c*)′; so the constructions given after Theorem 2.5 apply.

Uniqueness. Let X be an irreducible $\mathfrak{g}$ module satisfying the conditions of Proposition 2.8. The calculation in [18], before (5.3), shows that the representation μ of $\mathfrak{k}$ is strongly $\mathfrak{u}$-minimal in X ([18], Definition 3.13). Write μ_L for the trivial representation of $\mathfrak{l} \cap \mathfrak{k}$. By Theorem 3.14 of [18], μ_L occurs in $H^R(\mathfrak{u}, X)$ ($R = \dim \mathfrak{u} \cap \mathfrak{p}$) exactly as often as μ occurs in X. Let Y be an irreducible subquotient of $H^R(\mathfrak{u}, X)$ (as an $\mathfrak{l}$ module) containing μ_L. Since μ_L is one dimensional, the centralizer $U(\mathfrak{l})^{\mathfrak{l} \cap \mathfrak{k}}$ of $\mathfrak{l} \cap \mathfrak{k}$ in the enveloping algebra of $\mathfrak{l}$ acts by scalars on the $\mathfrak{l} \cap \mathfrak{k}$-type μ_L of Y. Write

$$\phi_L \colon U(\mathfrak{l})^{\mathfrak{l} \cap \mathfrak{k}} \to \mathbb{C}$$

for the corresponding homorphism. Recall from [18], (3.2) the homomorphism

$$\xi \colon U(\mathfrak{g})^{\mathfrak{k}} \to U(\mathfrak{l})^{\mathfrak{l} \cap \mathfrak{k}}.$$

Define

$$\phi\colon U(\mathfrak{g})^{\mathfrak{k}} \to \mathbb{C}, \qquad \phi = \phi_L \circ \xi.$$

By Theorem 3.5 of [18], $U(\mathfrak{g})^{\mathfrak{k}}$ acts on the $\mathfrak{k}$ type μ of X by the homomorphism ϕ. Thus X is determined by Y. To complete the uniqueness proof, we will show that Y must be the trivial representation of $\mathfrak{l}$; this is where we use hypothesis (b) on X.

Choose a maximally split θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{l}_0$; write

$$\mathfrak{t}_0^+ = \mathfrak{h}_0 \cap \mathfrak{k}, \quad \mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}, \quad \mathfrak{h}_0 = \mathfrak{t}_0^+ + \mathfrak{a}_0.$$

Write $W = W(\mathfrak{g}, \mathfrak{h})$ for the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$, and $\mathfrak{Z}(\mathfrak{g})$ for the center of $U(\mathfrak{g})$. Recall from [9] the Harish-Chandra isomorphism

$$\mathfrak{Z}(\mathfrak{g}) \xrightarrow{\chi} S(\mathfrak{h})^W. \tag{2.9)(a}$$

If $\lambda \in \mathfrak{h}^*$, then composition of χ with evaluation at λ gives

$$\chi_\lambda\colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}, \tag{2.9)(b}$$

with $\chi_\lambda = \chi_\delta$ if and only if $\lambda \in W \cdot \delta$.

Fix an Iwasawa decomposition

$$\mathfrak{l}_0 = (\mathfrak{l}_0 \cap \mathfrak{k}_0) \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$$

of $\mathfrak{l}_0$ (with $\mathfrak{a}_0$ as above), and let $\mathfrak{m}_0$ be the centralizer of $\mathfrak{a}_0$ in $\mathfrak{l}_0 \cap \mathfrak{k}_0$. Choose a positive root system $\Delta^+(\mathfrak{l}, \mathfrak{h})$ compatible with this decomposition; that is,

$$\Delta^+(\mathfrak{l}, \mathfrak{h}) = \Delta(\mathfrak{a}, \mathfrak{h}) \cup \Delta^+(\mathfrak{m}, \mathfrak{t}^+) \tag{2.10)(a}$$

for some choice of $\Delta^+(\mathfrak{m}, \mathfrak{t}^+)$. Define

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{l}, \mathfrak{h}) \cup \Delta(\mathfrak{u}, \mathfrak{h}). \tag{2.10)(b}$$

Let W_A denote the "little Weyl group" of $\mathfrak{a}_0$ in $\mathfrak{l}_0$. The irreducible representations of $\mathfrak{l}$ whose restrictions to $\mathfrak{l} \cap \mathfrak{k}$ contain μ_L are parametrized by W_A orbits in $\mathfrak{a}^*$; write ν for the parameter of Y. The parameter of the trivial representation is $\rho(\Delta^+(\mathfrak{l}, \mathfrak{h}))|_{\mathfrak{a}}$; so what we are trying to prove is

$$\nu = \overline{w}(\rho(\Delta^+(\mathfrak{l}, \mathfrak{h}))|_{\mathfrak{a}}), \quad \text{some } \overline{w} \in W_A. \tag{2.11}$$

Now $\mathfrak{Z}(\mathfrak{l})$ acts in Y by χ_γ^L (defined in analogy with (2.9)), with

$$\gamma = (\rho(\Delta^+(\mathfrak{m}, \mathfrak{t}^+), \nu) \in (\mathfrak{t}^+)^* + \mathfrak{a}^* = \mathfrak{h}^*.$$

Since Y occurs in $H^*(\mathfrak{u}, X)$, the Casselman-Osborne theorem ([3]) says that $\mathfrak{Z}(\mathfrak{g})$ acts in X by χ_λ, with

$$\lambda = \gamma + \rho(\mathfrak{u}).$$

On the other hand, hypothesis (b) of Proposition 2.8 says that $\mathfrak{Z}(\mathfrak{g})$ acts by χ_ρ. Therefore there is an element $w \in W$ such that

$$w(\rho(\Delta^+(\mathfrak{l})) + \rho(\mathfrak{u})) = (\rho(\Delta^+(\mathfrak{m}, \mathfrak{t}^+)), \nu) + \rho(\mathfrak{u}).$$

Write $\rho_A = \rho(\Delta^+(\mathfrak{l}, \mathfrak{h}))|_{\mathfrak{a}}$, $\rho_{T^+} = \rho(\Delta^+(\mathfrak{m}, \mathfrak{t}^+))$. Then

$$\rho(\Delta^+(\mathfrak{l}, \mathfrak{h})) = (\rho_{T^+}, \rho_A) \in (\mathfrak{t}^+)^* + \mathfrak{a}^*.$$

Since $\mathfrak{u}$ is θ-stable, $\rho(\mathfrak{u}) \in (\mathfrak{t}^+)^*$; so we get finally

$$w\big[(\rho_{T^+} + \rho(\mathfrak{u})), \rho_A\big] = \big[(\rho_{T^+} + \rho(\mathfrak{u})), \nu\big]. \tag{2.12}$$

By [20], 2.5.2.4, the left side is

$$\big[(\rho_{T^+} + \rho(\mathfrak{u})), \rho_A\big] - \sum_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \notin \Delta^+}} \alpha.$$

We now consider the inner product of both sides with $\rho(\mathfrak{u})$. Since $\rho(\mathfrak{u}) \in (\mathfrak{t}^+)^*$, we may ignore the ν and the ρ_A. Since $\rho(\mathfrak{u})$ is orthogonal to the roots of $\mathfrak{h}$ in $\mathfrak{l}$, it is orthogonal to ρ_{T^+}. So

$$\langle \rho(\mathfrak{u}), \rho(\mathfrak{u}) \rangle - \sum_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \notin \Delta^+}} \langle \alpha, \rho(\mathfrak{u}) \rangle = \langle \rho(\mathfrak{u}), \rho(\mathfrak{u}) \rangle.$$

So each root in the sum must belong to $\Delta^+(\mathfrak{l})$; so $w \in W(\mathfrak{l}, \mathfrak{h})$, and w fixes $\rho(\mathfrak{u})$. Now (2.120 becomes

$$w(\rho_{T^+}, \rho_A) = (\rho_{T^+}, \nu). \tag{2.13}$$

Since what we want to prove is (2.11), we may modify ν by W_A. Assume this has been done, in such a way that

$$\langle \alpha, \nu \rangle \geqslant 0 \tag{2.14}$$

whenever $\alpha \in \Delta(\mathfrak{n}, \mathfrak{h})$; that is, that ν is dominant for the restricted roots of $\mathfrak{a}$ in $\mathfrak{l}$. We claim that when $w = 1$; this will complete the proof. If α is a simple root of $\mathfrak{t}^+$ in $\mathfrak{m}$, then

$$\langle \check{\alpha}, w(\rho_{T^+}, \rho_A) \rangle = \langle \check{\alpha}, \rho_{T^+} \rangle = 1;$$

so α is also simple in $w(\Delta^+(\mathfrak{l}, \mathfrak{h}))$. Put

$$\overline{\Delta_2^+} = w(\Delta^+(\mathfrak{l}, \mathfrak{h})) - \Delta^+(\mathfrak{m}, \mathfrak{t}^+).$$

It follows that $W(\mathfrak{m}, \mathfrak{t}^+)$ permutes $\overline{\Delta_w^+}$. Let $w_0 \in W(\mathfrak{m}, \mathfrak{t}^+)$ be the element taking $\Delta^+(\mathfrak{m}, \mathfrak{t}^+)$ to $-\Delta^+(\mathfrak{m}, \mathfrak{t}^+)$. If $\alpha \in \overline{\delta_2^+}$, then

$$\begin{aligned}
\langle \alpha, \nu \rangle &= \tfrac{1}{2}\langle \alpha, (\rho_{T^+}, \nu)\rangle + \tfrac{1}{2}\langle \alpha, (-\rho_{T^+}, \nu)\rangle \\
&= \tfrac{1}{2}\langle \alpha, (\rho_{T^+}, \nu)\rangle + \tfrac{1}{2}\langle \alpha, (w_0\rho_{T^+}, \nu)\rangle \\
&= \tfrac{1}{2}\langle (\alpha + w_0\alpha), (\rho_{T^+}, \nu)\rangle \\
&= \tfrac{1}{2}\langle (\alpha + w_0\alpha), w(\rho_{T^+}, \rho_A)\rangle .
\end{aligned}$$

Since $\alpha \in \overline{\Delta_w^+}$ and this set is stable under w_0, the last term is positive. By (2.14), $\alpha \in \Delta(\mathfrak{a}, \mathfrak{h})$; so

$$\overline{\Delta_2^+} \subseteq \Delta(\mathfrak{a}, \mathfrak{h}).$$

Obviously this forces $w = 1$, as we wished to show.

□

The uniqueness part of Theorem 2.5 follows from Proposition 2.8, by Lemma 2.7.

3. The cohomology of $A_{\mathfrak{q}}$

DEFINITION 3.1: Let X be any module for $\mathfrak{g}$. Identify the exterior algebra $\Lambda^*\mathfrak{p}$ with the quotient of $\Lambda^*\mathfrak{g}$ by the ideal generated by $\mathfrak{k}$; thus $\mathrm{Hom}(\Lambda^*\mathfrak{p}, X)$ is identified with a subspace of $\mathrm{Hom}(\Lambda^*\mathfrak{g}, X)$. In this identification, the d map of $\mathfrak{g}$ cohomology preserves the subspace $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^*\mathfrak{p}, X)$, which therefore becomes a complex. Its cohomology groups are called $H^*(\mathfrak{g}, \mathfrak{k}, X)$, the *relative Lie algebra cohomology groups.*

For more about this definition, see [2] or [5].

PROPOSITION 3.2 (*see* [2]): *Suppose* $(\pi, \mathscr{H})$ *is an irreducible unitary representation of* G, *and* $\mathscr{H}^K$ *is its Harish-Chandra module.*

(a) $H^*_{\mathrm{ct}}(G, \mathscr{H}) \cong H^*(\mathfrak{g}, \mathfrak{k}, \mathscr{H}^K)$

(b) $H^*(\mathfrak{g}, \mathfrak{k}, \mathscr{H}^K)$ *is zero unless the Casimir operator acts by zero in* $\mathscr{H}^K$.

(c) *If the Casimir operator acts by zero in* $\mathscr{H}^K$, *then*

$$H^*(\mathfrak{g}, \mathfrak{k}, \mathscr{H}^K) \cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*\mathfrak{p}, \mathscr{H}^K).$$

Statements (b) and (c) make sense for any $\mathfrak{g}$ module in place of $\mathscr{H}^K$. In that generality, (b) is true when the Casimir operator acts by a scalar; and (c) is false in general (even for $(\mathfrak{g}, K)$ modules). Because of this proposition, we can compute the continuous cohomology of the unitary representation which ought to be attached to $A_{\mathfrak{q}}$, in terms of $A_{\mathfrak{q}}$ alone. The result, due to Zuckerman, is this.

THEOREM 3.3: *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ (see (2.2)), and put $R = \dim(\mathfrak{u} \cap \mathfrak{p})$. Then*

$$H^i(\mathfrak{g}, \mathfrak{k}, A_{\mathfrak{q}}) \cong H^{i-R}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k}, \mathbb{C})$$

$$\cong \mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(\Lambda^{i-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}).$$

Zuckerman's original proof of this theorem was very simple: since $A_{\mathfrak{q}} = \mathscr{R}^S_{\mathfrak{q}}(\mathbb{C})$ ([19], Definition 6.3.1), and $\mathscr{R}^i_{\mathfrak{q}}(\mathbb{C}) = 0$ for $i \neq S$, the spectral sequence of [19], Corollary 6.3.4 collapses to the isomorphism we want. However, a much more elementary argument can also be given. Since it is of some interest for the light it sheds on the structure of $\Lambda^* \mathfrak{p}$, we will give it here.

Write

$$\Delta(\mathfrak{u} \cap \mathfrak{p}) = \{\beta_1, \ldots, \beta_R\};$$

recall that R is the dimension of $\mathfrak{u} \cap \mathfrak{p}$. For each i, $1 \leqslant i \leqslant r$, choose a non-zero element X_i of $\mathfrak{u} \cap \mathfrak{p}$ of weight β_i, and a non-zero element Y_i of $\mathfrak{p}$ of weight $-\beta_i$. Choose also a basis $\{Z_1, \ldots, Z_m\}$ of $\mathfrak{l} \cap \mathfrak{p}$, consisting of weight vectors for $\mathfrak{t}$. If A and B are subsets of $\{1, \ldots, R\}$, and C is a subset of $\{1, \ldots, m\}$, put

$$X_A \wedge Y_B \wedge Z_C = \left(\bigwedge_{i \in A} X_i\right) \wedge \left(\bigwedge_{j \in B} Y_j\right) \wedge \left(\bigwedge_{k \in C} Z_k\right) \in \Lambda^{|A|+|B|+|C|}\mathfrak{p}.$$

These elements form a basis of $\Lambda^* \mathfrak{p}$ consisting of weight vectors for $\mathfrak{t}$.

LEMMA 3.4: *With notation as above, let $\lambda \in \mathfrak{t}^*$ be the weight of $X_A \wedge Y_B \wedge Z_C$. Then*

$$\langle \lambda, \rho(\mathfrak{u}) \rangle \leqslant \langle 2\rho(\mathfrak{u} \cap \mathfrak{p}), \rho(\mathfrak{u}) \rangle.$$

Equality holds if and only if $A = \{1, \ldots, R\}$ and $B = \emptyset$.

PROOF. If α is a root of $\mathfrak{t}$ in $\mathfrak{u}$, then

$$\langle \alpha, \rho(\mathfrak{u}) \rangle > 0;$$

and if β is a root of $\mathfrak{t}$ in $\mathfrak{l}$, then

$$\langle \beta, \rho(\mathfrak{u}) \rangle = 0.$$

The lemma is now obvious.

□

LEMMA 3.5: *Suppose $x \in \Lambda^* \mathfrak{p}$ is a weight vector for $\mathfrak{t}$, of weight $2\rho(\mathfrak{u} \cap \mathfrak{p}) + \delta$, with δ a weight of $\Lambda^*(\mathfrak{l} \cap \mathfrak{p})$. Then*

$$\mathrm{ad}(\mathfrak{u} \cap \mathfrak{k}) \cdot x = 0.$$

PROOF: If $U \in \mathfrak{u} \cap \mathfrak{k}$ is a weight vector of weight α, then $\mathrm{ad}(U)x$ has weight $2\rho(\mathfrak{u} \cap \mathfrak{p}) + \delta + \alpha$. Since

$$\langle \delta, \rho(\mathfrak{u}) \rangle = 0$$

$$\langle \alpha, \rho(\mathfrak{u}) \rangle > 0,$$

Lemma 3.4 says that $\Lambda^* \mathfrak{p}$ has no vectors of this weight.

□

PROPOSITION 3.6: *Let (π_L, F_L) be an irreducible representation of $\mathfrak{l} \cap \mathfrak{k}$, of highest weight δ, occurring in $\Lambda^*(\mathfrak{l} \cap \mathfrak{p})$; and let (π, F) be the irreducible representation of $\mathfrak{k}$ of highest weight $\delta + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ (if it exists). Then there is an isomorphism*

$$\mathrm{Hom}_{\mathfrak{k}}(F, \Lambda^i \mathfrak{p}) \cong \mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(F_L, \Lambda^{i-R}(\mathfrak{l} \cap \mathfrak{p})).$$

PROOF: Denote by V the one dimensional space $\Lambda^R(\mathfrak{u} \cap \mathfrak{p})$; and by F^0 the subspace of F annihilated by $\mathfrak{u} \cap \mathfrak{k}$. Then

$$F^0 \cong F_L \otimes V$$

as representations of $\mathfrak{l} \cap \mathfrak{k}$. By the Cartan-Weyl highest weight theory,

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{k}}(F, \Lambda^i \mathfrak{p}) &\cong \mathrm{Hom}_{\mathfrak{q} \cap \mathfrak{k}}(F^0, \Lambda^i \mathfrak{p}) \\ &\cong \mathrm{Hom}_{\mathfrak{q} \cap \mathfrak{k}}(F_L \otimes V, \Lambda^i \mathfrak{p}). \end{aligned}$$

Every weight of $F_L \otimes V$ is of the form $2\rho(\mathfrak{u} \cap \mathfrak{p}) + \delta$, with δ a weight of $\Lambda^*(\mathfrak{l} \cap \mathfrak{p})$. By Lemma 3.4, these occur in $\Lambda^* \mathfrak{p}$ only inside $V \otimes \Lambda^*(\mathfrak{l} \cap \mathfrak{p})$; so

$$\mathrm{Hom}_{\mathfrak{k}}(F, \Lambda^i \mathfrak{p}) \cong \mathrm{Hom}_{\mathfrak{q} \cap \mathfrak{k}}(F_L \otimes V, V \otimes \Lambda^{i-R}(\mathfrak{l} \cap \mathfrak{p})).$$

By Lemma 3.5, $V \otimes \Lambda^*(\mathfrak{l} \cap \mathfrak{p})$ is annihilated by $\mathfrak{u} \cap \mathfrak{k}$; so this becomes

$$\cong \operatorname{Hom}_{\mathfrak{l} \cap \mathfrak{k}}\left(F_L \otimes V, V \otimes \Lambda^{i-R}(\mathfrak{l} \cap \mathfrak{p})\right)$$

$$\cong \operatorname{Hom}_{\mathfrak{l} \cap \mathfrak{k}}\left(F_L, \Lambda^{i-R}(\mathfrak{l} \cap \mathfrak{p})\right).$$

□

In the preceding proof, we were a little careless about the question of the existence of π. There is no difficulty, however. As the proof shows, (π, F) can be taken to be the $\mathfrak{k}$ submodule of $\Lambda^*\mathfrak{p}$ generated by V times a copy (π_L, F_L) in $\Lambda^*(\mathfrak{l} \cap \mathfrak{p})$.

COROLLARY 3.7:

$$\operatorname{Hom}_{\mathfrak{k}}\left(\Lambda^i\mathfrak{p}, A_{\mathfrak{q}}\right) \cong \operatorname{Hom}_{\mathfrak{k}}\left(\Lambda^i\mathfrak{p}, \mu(\mathfrak{q})\right) \qquad \text{(notation (2.4))}$$

$$\cong \operatorname{Hom}_{\mathfrak{l} \cap \mathfrak{k}}\left(\Lambda^{i-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}\right).$$

PROOF: Let δ be the highest weight of a representation of $\mathfrak{k}$ occurring in $A_{\mathfrak{q}}$. By Theorem 2.5(c),

$$\delta = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta,$$

with $n_\beta \geqslant 0$. By Lemma 3.4, δ cannot occur in $\Lambda^*\mathfrak{p}$ unless all n_β are zero. This proves the first isomorphism. The second is a special case of Proposition 3.6.

□

To complete the proof of Theorem 3.3, we must show that the differential in the complex $\operatorname{Hom}_{\mathfrak{k}}(\Lambda^*\mathfrak{p}, A_{\mathfrak{q}})$ is zero. If we knew that $A_{\mathfrak{q}}$ arose from a unitary representation, this would follow from Proposition 3.2(c); and since this is the only case we really care about, the reader may wish to omit the rest of this section. We will in any case simply use the standard proof in the unitary case.

LEMMA 3.8: *$A_{\mathfrak{q}}$ admits a non-degenerate invariant Hermitian form $\langle\ ,\ \rangle$; that is, if $v, w \in A_{\mathfrak{q}}$, and $X \in \mathfrak{g}_0$, then*

$$\langle Xv, w\rangle = -\langle v, Xw\rangle$$

$$\langle v, w\rangle = \overline{\langle w, v\rangle}.$$

PROOF: Let Y be the space of all conjugate-linear, $\mathfrak{k}$-finite functionals on $A_{\mathfrak{q}}$:

$$Y = \left\{ \begin{array}{l} f: A_{\mathfrak{q}} \to \mathbb{C} \mid f(\lambda v) = \bar{\lambda} f(v)\ (\lambda \in \mathbb{C}), \text{ and} \\ f \text{ is non-zero on only finitely many representa-} \\ \text{tions of } \mathfrak{k} \end{array} \right\}.$$

Make Y into a $\mathfrak{g}_0$ module by

$$(X \cdot f)(v) = -f(X \cdot v),$$

and then into a $\mathfrak{g}$ module by complexification. We claim that $Y \cong A_{\mathfrak{q}}$; we will prove this using the characterization of Theorem 2.5. By general arguments, Y is irreducible, and $Y|_{\mathfrak{k}} \cong A_{\mathfrak{q}}|_{\mathfrak{k}}$. So (a) and (c) of the characterization are satisfied. Let

$$h\colon U(\mathfrak{g}) \to U(\mathfrak{g})$$

be the antiautomorphism determined by

$$\phi(X) = -X$$

$$\phi(\lambda X) = \bar{\lambda}\phi(X),$$

for $X \in \mathfrak{g}_0$, $\lambda \in \mathbb{C}$. Let $I \subseteq \mathfrak{Z}(\mathfrak{g})$ be the maximal ideal annihilating $A_{\mathfrak{q}}$; then $\phi(I)$ annihilates Y. By (b) of Theorem 2.5, I annihilates the trivial representation of $\mathfrak{g}$. Since that representation admits an invariant Hermitian form, $\phi(I) = I$. So I annihilates Y, establishing (b) of the characterization.

Choose an isomorphism $\psi\colon A_{\mathfrak{q}} \to Y$, and put

$$\langle v, w\rangle = \psi(v)(w).$$

Obviously this is a non-degenerate invariant form on Y, linear in v and conjugate linear in w. The restriction of $\langle\ ,\ \rangle$ to the $\mathfrak{k}$-type $\mu(\mathfrak{q})$ must still be non-degenerate; so if v_0 is a highest weight vector of $\mu(\mathfrak{q})$, then $\langle v_0, v_0\rangle \neq 0$. Modifying ψ by a scalar, we may assume $\langle v_0, v_0\rangle = 1$. Now we claim that $\langle\ ,\ \rangle$ is Hermitian. To see this, define

$$\tilde{\psi}\colon A_{\mathfrak{q}} \to Y, \tilde{\psi}(v)(w) = \overline{\langle w, v\rangle}.$$

We want to show that $\psi = \tilde{\psi}$. Since $A_{\mathfrak{q}}$ are Y are irreducible, and $\tilde{\psi}$ is a $\mathfrak{g}$-module map, we have

$$\tilde{\psi} = \lambda\psi$$

for some $\lambda \in \mathbb{C}$. But

$$\begin{aligned} 1 = \langle v_0, v_0\rangle &= \psi(v_0)(v_0) \\ &= \lambda\tilde{\psi}(v_0)(v_0) \\ &= \lambda\overline{\langle v_0, v_0\rangle} \\ &= \lambda, \end{aligned}$$

so $\lambda = 1$. □

A $\mathfrak{k}$-invariant form on an irreducible representation of K is necessarily definite; so we may assume that $\langle\ ,\ \rangle$ is positive definite on the $\mathfrak{k}$-type μ of $A_{\mathfrak{q}}$. The Killing form gives a natural positive definite Hermitian form on $\Lambda^*\mathfrak{p}$; so we get a natural Hermitian form on

$$\mathrm{Hom}_{\mathfrak{k}}\left(\Lambda^*\mathfrak{p}, A_{\mathfrak{q}}\right).$$

Since only μ contributes to this Hom (Corollary 3.7), this form is positive definite. That d is zero is now proved just as in [2], Proposition II.3.1. Theorem 3.3 then follows from Corollary 3.7.

4. Unitary representations with cohomology

THEOREM 4.1: *Let π be an irreducible unitary representation of G, and X the Harish-Chandra module of π. Suppose that $H^*_{\mathrm{ct}}(G, \pi) \neq 0$. Then there is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ (cf.* (2.2)) *such that $X \cong A_{\mathfrak{q}}$ (cf. Theorem* 2.5).

The main ingredient in the proof is the following result of Parthasarathy. Recall from Section 2 our Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$, and the positive root system $\Delta^+(\mathfrak{k})$.

LEMMA 4.2: (*Parthasarathy's Dirac operator inequality* - [2], *Lemma II*.6.11, *and* [13], (2.26)). *Let π be an irreducible unitary representation of G, and X the Harish-Chandra module of π. Fix a representation of $\mathfrak{k}$ occurring in X, of highest weight $\chi \in \mathfrak{t}^*$; and a positive root system $\Delta^+(\mathfrak{g})$ for $\mathfrak{t}$ in $\mathfrak{g}$. Write*

$$\rho = \rho(\Delta^+(\mathfrak{g})) \in \mathfrak{t}^*$$

$$\rho_c = \rho(\Delta^+(\mathfrak{k})) \in \mathfrak{t}^*$$

$$\rho_n = \rho(\Delta^+(\mathfrak{p})) = \rho - \rho_c \in \mathfrak{t}^*.$$

Fix an element $w \in W_K$ such that $w(\chi - \rho_n)$ is dominant for $\Delta^+(\mathfrak{k})$. Let c_0 denote the eigenvalue of the Casimir operator of $\mathfrak{g}$ in X. Then

$$\langle w(\chi - \rho_n) + \rho_c, w(\chi - \rho_n) + \rho_c \rangle \geqslant c_0 + \langle \rho, \rho \rangle.$$

Roughly speaking, this says that the length of the highest weight of a representation of $\mathfrak{k}$ occurring in a unitary representation must be at least the eigenvalue of the Casimir operator. For example, in a unitary spherical series representation, the eigenvalue of the Casimir is non-positive, since the trivial representation of $\mathfrak{k}$ occurs.

Here is a sketch of the proof of Theorem 4.1. By Proposition 3.2, X contains some $\mathfrak{k}$-type from $\Lambda^*\mathfrak{p}$, of highest weight μ; and the Casimir operator has eigenvalue zero in X. Kumaresan shows in [10] that any $\mathfrak{k}$-type of $\Lambda^*\mathfrak{p}$ satisfying the inequality of Lemma 4.2 with $c_0 = 0$, must be of the form $\mu(\mathfrak{q})$ (cf. (2.4)) for some θ-stable parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$. To show that $X \cong A_{\mathfrak{q}}$, we use the criterion of Proposition 2.8: roughly speaking, the bad $\mathfrak{k}$-types of (2.6) do not satisfy the inequality of Lemma 4.2. To carry this out, we need some calculations with roots and weights, drawn largely from [10].

LEMMA 4.3: *Suppose* γ, $\delta \in \mathfrak{t}^*$ *are dominant integral for* $\Delta^+(\mathfrak{k})$, *and* $\sigma \in W_K$. *Choose* σ', $\sigma'' \in W_K$ *so that* $\sigma\gamma - \delta$ *is dominant for* $\sigma'(\Delta^+(\mathfrak{k}))$, *and* $\gamma - \delta$ *is dominant for* $\sigma''(\Delta^+(\mathfrak{k}))$. *Then*

$$\langle \sigma\gamma - \delta + \sigma'\rho_c, \sigma\gamma - \delta + \sigma'\rho_c \rangle \geqslant \langle \gamma - \delta + \sigma''\rho_c, \gamma - \delta + \sigma''\rho_c \rangle.$$

Equality holds if and only if $\sigma\gamma - \delta \in W_k \cdot (\gamma - \delta)$.

PROOF: The "if" part is obvious; we prove the inequality and "only if" by induction on the length of σ, as in [10], Lemma 2.2. If $\sigma = 1$, there is nothing to prove; so suppose $\sigma > 1$, and the result is known for shorter Weyl group elements. Choose a reflection s_α ($\alpha \in \Delta^+(\mathfrak{k})$) so that

$$l(s_\alpha\sigma) < l(\sigma). \tag{4.4}$$

Choose $\tau \in W_K$ so that $s_\alpha\sigma\gamma - \delta$ is dominant for $\tau(\Delta^+(\mathfrak{k}))$. By induction, if suffices to show that

$$\langle \sigma\gamma - \delta + \sigma'\rho_c, \sigma\gamma - \delta + \sigma'\rho_c \rangle \geqslant \langle s_\alpha\sigma\gamma - \delta + \tau\rho_c, s_\alpha\sigma\gamma - \delta + \tau\rho_c \rangle,$$

with equality only if $\sigma\gamma - \delta$ is conjugate to $s_\alpha\sigma\gamma - \delta$. By [9], Lemma 13.4(C) and Proposition 21.3, it is enough to show that $s_\alpha\gamma - \delta$ is a weight of the finite dimensional representation F of $\mathfrak{k}$, of extremal weight $\sigma\gamma - \delta$. Now (4.4) implies that

$$m_1 = \langle \check{\alpha}, \sigma\gamma \rangle \leqslant 0$$

(since γ is dominant); and

$$m_2 = \langle \check{\alpha}, -\delta \rangle \leqslant 0$$

since δ is dominant. So the α-string of weights of F through $\sigma\gamma - \delta$ is

$$\sigma\gamma - \delta, \sigma\gamma - \delta + \alpha, \ldots (\sigma\gamma - \delta) + (m_1 + m_2)\alpha = s_\alpha(\sigma\gamma - \delta)$$

which contains

$$\sigma\gamma - \delta + m_1\alpha = s_\alpha\sigma\gamma - \delta.$$

□

LEMMA 4.5 (*Kumaresan's lemma*): *Under the hypotheses of Lemma* 4.2, *assume that the Casimir operator of* $\mathfrak{g}$ *acts by zero in* X. *Assume that there are elements* σ, $\tau \in W_K$, *and a subset* $A \subseteq \Delta^+(\mathfrak{g})$, *such that*

(a) $\sigma\chi - \rho_n$ *is dominant for* $\tau(\Delta^+(\mathfrak{k}))$

(b) $\sigma\chi - \rho_n + \tau\rho_c = \rho - 2\rho(A)$ (*notation after* (2.3)).

Then there is a θ*-stable parabolic subalgebra* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *of* $\mathfrak{g}$, *such that* $\chi = 2\rho(\mathfrak{u} \cap \mathfrak{p})$.

PROOF: Extend $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let W be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. The elements of $\Delta^+(\mathfrak{g})$ are the restriction to $\mathfrak{t}$ of a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ for $\mathfrak{h}$ in $\mathfrak{g}$; and the corresponding half sum of roots is just ρ, extended by zero on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$. Choose a subset $\bar{A} \subseteq \Delta^+(\mathfrak{g}, \mathfrak{h})$ whose restriction to $\mathfrak{t}$ is A. Then $\rho - 2\rho(A) \in \mathfrak{h}^*$ is a weight of the finite dimensional representation of $\mathfrak{g}$ of highest weight ρ; so

$$\langle \rho - 2\rho(\bar{A}), \rho - 2\rho(\bar{A}) \rangle \leqslant \langle \rho, \rho \rangle,$$

with equality only if

$$\rho - 2\rho(\bar{A}) = w\rho$$

for some $w \in W$. On the other hand,

$$\rho - 2\rho(A) = \rho - 2\rho(\bar{A})|_{\mathfrak{t}'}$$

so

$$\langle \rho - 2\rho(A), \rho - 2\rho(A) \rangle \leqslant \langle \rho - 2\rho(\bar{A}), \rho - 2\rho(\bar{A}) \rangle \leqslant \langle \rho, \rho \rangle.$$

Equality holds if and only if there is a w as above, and $w\rho$ is zero on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$. This is the same as requiring that w commute with θ ([18], Lemma 5.5). In that case, $w(\Delta^+(\mathfrak{g}, \mathfrak{h}))|_{\mathfrak{t}}$ is another positive system for the restricted roots $\Delta(\mathfrak{g}, \mathfrak{t})$; we write it as $w(\Delta^+(\mathfrak{g}))$. We may also regard w as acting on $\mathfrak{t}$ alone, since it commutes with θ. Thus

$$\langle \sigma\chi - \rho_n + \tau\rho_c, \sigma\chi - \rho_n + \tau\rho_c \rangle \leqslant \langle \rho, \rho \rangle, \tag{4.6(a)}$$

with equality if and only if

$$\sigma\chi - \rho_n + \tau\rho_c = w\rho. \tag{4.6(b)}$$

Now choose $\tau' \in W_K$ so that $\chi - \rho_n$ is dominant for $\tau'(\Delta^+(\mathfrak{k}))$. By Lemma 4.3,

$$\langle \chi - \rho_n + \tau'\rho_c, \chi - \rho_n + \tau'\rho_c \rangle \leqslant \langle \sigma\chi - \rho_n + \tau\rho_c, \sigma\chi - \rho_n + \tau\rho_c \rangle. \tag{4.6(c)}$$

Combining (4.6) (a) and (c) with Lemma 4.2, we deduce that equality must hold in (4.6) (a) and (c). Accordingly, we can find a new positive system $w(\Delta^+(\mathfrak{g}))$ and a $\sigma' \in W_K$ so that (4.6) (b) holds, and

$$\sigma\chi - \rho_n = (\sigma')^{-1}(\chi - \rho_n).$$

Therefore $\chi - \rho_n$ is dominant for $\sigma'\tau(\Delta^+(\mathfrak{k}))$; and

$$\chi - \rho_n + \sigma'\tau\rho_c = \sigma' w\rho.$$

The left side is dominant and regular for $\sigma'\tau(\Delta^+(\mathfrak{k}))$, so

$$(\sigma' w\Delta^+(\mathfrak{g})) \cap \Delta(\mathfrak{k}) = \sigma'\tau(\Delta^+(\mathfrak{k})).$$

If we set

$$\rho_n' = \rho((\sigma' w\Delta^+(\mathfrak{g})) \cap \Delta(\mathfrak{p}))$$

then we have

$$\chi = \rho_n + \rho_n'.$$

The conclusion of the lemma is now part (b) of the first theorem in [10]. (It is proved, incidentally, by a purely algebraic manipulation of roots and weights, in the spirit of the preceding arguments.)

□

LEMMA 4.7: *Fix a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ as in* (2.2), *and suppose δ is of the form* (2.6). *Then*

$$\langle \delta + 2\rho_c, \delta + 2\rho_c \rangle < \langle 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c, 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c \rangle.$$

PROOF: Chôose a positive root system $\Delta^+(\mathfrak{l})$ making $\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{k}))$ dominant; and put $\Delta^+(\mathfrak{g}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u})$ as usual. Define

$$\begin{cases} \text{(a)} & v = 2\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p})) \\ \text{(b)} & \tilde{\lambda} = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c - \rho + \tfrac{1}{2}v \\ \text{(c)} & \quad = \rho(\mathfrak{u}) + \rho(\Delta^+(\mathfrak{l} \cap \mathfrak{k})) \\ \text{(d)} & \quad = \rho - \rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p})). \end{cases} \tag{4.8}$$

We claim that if $\alpha \in \Delta^+(\mathfrak{g})$, then

$$\langle \alpha, \tilde{\lambda} \rangle \geqslant 0, \tag{4.9}$$

with strict inequality for $\alpha \in \Delta(\mathfrak{u})$. It suffices to prove this under the additional assumption that α is simple. If $\alpha \in \Delta^+(\mathfrak{l})$, then $\langle \alpha, \rho(\mathfrak{u}) \rangle = 0$; so (4.9) follows from (4.8) (c) and the choice of $\Delta^+(\mathfrak{l})$. If $\alpha \in \Delta(\mathfrak{u})$ is simple then α has a non-positive inner product with all positive roots in $\mathfrak{l}$; for they are positive combinations of simple roots distinct from α. Thus (4.9) follows from (4.8) (d) in this case. The computation leading to (5.3) in [18] shows that

$$\langle 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c, 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c \rangle - \langle \delta + 2\rho_c, \delta + 2\rho_c \rangle$$

$$= 2\langle \tilde{\lambda}, \sum_{\alpha \in B} \alpha \rangle + \langle 2\rho(B^c) - v, 2\rho(B) \rangle.$$

Here B is a non-empty subset of $\Delta(\mathfrak{u})$, and B^c is its complement in $\Delta^+(\mathfrak{g})$. By (4.9), the first term on the right is strictly positive. By Lemma 5.6 of [18], the second term on the right is non-negative.

□

PROOF OF THEOREM 4.1: By the main theorem of [10], there is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$, such that the representation of $\mathfrak{k}$ of highest weight $2\rho(\mathfrak{u} \cap \mathfrak{p})$ occurs in X. Choose such a $\mathfrak{q}$, with

$$\langle 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c, 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c \rangle$$

as small as possible. To prove that $X \cong A_{\mathfrak{q}}$, we use the characterization of Proposition 2.8. Condition (a) (that X contain $\mu(\mathfrak{q})$) follows from the choice of $\mathfrak{q}$. Condition (b) (on the infinitesimal character) follows from the non-vanishing of $H^*(\mathfrak{g}, \mathfrak{k}, X)$ by Wigner's lemma ([2], Theorem I.4.1). For condition (c)$'$, suppose δ satisfies (2.6). We can easily rewrite that as

$$\sigma\delta = 2\rho_n + \rho_c - 2\rho(B) - \sigma\rho_c; \tag{4.10}$$

here

$$B = B_0 \cup \Delta^+(\mathfrak{l} \cap \mathfrak{p}).$$

Define a new positive root system $\tau(\Delta^+(\mathfrak{k}))(\tau \in W_K)$ by

$$\tau\Delta^+(\mathfrak{k}) = \{\alpha \in \Delta(\mathfrak{k}) | \langle \alpha, \sigma\delta - \rho_n \rangle > 0\}$$

$$\cup \{\alpha \in \sigma\Delta^+(\mathfrak{k}) | \langle \alpha, \sigma\delta - \rho_n \rangle = 0\}.$$

Then $\sigma\delta - \rho_n$ is dominant for $\tau\Delta^+(\mathfrak{k})$; and if we put

$$C = \{\alpha \in \sigma\Delta^+(\mathfrak{k}) | \langle \alpha, \sigma\delta - \rho_n \rangle < 0\},$$

then

$$\sigma\rho_c - \tau\rho_c = 2\rho(C).$$

Since $\sigma\delta$ is dominant for $\sigma\Delta^+(\mathfrak{k})$, it is clear that

$$\begin{aligned} C &\subseteq \{\alpha \in \sigma\Delta^+(\mathfrak{k}) \mid \langle \alpha, \rho_n \rangle > 0\} \\ &\subseteq \Delta^+(\mathfrak{k}). \end{aligned}$$

Setting $A = B \cup C$, we have $A \subseteq \Delta^+(\mathfrak{g})$; and (4.10) becomes

$$\sigma\delta - \rho_n + \tau\rho_c = \rho - 2\rho(A).$$

By Kumaresan's lemma (Lemma 4.5), there is another θ-stable parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$, with $\delta = 2\rho(\mathfrak{u}' \cap \mathfrak{p})$. By Lemma 4.7,

$$\langle \delta + 2\rho_c, \delta + 2\rho_c \rangle < \langle 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c, 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho_c \rangle.$$

By the choice of $\mathfrak{q}$, this means that the representation of $\mathfrak{k}$ of highest weight δ cannot occur in X, verifying condition (c)′ of Proposition 2.8. □

5. Twisted coefficients

In this section, we will indicate (without detailed proofs) how to extend Theorem 4.1 to the case in which the cohomology is twisted by a finite dimensional representation of G. First, we must construct some representations. Fix a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ as in Section 2, and let $L \subseteq G$ be the connected subgroup with Lie algebra $\mathfrak{l}_0$. (L is closed in G; it is the centralizer of the element x used to construct $\mathfrak{q}$.) A one dimensional representation λ: $\mathfrak{l} \to \mathbb{C}$ is called *admissible* if it satisfies the following conditions:

$$\begin{aligned} &\text{(a)} \quad \lambda \text{ is the differential of a unitary character (also called } \lambda) \\ &\qquad \text{of } L \\ &\text{(b)} \quad \text{if} \quad \alpha \in \Delta(\mathfrak{u}), \quad \text{then} \quad \langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geqslant 0. \end{aligned} \tag{5.1}$$

Given $\mathfrak{q}$ and an admissible λ, define

$$\mu(\mathfrak{q}, \lambda) = \text{representation of } K \text{ of highest weight } \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p}). \tag{5.2}$$

Extend $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and choose $\Delta^+(\mathfrak{g}, \mathfrak{h})$ as in the proof of Lemma 4.5 (see also the discussion before (2.6)).

THEOREM 5.3: *Suppose* $\mathfrak{q}$ *is a* θ*-stable parabolic subalgebra of* $\mathfrak{g}$, *and* λ: $\mathfrak{l} \to \mathbb{C}$ *satisfies* (5.1). *Then there is a unique irreducible* $\mathfrak{g}$*-module* $A_{\mathfrak{q}}(\lambda)$ *with the following properties*:

(a) *The restriction of* $A_{\mathfrak{q}}(\lambda)$ *to* $\mathfrak{k}$ *contains* $\mu(\mathfrak{q}, \lambda)$ (*see* (5.2));

(b) $\mathfrak{Z}(\mathfrak{g})$ *acts by* $\chi_{\lambda+\rho}$ *in* $A_{\mathfrak{q}}(\lambda)$ (*compare* (2.9));

(c) *If the representation of* $\mathfrak{k}$ *of highest weight* δ *occurs in* $A_{\mathfrak{q}}(\lambda)$ *restricted to* $\mathfrak{k}$, *then*

$$\delta = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta,$$

with n_β *a non-negative integer.*

(Of course, $A_{\mathfrak{q}}$ is just $A_{\mathfrak{q}}(0)$.) This is proved in the same way as Theorem 2.5. There is also an analogue of Proposition 2.8, whose formulation we leave to the reader.

PROPOSITION 5.4 (*see* [2]): *Suppose* $(\pi, \mathscr{H})$ *is an irreducible unitary representation of* G, *and* $\mathscr{H}^K$ *its Harish-Chandra module. Let* F *be a finite dimensional irreducible representation of* G.

(a) $H^*_{\mathrm{ct}}(G, \mathscr{H} \otimes F) \cong H^*(\mathfrak{g}, \mathfrak{k}, \mathscr{H}^K \otimes F)$

(b) $H^*(\mathfrak{g}, \mathfrak{k}, \mathscr{H}^K \otimes F) = 0$ *unless the Casimir operator acts by the same scalars in* $\mathscr{H}^K$ *and in* F.

(c) *If the Casimir operator acts by the same scalars in* $\mathscr{H}^K$ *and in* F, *then*

$$H^*(\mathfrak{g}, \mathfrak{k}, \mathscr{H}^K \otimes F) \cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^* \mathfrak{p}, \mathscr{H}^K \otimes F).$$

THEOREM 5.5: *Let* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *be a* θ*-stable parabolic subalgebra of* $\mathfrak{g}$, *and* λ: $\mathfrak{l} \to \mathbb{C}$ *an admissible character* (5.1). *Put* $R = \dim \mathfrak{u} \cap \mathfrak{p}$. *Suppose* F *is a finite dimensional irreducible representation of* $\mathfrak{g}$, *of lowest weight* $-\gamma \in \mathfrak{h}^*$ *with respect to* $\Delta^+(\mathfrak{g}, \mathfrak{h})$ (*defined after* (5.2)). *Then*

$$\begin{aligned} H^i(\mathfrak{g}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) &\cong H^{i-R}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k}, \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(\Lambda^{i-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}) \end{aligned}$$

if $\gamma = \lambda|_{\mathfrak{h}}$; *and*

$$H^i(\mathfrak{g}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) = 0$$

otherwise.

This is proved in substantially the same way as Theorem 3.3. (The vanishing statement is an immediate consequence of Theorem 5.3(b) and Wigner's lemma.)

THEOREM 5.6: *Let π be an irreducible unitary representation of G, and F and irreducible finite dimensional representation of G. Write X for the Harish-Chandra module of π. Suppose $H^*_{ct}(G, \pi \otimes F) \neq 0$. Then there is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$, such that*

(a) *$F/\mathfrak{u}F$ is a one dimensional unitary representation of L; write $-\lambda$: $\mathfrak{l} \to \mathbb{C}$ for its differential.*
(b) *$X \cong A_{\mathfrak{q}}(\lambda)$.*

It is worth remarking that whenever $\mathfrak{q}$ satisfies (a), the corresponding λ automatically satisfies (5.1). The proof of Theorem 5.6 is a little harder than that of Theorem 4.1, since one must first check that Kumaresan's results in [10] can be generalized appropriately. Kumaresan's first result is generalized as follows. We assume fixed a Cartan $\mathfrak{t} \subseteq \mathfrak{k}$, a positive system $\Delta^+(\mathfrak{k})$, and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{t}$.

PROPOSITION 5.7: *Under the hypotheses of Theorem 5.6, let μ be the highest weight of a representation of $\mathfrak{k}$ occurring in both X and $F^* \otimes \Lambda\mathfrak{p}$. (Such a μ exists by Proposition 5.4). Then there are positive root systems $\Delta^+(\mathfrak{g})$, $\Delta^+(\mathfrak{g})'$, and a weight $\lambda \in \mathfrak{h}^*$ of F^*, such that*

(a) *λ is zero on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$*
(b) *λ is a highest weight of F^* with respect to both Δ^+ and $(\Delta^+)'$*
(c) *$\Delta^+(\mathfrak{g})$ contains $\Delta^+(\mathfrak{k})$*
(d) *$\mu = \lambda + \rho(\Delta^+(\mathfrak{p})) + \rho(\Delta^+(\mathfrak{p})')$.*

PROOF: The orthogonal Lie algebra $\mathfrak{so}(\mathfrak{p})$ has a natural representation $S = \mathrm{spin}(\mathfrak{p})$, of dimension $2^{[1/2 \dim \mathfrak{p}]}$. Since the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$ preserves the Killing form defining $\mathfrak{so}(\mathfrak{p})$, there is a natural map $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$; so S may be regarded as a representation of $\mathfrak{k}$. We refer to [2] for various basic facts concerning this representation, which will be used below. First of all, the exterior algebra $\Lambda^*\mathfrak{p}$ is isomorphic to one or two copies (according to whether n is even or odd) of $\mathrm{Hom}_{\mathbb{C}}(S, S)$. Write E_γ for the representation of $\mathfrak{k}$ of extremal weight $\gamma \in \mathfrak{t}^*$. By hypothesis,

$$\mathrm{Hom}_{\mathfrak{k}}\left(F^* \otimes \mathrm{Hom}_{\mathbb{C}}(S, S), E_\mu\right) \neq 0.$$

This space is naturally isomorphic to

$$\mathrm{Hom}_{\mathfrak{k}}\left(F^* \otimes S, E_\mu \otimes S\right);$$

write γ for the highest weight of a representation of $\mathfrak{k}$ occurring in both $F^* \otimes S$ and $E_\mu \otimes S$. Every highest weight of S is of the form $\rho(\Delta^+(\mathfrak{p}))$ for some positive system $\Delta^+(\mathfrak{g}) \supseteq \Delta^+(\mathfrak{k})$. As a highest weight of $F^* \otimes S$, γ is therefore of the form

$$\gamma = \rho\left(\Delta^+(\mathfrak{p})''\right) + \lambda = \rho_n'' + \lambda|_{\mathfrak{t}} \tag{5.8}$$

for some weight λ of F^* and some positive system $\Delta^+(\mathfrak{g})''$. Suppose E_γ

occurs in $E_\mu \otimes E_{\rho_n} \subseteq E_\mu \otimes S$ here $\Delta^+(\mathfrak{g})$ denotes a second positive system containing $\Delta^+(\mathfrak{k})$, and $\rho_n = \rho(\Delta^+(\mathfrak{p}))$. Choose $\tau \in W_K$ of minimal length so that $\mu - \rho_n$ is dominant for $\tau^{-1}(\Delta^+(\mathfrak{k}))$. By [14], p. 394, Corollaries 1 and 2,

$$\langle \gamma + \rho_c, \gamma + \rho_c \rangle \geqslant \langle \tau(\mu - \rho_n) + \rho_c, \tau(\mu - \rho_n) + \rho_c \rangle, \tag{5.9}$$

with equality if and only if $\gamma = \tau(\mu - \rho_n)$.

Write ν for the $\Delta^+(\mathfrak{g})''$-highest weight of F^*. Then the Casimir operator acts in F^* (or in F) by the scalar

$$\langle \nu + \rho'', \nu + \rho'' \rangle - \langle \rho'', \rho'' \rangle.$$

By Proposition 5.4(b), it acts in X by the same scalar. Now Lemma 4.2 (the Dirac operator inequality) gives

$$\langle \tau(\mu - \rho_n) + \rho_c, \tau(\mu - \rho_n) + \rho_c \rangle \geqslant \langle \nu + \rho'', \nu + \rho'' \rangle. \tag{5.10}$$

Finally, we know that

$$\langle \nu + \rho'', \nu + \rho'' \rangle \geqslant \langle \lambda + \rho'', \lambda + \rho'' \rangle \geqslant \langle \lambda|_{\mathfrak{t}} + \rho'', \lambda|_{\mathfrak{t}} + \rho'' \rangle. \tag{5.11}$$

The first inequality is [9], Lemma 13.4(c), and the second is clear (since ρ vanishes on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$). Equality holds if and only if $\nu = \lambda$, and λ vanishes on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$. Combining (5.8)–(5.11), we find that equality holds everywhere. This gives (a) of the lemma. Furthermore,

$$\begin{aligned} \lambda + \rho'' &= \gamma + \rho_c \\ &= \tau(\mu - \rho_n) + \rho_c. \end{aligned} \tag{5.12}$$

Now choose $\sigma \in W_K$ so that $\mu - \tau^{-1}\rho_n''$ is dominant for $\sigma(\Delta^+(\mathfrak{k}))$. Then

$$\tau\left(\mu - \tau^{-1}\rho_n'' + \sigma\rho_c\right) = \left(\lambda + \rho''\right) - \rho_n'' + \tau\rho_n + \tau\sigma\rho_c - \rho_c. \tag{5.13}$$

By Lemma 4.2, the left side has length at least $\langle \lambda + \rho'', \lambda + \rho'' \rangle$. Arguing as in the proof of Lemma 4.5, we deduce that the left side has length at most that. Furthermore, the equality implies that there is a positive system $w(\Delta^+(\mathfrak{g})'')$ ($w \in W(\mathfrak{g}, \mathfrak{h})$ commuting with θ) such that

$$w\rho'' - \tau\sigma\rho_c + \tau\rho_n. \tag{5.14}$$

Furthermore, w fixes λ; so that if we put

$$A = \Delta^+(\mathfrak{k}) \cap (-(\tau\sigma\Delta^+(\mathfrak{k})))$$

$$B = \Delta^+(\mathfrak{p})'' \cap (-(\tau\Delta^+(\mathfrak{p}))),$$

then every root in A and B is orthogonal to λ. We claim that τ fixes λ. Assume this for a moment, and apply τ^{-1} to (5.13). We get

$$\mu = \lambda + \rho_n + \tau^{-1}\rho_n''.$$

Setting $\Delta^+(\mathfrak{g})'' = \tau^{-1}\Delta^+(\mathfrak{g})''$, we get the conclusion of the lemma.

To show that τ fixes λ, put

$$C = \Delta^+(\mathfrak{k}) \cap (-(\tau\Delta^+(\mathfrak{k}))).$$

It suffices to show that $C \subseteq A$; for τ is a product of reflections in C, and we know that those in A fix λ. Suppose not. Fix $\alpha \in C$, $\alpha \notin A$. Since $\alpha \in C$, $\alpha \in \Delta^+(\mathfrak{k})$; no since $\alpha \notin A$, $\alpha \in \tau\sigma(\Delta^+(\mathfrak{k})) \subseteq w(\Delta^+(\mathfrak{g})'')$. By (5.14), this implies that

$$\langle \alpha, \tau\rho_n \rangle \geqslant 0.$$

Since $-\alpha \in \tau\Delta^+(\mathfrak{k})$,

$$\langle \alpha, \tau\mu \rangle \leqslant 0.$$

Therefore

$$\langle \alpha, \tau(\mu - \rho_n) \rangle \leqslant 0. \tag{5.15}$$

Since τ was chosen to have minimal length,

$$\tau^{-1}\Delta^+(\mathfrak{k}) = \{\alpha | \langle \alpha, \mu - \rho_n \rangle > 0\} \cup \{\alpha \in \Delta^+(\mathfrak{k}) | \langle \alpha, \mu - \rho_n \rangle = 0.\}.$$

Therefore

$$\Delta^+(\mathfrak{k}) = \{\alpha | \langle \alpha, \tau(\mu - \rho_n) \rangle 0\} \cup \{\alpha \in \tau\Delta^+(\mathfrak{k}) | \langle \alpha, \tau(\mu - \rho_n) \rangle = 0\}.$$

By (5.15), our assumption $\alpha \in \Delta^+(\mathfrak{k})$ forces $\alpha \in \tau\Delta^+(\mathfrak{k})$, contradicting $\alpha \in C$.

□

Here is Kumaresan's second result.

PROPOSITION 5.16: *Let $\Delta^+(\mathfrak{g})$, $\Delta^+(\mathfrak{g})'$ be two positive root systems for* $\mathfrak{t}$ *in* $\mathfrak{g}$ *(with $\Delta^+(\mathfrak{g}) \supseteq \Delta^+(\mathfrak{k})$; and suppose $\lambda \in \mathfrak{h}^*$ is $\Delta^+(\mathfrak{g})$-dominant. Let π be* *an irreducible unitary representation of G, and X its Harish-Chandra module. Assume that*

(a) *The Casimir operator of* $\mathfrak{g}$ *acts in X by* $\langle\lambda+\rho, \lambda+\rho\rangle - \langle\rho,\rho\rangle$.

(b) λ *is zero on the orthogonal complement of* $\mathfrak{t}$ *in* $\mathfrak{h}$.

(c) *The* $\mathfrak{k}$ *representation of highest weight*

$$\mu = \lambda|_{\mathfrak{t}} + \rho(\Delta^+(\mathfrak{p})) + \rho(\Delta^+(\mathfrak{p})')$$

occurs in X. Then there is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *of* $\mathfrak{g}$, *containing* $\mathfrak{h}$ *and the positive root vectors for* $\Delta^+(\mathfrak{g})$, *such that*

$$\mu = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u}\cap\mathfrak{p});$$

and $\mathfrak{q}$ *preserves the* λ *weight space of* F^*.

This is proved in exactly the same way as Kumaresan's Theorem 1(b) in [10]. Theorem 5.6 follows from Propositions 5.7 and 5.16 by the proof of Theorem 4.1; verification of this is left to the reader.

6. Some auxiliary results

In this section we collect several technical results which might be useful in applications of the main theorems to automorphic forms. The first provides a fairly simply way to identify $A_{\mathfrak{q}}(\lambda)$ in some cases (including those having non-zero cohomology).

PROPOSITION 6.1: *Suppose* $\mathfrak{q}$ *is a θ-stable parabolic subalgebra of* $\mathfrak{g}$, *and* λ: $\mathfrak{l} \to \mathbb{C}$ *is an admissible character* (5.1). *Assume in addition that* λ *is zero on the orthogonal complement of* $\mathfrak{t}$ *in* $\mathfrak{h}$. *Let* π *be an irreducible unitary representation of G, and X the Harish-Chandra module of* π. *Assume that*

(a) *The* $\mathfrak{k}$ *representation* $\mu(\mathfrak{q},\lambda)$ *of* (5.2) *occurs in X.*

(b) *The Casimir operator of* $\mathfrak{g}$ *acts by* $\langle\lambda|_{\mathfrak{h}}+\rho, \lambda|_{\mathfrak{h}}+\rho\rangle - \langle\rho,\rho\rangle$ *in X. (Here* ρ *is half the sume of the roots in any positive system compatible with* $\mathfrak{q}$.)

Then $X \cong A_{\mathfrak{q}}(\lambda)$.

This is clear from the proof of Theorem 5.6. To get a purely algebraic characterization of $A_{\mathfrak{q}}(\lambda)$ (omitting any discussion of unitarity), one could add

(c) X satisfies the Dirac operator inequality (of Lemma 4.2).

Then (a)–(c) characterize $A_{\mathfrak{q}}(\lambda)$. The formulation of Proposition 6.1 should be best for applications, however.

The condition that λ should vanish on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$ has a natural reformulation.

PROPOSITION 6.2: *In the setting of Theorem* 5.3, *choose an automorphism* σ *of* $\mathfrak{g}_0$ *which preserves* $\mathfrak{k}_0$ *and* $\mathfrak{h}$, *such that* $\sigma|_{\mathfrak{h}} = \theta$. *Then* (*if we let* σ *act on representations of* $\mathfrak{g}$) $\sigma \cdot A_{\mathfrak{q}}(\lambda) = A_{\mathfrak{q}}(\sigma\lambda)$; *and this is isomorphic to* $A_{\mathfrak{q}}(\lambda)$ *if and only if* λ *vanishes on the orthogonal complement of* $\mathfrak{t}$ *in* $\mathfrak{h}$.

This is an elementary consequence of Theorem 5.3. It indicates that one might hope to study the occurrence of $A_{\mathfrak{q}}(\lambda)$ in automorphic forms, when λ is zero on the complement of $\mathfrak{t}$ in $\mathfrak{h}$, using a σ-twisted trace formula. In particular, one might hope to verify the unitarity of such $A_{\mathfrak{q}}(\lambda)$. From the point of view of pure real group representation theory, one would then like a way to deduce unitarity in the general case. This is provided by the next proposition.

PROPOSITION 6.3: *In the setting of Theorem* 5.3, *let* $\mathfrak{a}_0 \subseteq \mathfrak{h}_0$ *be the split component of the center of* $\mathfrak{l}_0$; *and let* $P = MN$ *be a parabolic subgroup of* G, *with* M *the centralizer of* $\mathfrak{a}_0$ *in* G. *Then*

(a) M *is connected and reductive*, *and* $\mathfrak{m}_0 \supseteq \mathfrak{l}_0$.

(b) $\mathfrak{q}^M = \mathfrak{q} \cap \mathfrak{m} = \mathfrak{l} + \mathfrak{u}^M$ *is a* θ*-stable parabolic subalgebra of* $\mathfrak{m}$.

(c) $A_{\mathfrak{q}}(\lambda) \cong \mathrm{Ind}_P^G A_{\mathfrak{q}^M}(\lambda + \rho(\mathfrak{u}/\mathfrak{u}^M))$ (*normalized induction*).

Parts (a) and (b) are elementary. For (c), we use Theorem 5.3. It is easy to check conditions (a) and (b) of Theorem 5.3 for the induced representation. For (c), put

$$B = \{\alpha \in \Delta^+(\mathfrak{k}) | \alpha \notin \Delta(\mathfrak{m} \cap \mathfrak{k})\} = \Delta(\mathfrak{u} \cap \mathfrak{k}/\mathfrak{u}^M \cap \mathfrak{k}).$$

On the one hand, any highest weight of K occurring in the induced representation must be of the form

$$\delta = \tau + \sum_{\alpha \in B} m_\alpha \alpha,$$

with m_α a non-negative integer, and τ a highest weight of $M \cap K$ in $A_{\mathfrak{q}^M}(\lambda)$. By Theorem 5.3 for M,

$$\delta = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u}^M \cap \mathfrak{p}) + \rho(\mathfrak{u}/\mathfrak{u}^M) + \sum_{\beta \in \Delta(\mathfrak{u}^M \cap \mathfrak{p})} n_\beta \beta + \sum_{\alpha \in B} m_\alpha \alpha.$$

On the other hand, any root γ of $\mathfrak{h}$ in $\mathfrak{u}/\mathfrak{u}^M$ must not vanish on $\mathfrak{a}$. Therefore $\gamma \neq \theta\gamma$, and $\gamma|_{\mathfrak{t}}$ is both a compact and a noncompact root of $\mathfrak{t}$. Therefore

$$B = \Delta(\mathfrak{u} \cap \mathfrak{k}/\mathfrak{u}^M \cap \mathfrak{k}) = \Delta(\mathfrak{u} \cap \mathfrak{p}/\mathfrak{u}^M \cap \mathfrak{p})$$

$$\rho(\mathfrak{u}/\mathfrak{u}^M) = 2\rho((\mathfrak{u} \cap \mathfrak{p})/(\mathfrak{u}^M \cap \mathfrak{p})).$$

Our formula for δ is now

$$\delta = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u}^M \cap \mathfrak{p}) + 2\rho(\mathfrak{u} \cap \mathfrak{p}) - 2\rho(\mathfrak{u}^M \cap \mathfrak{p})$$

$$+ \sum_{\beta \in \Delta(\mathfrak{u}^M \cap \mathfrak{p})} n_\beta \beta + \sum_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{p}) - \Delta(\mathfrak{u}^M \cap \mathfrak{p})} m_\alpha \alpha$$

$$= \lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} r_\beta \beta.$$

So the induced representation satisfies (c) of Theorem 5.3.

□

Here is the character formula for $A_{\mathfrak{q}}(\lambda)$ on the fundamental Cartan.

PROPOSITION 6.4: *In the setting of Theorem* 5.3, *write H for the fundamental Cartan subgroup of G with Lie algebra $\mathfrak{h}_0$. Choose a positive root system $\Delta^+(\mathfrak{g})$ compatible with $\mathfrak{q}$. Identify W_K with the Weyl group of H in G, and thus with a subgroup of $W(\mathfrak{g}, \mathfrak{h})$. Then the character of $A_{\mathfrak{q}}(\lambda)$ on H is*

$$(-1)^R \left\{ \sum_{w \in W_K \cdot W(\mathfrak{l}, \mathfrak{h})} \det(w)\, e^{w(\lambda+\rho)-\rho} \Big/ \sum_{w \cap W(\mathfrak{g}, \mathfrak{h})} \det(w)\, e^{w\rho-\rho} \right\};$$

here $R = \dim \mathfrak{u} \cap \mathfrak{p}$.

If $\mathfrak{l} = \mathfrak{g}$, this is of course nothing but the Weyl character formula. If $\mathfrak{l} = \mathfrak{h}$, it is Harish-Chandra's formula for the character of a fundamental series representation. The general case may be reduced to these two by the results of [17], section 4.

The same results in [17] show that if the trivial representation of $\mathfrak{l}$ is at the end of a complementary series, then the same is true of $A_{\mathfrak{q}}(\lambda)$. This gives

PROPOSITION 6.5: *In the setting of Theorem* 5.3, *suppose that every simple noncompact factor of $\mathfrak{l}_0$ is isomorphic to $\mathfrak{su}(n, 1)$ or $\mathfrak{so}(n, 1)$. Then $A_{\mathfrak{q}}(\lambda)$ is (the Harish-Chandra module of) a unitary representation.*

It seems likely that most of the $A_{\mathfrak{q}}(\lambda)$ are isolated in the unitary dual; so the technique of proof in Propositions 6.3 and 6.5 cannot be extended very much. For $G = SL(n, \mathbb{R})$, all of the $A_{\mathfrak{q}}(\lambda)$ are unitarily induced (by Proposition 6.3) unless $n = 2m$ and $\mathfrak{l}_0 \cong \mathfrak{sl}(m, \mathbb{C}) +$ (one dimensional compact center). In that case, Speh has shown in [16] that such $A_{\mathfrak{q}}(\lambda)$ occur in the non-cuspidal discrete spectrum of $SL(n, \mathbb{R})$ modulo a congruence subgroup of $SL(n, \mathbb{Z})$. (In particular, *all* $A_{\mathfrak{q}}(\lambda)$ are unitary for $SL(n, \mathbb{R})$.) Analogous techniques may one day be available for other classical groups, but they seem to be far out of reach at present. When

G/K is Hermitian symmetric, and $\mathfrak{q}$ is compatible with a complex structure on G/K, then $A_{\mathfrak{q}}(\lambda)$ is a highest weight representation. Parthasarathy proves in [13] a converse to the Dirac operator inequality (Lemma 4.2) for such representations; so these $A_{\mathfrak{q}}(\lambda)$ are unitary. There may be some hope of extending this method to all $A_{\mathfrak{q}}(\lambda)$ when rank(G) = rank(K).

Baldoni-Silva and Barbasch have shown in [1] that all the $A_{\mathfrak{q}}(\lambda)$ are unitary when G has rank 1. Of course most of the cases are covered by Proposition 6.5, but the remaining ones require complicated and ingenious calculation.

Next we will describe how the representations $A_{\mathfrak{q}}(\lambda)$ fit into the Langlands classification of all irreducible Harish-Chandra modules ([2], Theorem 4.11). Fix $\mathfrak{q}$ and λ satisfying (5.1). We will construct a parabolic subgroup $P^{\mathrm{d}} \subseteq G$, with Langlands decomposition $P^{\mathrm{d}} = M^{\mathrm{d}}A^{\mathrm{d}}N^{\mathrm{d}}$; a (non-unitary) character $\nu^{\mathrm{d}} \in (\mathfrak{a}^{\mathrm{d}})^*$; and a discrete series representation $\sigma^{\mathrm{d}} \in (\hat{M}^{\mathrm{d}})$. We will have

$$\langle \operatorname{Re} \nu^{\mathrm{d}}, \alpha \rangle \geqslant 0, \text{ all roots } \alpha \text{ of } \mathfrak{a}^{\mathrm{d}} \text{ in } \mathfrak{n}^{\mathrm{d}}. \tag{6.6}$$

Then $A_{\mathfrak{q}}(\lambda)$ will be the unique "Langlands quotient" $J_{P^{\mathrm{d}},\sigma^{\mathrm{d}},\nu^{\mathrm{d}}}$ of the induced representation

$$\operatorname{Ind}_{P^{\mathrm{d}}}^{G}(\sigma^{\mathrm{d}} \otimes \nu^{\mathrm{d}} \otimes 1) = I_{P^{\mathrm{d}},\sigma^{\mathrm{d}},\nu^{\mathrm{d}}}.$$

The Langlands classification is usually phrased a little differently, however. Define

$$Z = \{ \alpha \text{ a root of } \mathfrak{a} \text{ in } \mathfrak{g} | \langle \operatorname{Re} \nu^{\mathrm{d}}, \alpha \rangle = 0 \}$$

$$A = \bigcap_{\alpha \in Z} (\ker \alpha) \subseteq A^{\mathrm{d}} \tag{6.8}$$

$$MA = \text{Langlands decomposition of centralizer of } A \text{ in } G$$

$$\nu = \nu^{\mathrm{d}}|$$

$$\sigma = \operatorname{Ind}_{P^{\mathrm{d}} \cap M}\left(\sigma^{\mathrm{d}} \otimes \nu^{\mathrm{d}}|_{(A^{\mathrm{d}} \cap M)} \otimes 1\right).$$

To make sense of the last statement, notice that

$$P^{\mathrm{d}} \cap M = M^{\mathrm{d}}(A^{\mathrm{d}} \cap M)(N^{\mathrm{d}} \cap M)$$

is a cuspidal parabolic subgroup of M. Because of (6.6), there is a unique parabolic subgroup of G having Levi factor MA and containing P^{d}; we write

$$P = MAN \supseteq P^{\mathrm{d}}$$

$$\operatorname{Ind}_P^G(\sigma \otimes \nu \otimes 1) = I_{P,\sigma,\nu}. \tag{6.9}$$

By the choice of P,

$$\langle \operatorname{Re} \nu, \alpha \rangle > 0, \text{ all roots } \alpha \text{ of } \mathfrak{a} \text{ in } \mathfrak{n}. \tag{6.10}$$

By induction by stages, $I_{P,\sigma,\nu} \cong I_{P^{\mathrm{d}},\sigma^{\mathrm{d}},\nu^{\mathrm{d}}}$. Since σ is unitarily induced from discrete series, it is tempered; and it turns out (for our choice of ν^{d}) that σ is irreducible. Therefore $I_{P,\sigma,\nu}$ is the kind of representation considered in [2], and it has a unique Langlands quotient $J_{P,\sigma,\nu}$; this is what we meant by $J_{P^{\mathrm{d}},\sigma^{\mathrm{d}},\nu^{\mathrm{d}}}$.

We turn now to the construction of P^{d}. Fix a maximally split θ-stable Cartan subgroup $H = T^+A^{\mathrm{d}}$ of L (the Levi factor of our θ-stable parabolic $\mathfrak{q}$), and an Iwasawa decomposition $L = (L \cap K)A^{\mathrm{d}}N^L$. Put

$$M^{\mathrm{d}}A^{\mathrm{d}} = \text{Langlands decomposition of centralizer of } A^{\mathrm{d}} \text{ in } G.$$

$$\nu^{\mathrm{d}} = \left(\tfrac{1}{2} \text{ sum of roots of } \mathfrak{a}^{\mathrm{d}} \text{ in } \mathfrak{n}^L\right) + \lambda|_{\mathfrak{a}^{\mathrm{d}}} \in (\mathfrak{a}^{\mathrm{d}})^*.$$

Now let P^{d} be any parabolic subgroup of G with Levi factor $M^{\mathrm{d}}A^{\mathrm{d}}$, satisfying (6.6); such subgroups certainly exist. It remains to define the discrete series representation σ^{d} of M^{d}. Choose a positive root system for $\mathfrak{t}^+$ in $\mathfrak{m}^{\mathrm{d}} \cap \mathfrak{l}$, and write

$$\rho^+ = \tfrac{1}{2} \text{ sum of positive roots of } \mathfrak{t}^+ \text{ in } \mathfrak{m}^{\mathrm{d}} \cap \mathfrak{l} \in (\mathfrak{t}^+)^*. \tag{6.12}$$

The main property of σ^{d} is

$$\text{Harish-Chandra parameter of } \sigma^{\mathrm{d}} = \rho^+ + \lambda|_{\mathfrak{t}^+} + \rho(\mathfrak{u}) \in (\mathfrak{t}^+)^*; \tag{6.13}$$

here $\rho(\mathfrak{u})$ denotes half the sum of the roots of $\mathfrak{t}^+$ in the nil radical $\mathfrak{u}$ of our θ-stable parabolic subalgebra $\mathfrak{q}$. The only remaining difficulty is that M^{d} need not be connected, so a discrete series need not be determined by its Harish-Chandra parameter. In analogy with (5.2), we define

$$\mu^{I^{\mathrm{d}}}(\mathfrak{q}, \lambda) = \text{representation of } M^{\mathrm{d}} \cap K \text{ of extremal weight}$$

$$(\lambda|_{T^+}) \otimes \left(\Lambda^{\dim \mathfrak{u} \cap \mathfrak{p}}\, \mathfrak{u} \cap \mathfrak{p}\right)|_{T^+}, \tag{6.14}$$

which turns out to be well defined. The remaining characteristic of σ^{d} is

$$\sigma^{\mathrm{d}} \text{ has lowest } M^{\mathrm{d}} \cap K\text{-type } \mu^{M^{\mathrm{d}}}(\mathfrak{q}, \lambda). \tag{6.15}$$

In fact (6.15) alone determines σ^{d} (without (6.13)); but (6.13) is perhaps useful information in its own right.

THEOREM 6.16: *Suppose* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *is a* θ*-stable parabolic subalgebra of* $\mathfrak{g}$, *and* $\lambda\colon \mathfrak{l} \to \mathbb{C}$ *is admissible* (*cf.* (5.1)). *Then the Langlands parameters for the representation* $A_{\mathfrak{q}}(\lambda)$ *are specified by* (6.6)–(6.15). *In particular,* A^{d} *is the split component of a maximally split Cartan subgroup of* L.

This is proved in [17], Theorem 4.23. (The reader who actually wishes to verify this may find it helpful to know that $A_{\mathfrak{q}}(\lambda)$ was denoted $X(\mathfrak{q}, \mathbb{C}_\lambda, \mu(\mathfrak{q}, \lambda))$ in [17].)

We can give a formula for the multiplicity of any irreducible representation of $\mathfrak{k}$ in $A_{\mathfrak{q}}(\lambda)$.

THEOREM 6.17: *Under the hypotheses of Theorem* 6.16, *define*

$$P_{\mathfrak{u}\cap\mathfrak{p}}\colon \mathfrak{t}^* \to \mathbb{N},$$

$P_{\mathfrak{u}\cap\mathfrak{p}}(\gamma) =$ *number of expressions of* γ *as* $\sum_{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{p})} n_\alpha \alpha$, *with* $n_\alpha \in \mathbb{N}$. *Then the representation of* $\mathfrak{k}$ *of highest weight* δ *occurs in* $A_{\mathfrak{q}}(\lambda)$ *with multiplicity*

$$\sum_{w\in W_K} \det(w) P_{\mathfrak{u}\cap\mathfrak{p}}\big(w(\delta + \rho_c) - \lambda|_{\mathfrak{t}} - 2\rho(\mathfrak{u}\cap\mathfrak{p})\big).$$

Here we have written ρ_c *for* $\rho(\Delta^+(\mathfrak{k}))$.

PROOF: This follows from [19], Theorem 6.3.12(d) by an easy calculation, if we recall that

$$A_{\mathfrak{q}}(\lambda) \simeq R^S_{\mathfrak{q}}(\mathbb{C}_\lambda).$$

□

When G/K is Hermitian symmetric, the cohomology of a representation acquires a bigrading, as follows. Write $\mathfrak{p}^+ \subseteq \mathfrak{p}$ for the holomorphic tangent space of G/K at the origin, and $\mathfrak{p}^-$ for the antiholomorphic tangent space. Then

$$\Lambda^i\mathfrak{p} \cong \bigoplus_{p+q=i} (\Lambda^p\mathfrak{p}^+) \otimes (\Lambda^q\mathfrak{p}^-), \tag{6.18(a)}$$

a K-invariant bigrading. If X is any $\mathfrak{g}$ module,

$$\operatorname{Hom}_{\mathfrak{k}}(\Lambda^i \mathfrak{p}, X) \cong \bigoplus_{p+q=i} \operatorname{Hom}_{\mathfrak{k}}(\Lambda^p \mathfrak{p}^+ \otimes \Lambda^q \mathfrak{p}^-, X). \tag{6.18)(b}$$

The differential d of Lie algebra cohomology splits as

$$\mathrm{d} = \partial + \bar{\partial},$$

with ∂ of (p, q) degree (1, 0), and $\bar{\partial}$ of degree (0, 1). If X is of the form $A_{\mathfrak{q}}(\lambda) \otimes$ (finite dimensional), then d is zero or the complex is exact (proof of Theorem 3.3); so in that case we have natural Dolbeaut cohomology spaces

$$H^i(\mathfrak{g}, \mathfrak{k}, X) \cong \bigoplus_{p+q=i} H^{p,q}(\mathfrak{g}, \mathfrak{k}, X). \tag{6.18)(c}$$

Obviously this decomposition corresponds (via Matsushima's formula) to the Hodge structure on the cohomology of locally symmetric spaces (for example); so it is of interest to compute it. If X is trivial, it is well known that only the (p, p) terms can be non-zero. The proof of Theorem 5.5 therefore computes the $H^{p,q}$ spaces; we have

PROPOSITION 6.19: *Suppose G/K is Hermitian symmetric, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$, $\lambda : \mathfrak{l} \to \mathbb{C}$ is admissible* (5.1), *and F is a finite dimensional irreducible representation of G. Assume that the lowest weight of F is $-\lambda|_{\mathfrak{h}}$. Put*

$$R^{\pm} = \dim(\mathfrak{u} \cap \mathfrak{p}^{\pm}).$$

Then

$$\begin{aligned} H^{p+R^+, p+R^-}(\mathfrak{g}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) &\cong H^{p,p}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k}, \mathbb{C}) \\ &\cong \operatorname{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(\Lambda^{2p}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}). \end{aligned}$$

This last space has dimension equal to the number of elements of $W(\mathfrak{l}, \mathfrak{h})/W_{L \cap K}$ of length p. If $p - q \neq R^+ - R^-$, then

$$H^{p,q}(\mathfrak{g}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) = 0.$$

7. Dirac oprators

We assume throughout this section that $\operatorname{rank}(G) = \operatorname{rank}(K)$. Fix a Cartan subgroup $T \subseteq K$, and a positive root system $\Delta^+(\mathfrak{g})$ for $\mathfrak{t}$ in $\mathfrak{g}$. Let $\lambda \in \hat{T}$

be a character whose differential (also called λ) has the property that

$$\begin{aligned} \langle\alpha, \lambda+\rho\rangle &\geqslant 0 \qquad (\alpha \in \Delta^{+}(\mathfrak{g})) \\ \langle\alpha, \lambda+\rho\rangle &> 0 \qquad (\alpha \in \Delta^{+}(\mathfrak{k})). \end{aligned} \tag{7.1}$$

Write

$$\rho_c = \rho(\Delta^{+}(\mathfrak{k})), \quad \rho_n = (\Delta^{+}(\mathfrak{p})).$$

$$\gamma = \lambda + \rho_n = \lambda - \rho_c. \tag{7.2}$$

By (7.1), γ is the highest weight of a representation E_γ of $\mathfrak{k}$.

Recall from the proof of Proposition 5.7 the spin representation S of $\mathfrak{k}$. Its weight are all the expressions of the form

$$\nu_A = \rho_n - 2\rho(A) \qquad (A | \Delta^{+}(\mathfrak{p})).$$

The subspaces

$$S^{+} = \text{span of the weights } \nu_A \text{ with } |A| \text{ even}$$

$$S^{-} = \text{span of weights } \nu_A \text{ with } |A| \text{ odd}$$

are invariant under $\mathfrak{k}$; they are called the half spin representations. Let X be any representation of $\mathfrak{g}$. Then Parthasarathy has defined Dirac operators

$$D^{\pm} : X \underset{\mathbb{C}}{\otimes} S \to X \underset{\mathbb{C}}{\otimes} S^{\pm};$$

see [11]. They are intertwining operators for the tensor product action of $\mathfrak{k}$; so we get operators

$$D_\gamma^{\pm} : \operatorname{Hom}_{\mathfrak{k}}\left(E_\gamma, X \otimes S^{\pm}\right) \to \operatorname{Hom}_{\mathfrak{k}}\left(E_\gamma, X \otimes S^{\mp}\right) \tag{7.3)(a}$$

when γ is as in (7.2). Put

$$\operatorname{Ind}_\gamma(X) = \dim\left(\ker D_\gamma^{+}\right) - \dim\left(\operatorname{coker} D_\gamma^{+}\right), \tag{7.3)(b}$$

the index of D_γ^{+} for X. Two examples are particularly important. First, suppose $\Gamma \subseteq G$ is a discrete torsion-free subgroup, and $X = C^\infty(\Gamma\backslash G)$ (regarded as a $\mathfrak{g}$ module by differentiating on the right). Then

$$\operatorname{Hom}_{\mathfrak{k}}\left(E_\gamma, X \otimes S^{\pm}\right) \cong C^\infty(\Gamma\backslash G) \underset{K}{\otimes} \left(E_\gamma^{*} \otimes S^{\pm}\right).$$

If we let $\mathscr{E}_\gamma^{\pm}$ denote the vector bundle on $\Gamma\backslash G/K$ defined by the K

representation $E_\gamma^* \otimes S^\pm$, then this space is just the space of sections of $\mathscr{E}_\gamma^\pm$. The Dirac operator D_γ is an elliptic first order differential operator; so $\mathrm{Ind}_\gamma(C^\infty(\Gamma\backslash G))$ is well defined, and computable by the index theorem. (For an appropriate normalization of Haar measure, the formula is

$$\mathrm{Ind}_\gamma(C^\infty(\Gamma\backslash G)) = \left\{ \prod_{\alpha\in\Delta^+(\mathfrak{g})} (\alpha, \lambda)/(\alpha, \rho) \right\} \times \{\mathrm{vol}(\Gamma\backslash G)\}(-1)^{1/2\dim G/K}; \tag{7.4}$$

see [8].) On the other hand, a version of Matsushima's formula computes the index one representation at a time. If we write (as in the introduction) m_π for the multiplicity of the irreducible representation π in $L^2(\Gamma\backslash G)$, then

$$\mathrm{Ind}_\gamma(C^\infty(\Gamma\backslash G)) = \sum_{\pi\in\hat{G}} m_\pi \,\mathrm{Ind}_\gamma(X_\pi); \tag{7.5}$$

here X_π is the Harish-Chandra module of π. This brings up the second case in which it is important to be able to compute the index: Harish-Chandra modules of irreducible unitary representations.

THEOREM 7.6: *Suppose* $\mathrm{rank}(G) = \mathrm{rank}(K)$, $T \subseteq K$ *is a Cartan subgroup*, $\Delta^+(\mathfrak{g})$ *is a positive system for* $\mathfrak{t}$ *in* $\mathfrak{g}$, *and* $\lambda \in \hat{T}$. *Assume that* $\langle\alpha, \lambda\rangle \geqslant 0$ *for all* $\alpha \in \Delta^+(\mathfrak{g})$; *and define* $\gamma = \gamma + \rho_n$. *Let X be the Harish – Chandra module of an irreducible unitary representation of G, and suppose* $\mathrm{Ind}_\gamma(X) \neq 0$ *(see* (7.3)(*b*)). *Then there is a θ-stable parabolic subalgebra* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *of* $\mathfrak{g}$, *with* $\mathfrak{t} \subseteq \mathfrak{l}$ *and* $\Delta(\mathfrak{u}) \subseteq \Delta^+(\mathfrak{g})$, *such that* λ *is admissible for* $\mathfrak{q}$ *(see* (5.1)), *and* $X \cong A_\mathfrak{q}(\lambda)$.

Notice that the hypothesis on λ is a little stronger than (7.1): this theorem is not as general as one would like. The work of Baldoni-Silva and Barbasch [1] shows that when λ is not dominant, more complicated representations than the $A_\mathfrak{q}(\lambda)$ can have non-zero index. They have found all representations with non-zero index in rank one; the general case is a very interesting open problem.

PROOF OF THEOREM 7.6: By [2], Lemma II.6.11,

$$\begin{aligned} D_\gamma^+ D_\gamma^- = D_\gamma^- D_\gamma^+ &= \langle \gamma + \rho_c, \gamma + \rho_c\rangle - \langle\rho, \rho\rangle - c_0 \\ &= \langle \lambda + \rho, \lambda + \rho\rangle - \langle\rho, \rho\rangle - c_0; \end{aligned}$$

here c_0 is the scalar by which the Casimir operator of $\mathfrak{g}$ acts in X. The non-vanishing of the index therefore implies that $c_0 = \langle\lambda + \rho, \lambda + \rho\rangle -$

$\langle \rho, \rho \rangle$. Let μ be a representation of K occurring in $E_\gamma \otimes S^*$ and in X; this exists by the non-vanishing of the index. Thus $\gamma = \lambda + \rho_n$, E_γ occurs in $\mu \otimes S$, and μ occurs in X. Now the argument for Proposition 5.7 applies, and establishes hypothesis (c) (on the form of μ) in Proposition 5.16. We observed above that hypothesis (a) (on the eigenvalue of the Casimir) holds; and (b) is vacuous since $\operatorname{rank}(G) = \operatorname{rank}(K)$. so Proposition 5.16 shows that $\mu = \mu(\mathfrak{q}, \lambda)$ (notation (5.2)) for some $\mathfrak{q}$ with the desired properties. By Proposition 6.1, $X \cong A_{\mathfrak{q}}(\lambda)$.

Proposition 7.7: *In the setting of Theorem* 7.6,

$$\operatorname{Ind}_\gamma(A_{\mathfrak{q}}(\lambda)) = (-1)^{\dim \mathfrak{u} \cap \mathfrak{p}}$$

Sketch of first proof. Let

$$\phi = \lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta$$

be the highest weight of a representation of $\mathfrak{k}$ occurring in $A_{\mathfrak{q}}(\lambda)$. To study $\operatorname{Ind}_\gamma(A_{\mathfrak{q}}(\lambda))$, we must determine whether E_γ occurs in $E_\phi \otimes S$. If it does, then

$$\begin{aligned} \gamma &= \phi + (\text{weight of spin}) \\ &= \phi - \rho_n + 2\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p})) + 2\rho(A_1) - 2\rho(A_2); \end{aligned}$$

here $A_1 \subseteq \Delta(\mathfrak{u} \cap \mathfrak{p})$, $A_2 \subseteq \Delta^+(\mathfrak{l} \cap \mathfrak{p})$. Since $\gamma = \lambda + \rho_n$, and

$$2\rho(\mathfrak{u} \cap \mathfrak{p}) = 2\rho_n - 2\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p})),$$

this says

$$\begin{aligned} \lambda + \rho_n &= \lambda + 2\rho_n - 2\rho\big(\Delta^+(\mathfrak{l} \cap \mathfrak{p}) + \textstyle\sum n_\beta \beta \\ &\quad - \rho_n + 2\rho\big(\Delta^+(\mathfrak{l} \cap \mathfrak{p}) + 2\rho(A_1) - 2\rho(A_2) \\ &= \lambda + \rho_n + \textstyle\sum n_\beta \beta + 2\rho(A_1) - 2\rho(A_2). \end{aligned}$$

Clearly this is impossible unless all n_β are zero (so that $E_\phi = \mu(\mathfrak{q}, \lambda)$), and A_1 and A_2 are empty. The weight of spin in question $(-\rho_n + 2\rho(\Delta^+(\mathfrak{l} \cap \mathfrak{p}))$ clearly lies in S^+ or S^-, depending on the parity of $\dim \mathfrak{u} \cap \mathfrak{p}$. Therefore either the domain or the target of D_γ^+ (see (7.3)(a)) is zero, and the other space is one dimensional. To check the formula for the index, we only have to show that E_γ really occurs in $S \otimes \mu(\mathfrak{q}, \lambda)$. This is fairly easy, and is left to the reader.

SECOND PROOF (assuming $A_{\mathfrak{q}}(\lambda)$ is unitary). Schmid has shown that the index for a unitary representation is the coefficient on $e^{\lambda+\rho}$ in the numerator of the character formula of $A_{\mathfrak{q}}(\lambda)$ on the compact Cartan subgroup. The result therefore follows from Proposition 6.4.

□

8. Vanishing theorems

Kumaresan's results in [10] imply an explicit vanishing theorem of the following kind.

THEOREM 8.1: *Suppose G is simple, and π is a non-trivial irreducible unitary representation of G. Let F be a finite dimensional representation of G. Then*

$$H^i_{\mathrm{ct}}(G, \pi \otimes E) = 0, \; i < r_G.$$

Here r_G is the minimum of $\dim(\mathfrak{u} \cap \mathfrak{p})$ over all proper θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$, and is given (for non-complex groups) by Table 8.2.

Table 8.2. Kumaresan Vanishing theorem for real groups

Group	Cartan label	Rank	r_G	L (up to covering)
$SL(n, \mathbb{R}), n \neq 4$	AI	$n-1$	$n-1$	$SL(n-2, \mathbb{R}) \times \Pi \times \mathbb{R}$
$SL(4, \mathbb{R})$	AI	3	3	$SL(2, \mathbb{R}) \times \Pi \times \mathbb{R}$ or $SL(2, \mathbb{C}) \times \Pi$
$SU^*(2n), n > 4$	AII	$n-1$	$2(n-1)$	$SU^*(2(n-1)) \times \Pi \times \mathbb{R}$
$SU^*(8)$	AII	3	6	$SU^*(6) \times \Pi \times \mathbb{R}$ or $(SL(4, \mathbb{C}) \times \Pi$
$SU^*(6)$	AII	2	3	$SL(3, \mathbb{C}) \times \Pi$
$SU(p, q), p \leqslant q$	AIII	p	p	$SU(p, q-1) \times \Pi$
$SO(p, q), p \leqslant q, 2 \leqslant q$	BDI	p	p	$SO(p, q-2) \times \Pi$
$SO^*(2n), n \geqslant 4$	DIII	$[\frac{1}{2}n]$	$n-1$	$SO^*(2(n-1)) \times \Pi$
$Sp(n, \mathbb{R}), n \geqslant 3$	CI	n	N	$Sp(n-1, \mathbb{R}) \times \Pi$
$Sp(p, q), p \leqslant q$	CII	p	$2p$	$SP(p, q-1) \times \Pi$
$(e_{6(6)}, Sp(4))$	EI	6	13	$SO(3, 5) \times \Pi \times \mathbb{R}$
$(e_{6(2)}, SU(6) \times SU(2))$	EII	4	8	$SO(4, 6) \times \Pi$
$(e_{6(-14)}, \mathrm{Spin}(10) \times \Pi)$	EIII	2	8	$SO(2, 8) \times \Pi$
$(e_{6(-26)}, f_4)$	EIV	2	6	$SU^*(6) \times \Pi$
$(e_{7(7)}, SU(8)$	EV	7	15	$e_{6(2)} \times \Pi$
$(e_{7(-5)}, \mathrm{Spin}(12) \times SU(2))$	EVI	4	12	$e_{6(2)} \times \Pi$
$(e_{7(-25)}, e_6 \times \Pi)$	EVII	3	11	$e_{6(-14)} \times \Pi$
$(e_{8(8)}, \mathrm{Spin}(16))$	EVIII	8	29	$e_{7(7)} \times \Pi$
$(e_{8(-24)}, e_7 \times SU(2))$	EXI	4	24	$e_{7(-5)} \times \Pi$
$(f_{4(4)}, Sp(3) \times Sp(1))$	FI	4	8	$Sp(3, \mathbb{R}) \times \Pi$ or $SO(4, 3) \times \Pi$
$(f_{4(-20)}, \mathrm{Spin}(9))$	FII	1	4	$Sp(2, 1) \times \Pi$
$(g_{2(2)}, SU(2) \times SU(2))$	G	2	3	$SL(2, \mathbb{R}) \times \Pi$

This is clear from Proposition 5.4 and Theorems 5.5 and 5.6. The calculation of r_G may require some comment, however. Clearly, what is wanted is an $\mathfrak{l}_0$ with dim $\mathfrak{l}_0 \cap \mathfrak{p}_0$ as large as possible; but given $\mathfrak{g}_0$, it is not a trivial matter to list all possible $\mathfrak{l}_0$. It is easy to list the θ-stable parabolics containing a fixed θ-stable Borel subalgebra; they are parametrized by the θ-stable subsets of the simple roots. However, there are in general many K conjugacy classes of θ-stable Borel subalgebras. The easiest way to get candidates for $\mathfrak{l}_0$ is to use the following conditions:

(a) $\mathfrak{l}$ is the Levi factor of a parabolic subalgebra of $\mathfrak{g}$.

(b) real rank $(\mathfrak{l}_0) \leqslant$ real rank $(\mathfrak{g}_0)$

(c) $\mathfrak{l}_0$ and $\mathfrak{g}_0$ have a common fundamental Cartan.

Using Helgason's tabulation (in [7]) of dimensions of symmetric spaces to compute dim $\mathfrak{l}_0 \cap \mathfrak{p}_0$, it is a simple matter to get for each simple group a short list of possible $\mathfrak{l}_0$ with dim $\mathfrak{l}_0 \cap \mathfrak{p}_0$ maximal. These can then be realized as θ-stable Levi factors by explicit calculations in the root system of a fundamental Cartan. (For complex groups, r_G is tabulated in [4] or [10].) The Levi factors with maximal dim $\mathfrak{l}_0 \cap \mathfrak{p}_0$ are also listed in the table. For exceptional groups, we list the Lie algebras $(\mathfrak{g}_0, K)$, with $\mathfrak{g}_0$ written as (root system type)$_{(\dim \mathfrak{p} - \dim \mathfrak{k})}$. Π denotes the circle group.

References

[1] M.W. Baldoni-Silva and D. Barbasch: The unitary spectrum for real rank one groups. *Invent. Math. 72* (1983) 27–55.

[2] A. Borel and N. Wallach: Continuous cohomology, discrete subgroups, representations of reductive groups, Princeton University Press, Princeton, New Jersey (1980).

[3] W. Casselman and M.S. Osborne: The $\mathfrak{n}$-cohomology of representations with an infinitesimal character. *Comp. Math. 31* (1975) 219–227.

[4] T. Enright: Relative Lie algebra cohomology and unitary representations of complex Lie groups. *Duke Math. J. 46* (1979) 513–525.

[5] A. Guichardet: Cohomologie des groupes topologiques et des algèbres de Lie, CEDIC-Fernand Nathan, Paris (1980).

[6] Harish-Chandra: Representations of semi-simple Lie groups I. *Trans. Amer. Math. Soc. 75* (1953) 185–243.

[7] S. Helgason: Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York (1978).

[8] R. Hotta and R. Parthasarathy: A geometric meaning of the multiplicities of integrable discrete classes in $L^2(\Gamma\backslash G)$. *Osaka J. Math. 10* (1973) 211–234.

[9] J. Humphreys: Introduction to Lie algebras and representation theory. Springer-Verlag, New York Heidelberg Berlin (1972).

[10] S. Kumaresan: The canonical $\mathfrak{k}$-types of the irreducible unitary $\mathfrak{g}$-modules with non-zero relative cohomology. *Invent. Math.* 59 (1980) 1–11.

[11] R. Parthasarathy: Dirac operator and the discrete series. *Ann. Math. 96* (1972) 1–30.

[12] R. Parthasarathy: A generalization of the Enright-Varadarajan modules. *Comp. Math. 36* (1978) 53–73.

[13] R. Parthasarathy: Criteria for the unitarizability of some highest weight modules. *Proc. Indian Acad. Sci.* 89 (1980) 1–24..

[14] K.R. Parthasarathy, R. Ranga Rao and V.S. Varadarajan: Representations of complex semi-simple Lie groups and Lie algebras. *Ann. Math. 85* (1967) 383–429.

[15] B. SPEH: Unitary representations of $GL(n, \mathbb{R})$ with non-trivial $(\mathfrak{g}, K)$ cohomology. *Invent. Math.* *71* (1983) 443–465.

[16] B. SPEH: Unitary representations of $SL(n, \mathbb{R})$ and the cohomology of congruence subgroups, In. Noncommutative Harmonic Analysis and Lie Groups, Lecture Notes in Mathematics 880, Springer-Verlag, Berlin Heidelberg New York (1981).

[17] B. SPEH and D. VOGAN: Reducibility of generalized principal series representations. *Acta Math.* *145* (1980) 227–299.

[18] D. VOGAN: The algebraic structure of the representations of semi-simple Lie groups I. *Ann. Math.* *109* (!979) 1–60.

[19] D. VOGAN: Representations of real reductive Lie groups, Birkhäuser, Boston-Vasel-Stuttgart (1981).

[20] G. WARNER: Harmonic analysis on semi-simple Lie groups I, Springer-Verlag, Berlin Heidelberg New York (1972).

[21] J. RAWNSLEY, W. SCHMID and J. WOLF: Singular unitary representations and indefinite harmonic theory, to appear in *J. Func. Anal.*, 1983.

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