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1. Introduction

The concept of a wreath product of permutation groups has been well documented in the literature. Kalužnin and Krasner introduced the idea in [6] in order to obtain groups with a certain universal property. Various other results were established in [6], for wreath products of finitely many groups. Wreath products with an infinite number of factors were discussed in [2] and [8]. The approach used there was to write the wreath product as a union of finite wreath products, with a suitable embedding rule.

Hall in [3] approached the problem of a wreath product of infinitely many permutation groups in a different manner. His construction involved defining the wreath product directly as a subgroup of the symmetric group on a suitably chosen set. The wreath product obtained was "restricted", in some sense. Moreover, Hall only considered wreath products of groups indexed by totally ordered sets. Hall used his results to construct various groups with special properties. The work of Kalužnin and Krasner was subsequently generalized in [4] by Holland who constructed "unrestricted" wreath products of permutation groups indexed by a partially ordered set. He showed, in particular, that every transitive permutation group could be embedded in an "unrestricted" wreath product of primitive permutation groups. Some applications of the results obtained appeared in [5]. The current paper generalizes some of Hall's results to wreath products of groups indexed by partially ordered sets. Unlike Holland, we obtain "restricted" wreath products. Some applications occur in [1]. We now describe the construction.

Let $(\Lambda, \geqslant)$ be a partially ordered set and, for each $\lambda \in \Lambda$, let G_λ be a permutation group defined on a set X_λ. In X_λ, select a fixed element, denoted by 1_λ. Let X denote the set of all "vectors" $(x_\lambda)_{\lambda \in \Lambda}$, where $x_\lambda \in X_\lambda$, and for all but finitely many $\lambda \in \Lambda$, $x_\lambda = 1_\lambda$. We write $X = Dr_{\lambda \in \Lambda} X_\lambda$, the direct product of the sets X_λ. For each $\lambda \in \Lambda$ we define an equivalence relation on X by:

$$x \equiv y (\operatorname{mod} \lambda) \quad \text{if and only if} \quad x_\mu = y_\mu \quad \text{for all} \quad \mu > \lambda .$$

Let $1 = (1_\lambda)_{\lambda \in \Lambda}$ and if $g \in G_\lambda$ define a permutation $\bar{g}$ on X by:
If $x \not\equiv 1 \pmod{\lambda}$ then $x\bar{g} = x$ and if $x \equiv 1 \pmod{\lambda}$ then $x\bar{g} = y$ where

$$y_\mu = \begin{cases} x_\mu & \text{if} \quad \mu \neq \lambda \\ x_\lambda g & \text{if} \quad \mu = \lambda. \end{cases}$$

Thus $\bar{g}$ only affects the λ-coordinate of x, and then only in one special case. Let $\overline{G}_\lambda = \{\bar{g} | g \in G_\lambda\}$, for each $\lambda \in \Lambda$. Then, as in [3], $\overline{G}_\lambda \cong G_\lambda$. The *restricted wreath product of the groups* G_λ, denoted by $wr_{\lambda \in \Lambda}(G_\lambda, X_\lambda)$, is then defined to be $\langle \overline{G}_\lambda | \lambda \in \Lambda \rangle$, a subgroup of the symmetric group on X. When it is clear which sets are being acted upon, we shall simply write $wr_{\lambda \in \Lambda} G_\lambda$ for the wreath product. We note that in general $wr_{\lambda \in \Lambda} G_\lambda$ depends on the choice of the elements 1_λ; however, just as in [3], if the G_λ act transitively on X_λ, for each $\lambda \in \Lambda$, then $wr_{\lambda \in \Lambda} G_\lambda$ is independent of the choice of the elements 1_λ.

Note that when Λ is totally ordered, our definition coincides with that of Hall. Also when Λ has no relations, then Lemma 2.4 implies the wreath product reduces to the direct product.

One particularly interesting kind of partially ordered set that can be used is the set of all subgroups of a group. Alternatively if G is an arbitrary group we can set $\Lambda = \Lambda(G) = \{H \leqslant G | H \text{ is finite}\}$ and define G_H to be H and regard G_H as acting on itself by right multiplication, so $X_H = H$ also. We then define $W(G) = wr_{H \in \Lambda} G_H$. Theorem 5.8 implies that for all groups G, $W(G)$ is locally finite; if G is also countable then so is $W(G)$. Thus our wreath product allows us to have some control over cardinality. Of course, the fewer finite subgroups a group G has, the simpler $W(G)$ is. For example, when G is torsion free $W(G) = 1$, and if G is a "Tarski Monster" (see [7], p. 30) then $W(G)$ is simply a direct product of cyclic groups of prime power order. Tarski monsters have recently been shown to exist by Ol'sanskii [9]. For further results concerning $W(G)$ the reader is referred to [1].

In the case when $\Lambda = \{1, 2\}$ and G_i acts on itself by right multiplication, then $W = wr\{G_1, G_2\}$ is simply the standard restricted wreath product of the groups G_1 and G_2. Furthermore, if G_i acts on X_i and Y is the orbit of 1_2 in X_2 then it is well known that $W = G_1^{(Y)}]G_2$ where as usual $G_1^{(Y)}$ is the set of functions from Y to G_1, the so called *base group*. The above isomorphism is as permutations groups on $X_1 \times X_2$.

The layout of the paper is as follows. In section 2 we obtain various preliminary results and commutator identities concerning conjugates of certain elements. We also obtain a generalized embedding lemma which says essentially that if $\Gamma \subseteq \Lambda$ then $wr_{\lambda \in \Gamma} G_\lambda \leqslant wr_{\lambda \in \Lambda} G_\lambda$. In Section 3 we obtain what can be regarded as a Kalužnin-Krasner type result which we state as:

THEOREM 3.5: *Let Γ be a finite partially ordered set and let Ω be the same set as Γ with some of the relations in Γ omitted. Suppose G_λ is finite and transitive on X_λ. Then $wr_{\lambda \in \Omega} G_\lambda$ is permutationally isomorphic with a subgroup of $wr_{\lambda \in \Lambda} G_\lambda$.*

A further result of this kind also appears in Section 3.

In Section 4 we obtain a result analogous to Lemma 2 of [3]. Thus we show that every element in a wreath product can be written in an essentially unique manner as a product of certain other elements chosen from certain canonical subgroups. The arguments used are similar to those used by Hall, but, as with much of this work, the partial ordering does create some subtle variations in the arguments used. In Section 5, we consider those subgroups which fix certain coordinates and show that under certain conditions such subgroups are normal. This gives a different characterization of the subgroups defined in section 4, and enables us to exhibit each wreath product as a split extension in a manner analogous to the usual representation of the standard wreath product as the base group extended by the top group. The final result of this section is our main theorem which is a generalization of Hall's theorem C in [3]. As applications of the result (Theorem 5.8) we can show, for example, that a wreath product of locally finite, locally solvable groups is also locally finite and locally solvable.

In Section 6 we discuss briefly what effect using a different equivalence relation has on the definition of the wreath product. For example, Theorem 5.8 also implies that a wreath product of p-groups is again a p-group. We give a very straightforward example of how a change in the equivalence relation makes the result above false.

The notation used is generally standard and is essentially the notation used in [10].

2. Preliminary results

In this section we shall obtain a generalization of Hall's segmentation law (see [3], p. 177) and a straightforward embedding lemma. We shall also obtain certain conjugacy results which indicate exactly how certain group elements act. These results will be of use throughout the rest of this paper. We shall use the notation introduced in the introduction.

LEMMA 2.1: *Let $\lambda_1, \ldots, \lambda_n \in \Lambda$ and suppose $g \in \langle \bar{G}_{\lambda_i} | i = 1, \ldots, n \rangle$. If $g \neq 1$, then there exists $y \in X$ such that $yg \neq y$ and $y_\mu = 1_\mu$ unless $\mu = \lambda_i$ for some i.*

PROOF: Since $g \neq 1$ there must be some $x \in X$ such that $xg \neq x$. The only coordinates of x that can be moved are the coordinates $\lambda_1, \ldots, \lambda_n$. Suppose $\lambda \neq \lambda_i$ and for all i we have either $\lambda < \lambda_i$ or λ and λ_i unrelated. Then the λ-coordinate of x is never moved. This coordinate also has no effect whatever on the action of g on x. Thus we may assume $x_\lambda = 1_\lambda$ and still have $xg \neq x$.

Suppose without loss of generality that the λ_1-coordinate of x is moved by g and let $y = (y_\mu)_{\mu \in \Lambda}$ be chosen so that

$$y_\mu = \begin{cases} x_\mu & \text{if } \mu = \lambda_i \text{ for some } i \text{ and } \mu \geqslant \lambda_1 \\ 1_\mu & \text{otherwise.} \end{cases}$$

Note that the action of g on the λ_1-coordinate of x is determined by those coordinates λ for which $\lambda > \lambda_1$. However if $\lambda > \lambda_1$ and $x_\lambda \neq 1_\lambda$ and $\lambda \neq \lambda_i$ for all i then the λ_1-coordinate of x never gets moved by g. Thus the coordinates λ with $\lambda > \lambda_1$ must have $x_\lambda = 1_\lambda$ and the preceding remarks then imply $yg \neq y$. This completes the proof.

A subset Γ of Λ will be called *full* if λ, $\mu \in \Gamma$ and $\lambda < \mu$ in Λ implies $\lambda < \mu$ in Γ. If G and H are permutation groups acting on sets X and Y respectively then we shall say G and H are *permutationally isomorphic* if there exist maps θ and ϕ so that

(i) θ is an isomorphism from G onto H.
(ii) ϕ is an injection from X into Y.
(iii) $(xg)\phi = (x\phi)(g\theta)$ for all $x \in X$ and $g \in G$.

This terminology differs slightly from that of Holland [4]. The following result is analogous to that of Hall ([3], lemma 3). However the proof is slightly different.

Lemma 2.2: *Suppose Γ is a full subset of Λ. Put $W = wr_{\lambda \in \Lambda}(G_\lambda, X_\lambda)$ and $V = wr_{\lambda \in \Gamma}(G_\lambda, X_\lambda)$. Then V is permutationally isomorphic with $\langle \overline{G}_\lambda | \lambda \in \Gamma \rangle \leqslant W$.*

Proof: The group V acts on the set $Y = Dr_{\lambda \in \Gamma} X_\lambda$ and we can identify Y with a subset of $X = Dr_{\lambda \in \Lambda} X_\lambda$ in a natural manner. Let ϕ: $Y \to X$ denote this embedding. If $\gamma \in \Gamma$ and $g \in G_\gamma$, let $\hat{g}$ be the map induced by g in V and let $\bar{g}$ be the map induced by g in W. We define θ: $V \to W$ by $(\hat{g})\theta = \bar{g}$ and then define $(\prod_{i=1}^n \hat{g}_i)\theta = \prod_{i=1}^n \bar{g}_i$ where $g_i \in G_{\lambda_i}$, say. We need to show that θ is well defined, which amounts to showing that if $g = \prod_{i=1}^n \hat{g}_i = 1_Y$ then $\prod_{i=1}^n \bar{g}_i = 1_X$. This follows by Lemma 2.1 however. Clearly θ is a homomorphism and is injective. Hence θ is an isomorphism of V onto $\langle \overline{G}_\gamma | \gamma \in \Gamma \rangle$. But the action of g on $x \in Y$ is the same as the action of $g\theta$ on $y = x\phi$. Thus

$$(xg)\phi = (x\phi)(g\theta) \quad \text{for all} \quad x \in Y \quad \text{and} \quad g \in V.$$

Hence V is permutationally isomorphic to $\langle \overline{G}_\gamma | \gamma \in \Gamma \rangle$.

In the sequel this result will be referred to as the (generalized) embedding lemma and will often be used implicitly.

We define a partial order $\ll$ on the subsets of Λ as follows. If Γ, $\Delta \subseteq \Lambda$ then we write $\Gamma \ll \Delta$ if and only if $\gamma < \delta$ whenever $\gamma \in \Gamma$ and $\delta \in \Delta$. If Γ is a full subset of Λ then we shall say that Γ is a *segment* of Λ if for each $\lambda \in \Lambda$ precisely one of the following holds:

(i) $\lambda \in \Gamma$
(ii) $\{\lambda\} \ll \Gamma$
(iii) $\Gamma \ll \{\lambda\}$
(iv) no element of Γ is related to λ.

A *segmentation* of Λ is a decomposition of Λ as a disjoint union of segments Λ_i, $\dot{\bigcup}_{i \in I} \Lambda_i$, where I is an index set. The set I inherits the ordering $\ll$ in a natural manner. We can then obtain in a manner analogous to Hall [3], the following segmentation law (or generalized associative law).

LEMMA 2.3: *Let I be an index set and for each $i \in I$, let Λ_i be a segment of Λ. Suppose $\Lambda = \dot{\bigcup}_{i \in I} \Lambda_i$ and for each $i \in I$, let $W_i = wr_{\lambda \in \Lambda} G_{\lambda_i}$ and $Y_i = Dr_{\lambda \in \Lambda_i} X_\lambda$. Then $wr_{i \in I}(W_i, Y_i) \cong wr_{\lambda \in \Lambda} G_\lambda$ as permutation groups.*

The proof of this result is omitted. Here the groups G_λ need not be transitive on X_λ, so the distinguished element of Y_i must be chosen to be $(1_\lambda)_{\lambda \in \Lambda_i}$.

We now start the task of establishing the commutator and conjugacy results that will be required later. For the sake of brevity, the proofs of some results are omitted.

LEMMA 2.4: *Suppose $\lambda, \mu \in \Lambda$ and λ and μ are unrelated. Then $[\bar{G}_\lambda, \bar{G}_\mu] = 1$.*

PROOF: The straightforward proof simply requires an analysis of the cases when $x \equiv 1 (\mathrm{mod}\ \lambda)$ and so on.

The next few results indicate how various elements interact with each other. We shall first require some notation which will be of use later. For each $\lambda \in \Lambda$ define the groups

$$V_\lambda = \langle \bar{G}_\mu | \mu \nless \lambda \rangle,$$

$$U_\lambda = \langle \bar{G}_\mu | \mu > \lambda \quad \text{or } (\mu \text{ and } \lambda \text{ are unrelated}) \rangle$$

$$T_\lambda = \langle \bar{G}_\mu | \mu > \lambda \rangle.$$

Thus $T_\lambda \leqslant U_\lambda \leqslant V_\lambda$.

Lemma 2.5: *Suppose $h \in U_\lambda$ and $g \in G_\lambda$, for some fixed $\lambda \in \Lambda$. Then*

$$(xh^{-1}\bar{g}h)_\mu = x_\mu \qquad \text{if} \quad \mu \neq \lambda.$$

$$(xh^{-1}\bar{g}h)_\lambda = \begin{cases} x_\lambda g & \text{if} \quad xh^{-1} \equiv 1 (\operatorname{mod} \lambda). \\ x_\lambda & \text{if} \quad xh^{-1} \not\equiv 1 (\operatorname{mod} \lambda). \end{cases}$$

Proof: Suppose $\lambda_i \in \Lambda$ for $i = 1, \ldots, n$ and that $\lambda_i > \lambda$ or λ_i is unrelated to λ, for each i. Let $h = \bar{h}_1 \ldots \bar{h}_n$ with $h_i \in G_{\lambda_i}$. Notice that $(xh^{-1})_\lambda = x_\lambda$. There are two cases.

(i) If $xh^{-1} \not\equiv 1 (\operatorname{mod} \lambda)$ then $(xh^{-1})\bar{g} = xh^{-1}$ and $x(h^{-1}\bar{g}h) = x(h^{-1}h) = x$.

(ii) If $xh^{-1} \equiv 1 (\operatorname{mod} \lambda)$ then, applying $\bar{g}$,

$$(xh^{-1}\bar{g})_\mu = (xh^{-1})_\mu \qquad \text{if} \quad \mu \neq \lambda \quad \text{and}$$

$$(xh^{-1}\bar{g})_\lambda = (xh^{-1})_\lambda g = x_\lambda g.$$

If h is now applied, certainly we have $(xh^{-1}\bar{g}h)_\lambda = x_\lambda g$, since h does not affect the λ-coordinate, and $(xh^{-1}\bar{g}h)_\mu = x_\mu$ if $\mu \neq \lambda$, since $\bar{h}_i$ will change the λ_i-coordinate if and only if $\bar{h}_i^{-1}$ changed the λ_i-coordinate. This completes the proof.

Note that Lemma 2.5 says that $\bar{g}^h$ can only affect the λ-coordinate, provided $h \in U_\lambda$. This is certainly not the case if $h \notin U_\lambda$. For example, if $\mu < \lambda$, $g \in G_\lambda$ and $h \in G_\mu$ then $\bar{g}^{\bar{h}}$ affects the μ-coordinate in general. A more general result of this type appears elsewhere (see [1]).

Lemma 2.6: *Let $g \in G_\lambda$, $h \in G_\tau$ and $h_i \in G_{\mu_i}$ (for $i = 1, \ldots, n$). Suppose $\mu_i > \lambda$ for each i and that λ and τ are unrelated. Then $[\bar{g}^{\bar{h}_1 \ldots \bar{h}_n}, \bar{h}] = 1$.*

Proof: Let $k = \bar{h}_1 \ldots \bar{h}_n$ and let $x \in X = Dr_{\gamma \in \Lambda} X_\gamma$. Then by lemma 2.5,

$$(xk^{-1}\bar{g}k)_\mu = x_\mu \qquad \text{if} \quad \mu \neq \lambda.$$

$$(xk^{-1}\bar{g}k)_\lambda = \begin{cases} x_\lambda g & \text{if} \quad xk^{-1} \equiv 1 \ (\operatorname{mod} \lambda) \\ x_\lambda & \text{otherwise.} \end{cases}$$

Applying $\bar{h}$ and noting that $\tau \neq \mu_i$ for any i, we have,

$$(xk^{-1}\bar{g}k\bar{h})_\mu = x_\mu \qquad \text{if} \quad \mu \neq \lambda, \quad \text{and} \quad \mu \neq \tau$$

$$(xk^{-1}\bar{g}k\bar{h})_\lambda = \begin{cases} x_\lambda g & \text{if} \quad xk^{-1} \equiv 1 \ (\operatorname{mod} \lambda) \\ x_\lambda & \text{otherwise.} \end{cases}$$

$$(xk^{-1}\bar{g}kh)_\tau = \begin{cases} x_\tau h & \text{if} \quad x \equiv 1 \ (\operatorname{mod} \tau) \\ x_\tau & \text{otherwise} \end{cases}$$

If we now apply $\bar{h}$ first we have:

$$(x\bar{h})_\mu = x_\mu \quad \text{if} \quad \mu \neq \tau$$

$$(x\bar{h})_\tau = \begin{cases} x_\tau h & \text{if} \quad x \equiv 1 \pmod{\tau} \\ x_\tau & \text{otherwise.} \end{cases}$$

Applying $\bar{g}^k$ gives

$$(x\bar{h}\bar{g}^k)_\mu = x_\mu \quad \text{if} \quad \mu \neq \tau \quad \text{and} \quad \mu \neq \lambda.$$

$$(x\bar{h}\bar{g}^k)_\lambda = \begin{cases} x_\lambda g & \text{if} \quad xk^{-1} \equiv 1 \pmod{\lambda} \\ x_\lambda & \text{otherwise} \end{cases}$$

$$(x\bar{h}\bar{g}^k)_\tau = \begin{cases} x_\tau h & \text{if} \quad x \equiv 1 \pmod{\tau} \\ x_\tau & \text{otherwise.} \end{cases}$$

Thus, comparing these, we see $\bar{h}\bar{g}^k = \bar{g}^k\bar{h}$ and the result follows.

Corollary 2.7: *Let $g \in G_\lambda$, $h \in G_\mu$ and suppose that λ and μ are unrelated. Let $k \in T_\lambda$ and $l \in T_\mu$. Then $[\bar{g}^k, \bar{h}^l] = 1$.*

Proof: Let $k = \bar{k}_1 \ldots \bar{k}_n$ and $l = \bar{l}_1 \ldots \bar{l}_n$ where $k_i \in G_{\tau_i}$, $l_i \in G_{\sigma_i}$, $\tau_i > \lambda$ and $\sigma_i > \mu$. Let $\{\sigma_{i_1}, \ldots, \sigma_{i_r}\}$ constitute the set of those elements which are unrelated to λ. Then all other σ_i must be bigger than λ. The result then follows by Lemma 2.6, the above observation and induction on r.

Corollary 2.8: *Suppose λ, $\tau \in \Lambda$ and $\lambda < \tau$. Suppose $a \in G_\lambda$ and h, $g \in G_\tau$. If $1_\tau h = 1_\tau g$ then $g^{-1}ag = h^{-1}ah$.*

Proof: The result follows from Lemma 2.5 and the fact that $x\bar{g}^{-1} \equiv x\bar{h}^{-1} \pmod{\lambda}$ for all $x \in X$.

Corollary 2.8 essentially says that the effect of conjugating by an element of $\overline{G_\tau}$ is determined by the effect of the corresponding element on 1_τ. The next lemma is not surprising when one recalls the base group representation of a wreath product of two groups.

Lemma 2.9: *Suppose λ, $\tau \in \Lambda$ and $\lambda < \tau$. If $1 \neq a$, $b \in G_\lambda$ and h, $g \in G_\tau$ and if $1_\tau g \neq 1_\tau h$ then $g^{-1}ag \neq h^{-1}bh$ and $[g^{-1}ag, h^{-1}bh] = 1$.*

Proof: The proof is omitted since it requires only a consideration of the cases and the use of Lemma 2.5, in a rather simplified form.

The next result is a variation of Lemma 2.9.

LEMMA 2.10: *Suppose* $\lambda, \mu, \tau \in \Lambda$ *and* $\mu < \lambda < \tau$. *Suppose* $g, h \in G_\tau$, $1 \neq a \in G_\mu$, $b \in G_\lambda$ *and* $1_\tau g \neq 1_\tau h$. *Then* $\overline{g^{-1}ag} \neq \overline{h^{-1}bh}$ *and* $[\overline{g^{-1}ag}, \overline{h^{-1}bh}] = 1$.

PROOF: Using the embedding lemma, we may assume $\Lambda = \{\lambda, \mu, \tau\}$. The result then follows by a straightforward consideration of the possible cases.

The list of commutator results is completed with the following result, whose proof is again omitted.

LEMMA 2.11: *Suppose* $\lambda, \mu, \tau \in \Lambda$. *Suppose* $\tau > \mu$, $\lambda > \mu$ *and* λ *and* τ *are unrelated. Let* $g \in G_\lambda$, $h \in G_\tau$ *and* $a, b \in G_\mu$. *If* $1_\tau h \neq 1_\tau$ *or* $1_\lambda g \neq 1_\lambda$ *then* $[\overline{g^{-1}ag}, \overline{h^{-1}bh}] = 1$.

3. Further embeddings

In the standard wreath product, $\Lambda \wr B$, provided B is finite, it is possible to define the diagonal and this is a subgroup of $A \wr B$ isomorphic to A. We can obtain analogous results in our more general setting.

If $g \in G_\lambda$ and the orbit of 1_τ in X_τ is finite and if $\tau > \lambda$, let $\Delta_\tau \bar{g}$ denote the product of all the distinct conjugates of $\bar{g}$ by elements of $\overline{G_\tau}$. Thus if $1_\tau a_1, \ldots, 1_\tau a_k$ denote the distinct images of 1_τ in X_τ (with $a_i \in G_\tau$) then corollary 2.8 implies

$$\Delta_\tau \bar{g} = \prod_{i=1}^{k} \bar{g}^{\bar{a}_i}.$$

This product is well defined by Lemma 2.9. The next result tells us what effect $\Delta_\tau \bar{g}$ has on $x \in X$.

LEMMA 3.1: *Suppose* $g \in G_\lambda$ *and the orbit of* 1_τ *in* X_τ *is finite, with* $\lambda < \tau$. *Let* $a_1, a_2, \ldots, a_k$ *be defined as above. Then*

$$[x(\Delta_\tau \bar{g})]_\mu = \begin{cases} x_\lambda g & \text{if} \quad x_\tau = 1_\tau a_i \quad \text{for some } i \text{ and} \\ & \qquad\;\; x_\gamma = 1_\gamma \quad \text{for all } \gamma > \lambda, \\ & \qquad\qquad\qquad\quad \text{with } \gamma \neq \tau \\ x_\mu & \text{otherwise.} \end{cases}$$

PROOF: Use Lemma 2.5.

In this section we shall establish an analogue of a theorem of Kalužnin and Krasner [6]. They proved that $A \wr B$ contains an isomorphic copy of every extention of A by B provided B is finite.

LEMMA 3.2: *Suppose $g \in G_\mu$, $h \in G_\lambda$ and μ and λ are unrelated. Suppose the orbit of 1_τ is finite and that $\tau > \lambda$ and $\tau > \mu$. Then*

$$(\Delta_\tau \bar{g})(\Delta_\tau \bar{h}) = (\Delta_\tau \bar{h})(\Delta_\tau \bar{g}).$$

PROOF: Use Corollary 2.7.

We can easily extend the definition of Δ_τ. Suppose $\Gamma \subseteq \Lambda$, and that $\Gamma \ll \{\tau\}$. Then $\Gamma \cup \{\tau\}$ can be segmented. If $1_\tau a_1, \ldots, 1_\tau a_k$ are the distinct images of 1_τ by elements $a_i \in G_\tau$ then for $h \in H \leqslant wr_{\gamma \in \Gamma} G_\gamma$ we see, as before, that $h^{\bar{a}_1}, \ldots, h^{\bar{a}_k}$ are the distinct conjugates of h and we define $\Delta_\tau h = \prod_{i=1}^k h^{\bar{a}_i}$. We define $\Delta_\tau H$ to be $\{\Delta_\tau h \mid h \in H\}$. In particular, if G_τ is finite and G_τ acts transitively on X_τ then $\Delta_\tau(wr_{\gamma \in \Gamma} G_\gamma)$ is simply the diagonal of $(wr_{\gamma \in \Gamma} G_\gamma) \wr G_\tau$. In the context of this paper the introduction of the diagonal in terms of the map Δ_τ seems more appropriate. Proposition 3.3 is hardly surprising in view of the preceding remarks.

PROPOSITION 3.3: *Suppose Γ is a full subset of Λ and let $\tau \in \Lambda$ be fixed. Suppose $\tau > \gamma$ for all $\gamma \in \Gamma$. If the orbit of 1_τ in X_τ is finite and $H \leqslant wr_{\gamma \in \Gamma} G_\gamma$ then*

(i) $H \cong \Delta_\tau H$

(ii) $[\Delta_\tau H, \bar{G}_\tau] = 1$.

PROOF: (i) Let $a_1, \ldots, a_k \in G_\tau$ be such that $1_\tau a_1, \ldots, 1_\tau a_k$ are all the distinct images of 1_τ in X_τ. Then for $g \in G_\gamma$ with $\gamma \in \Gamma$, $\Delta_\tau \bar{g} = \prod_{i=1}^k \bar{g}^{\bar{a}_i}$ and that Δ_τ is a homomorphism now follows from the observation that if $h \in G_\mu$ with $\mu \in \Gamma$ then $\Delta_\tau(gh) = \Delta_\tau(g)\Delta_\tau(h)$. This follows immediately from an application of one of Lemma 2.6, Corollary 2.7 or Lemma 2.9. To complete the proof that $\Delta_\tau H \cong H$ we note that as above $\Delta_\tau(wr_{\gamma \in \Gamma} G_\gamma)$ is the diagonal of $(wr_{\gamma \in \Gamma} G_\gamma) \wr G_\tau$ so $\Delta_\tau(wr_{\gamma \in \Gamma} G_\gamma) \cong wr_{\gamma \in \Gamma} G_\gamma$ from known results. Restricting Δ_τ to H proves the result.

(ii) To prove (ii) let $\bar{g} \in \bar{G}_\tau$ and $h \in H$. Then $h = \prod_{i=1}^n \bar{g}_i$, for certain g_i, and $\Delta_\tau(h) = \prod_{i=1}^n \Delta_\tau(\bar{g}_i)$, so it suffices to show $[\bar{g}, \Delta_\tau(\bar{h})] = 1$ when $\bar{h} \in \bar{G}_\lambda$ for $\lambda < \tau$. But $\Delta_\tau(\bar{h}) = \prod_{i=1}^k h^{\bar{a}_i}$ and $\Delta_\tau(\bar{h})^{\bar{g}} = \prod_{i=1}^k h^{\bar{a}_i \bar{g}}$. However $\bar{g}$ induces a permutation of the conjugates, which all commute, so

$$\Delta_\tau(\bar{h})^{\bar{g}} = \Delta_\tau(\bar{h}).$$

This completes the proof.

Our next result shows that the commutator actions of $K \leqslant wr_{\gamma \in \Gamma} G_\gamma$ and $\Delta_\tau K$ are the same.

LEMMA 3.4: *Suppose Γ is a full subset of Λ and $H, K \leqslant wr_{\gamma \in \Gamma} G_\gamma$. Suppose $\tau \in \Lambda$ and $\gamma < \tau$ for all $\gamma \in \Gamma$. Then $[h, l] = [h, \Delta_\tau l]$, for all $h \in H$, $l \in K$.*

PROOF: Let $h \in H$, $l \in K$. Then $\Delta_\tau l = \prod_{i=1}^{k} l^{\bar{a}_i}$ where the a_i are chosen as usual. Without loss of generality we may put $a_1 = 1$, and we set $\Omega_\tau(\bar{g}) = \prod_{i=2}^{k} \bar{g}^{\bar{a}_i}$ for $\bar{g} \in \bar{G}_\gamma$. So if $l = \prod_{i=1}^{n} \bar{g}_i$ then $\Delta_\tau(l) = l\Omega_\tau(\bar{g}_1)\ldots\Omega_\tau(\bar{g}_n)$ using Lemma 2.6, Corollary 2.7 and Lemma 2.9. But h commutes with all the $\Omega_\tau(\bar{g}_i)$ by the same results. Hence $[h, \Delta_\tau l] = [h, l]$ as required.

We come now to the main result in this section. We shall require some new terminology. Let Γ be a set on which two partial orders $\preccurlyeq$ and $\leqslant$ are defined. We say that the order $\leqslant$ *extends* the order $\preccurlyeq$ on Γ if whenever $\lambda, \mu \in \Gamma$ and $\lambda \preccurlyeq \mu$ then $\lambda \leqslant \mu$. If $(\Lambda, \leqslant)$ is a partially ordered set and $\Gamma \subseteq \Lambda$ then we may form a new partially ordered set $(\Lambda, \preccurlyeq)$ by simply suppressing some of the relations in $(\Lambda, \leqslant)$, and $\leqslant$ then extends $\preccurlyeq$. We shall say Γ is an *unfulfilled subset* of Λ when talking about the ordering $\preccurlyeq$ on Γ. To ease the notation we shall let Ω be a set with the same elements as Γ and assume that the ordering on Γ is the usual one and the ordering on Ω is $\preccurlyeq$.

THEOREM 3.5: *Let Γ be a finite partially ordered set and let Ω be an unfilfilled subset of Γ with $\Omega = \Gamma$. Let G_λ be transitive on X_λ and suppose G_λ is finite for each $\lambda \in \Gamma$. Then $wr_{\lambda \in \Omega} G_\lambda$ is permutationally isomorphic with a subgroup of $wr_{\lambda \in \Gamma} G_\lambda$.*

PROOF: The proof is by induction on the number of relations in $(\Gamma, \leqslant)$. Clearly we may assume $|\Gamma| > 1$ and that $(\Gamma, \leqslant)$ has $n \geqslant 1$ relations. Let $\Gamma = \{\lambda_1, \ldots, \lambda_r\}$. If $n = 1$ then we may assume $\lambda_1 < \lambda_2$ and λ_i is unrelated to λ_j for all other values of i and j. Since Ω is unfulfilled, $(\Omega, \preccurlyeq)$ has no relations so $wr_{\lambda \in \Omega} G_\lambda = Dr_{\lambda \in \Omega} G_\lambda$. Let $\lambda = \lambda_1$ and $\tau = \lambda_2$. Then in $(\Gamma, \leqslant)$, $\lambda < \tau$ so we can form $\Delta_\tau \bar{G}_\lambda$ and this is a group isomorphic with $\bar{G}_\lambda$ by Proposition 3.3(i).

Consider the action of $\Delta_\tau \bar{G}_\lambda$ on $x \in Dr_{\lambda \in \Gamma} X_\lambda$. By Lemma 3.1, if $1_\tau a_1, \ldots, 1_\tau a_k$ are the distinct images of 1_τ then, for $g \in G_\lambda$,

$$[x(\Delta_\tau \bar{g})]_\mu = \begin{cases} x_\lambda g & \text{if} \quad x_\tau = 1_\tau a_i \text{ for some } i \text{ and } \\ & \qquad x_\mu = 1_\mu \text{ for all } \mu > \lambda \\ & \qquad \text{with } \mu \neq \tau. \\ x_\mu & \text{otherwise.} \end{cases}$$

However, in this case, only $\tau > \lambda$ so $\Delta_\tau \bar{g}$ acts like $\bar{g}$ on $x = (x_\lambda)_{\lambda \in \Gamma}$, and as if λ and τ are unrelated. The action of $\bar{g} \in \bar{G}_\mu$ for $\mu \neq \lambda$ is unchanged. Hence $wr_{\gamma \in \Omega} G_\gamma$ is permutationally isomorphic with $\langle \bar{G}_\mu, \Delta_\tau \bar{G}_\lambda | \mu \neq \lambda, \mu \in \Gamma \rangle$.

Suppose that we know the result is true for $n = k$ and that $(\Gamma, \leqslant)$ has $k + 1$ relations. Suppose $\lambda, \tau \in \Gamma$ and $\lambda < \tau$. Let Ω be the same set as Γ, but let Ω have all the relations of Γ except that λ and τ are unrelated in Ω. Then Ω has k relations, so for any unfulfilled subset Σ of Ω we have, by induction, that $wr_{\gamma\in\Sigma}G_\gamma$ is permutationally isomorphic with a subgroup of $wr_{\gamma\in\Omega}\underline{G_\gamma}$.

Let $\hat{G}_\lambda = \Delta_\tau \bar{G}_\lambda \cong \bar{G}_\lambda$. Then as above, for $g \in G_\lambda$, $\hat{g} = \Delta_\tau \bar{g}$ acts on $x \in Dr_{\gamma\in\Gamma}X_\gamma$ as if γ and τ were unrelated. However the actions of $\bar{G}_\gamma$ for all $\gamma \neq \lambda$ are unchanged since all other relations are unchanged. It follows that if $H = \langle \bar{G}_\gamma | \gamma \neq \lambda \rangle$ then $\langle H, \hat{G}_\lambda \rangle \cong wr_{\gamma\in\Omega}G_\gamma$ as permutation groups. This completes the proof.

There is a slight variation of this when some of the G_μ are not transitive on the corresponding sets X_μ. The proof of the above result can be slightly modified and so we state the following theorem without proof.

Theorem 3.6: *Let Γ be a finite partially ordered set. Suppose G_τ is not transitive on X_τ for some $\tau \in \Gamma$ and that, in Γ, $\lambda < \tau$. Suppose Ω is the fulfilled subset of Γ, with $\Omega = \Gamma$, that has all the relations of Γ except that λ and τ are unrelated. Then $wr_{\mu\in\Omega}G_\mu$ is isomorphic, as a group, with $\langle \Delta_\tau \bar{G}_\lambda, \bar{G}_\mu | \mu \neq \lambda, \mu \in \Gamma \rangle$.*

The results obtained in Theorems 3.5 and 3.6 are the best possible in the sense that none of the G_λ can be infinite and Γ cannot be infinite. For example, let G be a cyclic group of order 2 and for each prime $p_i \neq 2$, let A_i be a cyclic group of order p_i. Let $A = Dr_{i\geqslant 1}A_i$ and consider the groups to be acting on themselves in the regular representation. Then $W = G \wr A$ contains no subgroup isomorphic with $G \times A$. To see this, let K be the base group of W. Then $W = KA$. Suppose $X \cong A$ and let $\pi = \{p_1, p_2, \dots\}$. Then X is a maximal π-group since if Y is a π-group containing X, then Y contains, for each prime p_i, a Sylow p_i-subgroup, Y_i, of W, which is also contained in X. Then $Y_i K/K \in Syl_{p_i} W/K$ and $YK/K = Dr_{i\geqslant 1} Y_i K/K = W/K$. Thus $W = YK$, whence $Y \cong A$. Then $Y \cong X$ and hence $X = Y$. (We have here used standard facts from the Sylow theory of locally finite groups.)

It follows that every group isomorphic with A complements K. We claim that for each such subgroup X, the centralizer $C_W(X) = X$. It suffices to show $C_K(X) = 1$, since $W = KX$ and $X \leqslant C_W(X)$. Suppose $x \in X$. Then $x = ka$ for some $k \in K$ and some $a \in A$. Clearly if p is a prime then x has order p if and only if a has order p. Thus if $l \in C_K(X)$ then

$$x^l = x$$

$$\Rightarrow k^l a^l = ka$$

$$\Rightarrow a^l = a \quad \text{since} \quad K \quad \text{is abelian.}$$

The previous remark then shows $l \in C_K(A) = 1$. Thus $C_K(X) = 1$ and $G \wr A$ contains no subgroup isomorphic with $G \times A$. Thus taking $\Gamma =$

$\{1, 2\}$, with the usual order, and $G_1 = G$, $G_2 = A$ we see that none of the G_λ can be infinite in Theorem 3.5. Taking $\Gamma = \{0, 1, 2, \dots\}$, with the order $0 < n$ for all $n = 1, 2, 3, \dots$ and m and n unrelated whenever m, $n \neq 0$, and putting $G_0 = G$ and $G_i = A_i$ for $i \geqslant 1$, we see that Γ cannot be infinite.

4. A uniqueness result

In this section we shall obtain a generalization of Lemma 2 in Hall's paper [3]. This method of expressing elements of $W = wr_{\lambda \in \Lambda}(G_\lambda, X_\lambda)$ in a unique manner as products of certain conjugates is sometimes useful since it gives a certain amount of control over which coordinates are moved. We shall use the notation already established. We shall be interested in considering the effects of conjugating by elements in the subgroups T_λ, U_λ, and V_λ introduced in Section 2. We also let $D_\lambda = \overline{G}_\lambda^{V_\lambda} \lhd V_\lambda$, for $\lambda \in \Lambda$. Clearly $\overline{G}_\lambda \leqslant D_\lambda$. Our first result shows that the effects of conjugating by elements from T_λ, U_λ, and V_λ are the same.

LEMMA 4.1: *With the above notation $D_\lambda = \overline{G}_\lambda^{T_\lambda} = \overline{G}_\lambda^{U_\lambda}$.*

PROOF: Clearly $\overline{G}_\lambda^{T_\lambda} \leqslant \overline{G}_\lambda^{U_\lambda} \leqslant D_\lambda$. Suppose $k \in D_\lambda$ and $k = \overline{g}^{\overline{h}_1 \dots \overline{h}_n}$ where $g \in G_\lambda$ and $h_i \in G_\mu$ for certain G_μ with $\mu \not< \lambda$. Suppose that $h_i \in G_\lambda$ for some i. Then

$$k = \left(\overline{h}_i^{-1}\right)^{\overline{h}_{i+1} \dots \overline{h}_n} \overline{g}^{\overline{h}_1 \dots \overline{h}_{i-1} \overline{h}_{i+1} \dots \overline{h}_n \overline{h}_i} \overline{h}_i^{\overline{h}_{i+1} \dots \overline{h}_n} \in \overline{G}_\lambda^{U_\lambda}.$$

In this manner we can write every element of D_λ as an element of $\overline{G}_\lambda^{U_\lambda}$, whence $\overline{G}_\lambda^{U_\lambda} = D_\lambda$. The fact that $\overline{G}_\lambda^{U_\lambda} = \overline{G}_\lambda^{T_\lambda}$ follows from Lemma 2.6. The result follows.

LEMMA 4.2: *For each $\lambda \in \Lambda$, $D_\lambda \cap U_\lambda = 1$.*

PROOF: Let $g = \overline{g}_1^{h_1} \dots \overline{g}_n^{h_n} \in D_\lambda \cap U_\lambda$, where $g_i \in G_\lambda$, $h_i \in T_\lambda$ and suppose $g \neq 1$. Then, for some $x \in X = Dr_{\lambda \in \Lambda} X_\lambda$, $xg \neq x$. Lemma 2.5 implies

$$(xg)_\lambda \neq x_\lambda.$$

However elements of U_λ are generated by elements of $\overline{G}_\mu$ for $\mu > \lambda$ or μ and λ unrelated. In particular elements of $\overline{G}_\mu$ can never alter the λ-coordinate of x, which is a contradiction.

We can now obtain the uniqueness part of the result required. First, we note that Szpilrajn [11] has shown that a partial order defined on a set can always be extended to a total order. (This theorem is actually an unpublished result of Banach, Kuratowski and Tarski.) Thus we shall let $\preccurlyeq$ be an extension of the order $\leqslant$ on Λ, so that $(\Lambda, \preccurlyeq)$ is totally ordered.

LEMMA 4.3: *Let* $\Gamma = \{\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m\}$ *and* $g_1 \ldots g_n = h_1 \ldots h_m$ *where* $1 \neq g_i \in D_{\lambda_i}$ *and* $1 \neq h_i \in D_{\mu_i}$. *Suppose* $\lambda_i \prec \lambda_j$ *and* $\mu_i \prec \mu_j$, *whenever* $i < j$. *Then* $n = m$ *and* $g_i = h_i$ *for all* i.

PROOF: We consider three cases. First suppose $\lambda_1 < \mu_1$. Then $g_1 = h_1 \ldots h_m g_n^{-1} \ldots g_2^{-1}$. Hence $g_1 \in D_{\lambda_1} \cap U_{\lambda_1}$ (we cannot have $\mu_i < \lambda_1$ otherwise $\mu_i < \mu_1$, a contradiction). By Lemma 4.2, $g_1 = 1$, a contradiction. Now suppose $\lambda_1 = \mu_1$. Then

$$h_1^{-1} g_1 = h_2 \ldots h_m g_n^{-1} \ldots g_2^{-1} \in D_{\lambda_1} \cap U_{\lambda_1} = 1.$$

Hence $g_1 = h_1$ so $g_2 \ldots g_n = h_2 \ldots h_m$ and the result follows by induction. Finally suppose λ_1 and μ_1 are unrelated in the ordering $\leqslant$. If $\lambda_1 \prec \mu_i$ for all i then we obtain a contradiction as in the first case. Thus we may assume $\mu_i < \lambda_1$ for some i. Choose i as small as possible so that $\mu_i < \lambda_1$. Now

$$h_i^{-1} = h_{i+1} \ldots h_m g_n^{-1} \ldots g_1^{-1} h_1 \ldots h_{i-1} \in D_{\mu_i}.$$

On the other hand, the choice of i implies $h_i^{-1} \in U_{\mu_i}$. Hence $h_i^{-1} \in D_{\mu_i} \cap U_{\mu_i} = 1$, again a contradiction. This completes the proof.

We can now obtain the extension of Hall's result that is required.

PROPOSITION 4.4: *Every element of* W *can be written uniquely in the form* $g = g_1 \ldots g_n$ *where* $g_i \in D_{\lambda_i}$ *and* $\lambda_i \prec \lambda_j$ *whenever* $i < j$.

PROOF: The uniqueness part of the assertion follows from Lemma 4.3. To prove existence, let $g = \bar{h}_1 \ldots \bar{h}_m \in W$ where $h_i \in G_{\mu_i}$ say. Choose $\mu_i \in \Gamma = \{\mu_1, \ldots, \mu_m\}$ so that μ_i is minimal and so that i is as small as possible with μ_i minimal. Then, for $j < i$, $\mu_i \prec \mu_j$ and if $k = (\bar{h}_1 \ldots \bar{h}_{i-1})^{-1} \in U_{\mu_i}$ we have,

$$g = \bar{h}_i^k \left(k^{-1} \bar{h}_{i+1} \ldots \bar{h}_m\right).$$

The result then follows by induction on m, with applications of Lemma 2.6 and Corollary 2.7 where necessary.

The results obtained above can be used to obtain a concise formula for the order of a finite wreath product, in terms of the groups G_λ and sets X_λ. The details will appear elsewhere.

5. Subgroups fixing certain coordinates

The aim of this section is to obtain a structural result concerning wreath products over arbitrary partially ordered sets. This should be regarded as analogous to theorem C in Hall's paper [3], although the result established here requires a rather different proof. The results already established in Section 3 could be used to obtain a somewhat restricted version of the main result of this section, but further machinery is necessary to obtain the best possible result. At the same time we obtain certain canonical subgroups of a wreath product.

We use the notation introduced earlier. Thus Λ will denote an arbitrary partially ordered set. For each $\lambda \in \Lambda$, G_λ will be a group acting on a set X_λ and $W = wr_{\lambda \in \Lambda}(G_\lambda, X_\lambda)$.

If $\Gamma \subseteq \Lambda$ then the set of elements of W fixing all coordinates in Γ will be denoted by H_Γ. Thus $H_\Gamma = \{ g \in W | (xg)_\gamma = x_\gamma \text{ for all } \gamma \in \Gamma \}$. The following result is then clear.

Lemma 5.1: *For all subsets* Γ *of* Λ, H_Γ *is a subgroup of* W.

In general, for $\Gamma \subseteq \Lambda$, H_Γ need not be a normal subgroup of W. For example, if $\Lambda = \{1, 2\}$ with the usual order and if $h \in G_1$, $g \in G_2$ (where G_i acts on itself in the right regular representation for $i = 1, 2$) then $\bar{g}$ fixes the first coordinate but $\bar{g}^h$ does not. We shall obtain conditions on Γ which ensure that $H_\Gamma \lhd W$.

A subset Γ of Λ is called a *filter with respect to* $\lambda \in \Lambda$ if $\mu \in \Gamma$ whenever $\mu \in \Lambda$ and $\mu > \lambda$. Γ is called a *filter* if whenever $\lambda \in \Gamma$ and $\mu > \lambda$ then $\mu \in \Gamma$.

Lemma 5.2: *Let* Γ *be a filter with respect to* $\lambda \in \Lambda$. *Then* $\bar{G}_\lambda \leqslant N_W(H_\Gamma)$.

Proof: If $\Gamma = \emptyset$ then $H_\Gamma = W$, so the result is clear. If $\lambda \notin \Gamma$ then elements of $\bar{G}_\lambda$ fix all coordinates but the λ-coordinate, by definition. Hence $\bar{G}_\lambda \leqslant H_\Gamma$ in this case. If $\lambda \in \Gamma$ and $h \in H_\Gamma$ then for $g \in G_\lambda$, we need to show $\bar{g}^{-1} h \bar{g}$ fixes all coordinates in Γ. There are several cases.

If $x \not\equiv 1 \pmod{\lambda}$ then $x\bar{g}^{-1} = x$ so $x\bar{g}^{-1}h\bar{g} = xh\bar{g}$. Also since Γ is a filter with respect to λ, $x \not\equiv 1 \pmod{\lambda}$ implies $xh \not\equiv 1 \pmod{\lambda}$. Thus $xh\bar{g} = xh$. Hence if $x \not\equiv 1 \pmod{\lambda}$ then $x\bar{g}^{-1}h\bar{g} = xh$ and all Γ-coordinates are fixed.

If $x \equiv 1 \pmod{\lambda}$ then

$$(x\bar{g}^{-1})_\tau = \begin{cases} x_\tau & \text{if } \tau \neq \lambda \\ x_\lambda g^{-1} & \text{if } \tau = \lambda \end{cases}.$$

For $\tau \neq \lambda$ and $\tau \in \Gamma$ we therefore have

$$\begin{aligned} (x\bar{g}^{-1}h\bar{g})_\tau &= (x\bar{g}^{-1}h)_\tau && \text{since } \bar{g} \text{ fixes the } \tau\text{-coordinate.} \\ &= (x\bar{g}^{-1})_\tau && \text{since } h \text{ fixes the } \tau\text{-coordinate} \\ &= x_\tau. \end{aligned}$$

For $\tau = \lambda$, $(x\bar{g}^{-1}h\bar{g})_\lambda = x_\lambda$ since $x \equiv 1 \pmod{\lambda}$ implies $x\bar{g}^{-1} \equiv 1 \pmod{\lambda}$ and $x\bar{g}^{-1}h \equiv 1 \pmod{\lambda}$. This completes the proof.

COROLLARY 5.3: *Suppose $\Gamma \subseteq \Lambda$ and that Γ is a filter. Then $H_\Gamma \lhd W$.*

PROOF: If $\lambda \in \Gamma$ then Γ is a filter with respect to λ, so $\bar{G}_\lambda \leqslant N_W(H_\Gamma)$ by Lemma 5.2. If $\lambda \notin \Gamma$ then $\bar{G}_\lambda \leqslant H_\Gamma \leqslant N_W(H_\Gamma)$. The result follows.

Note that even when Γ is a filter, H_Γ need not be normal in Sym X, the symmetric group on X. Before giving some examples of filters, we shall obtain a partial converse of Corollary 5.3.

LEMMA 5.4: *Let $\Gamma \subseteq \Lambda$. If $H_\Gamma \lhd W$ then Γ is a filter, provided the orbit of 1_μ is not trivial for all $\mu \notin \Gamma$.*

PROOF: Suppose $H_\Gamma \lhd W$, but Γ is not a filter. Then there exists $\lambda \in \Gamma$ and some $\mu > \lambda$ with $\mu \notin \Gamma$. Then $\bar{G}_\mu \leqslant H_\Gamma$ and hence $\bar{G}_\mu^W \leqslant H_\Gamma$ since $H_\Gamma \lhd W$. Choose $g \in G_\mu$ so that $1_\mu g \neq 1_\mu$ and choose $h \in G_\lambda$, $y \in X_\lambda$ so that $yh \neq y$. Then if $x \in X$ is chosen so that

$$x_\tau = \begin{cases} 1_\tau & \text{if} \quad \tau \neq \lambda \\ y & \text{if} \quad \tau = \lambda \end{cases}$$

we have

$$\left(x\bar{h}^{-1}\bar{g}\bar{h}\right)_\tau = \begin{cases} yh^{-1} & \text{if} \quad \tau = \lambda \\ 1_\mu g & \text{if} \quad \tau = \mu \\ 1_\tau & \text{otherwise.} \end{cases}$$

In particular, $\bar{h}^{-1}\bar{g}\bar{h}$ does not fix the λ-coordinate, a contradiction. The result is proved.

To obtain examples of filters, let $\lambda \in \Lambda$ and define the sets Γ_λ, Ω_λ, and Δ_λ as follows:

$$\Gamma_\lambda = \{\mu \in \Lambda | \mu > \lambda\}.$$

$$\Omega_\lambda = \{\mu \in \Lambda | \mu \geqslant \lambda\}.$$

$$\Delta_\lambda = \{\mu \in \Lambda | \lambda \geqslant \mu\}.$$

The following result is then routine.

LEMMA 5.5: *For each $\lambda \in \Lambda$, the sets Γ_λ, Ω_λ and $\bar{\Delta}_\lambda = \Lambda - \Delta_\lambda$ are all filters.*

It then follows from Corollary 5.3 that the subgroups H_{Γ_λ}, H_{Ω_λ} and $H_{\bar{\Delta}_\lambda}$ are all normal in W. We can form filters in other ways also. Let

$\Gamma \subseteq \Lambda$ and let $\Gamma^* = \bigcup_{\lambda \in \Gamma} \Omega_\lambda$. Then Γ^* is a filter called the *filter generated by* Γ. This method of forming filters enables us to decompose wreath products into semi-direct products, as follows. We first introduce the notation that if $\Gamma \subseteq \Lambda$ then $W_\Gamma = \langle \overline{G}_\gamma | \gamma \in \Gamma \rangle$.

LEMMA 5.6: *Let* Γ *be a filter. Then*
(i) $W = H_\Gamma H_{\overline{\Gamma}}$ *and* $H_\Gamma \cap H_{\overline{\Gamma}} = 1$.
(ii) $H_\Gamma = (W_{\overline{\Gamma}})^W \lhd W$.
(iii) $H_{\overline{\Gamma}} = W_\Gamma$.

PROOF: (i) Since Γ is a filter, $H_\Gamma \lhd W$. Also if $\gamma \notin \Gamma$ then $\overline{G}_\gamma \leqslant H_\Gamma$. Hence

$$V = \langle \overline{G}_\gamma | \gamma \notin \Gamma \rangle^W \leqslant H_\Gamma$$

and

$$W_\Gamma = \langle \overline{G}_\gamma | \gamma \in \Gamma \rangle \leqslant H_{\overline{\Gamma}}.$$

Moreover $W = VW_\Gamma$ so $W = H_\Gamma H_{\overline{\Gamma}}$. Clearly $H_\Gamma \cap H_{\overline{\Gamma}} = 1$, so (i) is proved.

(ii) and (iii) follow from the observation in (i), the facts that $W = VH_{\overline{\Gamma}} = H_\Gamma W_\Gamma = H_\Gamma H_{\overline{\Gamma}}$ and the Dedekind law.

Notice that in general $W_\Gamma \neq H_{\overline{\Gamma}}$. It is always the case that $W_\Gamma \leqslant H_{\overline{\Gamma}}$, but when $\Lambda = \{1, 2, 3\}$ with the usual order, and $\Gamma = \{2\}$ then $W_\Gamma = \overline{G}_2 < \overline{G}_2^{\overline{G}_3} \leqslant H_{\overline{\Gamma}}$. We now obtain a slightly different characterisation of certain of the subgroups H_Γ. We recall that if $\lambda \in \Lambda$ then $T_\lambda = \langle \overline{G}_\mu | \mu > \lambda \rangle$ and $D_\lambda = \overline{G}_\lambda^{T_\lambda}$. We shall write $\overline{\lambda}$ for the complement of $\{\lambda\}$ in Λ.

LEMMA 5.7: *For all* $\lambda \in \Lambda$, $D_\lambda = H_{\overline{\lambda}}$.

PROOF: By Lemma 2.5, $D_\lambda \leqslant H_{\overline{\lambda}}$. Suppose $g \in H_{\overline{\lambda}}$. Then, by Proposition 4.4., $g = g_1 \dots g_n$, where $1 \neq g_i \in D_{\lambda_i}$ for certain $\lambda_i \in \Lambda$ $(i = 1, \dots, n)$. By choosing $x \in X$ correctly we may suppose $xg_1 \neq x$. Consequently the λ_1-coordinate of x is moved, again by Lemma 2.9, and hence g moves the λ_1-coordinate of x. Hence $\lambda_1 = \lambda$.

Suppose $n \geqslant 2$. If λ_2 is unrelated to λ_1 then $g = g_2 g_1 g_3 \dots g_n$ by Corollary 2.7 and, as above, $\lambda = \lambda_2$, a contradiction. Thus $\lambda_1 \leqslant \lambda_2$. But we can then choose $x \in X$ so that $x_{\lambda_2} \neq 1_{\lambda_2}$ and $x \neq xg_2 = xg_1g_2$. Thus, as above, $\lambda_2 = \lambda = \lambda_1$, again a contradiction. It follows that $n = 1$ and $g \in D_{\lambda_1} = D_\lambda$. Thus $H_{\overline{\lambda}} \leqslant D_\lambda$ and the proof is complete.

Suppose now that λ is minimal in Λ. Then $\overline{\lambda}$ is a filter and $H_{\overline{\lambda}} = D_\lambda \lhd W$. Thus $H_{\overline{\lambda}}$ is the base group of $\overline{G}_\lambda \wr T_\lambda$ and is simply the direct product of certain conjugates of $\overline{G}_\lambda$ by elements of T_λ. These observations will be useful in the proof of the next theorem, which is the analogue of theorem C in Hall [3] that we seek.

THEOREM 5.8: *Suppose $\mathfrak{X}$ is a class of groups which is closed with respect to forming* (i) *direct powers and* (ii) *split extensions. Then*

(i) *If Λ is finite and $G_\lambda \in \mathfrak{X}$ for all $\lambda \in \Lambda$ then $W \in \mathfrak{X}$.*

(ii) If $G_\lambda \in L\mathfrak{X}$, for all $\lambda \in \Lambda$, then $W \in L\mathfrak{X}$.

PROOF: (i) The result is proved by induction on $|\Lambda|$, the case $|\Lambda| = 1$ being clear. Suppose the result is true for sets Ω with $|\Omega| < |\Lambda|$, and let Γ be the set of minimal elements in Λ. Then $\bar{\Gamma}$ is a filter and Lemma 5.6 (i) implies $W = H_{\bar{\Gamma}} H_\Gamma$. Also $H_\Gamma = W_{\bar{\Gamma}}$ in this case. Since $|\bar{\Gamma}| < |\Lambda|$, it follows that $W_{\bar{\Gamma}} \in \mathfrak{X}$. We claim also that $H_{\bar{\Gamma}} = Dr_{\gamma \in \Gamma} D_\gamma$. For, if $\gamma \in \Gamma$, $D_\gamma = \bar{G}_\gamma^W \leqslant H_{\bar{\Gamma}}$. On the other hand, $\langle \bar{G}_\gamma | \gamma \in \Gamma \rangle \leqslant \Pi_{\gamma \in \Gamma} D_\gamma \lhd W$, so Lemma 5.6 (ii) implies

$$H_{\bar{\Gamma}} = \langle \bar{G}_\gamma | \gamma \in \Gamma \rangle^W \leqslant \prod_{\gamma \in \Gamma} D_\gamma .$$

Thus $H_{\bar{\Gamma}} = \Pi_{\gamma \in \Gamma} D_\gamma$. Lemma 5.7 now implies, for $\gamma, \omega \in \Gamma$

$$[D_\gamma, D_\omega] = [H_{\bar{\gamma}}, H_{\bar{\omega}}] \leqslant H_{\bar{\gamma}} \cap H_{\bar{\omega}} = 1.$$

Hence the product is actually direct and the claim follows. Now the remark preceding Theorem 5.8 implies $H_{\bar{\Gamma}}$ is a direct product of finitely many $\mathfrak{X}$-groups, and hence is an $\mathfrak{X}$-group. Hence $W = H_{\bar{\Gamma}} H_\Gamma \in \mathfrak{X}$, since $\mathfrak{X}$ is split extension closed.

Part (ii) follows from part (i). This completes the proof.

For further illustration of how useful the subgroups H_Γ can be, the reader is referred to [1].

6. Concluding remarks

When the wreath product was originally defined in Section 1, it was mentioned that one could use other equivalence relations. Here we give an example to show that the use of a different equivalence relation does not give such nice results as we have obtained. We shall use the notation introduced in Section 1, but introduce a new equivalence relation on X by defining:

$$x \equiv y \pmod{\lambda} \qquad \text{if and only if} \quad x_\mu = y_\mu \quad \text{whenever} \quad \mu \nleqslant \lambda.$$

Such a relation can be defined for each $\lambda \in \Lambda$. If $g \in G_\lambda$ we now define $\bar{g} \in \bar{G}_\lambda$ as before. Thus (with the new relation) $x\bar{g} = x$ whenever $x \not\equiv 1 \pmod{\lambda}$ whereas if $x \equiv 1 \pmod{\lambda}$ then

$$(x\bar{g})_\mu = \begin{cases} x_\mu & \text{if} \quad \mu \neq \lambda \\ x_\lambda g & \text{if} \quad \mu = \lambda . \end{cases}$$

As before we put $\bar{G}_\lambda = \{\bar{g} | g \in G_\lambda\}$ and define the wreath product to be $\langle \bar{G}_\lambda | \lambda \in \Lambda \rangle$. We now show that the wreath product, defined in this manner, of two groups of order 2 can be isomorphic with the symmetric group on three symbols.

Let $\Lambda = \{1,2\}$ and suppose 1 and 2 are unrelated. Let $G_1 = \langle g \rangle$ and $G_2 = \langle h \rangle$ be groups of order 2 acting on $X_1 = X_2 = \{1, 2\}$ in the same manner as the permutation group $\langle (1, 2) \rangle$ acts. It is easy to see that $\bar{g}$, $\bar{h}$ and $\overline{gh}$ are all distinct and all fix the ordered pair (2, 2). Thus the wreath product in this case essentially only acts on a three element set and hence is isomorphic with the symmetric group on three symbols. In particular, Theorem 5.8 is not valid for this wreath product. This illustrates how crucial the choice of equivalence relation is in the definition of a generalized wreath product.

To conclude, we remark that the wreath product as defined in Section 1 enables us to construct numerous, apparently new, examples. In [1], for example, a canonical method is given for constructing certain FC-groups.

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