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REMARKS ON p-TORSION OF ALGEBRAIC SURFACES

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This paper is divided into two independent parts. In the first part, we show that if Y is a nonsingular model of a weighted complete intersection surface with only rational double points as singularities, then $\mathrm{Pic}(Y)$ is torsion-free. In the second part, we give an example of a surface with torsion-free crystalline cohomology, but for which the Hodge-de Rham spectral sequence is non-degenerate. At present, this is the only known example of this phenomenon.

The first part was inspired by a question of P. Blass, and the second part by a question of L. Illusie. I should like to thank both of them, as well as N. Katz and M. Raynaud, for their encouragement.

Throughout the paper, we work over an algebraically closed field k of characteristic p. All surfaces considered will be reduced, irreducible, and complete, unless otherwise stated.

1. The Picard group of a weighted complete intersection surface

For the definition and basic properties of weighted complete intersections, see [9].

THEOREM: *Let X be a weighted complete intersection surface with only rational double points as singularities. Let Y be a nonsingular model of X. Then* $\mathrm{Pic}(Y)$ *and* $\mathrm{Pic}(X)$ *are torsion-free.*

PROOF: First, notice that it is enough to prove the theorem for one nonsingular model of X. Therefore, we may assume that Y is a minimal resolution of the singularities of X.

LEMMA 1: (*Artin*). *Let X be a surface with only rational double points as singularities, let $g: Y \to X$ be a minimal resolution of the singularities of X, and let $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf on Y. If $D \cdot E = 0$ for all components E of the exceptional divisors obtained by resolving the singularities of X, then there exists an invertible sheaf $\mathcal{L}'$ on X such that $\mathcal{L} = g^*\mathcal{L}'$.*

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PROOF: See [1], Cor. 2.6.

COROLLARY: *If* $g: Y \to X$ *is as in Lemma* 1, *then* $g^*: \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *is an isomorphism on torsion.*

By Lemma 1 and its corollary, it is enough to show that $\mathrm{Pic}(X)$ is torsion-free.

LEMMA 2: *Let* ℓ *be a prime number*, $\ell \neq p$. *Then* $\mathrm{Pic}(X)$ *is* ℓ*-torsion-free.*

PROOF: (This is presumably well-known, but a proof does not appear in the literature.) In the proof of Theorem 3.7 of [9], Mori constructs a finite morphism $f: \tilde{X} \to X$, where $\tilde{X}$ is an ordinary complete intersection, and $f^*: \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ is an injection. Therefore it is enough to show that if X is an ordinary complete intersection of dimension 2 (with arbitrary singularities) then $\mathrm{Pic}(X)$ is ℓ-torsion-free.

For this, we follow the argument of Hartshorne [5,IV.3]. (See also [11].) Let P be the ambient projective space in which X is a complete intersection, let U be an open neighborhood of X in P, let $\hat{P}$ be the completion of P along X, and let X_n be the $(n-1)$st infinitesimal neighborhood of X in P. Then by the argument of Hartshorne, $\mathrm{Pic}(P) \simeq \mathrm{Pic}(U) \simeq \mathrm{Pic}(\hat{P}) \simeq \varprojlim \mathrm{Pic}(X_n)$. We now show, by induction on n, that the ℓ-torsion part of $\mathrm{Pic}(X_n)$ is isomorphic to the ℓ-torsion part of $\mathrm{Pic}(X)$, and since $\mathrm{Pic}(P) \simeq \mathbb{Z}$, this will conclude the proof of Lemma 2. We use the exact sequence [5, op. cit.]

$$0 \to I^n/I^{n+1} \to \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 0$$

where I is the ideal sheaf defining X as a subscheme of P. Taking cohomology, we get an exact sequence

$$H^1(X, I^n/I^{n+1}) \to H^1(X_{n+1}, \mathcal{O}^*_{X_{n+1}}) \to H^1(X_n, \mathcal{O}^*_{X_n}) \to H^2(X, I^n/I^{n+1}).$$

But $H^1(X, I^n/I^{n+1}) = 0$ [5, op. cit.], and $H^2(X, I^n/I^{n+1})$ is a p-torsion group (or torsion-free, if $p = 0$). Thus the map $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ is an isomorphism on ℓ-torsion. this completes the proof of Lemma 2.

We again assume that X is a weighted complete intersection surface with only rational double points as singularities.

LEMMA 3: $\mathrm{Pic}(X)$ *is p-torsion free.*

PROOF: (Compare [SGA7], Exp. XI, Th. 1.8.) Suppose $\mathcal{L}$ is a nontrivial line bundle on X which is killed by p. We construct a global Kahler

1-form on X, using the d log map. The best description of this map for our purposes is found in [4], p. 220. Let $\{f_{ij}\}$ be a 1-cocycle representing the class of $\mathscr{L}$ in $H^1(X, \mathcal{O}_X^*)$, then $\{f_{ij}^p\}$ is a coboundary, so we may write $f_{ij}^p = g_i/g_j$, $g_i \in H^0(U_i, \mathcal{O}_X^*)$. Then dg_i/g_i is a global section of the sheaf of Kahler differentials Ω_X^1 on X. Since d log is an injective map from ${}_p\mathrm{Pic}(Y)$ to $H^0(Y, \Omega_Y^1)$ (${}_p\mathrm{Pic}(Y)$ denotes the kernel of multiplication by p on $\mathrm{Pic}(Y)$), and since the map is functorial, we see that d log: ${}_p\mathrm{Pic}(X) \to H^0(X, \Omega_X^1)$ is injective also.

Therefore it is enough to show that $H^0(X, \Omega_X^1) = 0$. For this, we use the exact sequence of locally free sheaves on P

$$0 \to \mathcal{O}_P \to \bigoplus_{i=0}^{n} \mathcal{O}_P(e_i) \to T_p \to 0$$

[9, Remark 2.4], where P is the ambient weak projective space of dimension n in which X is a weighted complete intersection, $(e_0, \ldots, e_n)$ are the weights of the variables, and T_P denotes the tangent bundle of P. Dualizing, and restricting to X, we get

$$0 \to \Omega_P^1|_X \to \oplus_{i=0}^{n} \mathcal{O}_X(-e_i) \to \mathcal{O}_X \to 0.$$

From this, and [9, Remark 2.2 and Prop. 3.3], it is easy to see that

$$H^0(X, \Omega_X^1|_P) = 0.$$

Next, we use the exact sequence

$$0 \to \bigoplus_{j=1}^{s} \mathcal{O}_X(-f_j) \to \Omega_p^1|_X \to \Omega_X^1 \to 0.$$

where f_j, $1 \leqslant j \leqslant s = n-2$, are the degrees of the defining equations of X as a weighted complete intersection in P. Taking cohomology, we find an exact sequence

$$H^0(X, \Omega_P^1|_X) \to H^0(X, \Omega_X^1) \to \bigoplus_{j=1}^{s} H^1(X, \mathcal{O}_X(-f_j)).$$

We know from above that $H^0(X, \Omega_P^1|_X) = 0$, and $H^1(X, \mathcal{O}_X(-f_j)) = 0$ by [9, Prop 3.3]. Therefore $H^0(X, \Omega_X^1) = 0$ and the proof of Lemma 3 is complete.

Since $\mathrm{Pic}(X)_{\mathrm{tors}} \simeq \mathrm{Pic}(Y)_{\mathrm{tors}}$, and these groups are finite, Lemmas 2 and 3 imply the theorem.

REMARK: The motivating examples to which our theorem applies are the generic Zariski surfaces introduced by P. Blass in [2] and [3]. Blass uses the phrase "generic Zariski surface" in two different senses in these two papers, but in both cases, it refers to the nonsingular model of a weighted hypersurface with only rational double points, to which our theorem applies. Incidentally, it is not known if $H^0(Y, \Omega^1_Y) = 0$ where Y is a generic Zariski surface.

2. Raynaud surfaces with torsion-free crystalline cohomology

We use freely the results and notations of [6]. Let X be a quasi-elliptic Raynaud surface over a algebraically closed field of characteristic 3 (or, in the notation of [7], a generalized Raynaud surface of type (3,1,d)). Then there is a map $f\colon X \to C$, where C is a smooth curve, and all fibres are curves of arithmetic genus 1 with a single cusp. Let $\mathscr{L} = R^1 f_* \mathscr{O}_X$, a locally free shear of rank one on C. We know that $\mathscr{L}^6 \simeq K_C$.

PROPOSITION: *Let X be a Raynaud surface over a curve C of genus $g > 1$. Then $h^0(Z^1) = h^0(K_C) + h^0(\mathscr{L}^3)$, where Z^1 is the sheaf of closed 1-forms on X, and $h^0(\Omega^1_X) = h^0(Z^1) + \dim \ker g : H^0(C, \mathscr{L}^5) \to H^1(C, \mathscr{L}^3)$.*

PROOF: This is Theorem 4.5 of [6]. The description of the map g is not needed in the present paper.

THEOREM: *Suppose $f\colon X \to C$ is a Raynaud surface, $g(C) > 1$, $\mathscr{L} = R^1 f_* \mathscr{O}_X$. If $H^0(C, \mathscr{L}^3) = 0$, then the crystalline cohomology of X is torsion-free.*

PROOF: By Serre duality, $H^1(C, \mathscr{L}^3) = 0$. Then the dimension of the image of $d\colon H^0(\Omega^1_X) \to H^0(\Omega^2_X)$ is $h^0(\mathscr{L}^5)$. By Serre duality, $h^0(\mathscr{L}^5) = h^1(\mathscr{L})$, and by the Leray spectral sequence $h^1(C, \mathscr{L}) = h^2(X, \mathscr{O}_X)$. Again by Serre duality, $h^2(X, \mathscr{O}_X) = h^0(X, \Omega^2)$, so d is surjective. Moreover, by the above proposition, the kernel of d (which is $H^0(X, Z^1)$) consists of 1-forms pulled up from the base so $\dim \ker d = g$.

From the Leray spectral sequence, we get an exact sequence

$$0 \to H^1(C, \mathscr{O}_C) \to H^1(X, \mathscr{O}_X) \to H^0(C, \mathscr{L}) \to 0.$$

Since $H^0(C, \mathscr{L}^3) = 0$, $H^0(C, \mathscr{L}) = 0$ also. Hence, $H^1(X, \mathscr{O}_X) \simeq H^1(C, \mathscr{O}_C)$ so all of $H^1(X, \mathscr{O}_X)$ lives forever in the Hodge-de Rham spectral sequence. So we find that $h^1_{DR}(X) = 2g$. Since $\mathrm{Jac}(C) = \mathrm{Alb}(X)$ (the fibres of f are rational curves), we see that $h^1_{DR}(X) = B_1$, where B_1 is the first Betti number of X. It is now easy to see that $H^i_{\mathrm{cris}}(X/W)$ is torsion-free, using the universal coefficient theorem and Poincare duality for crystalline cohomology. In fact, one can show (using the methods of [8]) that $\mathrm{d}: H^2(W\mathscr{O}_X) \to H^2(W\Omega^1_X)$ is injective, hence $H^2_{\mathrm{cris}}(X/W)$ is isomorphic to $W^2(-1)$ as an F-crystal.

Notice also that if X is as in the theorem, then the Hodge-de Rham spectral sequence is non-degenerate, since $h^0(X, \Omega^2) = h^1(C, \mathscr{L})$ and since $h^0(\mathscr{L}) = 0$, the Riemann-Roch theorem implies $h^1(\mathscr{L}) = 2(g-1)/3 > 0$.

We now want to exhibit a surface satisfying the hypothesis of the theorem. We know that if $(C, \mathscr{L})$ is a pair consisting of a smooth complete curve C, and a line bundle $\mathscr{L}$ on C, then there is a Raynaud surface X together with a map $f: X \to C$ such that $R^1 f_* \mathcal{O}_X = \mathscr{L}$ if and only if there is a nowhere vanishing section dt of $\Omega_C^1 \otimes \mathscr{L}^{-6}$ killed by the Cartier operator $C: \Omega_C^1 \otimes \mathscr{L}^{-6} \to \Omega_C^1 \otimes \mathscr{L}^{-2}$. We say that the triple $(C, \mathscr{L}, \mathrm{d}t)$ is a Tango curve, or, more precisely, a curve with Tango structure. (see [6], Section 1.)

LEMMA: *If $(C, \mathscr{L}, \mathrm{d}t)$ is a curve with tango structure, then $(C, \mathscr{L} \otimes T, \mathrm{d}t)$ is also a curve with Tango structure where T is a line bundle of order* 2 *in* Pic(C).

PROOF: Obvious. □

Now observe that since $\mathscr{L}^6 \simeq K_C$, $\mathscr{L}^3$ is a theta characteristic in the sense of Mumford [10]. Replacing $\mathscr{L}$ by $\mathscr{L} \otimes T$, and letting T run through all elements of Pic(C) of order 2, we see that all theta characteristics of C are of the form $\mathscr{L}^3$, where $(C, \mathscr{L}, \mathrm{d}t)$ is a Tango curve. Therefore, to get a Raynaud surface $f: X \to C$ such that $H^0(C, \mathscr{L}^3) = 0$, it is enough to find a Tango curve C and a theta characteristic $\mathscr{M}$ on C such that $H^0(C, \mathscr{M}) = 0$. However, any hyperelliptic Tango curve will do for this. (See [10], p. 191. Take $\mathscr{M} = b_s^{(-1)}$.) Examples of hyperelliptic Tango curves are given in [6], p. 481. An interesting open problem is to find an *ordinary* Tango curve satisfying the hypothesis of the theorem. (If $(C, \mathscr{L}, \mathrm{d}t)$ is a curve with Tango structure such that $H^0(C, \mathscr{L}) \neq 0$, then C cannot be ordinary, since it gives rise to a *holomorphic* differential on C killed by the Cartier operator. Thus, the examples in [6] are not ordinary.)

Notice that the observation that $\mathscr{L}^3$ is a theta characteristic implies that the moduli space of Raynaud surfaces X (together with the map $f: X \to C$) in characteristic three is disconnected, since a Raynaud surface such that $\mathscr{L}^3$ is an even theta characteristic cannot be deformed into one such that $\mathscr{L}^3$ is an odd theta characteristic by [10, Section 1].

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