COMPOSITIO MATHEMATICA

MARC N. LEVINE
V. SRINIVAS

Zero cycles on certain singular elliptic surfaces

Compositio Mathematica, tome 52, nº 2 (1984), p. 179-196
<http://www.numdam.org/item?id=CM_1984__52_2_179_0>

Compositio Mathematica 52 (1984) 179–196

ZERO CYCLES ON CERTAIN SINGULAR ELLIPTIC SURFACES

Marc N. Levine and V. Srinivas

If X is a quasi-projective surface over an algebraically closed field, we let $A_0(X)$ denote the subgroup of the Grothendieck group $K_0(X)$ generated by sheaves with zero dimensional support contained in the smooth locus of X. When X is affine, the cancellation theorem of Murthy and Swan [13] shows that a projective module on X is determined by its class in $K_0(X)$; in particular, if $A_0(X)=0$, then every projective is a direct sum of rank one projectives. The vanishing of $A_0(X)$ thus has important consequences for questions of embedding dimension, complete intersections, set-theoretic complete intersections, etc.

One method of analyzing $A_0(X)$ is via resolution of singularities. For instance, suppose X is normal, and let $f: Y \to X$ be a desingularization of X. The map f induces a surjection $f^*: A_0(X) \to A_0(Y)$, and, conjecturally, the kernel of f^* is essentially determined by the formal neighborhood of the exceptional divisor in Y. In particular, if X has only rational singularities, then f^* should be an isomorphism. At present, this conjecture has only been proved in some special cases, so we must resort to other methods.

One technique for studying $A_0(X)$ for a singular surface X is to exhibit a pencil of rational curves whose general member is contained entirely in the smooth locus of X. For instance if X is an irrational (birationally) ruled surface with an isolated rational singularity then the ruling on X has this property. If X has a unique singular point P, and $U = X$-$[P]$, we define the logarithmic Kodaira dimension of P on X to be the logarithmic Kodaira dimension of the non-complete variety U in the sense of Iitaka [6]. It turns out that this is really an invariant of the local ring $\mathcal{O}_{X,P}$. We denote the logarithmic Kodaira dimension by $\bar{\kappa}$. By results of Miyanishi, Sugie, Fujita and others (see [3,9,16] for example) one can find a rational pencil on X whose general member is contained in U iff $\bar{\kappa} = -\infty$. M.P. Murthy and N. Mohan Kumar ([10,18]) attempt to classify the algebraic local rings within given analytic isomorphism classes which correspond to the local ring of a rational singular point on a rational surface. They show that the Chow group $A_0(X)=0$ for a rational surface X with a rational double point of type A_n ($n \neq 7,8$) or $D_n (n \neq 8)$, since in these cases $\bar{\kappa} = -\infty$ (in fact there is only 1 isomorphism class of algebraic local ring within the given analytic isomorphism class which can occur on a rational surface). They also study local rings on rational surfaces analytically isomorphic to $k[[x^n, x^{n-1}y, \dots, xy^{n-1}, y^n]] \subseteq k[[x, y]]$, and classify all those with $\bar{\kappa} < 2$. In case $\bar{\kappa} = 0$ or 1, they prove the existence of a pencil of elliptic curves whose general members are contained in the smooth locus. Bloch (unpublished) has shown using K-theory that $A_0(X) = 0$ for a rational surface with a rational double point, without classifying the local rings themselves.

In this paper, we consider normal projective surfaces X which carry a certain type of elliptic pencil (like the ones studied by Murthy and Mohan Kumar). We use linear equivalence on the general members of the pencil to study the Chow group $A_0(X)$. This is inspired by the paper [2] of Bloch, Kas and Lieberman, where the authors show that the Chow group of an elliptic surface with $p_g = 0$ is trivial. Their idea is to consider the Jacobian fibration associated to the given fibration. A choice of a multisection of the given fibration determines a generically finite morphism from the given surface to the Jacobian fibration. Using Abel's theorem on the fibres and the divisibility of the "transcendental" part of the Chow group, they reduce the problem to the case of Jacobians. They then use geometric arguments to prove the result in this special case.

In our situation, it turns out that the Jacobian fibration associated to our special type of elliptic pencil is a *smooth* surface even though the given surface is singular. Further, we are able to construct an Abel *morphism* from the singular surface to the Jacobian that on general (smooth) fibres is "multiplication by n". Using Abel's theorem on such fibres and the contravariant functoriality of A_0 (valid even in the singular case), we deduce our Main Theorem.

In contrast we remark that if X is a normal projective surface over $\mathbf{C}$ with $H^2(X, \mathcal{O}_X) \neq 0$, then $A_0(X)$ is infinite dimensional, in the sense that there do not exist curves $C_1, C_2, \dots, C_n$ contained in the smooth locus such that 0-cycles supported on the union of the C_i generate $A_0(X)$. Thus on a rational, Enriques or Godeaux surface with a non-rational singularity, the Chow group is infinite dimensional (see [17] for these facts).

The paper is organised as follows. In §1 we prove Proposition 6, which is the main technical tool giving the existence and smoothness of the relative Jacobian, and the existence of an Abel morphism with the desired properties. In §2 we use this to prove our Main Theorem, stated below.

MAIN THEOREM: *Let Y be a smooth, projective surface over an algebraically closed field of characteristic $\neq 2$. Let $f: Y \to C$ be a morphism to a smooth curve whose general fibres are smooth elliptic curves. Let $P_1, P_2, \dots, P_n$ be points of C such that $f^{-1}(P_i)$ is a reduced irreducible rational curve with a node Q_i. Let $X \to C$ be the singular (projective) surface obtained by blowing up Y at the Q_i and blowing down the strict transforms* *of the* $f^{-1}(P_t)$. *Then the map* $A_0(X) \to A_0(Y)$ *is an isomorphism (we identify* $A_0(Y)$ *with* A_0 *of the blow up of the* P_t*).*

The reader will note that the Main Theorem provides many examples of singular surfaces with a rational singularity such that the smooth surface obtained by blowing up the singularities has infinite dimensional Chow group. As far as the authors are aware, in all cases where the effect of a rational singularity on the Chow group of a surface has been computed in the literature, it is assumed that the Chow group of the corresponding smooth surface is finite dimensional.

In §3 we discuss some examples of singular surfaces to which our main Theorem applies. We consider many of the singularities with $\bar{\kappa} = 0$, 1 of Murthy and Mohan Kumar which come from the classical Halphen pencils on the projective plane. We construct an Enriques surface with the given type of singularity, which thus has trivial Chow group. Finally, we discuss an amusing example of an affine elliptic surface X with 2 singular points, P, Q such that $A_0(X)$ is infinite dimensional but the surfaces obtained by resolving any *one* of the singularities has $A_0 = 0$: in particular it is of interest to prove our result for many singularities at once. since in general one cannot assess the contributions of the singularities one at a time.

The authors would like to thank Spencer Bloch for stimulating discussions on this work, and Rick Miranda for showing us how to construct the Halphen pencils. We would also like to thank the referee for his helpful suggestions.

1. Construction of the Jacobian and the Abel morphism

We fix regular k-schemes B and X, a proper of finite type $f: X \to B$, and closed points $b_1, \dots, b_s$ in B such that:

(i) $f^{-1}(B - \{b_1, \dots, b_s\})$ is smooth over $B - \{b_1, \dots, b_s\}$ and the fibres are smooth curves of genus 1, and
(ii) the singular fibre $C_i = f^{-1}(b_i)$ is a reduced irreducible rational curve with a single ordinary node $p_i \in C_i$, for $i = 1, \dots, s$.

Let $u: Y \to X$ be the blow up of X at all of the p_i, and let $E = \cup E_i$ be the exceptional locus. Let F_i be the proper transform of C_i. Then each F_i is a smooth rational curve with self-intersection $F_i^2 = -4$; we blow down the union of the F_i to yield a morphism $Y \xrightarrow{v} Z$ to a normal surface Z with singular points $q_i = v(F_i)$. Both Y, Z are B-schemes (via morphisms $g: Z \to B$ and $h: Y \to B$). We note that the fibres $D_i = Z_{b_i} = v(E_i)$ are not reduced: more specifically:

(iii) $(D_i)_{\text{red}}$ is a rational curve with an ordinary node at q_i,
(iv) as a Weil divisor, $D_i = 2\,(D_i)_{\text{red}}$.

Our main purpose in this section is to analyse the Jacobian fibration associated to $Z \to B$ and to construct an "Abel morphism" from Z to its family of Jacobians. We first show that the relative Picard group of Z over B is represented by an algebraic space over B by appealing to the criterion of Artin [1].

LEMMA 1: *The morphism $g: Z \to B$ is "cohomologically flat in dimension 0" i.e. for all B-schemes $T \to B$ we have $(g_T)_*\mathcal{O}_{Z_T} = \mathcal{O}_T$.*

PROOF: As the assertion is local on B, we may assume that $B = \operatorname{Spec} R$ for some discrete valuation ring R with quotient field K and residue field $k = k(b)$ where $b \in B$ is the closed point. If the fiber Z_b is smooth the result follows easily from the fact that smooth morphisms are preserved under base change, and Zariski's Main theorem (see [4]). We may therefore assume that $b = b_i$ for some i. To simplify notation we drop the subscript i in the remainder of the proof of the lemma.

We have $R^1u_*\mathcal{O}_Y = 0$ (from the formal function theorem, for example [4]). The spectral sequence $R^iu_*(R^jf_*\mathcal{O}_Y) \Rightarrow R^{i+j}h_*\mathcal{O}_Y$ yields the exact sequence of terms in low degree

$$0 \to R^1f_*(u_*\mathcal{O}_Y) \to R^1h_*\mathcal{O}_Y \to f_*R^1u_*\mathcal{O}_Y \to 0,$$

hence $R^1_{\mathrm{h}}{}_*(\mathcal{O}_Y) = R^1f_*(u_*\mathcal{O}_Y) = R^1f_*\mathcal{O}_X$.

Since q is a rational singularity, we also have $R^1v_*(\mathcal{O}_Y) = 0$. Hence a similar spectral sequence argument gives $R^1g_*\mathcal{O}_Z = R^1_h{}_*\mathcal{O}_Y = R^1f_*\mathcal{O}_X$. Also, since $H^0(C, \mathcal{O}_C) = k = H^1(C, \mathcal{O}_C)$, $H^0(X_K, \mathcal{O}_{X_K}) = H^1(X_K, \mathcal{O}_{X_K}) = K$, we have $R^1f_*\mathcal{O}_X = R$, hence $R^1g_*\mathcal{O}_Z = R$.

By Mumford [11], § 5, there is a complex of free R-modules

$$K.: \qquad 0 \to K_0 \xrightarrow{\alpha} K_1 \to 0$$

such that for any R-scheme $T \xrightarrow{s} \operatorname{Spec}(R)$, the cohomology sheaf $\mathcal{H}^i(s^*(K.))$ is isomorphic to the sheaf $R^ig_{T*}(\mathcal{O}_{Z_T})$. Applying this to the identity map on R, we find that $H^1(K.) = R^1g_*(\mathcal{O}_Z) = R$, which allows us to split K_1 as

$$K_1 = R^1g_*(\mathcal{O}_Z) \oplus \alpha(K_0).$$

This gives a splitting of K_0 as

$$K_0 = ker(\alpha) \oplus \alpha(K_0).$$

Also, $\ker \alpha = H^0(K.) = g_*(\mathcal{O}_Z) = R$. Hence we have, for any R-scheme $T \xrightarrow{s} \operatorname{Spec}(R)$,

$$g_{T*}(\mathcal{O}_{Z_T}) = H^0(s^*(K.)) = s^*(\ker \alpha) = \mathcal{O}_T. \qquad \text{Q.E.D.}$$

The representability theorem of Artin [1] immediately yields

COROLLARY 2: *The relative Picard group* Pic (Z/B) *is represented by an algebraic space* α: $\operatorname{Pic}_B(Z) \to B$ *over* B.

LEMMA 3: *The restriction map* $H^1(D_i, \mathcal{O}^*_{D_i}) \to H^1((D_i)_{\text{red}}, \mathcal{O}^*_{(D_i)_{\text{red}}})$ *is an isomorphism for each* $i = 1, \ldots, s$.

PROOF: We again drop the i's to simplify notation. We have

$$0 \to \mathcal{I} \to \mathcal{O}_D \to \mathcal{O}_{D_{\text{red}}} \to 0,$$

where $\mathcal{I}$ is the ideal sheaf of D_{red} on D. From the proof of Lemma 1, we know that $R^1 g_* \mathcal{O}_Z$ and $g_* \mathcal{O}_Z$ are invertible sheaves on B. Thus the Base Change theorem (see [9], for example) yields

$$H^1(D, \mathcal{O}_D) = k = H^1(D_{\text{red}}, \mathcal{O}_{D_{\text{red}}}) \quad \text{and} \quad H^0(D, \mathcal{O}_D) = H^0(D_{\text{red}}, \mathcal{O}_{D_{\text{red}}}).$$

Hence all cohomology groups of $\mathcal{I}$ vanish. The lemma follows at once from the cohomology sequence associated to

$$0 \to \mathcal{I} \to \mathcal{O}^*_D \to \mathcal{O}^*_{D_{red}} \to 0. \qquad \text{Q.E.D.}$$

PROPOSITION 4: $\operatorname{Pic}_B(Z)$ *is a separated algebraic space.*

PROOF: Since B is separated the assertion is local on B; we may replace B with Spec R, $R = \mathcal{O}_{B,b}$. If $Z_{k(b)}$ is smooth, the result is well known and easy to prove, so we may assume $b = b_i$ for some i. Once more we suppress the i. Let $S \leftarrow R$ be a D.V.R. with quotient field K_0, residue field k_0. By the discrete valuative criterion for separatedness, we need only verify

(*) if L is an invertible sheaf on Z_S that restricts to the trivial sheaf on Z_{K_0}, then L is trivial restricted to Z_k.

We first consider the case when Spec S maps to the closed point of Spec R i.e. $R \not\subset S$. Then

$$Z_S = z_{k \times k} \operatorname{Spec} S = D_{\times k} \operatorname{Spec} S.$$

By Lemma 3 we need only show that L restricted to $D_{\text{red}} \times \operatorname{Spec}(s)$ is trivial, but this follows from the well known isomorphism $\mathrm{Pic}_k(D_{\mathrm{red}}) = \mathbb{G}_m \times \mathbb{Z}$ (as schemes).

Next suppose that S dominates R. The sheaf L gives rise to an R-morphism $\tau: \mathrm{Spec}(S) \to \mathrm{Pic}_R(Z)$ which generically factors through the identity section I of $\mathrm{Pic}_R(Z)$. Suppose the closure $\bar{I}$ of I in $\mathrm{Pic}_R(Z)$ is separated over $\mathrm{Spec}(R)$. Then $\bar{I} = I$, and hence τ factors through I. In particular $\tau(\mathrm{Spec}(k_0)) = id$, and hence L restricted to Z_{k_0} is trivial, as desired. Thus we need only show that $\bar{I}$ is separated over $\mathrm{Spec}(R)$. If not, there is a section $\sigma: \mathrm{Spec}(R) \to \mathrm{Pic}_R(Z)$ such that $\sigma(\mathrm{Spec}(K)) = id$, $\sigma(\mathrm{Spec}(k)) \neq id$ (K is the quotient field of R, and k the residue field). Since $H^2(\mathrm{Spec}(R), \mathbb{G}_m) = 0$, such a section gives rise to an invertible sheaf M on Z with $M \otimes K$ trivial, and $M \otimes k$ not trivial. In other words, we may assume $S = R$.

We claim that the Weil divisor D_{red} is not a Cartier divisor on Z. For if D_{red} were a Cartier divisor, the pullback $v^*(D_{\mathrm{red}})$ would be a Cartier divisor on Y such that $v^*(D_{\mathrm{red}}) \cdot F = 0$ (here $F = v^{-1}(q)$). Furthermore we could express $v^*(D_{\mathrm{red}})$ as an integral linear combination $v^*(D_{\mathrm{red}}) = E + nF$ where $E = u^{-1}(p) = v^{-1}[\mathrm{D}_{\mathrm{red}}]$, the strict transform of D_{red}. As $E \cdot F = 2$ and $F^2 = -4$, this is impossible.

We now return to the sheaf L on Z. Since $L \otimes K$ is trivial there is a K-rational, nowhere vanishing section $s_K \in H^0(Z \otimes K, L \otimes K)$. Multiplying s_K by a suitable power of a uniformizing parameter t of R, we may assume that s_K extends to a section s_R of L. Thus the divisor of s_R is of the form $(s_R) = n.(D_{\mathrm{red}})$. Choosing the proper power of t we may further assume $n = 1$ or 0. But $n = 1$ is impossible since (s_R) is Cartier. Hence $n = 0$. As L is invertible and Z is normal, L is trivial. Q.E.D.

Let $\mathrm{Jac}_B(Z)$ denote the connected component of the identity in $\mathrm{Pic}_B(Z)$. We first make a local analysis of $\mathrm{Jac}_B(Z)$. Let $K = k(B)$. Since $\mathrm{Jac}_K(Z_K)$ is the Jacobian of the elliptic curve Z_K, $\mathrm{Jac}_K(Z_K)$ is complete, and hence $\mathrm{Jac}_B(Z)$ is the closure of $\mathrm{Jac}_K(Z_K)$ in $\mathrm{Pic}_B(Z)$. Similarly, if $b \in B$, and R is the completion of the local ring of b in B, then $\mathrm{Jac}_R(Z_R)$ is the closure in $\mathrm{Pic}_R(Z_R)$ of $\mathrm{Jac}_{\hat{K}}(Z_{\hat{K}})$, where $\hat{K}$ is the quotient field of R. Furthermore, by functoriality, we have

$$\mathrm{Pic}_B(Z) \otimes R = \mathrm{Pic}_R(Z_R); \qquad \mathrm{Pic}_B(Z) \otimes \hat{K} = \mathrm{Pic}_{\hat{K}}(Z_{\hat{K}}),$$

and also $\mathrm{Jac}_{\hat{K}}(Z_{\hat{K}}) = \mathrm{Jac}_K(Z_K) \otimes \hat{K}$. Thus $\mathrm{Jac}_B(Z) \otimes R = \mathrm{Jac}_R(Z_R)$ (i.e. the connected component of the identity remains connected under completion). Hence for our local analysis we may assume B is the spectrum of a complete local ring. As the local analysis of $\mathrm{Jac}_B(Z)$ is well understood when Z is smooth over B, we may assume that $B = \mathrm{Spec}\, \mathcal{O}_{B,b}$ where $b = b_i$ for some i. We will denote $\mathrm{Jac}_B(Z)$ by $\mathrm{Jac}(Z)$, C_i by C, etc.

In the sequel we will be constructing various surfaces as quotients of group schemes by finite group actions. In this vein we will require the following lemma.

LEMMA 5. *Let X be a normal variety, with a proper map to a variety Y, and let X^0 be an open subset of X such that $X - X^0$ is of dimension zero. Suppose $T^0: X^0 \to X^0$ is an automorphism over Y. Then T^0 extends to an automorphism T of X over Y.*

PROOF: Let T be the extension of T^0 to a rational map of X over Y. If x is in X^0, then the total transform $T(x)$ is just $T^0(x)$, hence $T^{-1}(X - X^0)$ is contained in the finite set $X - X^0$. Thus T^{-1} is a morphism by Zariski's Main Theorem. Similarly, T is a morphism. Q.E.D.

We will use Lemma 5 in the following situation: X will be a normal surface proper over a smooth curve Y via $f: X \to Y$, we will have an open subset X^0 as above, together with a section $\alpha: Y \to X^0$ to f which makes X^0 into a group scheme over Y with identity section α. If $\beta: Y \to X^0$ is another section, then translation by β, $T^0_\beta: X^0 \to X^0$, extends by the above to an automorphism $T_\beta: X \to X$.

PROPOSITION 6: *The closed fibre* $\mathrm{Jac}(Z) \otimes k = \mathbb{G}_m$. *Further* $\mathrm{Jac}(Z)$ *has a smooth completion* $\mathrm{Jac}(Z)^*$ *such that*:

(*i*) $\mathrm{Jac}(Z)^* - \mathrm{Jac}(Z)$ *is a single point* $a \in \mathrm{Jac}(Z)^* \otimes k$;

(*ii*) $\mathrm{Jac}(Z)^* \otimes K$ *is a smooth elliptic curve*; $\mathrm{Jac}(Z)^* \otimes k$ *is a rational curve with a single ordinary node at* a;

(*iii*) $\mathrm{Jac}(Z)^*$ *is a proper B-scheme* (*this is what we mean by a completion*).

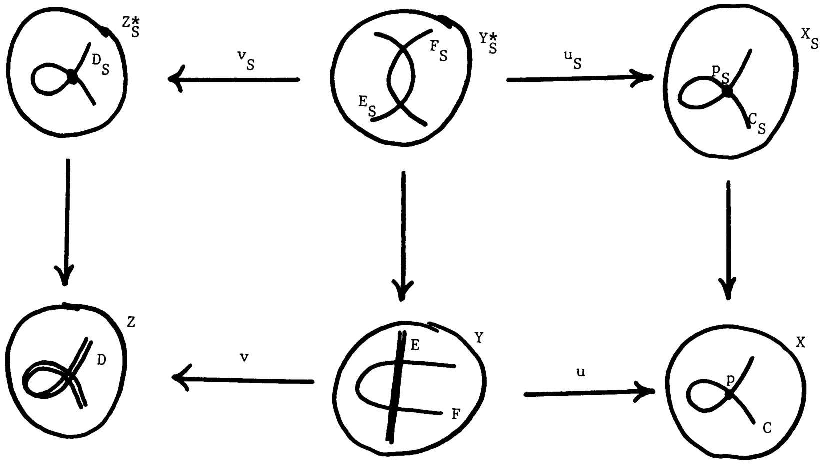
Further, let $S = R[u]/(u^2 - t)$ where t is the local parameter in R. Let $g_S^*: Z_S^* \to \mathrm{Spec}\, S$ be the normalisation of the fibre product $Z_S = Z \times_R S$. Then there is a section $s: \mathrm{Spec}\, S \to Z_S^*$ of the map g_S^* with image in the regular locus of Z_S^*. Furthermore, for any such section there is an S-morphism $w_s: \mathrm{Jac}(Z)^* \otimes S \to Z_S^*$ which is an isomorphism satisfying $w_s(0_S) = s$ (where 0_S is the 0-section of $\mathrm{Jac}(Z) \otimes S$), and such that $w_s \otimes K_S: \mathrm{Jac}(Z)^* \otimes K_S \to Z_S$ is the canonical isomorphism of Z_{K_S} with its Jacobian (determined by the given K_S-point as base point). Finally w_s is uniquely determined by s.

PROOF: Let Y_S^* denote the normalisation of Y_S. The commutative diagram

$$\begin{array}{ccccccccc}
j^{-1}(D)_{\mathrm{red}} = D_S & \subset & Z_S^* & \xleftarrow{v_S} & Y_S^* & \xrightarrow{u_S} & X_S & \supset & C_S = h^{-1}(C) \\
\downarrow & & \downarrow j & & i \downarrow & & h \downarrow & & \downarrow \\
D & \subset & Z & \xleftarrow{v} & Y & \xrightarrow{u} & X & \supset & C
\end{array}$$

identifies Y_S^* with the blow up of X_S at the ordinary double point

$p_S = j^{-1}(p)$, and identifies Z_S^* with the blow down of Y_S^* along $F_S = i^{-1}(F) = u_S^{-1}(C_S)$. The picture is as follows:



Since X-$\{p\}$ is smooth over R, $X - \{p\}$ admits a section 0_X: Spec $R \to X - \{p\}$ making $x - \{p\}$ into a commutative group scheme over R. The fibre $Y_S^* \otimes k$ has 2 singular points p_1, p_2; the induced 0-section of $Y_S^* - \{p_1, p_2\} = Y_S^0$ makes Y_S^0 into a commutative group scheme over S.

We claim that the subgroup of 2-torsion sections $G_2 \subseteq Y_S^0$ is finite and etale over S. In fact G_2 is the kernel of the morphism $Y_S^0 \to Y_S^0$ given by multiplication by 2, so by Hensel's lemma it is enough to show that $Y_S^0 \otimes k$ has 4 distinct 2-torsion points. From Kodaira [7] $Y_S^0 \otimes k = \mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$ which has the required 2-torsion subgroup. Let $\{0, \alpha_1, \alpha_2, \alpha_3\}$ be the corresponding 4 2-torsion sections of Y_S^0. By construction, 0 passes through F_S; label the α_i so that α_1 meets F_S (and is disjoint from E_S) and α_2, α_3 meet E_S (and avoid F_S).

We now define an S-isomorphism $c: Z_S^* \to X_S$ as follows. Let c be the composite $u_S \circ T_{\alpha_2} \circ v_S^{-1}$, where T_s^0 denotes translation by s for any section s: Spec $S \to Y_S^0$, and T_s is the corresponding automorphism of Y_S^* given by Lemma 5. As F_S is the exceptional divisor of v_S, and $T_{\alpha_2}(F_S) = E_S$ which is the exceptional divisor of u_S, the composition c (considered as a birational transformation over S) has neither fundamental locus on Z_S^* nor exceptional locus on X_S. Hence by Zariski's Main theorem and the normality of Z_S^* and X_S, we see that c is an isomorphism.

Let ?: Spec $S \to$ Spec S be the involution of S over R. Lift ? to involutions $$: $Z_S \to Z_S$ and $: $X_S \to X_S$; this is done by writing $Z_S = Z \times_R S$, and letting $$ be the identity on the first factor (and similarly for $ on X_S). Clearly $$ lifts to an involution on Z_S^* which we denote by the same symbol. Thus $Z = Z_S^*/\langle\$\$\rangle$ and $X = X_S/\langle\$\rangle$. The isomorphism c

induces a second involution $\$+: X_S \to X_S$ given by $\$+ = c \circ \$\$ \circ c^{-1}$, and we have $Z = X_S/\langle \$+ \rangle$.

We now give an explicit description of $\$+$. The fibre $X_S \otimes k$ has only the two 2-torsion points corresponding to the sections $0_{X_S} \otimes k$ and $u_S(\alpha_1)$. Hence $u_S(\alpha_2)$ and $u_S(\alpha_3)$ must specialise to the singular point on the fibre. Since X is smooth over k, $X \to \operatorname{Spec} R$ has no sections through the singular point of the closed fibre. Thus $u_S(\alpha_2)$ and $u_S(\alpha_3)$ cannot be rational over K (quotient field of R) and are thus conjugate under the involution $\$$. If we identify X_S and Z_S^* via the birational map $v_S u_S^{-1}$, we have $c = c^{-1} = T_{\alpha_2}$. Hence $\$\$ \circ c^{-1} = T_{\alpha_3}$ and $\$+ = c \circ \$\$ \circ c^{-1} = T_{\alpha_2+\alpha_3} = T_{\alpha_1}$. From this description of $\$+$ we see that the action of $\$+$ on $\operatorname{Jac}_S(X_S) = \operatorname{Jac}(X) \times_R S$ is trivial. Thus

$$\operatorname{Jac}(Z) = \operatorname{Jac}_R(X_S/\langle \$+ \rangle) = \operatorname{Jac}_S(X_S)/\langle \$+ \rangle = \operatorname{Jac}(X).$$

As $\operatorname{Jac}(X) = X - \{p\}$, we may take $\operatorname{Jac}(Z)^*$ to be X, from which (i), (ii) are clear. We may choose the isomorphism $\operatorname{Jac}(X) = X - \{p\}$ to restrict to the canonical isomorphism of $\operatorname{Jac}_K(X_K)$ with X_K induced by the section $0_X \otimes K$. If we let $s: \operatorname{Spec} S \to Z_S^*$ be the section $u_S(\alpha_2)$ then $C \circ S$ is the zero section 0_X. Hence the composition $w_s = \beta_S \circ c: Z_S^* \to \operatorname{Jac}(Z)^* \otimes S$ (where $\beta: X \to \operatorname{Jac}(X)$ is the identification given by 0_X) satisfies all the requirements of the proposition. Finally, given some other section $s': \operatorname{Spec} S \to Z_S^*$ define $w_{s'}(x) = w_s(x - s')$. This satisfies all the requirements of the proposition. Q.E.D.

We now return to the general setting, B will be an arbitrary k-scheme of dimension 1. Let $d: \operatorname{Jac}_B(Z) \to B$ be the structure map. Let R_i denote the completion of the local ring of b_i in B. We let $\operatorname{Jac}_B(Z)^*$ denote the completion of $\operatorname{Jac}_B(Z)$ obtained by completing $\operatorname{Jac}_{R_i}(Z \otimes R_i) = \operatorname{Jac}_B(Z) \otimes R_i$ via Proposition 6, for each $i = 1, \ldots, s$.We take a finite Galois cover $r: B' \to B$ such that:

(i) over R_i, $B' \otimes_B R_i$ is a disjoint union of degree two ramified extensions of R_i, for each $i = 1, \ldots, s$.
(ii) the normalisation $Z_{B'}$ of $Z \times_B B'$ admits a section $s: B' \to Z_{B'}$ whose image is contained in the smooth locus of $Z_{B'}$.

Such a B' can always be arranged: for instance take B' to be the Galois closure of a generic hyperplane section of Z which intersects all the $(D_l)_{\text{red}}$ transversally. In section $s: B' \to Z_{B'}$ induces the canonical isomorphism over $k(B')$, $\beta_1: Z_{B'} \otimes k(B') \to \operatorname{Jac}_B(Z)^* \otimes k(B')$. Let $U = B' - r^{-1}\{p_1, \ldots p_s\}$. Then Z_U is smooth over U, with elliptic curves as fibres; hence β_1 extends to a U-isomorphism $\beta_U: Z_U \to \operatorname{Jac}_B(Z)^* \times_B U$. By proposition 6, β_U extends to a B'-isomorphism $\beta: Z_{B'} \to \operatorname{Jac}_B(Z)^* \times_B B'$.

Let G be the Galois group of B'/B. We define a G-action on $Z_{B'}$ by lifting to $Z_{B'}$ the action on $Z \times_B B'$ which is trivial on the first factor. Similarly define a G-action on $\mathrm{Jac}_B(Z)^* \times_B B'$. The isomorphism β determines a G-action on $\mathrm{Jac}_B(Z)^* \times_B B'$. Let α' denote the G-action described above on $Z_{B'}$ and let α denote the action on $\mathrm{Jac}_B(Z)^* x_B B'$. The new action α'' on $\mathrm{Jac}_B(Z)^* \times_B B'$ is the composite $\beta \circ \alpha' \circ \beta^{-1}$. If we let α_0 be the action of G on $\mathrm{Jac}_{B'}(\mathrm{Jac}_B(Z) \times_B B')$ induced by α'', then we have relations

$$Z = (\mathrm{Jac}_B(Z)^* \times_B B')/\alpha''; \qquad Z_{k(B)} = (\mathrm{Jac}_B(Z)^* \otimes_B k(B'))/\alpha'';$$

$$\mathrm{Jac}_B(Z) = (\mathrm{Jac}_{B'}(\mathrm{Jac}_B(Z)^* \times_B B'))/\alpha_0;$$

$$\mathrm{Jac}_{R(B)}(Z_{k(B)}) = (\mathrm{Jac}_{k(B')}(\mathrm{Jac}_B(Z)^* \times_B k(B')))/\alpha_0.$$

As the $k(B)$-rational 0_B in $\mathrm{Jac}_B(Z)^* \times_B k(B)$ gives a canonical identification of $\mathrm{Jac}_{k(B)}$ $(\mathrm{Jac}_B(Z)^* \times_B k(B))$ with $\mathrm{Jac}_B(Z)^* \times_B k(B)$, we have the isomorphism $\mathrm{Jac}_{k(B)}(Z_{k(B)}) = (\mathrm{Jac}_{k(B)}(Z)^* \times_{k(B)} k(B'))/\alpha$. Thus the induced action $\alpha_0 \otimes k(B')$ is the trivial action $\alpha \otimes k(B')$. From this we see that α_0 is the trivial action α, and that α'' has the form

$$\alpha''(g) = T_{y_g} \times_B \alpha,$$

where $g \to y_g$ is a representation of G on a group of torsion sections of $\mathrm{Jac}_B(Z) \times_B B'$.

We collect our results in the following proposition.

Proposition 7: *Let $f\colon X \to B$ be a family of curves satisfying (i), (ii) at the beginning of* §2., *and let $g\colon Z \to B$ be the associated singular family. Then there is a Galois cover $r\colon B' \to B$ with group G, and a representation of G, $\beta\colon G \to (\mathrm{Jac}_B(Z) \times_B B')_n$ of G into the n-torsion subgroup of the group of B'-sections of $\mathrm{Jac}_B(Z) \times_B B'$ such that $Z = (\mathrm{Jac}_B(Z)' \times_B B')/\beta \times \alpha$, where α is the standard representation of G on B'.*

Corollary 8: *Let $g\colon Z \to B$ as above. Then there is an integer n, and finite dominant B-morphisms $u\colon Z \to \mathrm{Jac}_B(Z)^*$, ${}^t u\colon \mathrm{Jac}_B(Z)^* \to Z$ such that $u \circ {}^t u = n^2 \cdot \mathrm{Id}_J$ (where Id_J is the identity map on $\mathrm{Jac}_B(Z)$) and if A is a 0-cycle supported on a smooth fibre $Z_B \otimes k(b)$, then the divisor $u^*(u_* A) - n^2 \cdot A$ is linearly equivalent to zero on the elliptic curve $Z \otimes k(b)$.*

Proof: Let $r\colon B' \to B$, n, G, β, α be as in Proposition 7. The map $n \cdot \mathrm{Id}_J\colon \mathrm{Jac}_B(Z)^* \times_B B' \to \mathrm{Jac}_B(Z)^* \times_B B'$ is invariant under the action $\beta \times \alpha$, hence descends to the desired maps u, ${}^t u$. Clearly $n \cdot \mathrm{Id}_J$ composed with itself is $n^2 \cdot \mathrm{Id}_J$; hence we deduce that $u \circ {}^t u$ is multiplication by n^2 since this holds after the base change $B' \to B$. To verify the other claim, we may replace $k(b)$ by its algebraic closure $\overline{k(b)}$ and identify $Z \otimes \overline{k(b)}$ with $\mathrm{Jac}_B(Z)^* \times_B \overline{k(b)}$ by the choice of a base point 0 on $Z_{\overline{k(b)}}$. Then u becomes just $n \cdot \mathrm{Id}_{J \otimes \overline{k(b)}}$. Let J_n be the n-torsion subgroup of $Z_{\overline{k(b)}}$, and let $\sim$ denote linear equivalence. We have

$$\begin{aligned} u^*(u_* A) - n^2 \cdot A &= \left(\sum_{a \in J_n} T_a(A) \right) - n^2 \cdot A \\ &= \sum_{a \in J_n} T_a(A) - A \\ &\sim \sum_{a \in J_n} \deg(A) \cdot ((a) - (0)) \\ &\sim \deg(A) \cdot \left(\left(\sum_{a \in J_n} a \right) - (0) \right) \\ &= 0. \end{aligned} \qquad \text{(Q.E.D).}$$

2. Proof of the Main Theorem

We return to the notation of the statement of the theorem. Thus X is a smooth projective surface over an algebraically closed field k and $f: X \to C$ is a morphism onto a smooth curve whose general fibres are smooth elliptic curves (and we exclude $\mathrm{char} \cdot k = 2$). We are also given points $P_1, \dots, P_n$ of C such that the fibre $f^{-1}(P_i)$ is a reduced irreducible rational curve with one node Q_i for each i. We construct the singular surface Z by blowing up the Q_i and blowing down the strict transforms of the nodal curves $f^{-1}(P_i)$. Let $g: Z \to C$ be the resulting morphism.

Let $B \subset C$ be an open subset containing all the P_i such that the fibres $f^{-1}(b)$ for $b \in B - \{P_1, \dots, P_n\}$ are smooth. Let $Z^0 = g^{-1}(B)$. Then by Corollary 8, there is a smooth proper B-scheme $h: J^0 \to B$, an integer d, and a dominant surjective morphism $u: Z^0 \to J^0$ (over $B0$ such that if $b \in B - \{P_1, \dots, P_n\}$, and A is a 0-cycle supported on $g^{-1}(b)$, then $u^*(u_* A) - d.A$ is linearly equivalent to 0 on the elliptic curve $g^{-1}(b)$. Since A_0 of a (possibly singular) surface is unchanged when we blow up a smooth point, we may assume that for some smooth surface J there is a diagram (where α, β are inclusions)

$$\begin{array}{ccc} \mathrm{J}^0 & \xrightarrow{\alpha} & \mathrm{J} \\ \downarrow & & \downarrow \\ \mathrm{B} & \xrightarrow{\beta} & \mathrm{C} \end{array}$$

and

we have a C-morphism $v: Z \to J$ extending u. Let $r: W \to Z$ be the blow up of all the singular points; the Main Theorem asserts that $r^*: A_0(Z) \to A_0(W)$ is an isomorphism. We denote the composite C-morphism $v \circ r$ by w.

Since W, J are smooth varieties we have a map $w_*: A_0(W) \to A_0(J)$ such that we have a commutative triangle

$$\begin{tikzcd}
A_0(W) \arrow[rr, "w_*"] & & A_0(J) \arrow[dl, "v^*"] \\
& A_0(Z) \arrow[ul, "r^*"] &
\end{tikzcd}$$

where all the triple composites $A_0(W) \xrightarrow{r^*v^*w_*} A_0(W)$, etc. are just multiplication by d. In particular ker r^* is d-torsion. Hence we are done by the following lemma.

LEMMA 11: *Let X be a normal surface over an algebraically closed field, and let $f: Y \to X$ be a resolution of singularities. Then the kernel of $f^*: A_0(X) \to A_0(Y)$ is divisible.*

PROOF. Let K denote $\ker f^*$. If nK, ${}_nA_0(X)$, ${}_nA_0(Y)$ are the respective n-torsion subgroups, we have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & {}_nK \arrow[r] \arrow[d] & {}_nA_0(X) \arrow[r] \arrow[d] & {}_nA_0(Y) \arrow[d] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & A_0(X) \arrow[r] \arrow[d] & A_0(Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & A_0(X) \arrow[r] \arrow[d] & A_0(Y) \arrow[r] & 0 \\
 & (K/nK) \arrow[r] \arrow[d] & 0 & & \\
 & 0 & & &
\end{tikzcd}$$

Thus we have to prove that ${}_nA_0(X) \twoheadrightarrow {}_nA_0(Y)$ for all n. We now appeal to a deep theorem of Roitman [15] which states that for any smooth variety Y, there is an isomorphism (for each n) ${}_nA_0(Y) \to {}_n\mathrm{Alb}(Y)$ where $\mathrm{Alb}(Y)$ is the Albanese variety of Y (in [15], Roitman proves this only when n is relatively prime to the characteristic; Milne [8] covers the case of p-torsion in characteristic $p > 0$). Thus it suffices to prove the following claim: if $C \subset X$ is a smooth hyperplane section contained in the smooth locus of X, the induced composite map from the Jacobian of C

$$J(C) \to A_0(X) \to A_0(Y) \to \mathrm{Alb}(Y)$$

is surjective on n-torsion subgroups for each n. We have surjections (ignoring twisting by roots of unity)

$$\mathrm{Hom}\big(H^1_{et}(C, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z}\big) \to {}_nJ(C)$$

$$\mathrm{Hom}\big(H^1_{et}(Y, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z}\big) \to {}_n\mathrm{Alb}(Y).$$

Hence it suffices to prove that we have an injection for each n

$$H^1_{et}(Y, \mathbb{Z}/n\mathbb{Z}) \to H^1_{et}(C, \mathbb{Z}/n\mathbb{Z}).$$

Geometrically this just means that if $g: U \to Y$ is an etale Galois $(\mathbb{Z}/n\mathbb{Z})$-covering (which is irreducible) then $g^{-1}(C)$ is connected (we use the same letter to denote $C \subset X$ and its inverse image in Y). Let V be the normalisation of X in the function field of U. Then $h: V \to X$ is finite, and $C \subset X$ is ample, hence so is $h^{-1}(C) \subseteq V$. In particular $h^{-1}(C) = g^{-1}(C)$ is connected. Q.E.D.

3. Examples

(1) We describe the Halphen pencils of elliptic curves on $\mathbb{P}^2$ which lead to singular rational surfaces with a singularity of the type considered in the Main Theorem, or to a similar one where the singular fibre is an ordinary cusp. Our argument does not work for the case of cusps since the 2-torsion subgroup of $\mathbb{G}_a$ is trivial.

Let $E \subset \mathbb{P}^2$ be an irreducible cubic with the usual group law, and let $P_1, \ldots, P_9$ be 9 points on the smooth locus of E such that the only relation satisfied by the P_i in the group of smooth points is $m\ (P_1 + P_2 + \ldots P_9) = 0$, where $m > 1$ will remain fixed during the rest of this discussion. In particular, there is a curve $D \subseteq \mathbb{P}^2$ of degree $3m$ such that the intersection cycle $(D \cdot E) = m(P_1 + \ldots P_9)$. We claim that D can be chosen so that $P_1, \ldots, P_9$ have multiplicitly m on D. To see this, consider first the problem of finding a curve of degree $3m$ with an m-tuple point at each of 9 given general points. In general if we consider a linear system on $\mathbb{P}^2$ and require its members to have an m-tuple point at a given point, this imposes $m(m+1)/2$ conditions on the linear system. Thus "in general" the dimension of the linear system of plane curves of degree $3m$ with 9 prescribed m-tuple points is $(3m+2)(3m+1)/2 - 1 - 9 \cdot m(m+1)/2 = 0$. Since one such curve is just the m-fold cubic through the 9 points, in general there is no other member. However, we impose m-tuple points at

$P_1,\ldots,P_8$ and for some local parameters x, y at P_9 such that $x=0$ determines E locally near P_9 we require that the local defining function of the curve D have no nonzero terms of order $<m$ except the term with y^{m-1}. This imposes one less condition than before and so there exists a member distinct from the m-fold curve E. This curve cannot contain E as one of its components since this would impose other relations on the P_i. The intersection cycle $(D\cdot E)=m(P_1+\ldots+P_8)+(m-1)P_9+Q$ for some point Q. But this means that $Q=P_9$ in the group structure on E, i.e. $Q=P_9$. But the condition that D and E have m-fold contact at P_9 is just the vanishing of the y^m term, so D automatically has an m-tuple point at P_9.

We note that for any curve D of degree $3m$ with m-tuple points at the P_i we either have $D=mE$ or D is reduced and irreducible. For otherwise we would have more relations satisfied by the P_i in the group law on E. Consider the pencil spanned by mE and any curve of degree $3m$ with m-tuple points at the P_i. If D is a member different from mE we have $p_a(D)=p_a(D')+9m(m-1)/2$ where D' is the strict transform of D under the blow up of $\mathbb{P}^2$ at the P_i. But $p_a(D)=(3m-1)(3m-2)/2$, giving $p_a(D')=1$. Thus the general member of the pencil is an elliptic curve with 9 m-tuple points which are resolved by one blow up, and the degenerate members (apart from mE) are irreducible rational curves with 9 m-tuple points and 1 double point (perhaps infinitely near) which is at worst an ordinary cusp. Further the strict transform of the pencil on the blow up W of $\mathbb{P}^2$ at the P_i has no base locus and exhibits W as an elliptic surface over $\mathbb{P}^1$ whose fibres are irreducible, with one fibre of multiplicity m, and the only other singular fibres are irreducible rational curves with a node (from which we can construct a singularity of the type considered in the Main Theorem) or an ordinary cusp. The first case of interest is $m=2$, where the singular fibre of the elliptic fibration is the blow up at 9 singular points of a sextic plane curve with 10 double points which are nodes or ordinary cusps (one of the double points can be infinitely near). In this case we can also proceed by taking the generic projection to the plane of a smooth rational sextic in $\mathbb{P}^6$, which gives a rational plane sextic with 10 nodes. This is exhibited in a (perhaps degenerate) Halphen pencil by taking the pencil generated by the sextic and twice the cubic through $P_1,\ldots,P_9$ where the P_i are any subset of 9 of the 10 nodes. In the classification of Mohan Kumar and Murthy, this is precisely the case of Kodaira dimension 0; the case $m>2$ corresponds to $\kappa=1$.

(2) It is well known that an Enriques surface is obtained by the following construction (see [5] for details). Let $C\subset\mathbb{P}^1\times\mathbb{P}^1$ be an irreducible curve of type (4,4), with 2 tacnodes P_0, P_∞ (say on $\mathbb{P}^1\times\{0\}$, $\mathbb{P}^1\times\{\infty\}$ respectively), such that the vertical lines $\mathbb{P}^1\times\{0\}$, $\mathbb{P}^1\times\{\infty\}$ are the tangents to the branches of C through P_0, P_∞ respectively. Let $D_i=\mathbb{P}^1\times\{i\}$, $i=0,\ \infty$. The Enriques surface is the double cover of $\mathbb{P}^1\times\mathbb{P}^1$ branched along the divisor $C+D_0+D_\infty$ (which yields a singular

surface; we then resolve the singularities). If $D_t = \mathbb{P}^1 \times \{t\}$ for $t \in \mathbb{P}^1$, then the general D_t meets the branch locus transversally at 4 points. Hence the D_t determine an elliptic fibration of the Enriques surface. Singular fibres occur when 2 or more of the expected 4 points of intersection of D_t and C come together. Thus to get a nodal curve we want $D_t \cdot C$ to consist of one point with multiplicity 2, and two simple points, i.e. we want the fibre over t of the projection $p_2 : C \to \mathbb{P}^1$ onto the second factor to contain one simple ramification point. If $(y_0 : y_1)$ and $(x_0 : x_1)$ give homogenous coordinates on the first and second factor of $\mathbb{P}^1 \times \mathbb{P}^1$, consider the curve C given by

$$x_0^4 y_1^4 + x_1^4 y_0^4 + x_1 y_1^3 x_0^3 y_0 + x_1^2 x_0^2 y_1^4 + x_1^2 x_0^2 y_0^4 = 0.$$

In local coordinates $x_0 = y_0 = 1$ this is

$$y^4 + x^4 + xy^3 + x^2 y^4 + x^2 = 0,$$

and we see that the origin is analytically isomorphic to $y^4 = x^2$ (char $k \neq 2$) which is a tacnode. Similarly if $x_1 = y_1 = 1$ is used to give a local description then we obtain

$$x^4 + y^4 + x^3 y + x^2 + x^2 y^4 = 0$$

which again has a tacnode at the origin. We will henceforth only use the second local description given by $x_1 = y_1 = 1$; let x, y denote the corresponding coordinates on the affine plane. To look for ramification of the projection $p_2 : C \to \mathbb{P}^1$ we must locate the zeroes of the x-partial derivative of the defining function of C. Thus we have the equations

$$\begin{aligned} &(\mathrm{i})\ldots \quad y^4(1 + x^2) + x^3 y + x^2(1 + x^2) = 0 \\ &(\mathrm{ii})\ldots \quad y^4(2x) + 3x^2 y + (2x + 4x^3) = 0. \end{aligned}$$

Treating these as linear equations in y, y^4 over $k(x)$ we obtain

$$(*)\ldots \qquad y = \frac{-2(1 + x^2)^2}{x(x^2 + 3)}, \qquad y^4 = \frac{x^2(x^2 - 1)}{x^2 + 3}$$

Hence we obtain the equation for x

$$(**)\ldots \qquad 16(1 + x^2)^8 = x^6(x^2 - 1)(x^2 + 3)^3.$$

Our analysis breaks down if $x = 0$ or $x^2 + 3 = 0$. For other x satisfying $(**)$ we have a unique solution (x, y) by $(*)$ of the equations (i), (ii). Since the points with infinite y-coordinate on c have $x = \infty$ or $x^2 + 1 = 0$, we conclude that for "most" of the solutions of $(**)$ the corresponding fibre of p_2 has a unique ramification point. This ramification point will be simple unless the second x-partial derivative of the defining function also vanishes at the given point. For this to happen (x, y) must also satisfy

$$\text{(iii)}\ldots \qquad 2y^4 + 6xy + 2 + 12x^2 = 0.$$

Combining (ii), (iii) yields (for $x \neq 0$) $y = -8x/3$. This rules out at most 4 points (other than the 2 tacnodes). Hence we see that the Enriques surface obtained by taking the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along the divisor $C + D_0 + D$ (where D_0, D_∞ are the vertical lines through the 2 tacnodes) has an elliptic pencil with many fibres which are reduced, irreducible rational curves with a node. This concludes our discussion of this example.

(3) The following finite dimensionality result is proved in [17]. Let X be a normal projective surface. We say that the Chow group $A_0(X)$ is infinite dimensional if there do not exist curves $C_1, \ldots, C_n$ completely contained in the smooth locus of X such that the 0-cycles supported on C_i generate $A_0(X)$. Equivalently, arbitrarily small neighbourhoods of the singular locus have nontrivial Chow group. It is shown in [17] that if X is projective over $\mathbb{C}$ and $H^2(X, \mathcal{O}_X) \neq 0$, then $A_0(X)$ is infinite dimensional. We use this result to give an example of an affine elliptic ruled surface X with 2 singular points P, Q such that the surfaces X_P, X_Q obtained by resolving only P or only Q respectively satisfy $A_0(X_P) = A_0(X_Q) = 0$; yet $A_0(X)$ is infinite dimensional.

Our example X is the complete intersection in $\mathbb{A}^4$ (considered over $\mathbb{C}$) of the surfaces $x^3 + y^3 + z^3 = 0$, $w^2 - z^2 = y^2 + 1$ where x, y, z, w are affine coordinates. Then X is the double cover of the cone $x^3 + y^3 + z^3 = 0$ branched along the space curve where the cone meets $y^2 + z^2 + 1 = 0$. We claim that the branch curve is smooth, and note that it does not contain the vertex of the cubic cone. To check smoothness we compute that the Jacobian matrix of partial derivatives has minors $6x^2y$, $6x^2z$, $6yz(y+z)$. For all of them to vanish at least 2 of x, y, z must vanish, or $x = 0, y = z$; thus $x^3 + y^3 + z^3 = 0$ forces $x = y = z = 0$ which is not on the quadric $y^2 + z^2 + 1 = 0$. Thus X has exactly 2 singular points which are each analytically isomorphic to the vertex of the cubic cone. The inverse image of a ruling is a conic (it is the double cover of a line branched at 2 points; or else, use the equations given) which is either smooth (if the 2 branch points are distinct) or a union of two lines meeting at a point lying over the coincident branch points. Thus X is a ruled surface, and if we blowup the 2 singular points P, Q we obtain sections of the ruling (by conics) so that the base of the ruling is the elliptic curve in $\mathbb{P}^2$ with equation $x^3 + y^3 + z^3 = 0$ (where x, y, z are regarded as homogeneous coordinates).

Let $\overline{X}$ denote the double cover of the projective cone over the plane curve $x^3 + y^3 + z^3 = 0$, branched along the closure of the branch curve of the double over X. Let Z be the blow up of P, Q on $\bar{X}$ so that Z is a smooth elliptic ruled surface with a ruling given by the inverse images of the ruling of the blown up cubic cone. In particular reducible fibres correspond precisely to the rulings tangent to the branch curve, and have exactly 2 components which are exceptional rational curves of the first kind, meeting transversally at one point. Let C_P, C_Q be the two exceptional sections obtained on Z as the inverse images of P, Q respectively. Then C_P, C_Q never meet the same component of any reducible fibre of the ruling. Consider the minimal model obtained from Z by blowing down the component of each reducible fibre which meets C_Q. Then if C is the image of C_P, C' the image of C_Q, the sections C, C' are disjoint. Hence (exercise in [4], V) the minimal model is the projective bundle associated to the decomposable vector bundle $\mathcal{O}_C \oplus \mathcal{O}_C(C)$. But C is the exceptional elliptic curve obtained by blowing up a singularity analytically isomorphic to the vertex of the cubic cone; so C has the same normal bundle. Thus our minimal model is isomorphic to the blow up of the cubic cone at the vertex. In particular the singular surface X_P is obtained from the projective cubic cone by blowing up the 6 points of intersection of a smooth hyperplane section and a smooth quadric section, and removing the strict transform of the quadric section. Note that the involution of X as the double cover interchanges the points P, Q so that the surfaces X_P, X_Q are isomorphic. Now we claim that $A_0(\bar{X}_P)$ is isomorphic to $A_0(Z)$ i.e. to the Albanese variety of Z which is just the elliptic curve over which Z is ruled. Once we have this it follows that $A_0(X_P) = 0$, since the strict transform of the quadric section which we remove from $\bar{X}_P$ to get X_P maps onto the Albanese variety of Z. To prove that $A_0(\bar{X}_P)$ is isomorphic to $A_0(Z)$ it suffices to show that A_0 of the projective cone is isomorphic to A_0 of the blow up of the vertex. Equivalently, it suffices to show that A_0 of the affine cubic cone is 0. This is well known; we reproduce the elegant and simple proof of Ojanguren [14] below.

Let C denote the smooth plane cubic, and let Y be the affine cone over C. Let $p \in Y$ be a smooth point and $O \in Y$ the vertex of the cone; let $f: Y\text{-}\{O\} \to C$ be the projection. Let $f(p) = P \in C$. Choose a point $Q \in C$ such that the tangent line to C at Q contains P, i.e. $P + 2Q = O$ in the group law. The cone over the tangent to C at Q is a plane section of the cone passing through the vertex such that the intersection consists of two concurrent lines L_1, L_2 where $p \in L_1$ and L_2 has multiplicity 2. The line L in the plane spanned by L_1, L_2 through p and parallel to L_2 (the lines L_i are affine so "parallel" makes sense) is a line in the ambient affine space whose intersection with the cubic cone is just the point p with multiplicity 1. Since K_0 of affine space is trivial we deduce that $A_0(Y) = 0$.

Thus we have shown that $A_0(X_P) = A_0(X_Q) = 0$. We now verify that $A_0(X)$ is infinite dimensional. To do this we need only show that

$H^2(\overline{X}, \mathcal{O}_{\overline{X}})$ is nonzero. The Leray spectral sequence for the map $g: Z \to \overline{X}$ yields the exact sequence

$$H^1(Z, \mathcal{O}_Z) \to H^0(\overline{X}, R^1 g_* \mathcal{O}_Z) \to H^2(\overline{X}, \mathcal{O}_{\overline{X}}) \to H^2(Z, \mathcal{O}_Z)$$

where the last term is 0 since $p_g(Z) = 0$. We have a surjection $H^0(\overline{X}, R^1 g_* \mathcal{O}_X) \to H^1(C_P, \mathcal{O}_{C_P}) \oplus H^1(C_Q, \mathcal{O}_{C_Q})$ which is 2-dimensional while $H^1(Z, \mathcal{O}_Z)$ has dimension 1. The result follows. Note that the infinite dimensionality of $A_0(X)$ follows immediately from that of $\overline{X}$.

References

[1] M. Artin: Algebraization of formal moduli. In: D.C. Spencer and S. Iyanaga (eds.), *Global Analysis: Papers in Honor of K. Kodaira*, Princeton: Princeton Univ. Press (1969).

[2] S. Bloch, D. Kas and D. Lieberman, Zero-cycles on surfaces with $p_g = 0$, *Comp. Math. 33* (1976) 135–145.

[3] T. Fujita, On Zariski problem, *Proc. Japan Acad. Ser. A55* (1979) 106–110.

[4] R. Hartshorne, Algebraic Geometry, GTM No. 52 New York: Springer-Verlag (1977).

[5] E. Horikawa, On the periods of Enriques surfaces I, *Math. Ann. 234* (1978) 73–88.

[6] S. Iitaka, Logarithmic Kodaira dimension of algebraic varieties. In: *Complex Analysis and Algebraic Geometry*. Tokyo: Iwanami Sholten (1977).

[7] K. Kodaira, *Collected works III*, Princeton: Princeton Univ. Press (1970).

[8] J.S. Milne, Zero cycles on algebraic varieties in nonzero characteristic: Rojtman's theorem, *Comp. Math. 47* (1982) 271–287.

[9] M. Miyanishi and T. Sugie, Affine surfaces containing a cylinder-like open set, *J. Math. Kyoto Univ.* 20 (1980) 11–42.

[10] M. Mohan Kumar, Rational double points on a rational surface, *Invent. Math.* 82 (1981).

[11] D. Mumford, *Abelian Varieties*. London: Oxford Univ. Press (1970).

[12] M.P. Murthy and N. Mohan Kumar, Curves with negative self intersection on a rational surface (to appear).

[13] M.P. Murthy and R.G. Swan, Vector bundles over affine surfaces, *Invent. Math.* 36 (1976) 125–165.

[14] M. Ojanguren, Fibrati vettoriai sulle superfici cubische *Archiv Math. 30* (1978) 143–145.

[15] A.A. Roitman, The torsion of the group of zero cycles modulo rational equivalence, *Ann. Math. 111(3)* (1980) 553–570.

[16] K.P. Russell, On affine ruled rational surfaces, *J. Pure Appl. Alg. 23* (1982) 243–250.

[17] V. Srinivas, Infinite dimensionality of the Chow group of zero cycles on a normal surface with $H^2(X, O_X) \neq 0$" (to appear).

(Oblatum 21-VI-1982)

University of Pennsylvania
Philadelphia, PA 19104
USA