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A STRUCTURE THEOREM FOR LIE ALGEBRAS OF UNBOUNDED DERIVATIONS IN C^*-ALGEBRAS

Palle E.T. Jørgensen

Abstract

Let G be a compact abelian group, and let τ by a faithful, ergodic action of G on a C^*-algebra $\mathfrak{A}$ with unit. Assume that every finite-rank subgroup of $\hat{G}$ is finitely generated. Let G_0 be the subgroup of elements in G which act trivially on the centre $\mathscr{Z}$ in $\mathfrak{A}$. Let $\mathfrak{A}_\infty \subset \mathfrak{A}$ be the ring of C^∞-elements for the G-action, and let $\mathscr{L}$ be the Lie algebra of all *-derivations in $\mathfrak{A}_\infty$, i.e., an element δ in $\mathscr{L}$ is an unbounded *-derivation in $\mathfrak{A}$ which is defined on $\mathfrak{A}_\infty$, and maps $\mathfrak{A}_\infty$ into itself. Let $\mathscr{A}$[resp., $\mathscr{L}_0$] be the approximately inner [resp., the derivations in the commutant of $\tau(G_0)$] elements in $\mathscr{L}$. Then we have the canonical (unique) decomposition, $\mathscr{L} = \mathscr{L}_0 + \mathscr{A}$, where $\mathscr{A}$ is an ideal in $\mathscr{L}$. Let $\mathscr{L}_{00} = \{\delta \in \mathscr{L}: \delta\tau(g) = \tau(g)\delta,\ g \in G\}$. If $\hat{G}_0$ is torsion free, then the Lie algebra $\mathscr{L}_{00}$ is maximal abelian in $\mathscr{L}$.

The direct decomposition of $\mathscr{L}$ represents joint work with Bratteli and Elliott, while the Lie theory is considered for the first time in the present paper. The linear mapping,

$$\delta \to \int_{G_0} \tau(g_0)\cdot\delta\cdot\tau\left(g_0^{-1}\right)\mathrm{d}g_0 = \delta^{G_0}$$

preserves the Lie bracket on $\mathscr{L}$, viz., $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$. That is,

$$[\delta_1, \delta_2]^{G_0} = \left[\delta_1^{G_0}, \delta_2^{G_0}\right] \qquad \text{for all } \delta_1, \delta_2 \in \mathscr{L}.$$

1. Introduction

In [5], Sakai posed the following problem: Is there a simple C^*-algebra $\mathfrak{A}$, a dense *-subalgebra $\mathscr{D}$ in $\mathfrak{A}$, and a finite set of *-derivations $\delta_1, \ldots, \delta_n$ which generate continuous $\mathbb{R}$-actions on $\mathfrak{A}$, are not approximately bounded, but which satisfy the following? $\mathscr{D} \subset D(\delta_i)$, $\delta_i(\mathscr{D}) \subset \mathscr{D}$, and $\mathscr{D}$ is a core for each δ_i. Sakai asked if it is possible to choose this structure such that every *-derivation, δ: $\mathscr{D} \to \mathfrak{A}$, admits a unique decomposition,

$$\delta = \sum_{i=1}^{n} k_i\delta_i + \tilde{\delta}$$

where $\tilde{\delta}$: $\mathscr{D} \to \mathfrak{A}$ is an approximately bounded *-derivation. The problem was solved, recently, in the affirmative ([1,8]) for all $n = 1, 2, \ldots$. It was shown, in addition [1], that the solution is a special instance of a more general structure on the unbounded *-derivations associated to ergodic actions of compact abelian groups G on C^*-algebras.

Associated to every such action τ, we have a dense *-subalgebra $\mathfrak{A}_F$ of G-finite elements in $\mathfrak{A}$.

An element A in $\mathfrak{A}$ is said to be G − finite if the set $\{\tau(g)(A): g \in G\}$ spans a finite dimensional linear subspace in $\mathfrak{A}$. It is easy to show that $\mathfrak{A}_F$ is the span of the spectral subspaces for the action.

It is known that $\mathfrak{A}_F$ is contained in the algebra $\mathfrak{A}_\infty$ of all smooth elements for the G-action. (By definition, an element A in $\mathfrak{A}$ falls in $\mathfrak{A}_\infty$ iff, for every continuous one-parameter group $(g(t))_{t\in\mathbb{R}} \subset G$, the mapping, $t \to \tau(g(t))(A)$ is of class C^∞ from $\mathbb{R}$ into $\mathfrak{A}$. If δ_g denotes the infinitesimal generator of the one-parameter group of *-automorphisms, $\tau(g(t))$, then the condition is that A is in the domain of δ_g^n for all $n = 1, 2, \ldots$, and for all choices of continuous one-parameter groups g in G.)

2. Extending derivations from $\mathfrak{A}_F$ to $\mathfrak{A}_\infty$

A C^*-algebraic dynamical system is a triple $(\mathfrak{A}, G, \tau)$ where $\mathfrak{A}$ is a C^*-algebra, G is a compact group, and τ is a strongly continuous action of G by *-automorphisms on $\mathfrak{A}$. A linear mapping δ with dense domain $D(\delta)$ in $\mathfrak{A}$ is said to be a *-derivation if $D(\delta)$ is a *-subalgebra, and $\delta(AB) = \delta(A)B + A\delta(B)$, $\delta(A^*) = \delta(A)^*$, for A, $B \in D(\delta)$. A special class of *-derivations are those which occur as infinitesimal generators of the one-parameter groups, $t \to \tau(g(t))$ where $(g(t)) \subset G$ is a continuous one-parameter group in G. In this paper we restrict attention to compact abelian groups G. Let Δ be the set of all *-derivations which occur as infinitesimal generators of one-parameter *-automorphism groups $\tau(g(t))$. For a finite subset $\{\delta_1, \ldots, \delta_n\} \subset \Delta$, consider the norm, $A \to \|A\| + \|\delta_1 \ldots \delta_n(A)\|$, on $\mathfrak{A}_\infty$. These norms on $\mathfrak{A}_\infty$ generate a complete locally convex topology which turns $\mathfrak{A}_\infty$ into a locally convex ring. For more details on C^∞-elements, see [2] and [4].

We begin by recalling the relationship between *-derivations defined on $\mathfrak{A}_F$, and the naturally extended derivations defined on the larger algebra $\mathfrak{A}_\infty$.

THEOREM 2.1 ([1]): *Let $(\mathfrak{A}, G, \tau)$ be a C^*-dynamical system satisfying the following conditions,*

(*i*) *$\mathfrak{A}$ is a C^*-algebra with unit;*

(*ii*) *G is a compact abelian group;*

(*iii*) *every finite-rank subgroup of the dual group $\hat{G}$ is finitely generated;*

(*iv*) *the action τ is faithful and ergodic.*

Then the following conditions (a) through (c) hold:

(a) If the Lie group dimension d of G is finite, and if $n > d/2 + 1$, then it follows that every *-derivation, δ: $\mathfrak{A}_F \to \mathfrak{A}$ extends by closure to a *-derivation, $\bar{\delta}$: $\mathfrak{A}_n \to \mathfrak{A}$. If $d = \infty$, then $\bar{\delta}$ is defined at last on $\mathfrak{A}_\infty$.
(b) If a given *-derivation, δ: $\mathfrak{A}_F \to \mathfrak{A}$ (alias, $\bar{\delta}$: $\mathfrak{A}_\infty \to \mathfrak{A}$) is known to map into $\mathfrak{A}_\infty$, then the implemented derivation, δ_∞: $\mathfrak{A}_\infty \to \mathfrak{A}_\infty$ is sequentially continuous in the complete locally convex topology on the ring $\mathfrak{A}_\infty$.
(c) Assuming that $\bar{\delta}$ is defined on some $\mathfrak{A}_n$, then it follows that, for all $A \in \mathfrak{A}_n$, there is a net $\{A_\nu\} \subset \mathfrak{A}_F$ such that $\|A - A_\nu\| \to 0$, and $\|\bar{\delta}(A) - \delta(A_\nu)\| \to 0$, i.e., $\mathfrak{A}_F$ is a core.

PROOF: The reader is referred to [1] for details for parts (a) and (b). Here we sketch the proof of (c). If δ is given as in (c), then the problem is to show that $\mathfrak{A}_F$ is dense in $\mathfrak{A}_n$ relative to the graph topology of $\bar{\delta}$. Recall that $\mathfrak{A}_n$ is defined to be the *-algebra of elements A such that the mapping, $g \to \tau(g)(A)$, is a class C^n from G into $\mathfrak{A}$. But in view of the core theorem [4, Theorem 1.3], we know that $\mathfrak{A}_F$ is dense on $\mathfrak{A}_n$ relative to the $\mathfrak{A}_n$ locally convex topology. In view of [1, Corollary 3.3] there is some seminorm p such that $\bar{\delta}$ is bounded relative to p on $\mathfrak{A}_n$.

There is some seminorm p for the $\mathfrak{A}_n$ topology such that

$$\|\bar{\delta}(A)\| \leqslant \text{Const.}\, p(A) \qquad \text{for all } A \in \mathfrak{A}_n.$$

Since the $\mathfrak{A}_n$-topology is stronger than the p-seminorm topology, it follows that $\mathfrak{A}_F$ is also dense relative to p. We may choose, therefore, a net $\{A_\nu\} \subset \mathfrak{A}_F$ such that $p(A - A_\nu) \to 0$. Since $\|\bar{\delta}(A) - \delta(A_\nu)\| \leqslant$ Const. $p(A - A_\nu)$, and $\|A - A_\nu\| \leqslant p(A - A_\nu)$, the desired conclusion follows.

3. Ergodic actions. The main theorem

If the action τ in the system $(\mathfrak{A}, G, \tau)$ is ergodic, then the minimal eigen-spaces (defined for $\gamma \in \hat{G}$)

$$\mathfrak{A}^\tau(\gamma) = \{A \in \mathfrak{A} : \tau(g)(A) = \langle \gamma, g \rangle A, g \in G\}$$

are one- or zero dimensional. As is well known [3,6,7] it is no loss of generality to assume that τ is faithful. Then each $\mathfrak{A}^\tau(\gamma)$ is spanned by a unitary element $U(\gamma)$.

The relation,

$$U(\gamma_1)U(\gamma_2) = \rho(\gamma_1, \gamma_2)U(\gamma_2)U(\gamma_1)$$

determines a scalar function ρ on $\hat{G} \times \hat{G}$ which is independent of the phase for the U's. The function ρ can easily be checked to be an anti-symmetric bi-character. Conversely, every anti-symmetric bi-character on some discrete abelian group gives rise in a natural way ([3]) to an ergodic C^*-action. Slawny [6] showed that the C^*-algebra $\mathfrak{A}$ of the system is simple iff the bi-character is *non-degenerate*, i.e., for all $\gamma \neq 0$ in $\hat{G}$ there is some ξ such that $\rho(\gamma, \xi) \neq 1$.

THEOREM 3.1.: *Let G be a compact abelian group, and assume that every finite-rank subgroup of $\hat{G}$ is finitely generated. Let G_0 be the subgroups of elements g in G such that $\tau(g)$ acts trivially on the centre $\mathscr{Z}$ of $\mathfrak{A}$. Let τ be a strongly continuous action of G by *-automorphisms on a unital C^*-algebra $\mathfrak{A}$. Assume that the action is faithful and ergodic.*

*Consider the following Lie subalgebras, $\mathscr{L}_0$, $\mathscr{A}$, $\mathscr{L}_{00}$, and $\mathscr{B}$ of the Lie algebra $\mathscr{L}$ of all *-derivations in the ring $\mathfrak{A}_\infty$ of C^∞-elements for the G-action*:

$$\mathscr{L}_0 = \{ \delta \in \mathscr{L}\colon \delta \textit{ commutes with } \tau(G_0)\}$$

$$\mathscr{L}_{00} = \{ \delta \in \mathscr{L}\colon \delta \textit{ commutes with } \tau(g) \textit{ for all } g \in G \}$$

$$\mathscr{A} = \{ \delta \in \mathscr{L}\colon \delta \textit{ is approximately inner on } \mathfrak{A}_\infty \}$$

$$\mathscr{B} = \{ \delta \in \mathscr{L}\colon \delta|_{\mathscr{Z}} \equiv 0\}$$

and

$$\mathscr{C} = \{ \delta \in \mathscr{L}\colon \delta(\mathscr{Z} \cap \mathfrak{A}_F) \subset \mathscr{Z} \}.$$

Then

(*a*) *$\mathscr{L} = \mathscr{L}_0 + \mathscr{A}$ is a canonical direct decomposition.*
(*b*) *The Lie subalgebra, $\mathscr{L}_{00}$ in $\mathscr{L}$ is maximal abelian, assuming that $\hat{G}_0$ is torsion-free.*
(*c*) *$\mathscr{A} \subset \mathscr{B}$ and $\mathscr{L}_0 \cap \mathscr{B}$ is abelian.*
(*d*) *For the normalizer, $N_{\mathscr{A}} = \{ \delta \in \mathscr{L}\colon [\delta, \mathscr{A}] \subset \mathscr{A} \}$ we have the identity, $N_{\mathscr{A}} = \mathscr{C}$.*

COROLLARY 3.2: *Let the C^*-dynamical system be as in Theorem* 3.1. *Then we have the identity*

$$N_{\mathscr{A}} = \mathscr{L} = \mathscr{C}.$$

*That is, $\mathscr{A}$ is an ideal in $\mathscr{L}$; and every δ in $\mathscr{L}$ restricts to *-derivation in $\mathscr{Z}$ with domain $\mathscr{Z} \cap \mathfrak{A}_\infty$.*

PROOF: We have $\mathscr{A} \subset \mathscr{B} \subset \mathscr{C}$, and $\mathscr{L}_0 \subset \mathscr{C}$. Hence, $\mathscr{L} \subset \mathscr{C}$, in view of (a) in the theorem. The corollary now follows from (d) in the theorem.

The following known Lemma (cf., [1]) will be used in the sequal without mentioning.

Let ρ be the anti-symmetric bi-character on $\hat{G} \times \hat{G}$ which is associated to the given system $(\mathfrak{A}, G, \tau)$, and let Γ be the kernel of ρ, viz.,

$$\Gamma = \{\gamma \in \hat{G} : \rho(\gamma, \xi) = 1, \xi \in \hat{G}\}.$$

The following lemma is implicit in [10] and in [1].

Lemma 3.3: *Let $(\mathfrak{A}, G, \tau)$ be a C^*-dynamical system as above with G acting faithfully and ergodically.*

(*a*) *Then, if ρ, Γ, $\mathscr{Z}$, and G_0, are defined as before, we have*

$$\mathscr{Z} = \mathfrak{A}^{G_0} = \{A \in \mathfrak{A} : \tau(g_0)(A) = A, g_0 \in G_0\},$$

$$\Gamma = G_0^{\perp} := \{\gamma \in \hat{G} : \langle \gamma, g_0 \rangle = 1, g_0 \in G_0\},$$

and $\mathscr{Z}$ is the C^*-algebra generated by the $\mathfrak{A}^{\tau}(\gamma)$'s for γ in Γ.

(*b*) *An element A in $\mathfrak{A}$ falls in the centre $\mathscr{Z}$ iff, for all $\xi \in \hat{G}_0 \setminus \{0\}$ (or equivalently, for all $\xi \in \hat{G} \setminus \Gamma$), we have*

$$\int_{G_0} \tau(g_0)(A) \langle \xi, g_0 \rangle \mathrm{d} g_0 = 0.$$

Definition 3.3: Let $(c^{\gamma}(\xi))$ be a system of elements in $\mathscr{Z}$ which is indexed by $\hat{G} \times \hat{G}$. We say that the system is *admissible* if conditions (i) through (iii) below are satisfies:

(i) For all $\gamma \in \hat{G}$, there is some $A(\gamma) \in \mathfrak{A}$ such that

$$U(\gamma)^* \int_{G_0} \overline{\langle \gamma + \xi, g_0 \rangle} \tau(g_0)(A(\gamma)) \mathrm{d} g_0 = c^{\gamma}(\xi) U(\xi)$$

(ii) $c^{\gamma_1}(\xi)(\rho(\xi, \gamma_2) - 1) = c^{\gamma_2}(\xi)(\rho(\xi, \gamma_1) - 1)$, and

(iii) $c^{-\gamma}(\xi)\rho(\xi, \gamma) = c^{\gamma}(-\xi)^*$ for all $\gamma, \xi \in \hat{G}$.

Lemma 3.4: *Let $(\mathfrak{A}, G, \tau)$ be a C^*-dynamical system where the group G is compact abelian, and τ is a faithful ergodic action of G on a unital C^*-algebra $\mathfrak{A}$. For every $\gamma \in \hat{G}$, let $\{c^{\gamma}(\xi)\}_{\xi \in \hat{G}}$ be an admissible system of coefficients in the centre $\mathscr{Z}$, defined relative to a family of eigen-unitaries $\{U(\xi)\}_{\xi \in \hat{G}}$. Then the linear mapping, $\delta: \mathfrak{A}_F \to \mathfrak{A}$ defined by*

$$\delta(U(\gamma)) = \sum_{\xi \in \hat{G}/\Gamma} c^{\gamma}(\xi) U(\gamma) U(\xi)$$

*is a *-derivation. Conversely, every *-derivation, $\delta: \mathfrak{A}_F \to \mathfrak{A}$, is associated in this manner with an admissible system $(c^{\gamma}(\xi))$ of coefficients in $\mathscr{Z}$.*

PROOF: The proof is carried out in [1; proof of Theorem 2.1] for the special case of system $(c^{\gamma}(\xi))$ with scalar values. The modifications needed when $(c^{\gamma}(\xi))$ is centre-valued are only small, and are left to the reader.

PROOF OF THEOREM 3.1: Since the action is faithful, each of the eigen-spaced $\mathfrak{A}^{\tau}(\gamma)$, for $\gamma \in \hat{G}$, is one-dimensional. We may choose unitary elements $U(\gamma)$ in $\mathfrak{A}^{\tau}(\gamma)$ such that

$$\mathfrak{A}^{G_0}(\gamma) = U(\gamma)\mathscr{Z} \tag{1}$$

where $\mathfrak{A}^{G_0}(\gamma)$ is the eigen-space for the restriction of τ to G_0.

By definition, an element δ in $\mathscr{L}$ is a *-derivation which is defined on $\mathfrak{A}_{\infty}$ and takes values in $\mathfrak{A}_{\infty}$. We proved in Theorem 2.1 (alias [1]) that the algebra $\mathfrak{A}_F$ of G-finite elements will always be a core for such a *-derivation δ.

We may consider the action of the subgroup G_0 which is obtained by restricting τ to G_0. The corresponding Fourier analysis is then based on the dual group $\hat{G}_0$, and, in view of Pontryagin-duality, we have,

$$\hat{G}_0 \approx \hat{G}/\Gamma$$

We then arrive at the formal Fourier expansion for $\delta(U(\gamma))$, given as follows,

$$\delta(U(\gamma)) = \sum_{\xi \in \hat{G}/\Gamma} c^{\gamma}(\xi)U(\gamma)U(\xi) \tag{2}$$

where the coefficients $c^{\gamma}(\xi)$ fall in $\mathscr{Z}$, and the summation is carried over the discrete group $\hat{G}/\Gamma$ of Γ-costs in $\hat{G}$.

We recall from [1] that this Fourier expansion is norm convergent when we taken Cesàro means over finitely generated subgroups of $\hat{G}/\Gamma$, and then follow by a certain inductive limit procedure. (The reader is referred to [1] for details.) If we define

$$\delta_0(A) = \int_{G_0} \tau(g_0)\big(\delta\big(\tau\big(g_0^{-1}\big)(A)\big)\big)\mathrm{d}g_0,$$

then the integral is convergent for A in $\mathfrak{A}_F$, and defines a *-derivation, δ_0: $\mathfrak{A}_F \to \mathfrak{A}$ satisfying $\delta_0(U(\gamma)) = c^{\gamma}(0)U(\gamma)$. It follows that the difference, $\delta - \delta_0$ is also a *-derivation, defined on $\mathfrak{A}_F$. For $\tilde{\delta} = \delta - \delta_0$, we have

$$\tilde{\delta}(U(\gamma)) = \sum_{\substack{\xi \in \hat{G}/\Gamma \\ \xi \neq 0}} c^{\gamma}(\xi)U(\gamma)U(\xi)$$

and moreover, $\tilde{\delta}(U(\gamma))$ is precisely the Cesàro-inductive limit of a net of commutators,

$$[H_\nu, U(\gamma)] = H_\nu U(\gamma) - U(\gamma) H_\nu$$

where H_ν is a net of elements in $\mathfrak{A}_\infty$ which is constructed from the following (second) formal Fourier expansion,

$$H = \sum_{\xi \in \hat{G}/\Gamma} \mathrm{d}(\xi) U(\xi). \tag{3}$$

The coefficients $\mathrm{d}(\xi)$ in (3) belong to $\mathscr{Z}$ and are obtained as a unique solution to the following system of equations,

$$c^\gamma(\xi) = \mathrm{d}(\xi)(\rho(\xi, \gamma) - 1).$$

Indeed, if $\xi \in \hat{G}/\Gamma$, $\xi \neq 0$, then there is some $\gamma \in \hat{G}$ such that $\rho(x, \gamma) - 1 \neq 0$. Hence, we may set

$$\mathrm{d}(\xi) = c^\gamma(\xi)(\rho(\xi, \gamma) - 1)^{-1}$$

and this solution is independent of the particular γ in $\hat{G}$ which was chosen. To see this, the reader may invoke the compatibility relation,

$$c^{\gamma_1}(\xi)(\rho(\xi, \gamma_2) - 1) = c^{\gamma_2}(\xi)(\rho(\xi, \gamma_1) - 1) \tag{4}$$

established in [1]. See also Lemma 3.4.

The decomposition, $\delta = \delta_0 + \tilde{\delta}$ results from taking out the term indexed with $\xi = 0$ in the Fourier expansion (2). For the element $c^\gamma(0)$, we have the formula

$$c^\gamma(0) = U(\gamma)^* \delta_0(U(\gamma)).$$

To check uniqueness, it suffices to verify that $\mathscr{L}_0 \cap \mathscr{A} = 0$. let $\delta \in \mathscr{L}_0$ be given, and assume that δ is also approximately inner. For $\gamma \in \hat{G}$, consider the projection,

$$Q_\gamma(A) = \int_{G_0} \overline{\langle \gamma, g_0 \rangle} \tau(g_0)(A) \mathrm{d} g_0.$$

We have $Q_\gamma(\mathfrak{A}) = \mathfrak{A}^{G_0}(\gamma) = U(\gamma)\mathscr{Z}$, cf., Lemma 3.2, and, in particular, formula (1). Let $\{H_\nu\}$ be a net of elements in $\mathfrak{A}$ such that $\lim_\nu \|[H_\nu, A] - \delta(A)\| = 0$ for all A in $\mathfrak{A}_F$. (Note that the convergence is pointwise *not* uniform.) We have,

$$Q_\gamma(\delta(U(\gamma))) = \delta(Q_\gamma(U(\gamma))) = \delta(U(\gamma))$$

and

$$Q_\gamma([H_\nu, U(\gamma)]) = \left[\int_{G_0} \tau(g_0)(H_\nu)\mathrm{d}g_0, U(\gamma)\right] = 0, \qquad \text{since}$$

$$\int_{G_0} \tau(g_0)(H_\nu)\mathrm{d}g_0 = Q_0(H_\nu) \in \mathfrak{A}^{G_0} = \mathscr{Z}.$$

But $\mathfrak{A}_F$ is a core for δ, and we may conclude that δ is the zero derivation. This finishes the proof of (a). (We note that the statement (a), with proof, is also contained in [1]. The present proof differens from that of [1] at one point. We have sketched the details since they are needed.)

In view of Lemma 3.3, $\mathscr{Z}$ is generated by the $U(\gamma)$'s for $\gamma \in \Gamma$. hence, a more precise definition of $\mathscr{B}$ is the following,

$$\mathscr{B} = \{\delta \in \mathscr{L}: \delta(U(\gamma)) = 0, \gamma \in \Gamma\}.$$

But there is no ambiguity in view of Theorem 2.1 (alias Corollary 3.5 and the Remark following Theorem 5.3 in [1]). To check that $\mathscr{L}_0 \cap \mathscr{B}$ is abelian, consider δ_1 and δ_2 in $\mathscr{L}_0 \cap \mathscr{B}$. We have

$$\delta_i(U(\gamma)) = c_i^\gamma(0)U(\gamma)$$

for $i = 1, 2$, $\gamma \in \hat{G}$, where $c_i^\gamma(0) \in \mathscr{Z}$. Hence,

$$\begin{aligned}\delta_1(\delta_2(U(\gamma))) &= \delta_1(c_2^\gamma(0)U(\gamma))\\ &= \delta_1(c_2^\gamma(0))U(\gamma) + c_2^\gamma(0)\delta_1(U(\gamma))\\ &= c_2^\gamma(0)c_1^\gamma(0)U(\gamma).\end{aligned}$$

By symmetry of the argument, we have

$$\delta_2(\delta_1(U(\gamma))) = c_1^\gamma(0)c_2^\gamma(0)U(\gamma).$$

We used first that $\delta_1 \in \mathscr{B}$, and then $\delta_2 \in \mathscr{B}$. Clearly, the commutator $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$ must vanish on $\mathfrak{A}_F$, and therefore on $\mathfrak{A}_\infty$. Use Theorem 2.1 again.

In view of [1; Theorem 2.1], an element δ in $\mathscr{L}$ falls in $\mathscr{L}_{00}$ iff $\delta(U(\gamma)) = c^\gamma(0)U(\gamma)$ for all $\gamma \in \hat{G}$. In view of [9; Remark 2.1], δ is then the infinitesimal generator of $\tau(g(t))$ for some continuous one-parameter group $\{g(t)\} \subset G$, and $\gamma \to ic^\gamma(0) = h(\gamma)$ is in $\mathrm{Hom}(\hat{G}, \mathbb{R})$.

Conversely, every derivation δ satisfying $\delta(U(\gamma)) = -ih(\gamma)U(\gamma)$ for some $h \in \mathrm{Hom}(\hat{G}, \mathbb{R})$ is in $\mathscr{L}_{00}$.

Clearly $\mathscr{L}_{00}$ is abelian.

Suppose $\delta \in \mathscr{L}$ commutes with all derivations δ_0 in $\mathscr{L}_{00}$. It follows that δ must commute with the infinitesimal generator of every one-parameter group, $t \to \tau(g_0(t))$, obtained by restricting τ to some continuous one-parameter subgroup $(g_0(t))_{t \in \mathbf{R}} \subset G$. We have the formula $G = \mathbb{T}^{\Omega} \times H$ for some compact abelian torsion group H; (cf., the discussion in [1] preceding Theorem 5.1). It follows then that δ commutes with the component $\mathbb{T}^{\Omega} \times (1)$ in G.

Recall the known $1 - 1$ correspondence, $\mathrm{Hom}(\hat{G}, \mathbb{R}) \leftrightarrow \{ g_0 \colon g_0(t)$ is a continuous one-parameter subgroup $\subset G \}$, given by $h_0 \leftrightarrow g_0$, through the identity

$$\exp(ith_0(\xi)) = \langle \xi, g_0(t) \rangle \qquad \text{for } t \in \mathbb{R}, \xi \in \hat{G}. \tag{5}$$

Since $\hat{G}_0$ is torsion free we may conclude that δ commutes with $\tau(G_0)$. In view of (1) we therefore have

$$\delta(U(\gamma)) = C(\gamma)U(\gamma), \quad \gamma \in \hat{G},$$

where $C(\gamma) \in \mathscr{Z}$. Since δ is a derivation, the mapping, $\gamma \to iC(\gamma)$, falls in $\mathrm{Hom}(\hat{G}, \mathscr{Z})$. Using (5), we may factor out the torsion part $\hat{H}$ in $\hat{G}$. Using that δ commutes with $\tau(\mathbb{T}^{\Omega})$ and the above characterization of $\mathscr{L}_{00}$ we finally conclude that, in fact, $\delta \in \mathscr{L}_{00}$.

In part (c) of the theorem, we first prove the inclusion, $\mathscr{A} \subset \mathscr{B}$. So let $\delta \in \mathscr{A}$ be given, and let $\gamma \in \Gamma$. Choose some net $\{ H_\nu \}$ such that $adH_\nu \to \delta$. We have $[H_\nu, \mathrm{U}(\gamma)] = 0$, and hence $\delta(U(\gamma)) = 0$.

To check, finally, $[\mathscr{A}, \mathscr{C}] \subset \mathscr{A}$, we consider a pair of elements δ_1 and δ_2 in $\mathscr{L}$. Using the formulas,

$$\delta_i(U(\gamma)) = \sum_{\xi \in \hat{G}/\Gamma} c_i^{\gamma}(\xi) U(\gamma) U(\xi)$$

for $i = 1, 2$, $\gamma \in \hat{G}$, we arrive at the following Fourier expansion for the commutator $[\delta_1, \delta_2]$. In the formula, there is a double summation over elements ξ, φ in the discrete group $\hat{G}/\Gamma$. We introduce a scalar phase-factor, $\beta(\xi, \varphi) \in S^1$, determined by the identity

$$U(\xi)U(\varphi) = \beta(\xi, \varphi)U(\xi + \varphi).$$

A direct calculation yields,

$$\begin{aligned}
[\delta_1, \delta_2](U(\gamma)) &= \sum_{\xi} \left(\delta_1(c_2^{\gamma}(\xi)) - \delta_2(c_1^{\gamma}(\xi)) \right) U(\gamma) U(\xi) \\
&\quad + \sum_{\varphi} \left(\sum_{\xi} \left(c_2^{\gamma}(\xi) c_1^{\gamma+\xi}(\varphi - \xi) - c_1^{\gamma}(\xi) c_2^{\gamma+\xi}(\varphi - \xi) \right) \times \beta(\xi, \varphi - \xi) \right) \\
&\quad \times U(\gamma) U(\varphi).
\end{aligned} \tag{6}$$

If $\delta_1 \in \mathscr{C}$ and $\delta_2 \in \mathscr{A}$, we have $\delta_1(c_2^\gamma(0)) = 0$, since $c_2^\gamma(0) = 0$. Moreover, $\delta_2(c_1^\gamma(0)) = 0$, since $\delta_2|_{\mathscr{Z}} \equiv 0$. It remains to consider the double sum. To verify that $[\delta_1, \delta_2] \in \mathscr{A}$, we must check that the term in the double-sum corresponding to $\varphi = 0$ $(\in \hat{G}/\Gamma)$ vanishes identically. This amounts to the verification of,

$$\sum_{\xi} c_2^\gamma(\xi) c_1^{\gamma+\xi}(-\xi) \beta(\xi, -\xi)$$

$$= \sum_{\xi} c_1^\gamma(-\xi) c_2^{\gamma-\xi}(\xi) \beta(-\xi, \xi). \tag{7}$$

But this identity follows from the formulas, $\beta(\xi, -\xi) = \beta(-\xi, \xi) = 1$, combined with the consistency relations (4),

$$c_2^\gamma(\xi)(\rho(\xi, \gamma - \xi) - 1) = c_2^{\gamma-\xi}(\xi)(\rho(\xi, \gamma) - 1)$$

and

$$c_1^{\gamma+\xi}(-\xi)(\rho(-\xi, \gamma) - 1) = c_1^\gamma(-\xi)(\rho(-\xi, \gamma + \xi) - 1).$$

Note that formula (7) holds without any restrictions on the derivation pair δ_1, δ_2.

Conversely, let $\delta \in \mathscr{N}_{\mathscr{A}}$. Then it can be shown from (5) that $\delta(D(\delta) \cap \mathscr{Z}) \subset \mathscr{Z}$. On the other hand we know that this is satisfied for all $\delta \in \mathscr{L}$. We therefore have $\mathscr{N}_{\mathscr{A}} = \mathscr{C} = \mathscr{L}$. (O. Bratteli pointed out to us that the inclusion $\delta(D(\delta) \cap \mathscr{Z}) \subset \mathscr{Z}$ can easily be verified directly from the Leibnitz derivation formula for δ.)

COROLLARY 3.5: *Let $(\mathfrak{A}, G, \tau)$ be a C^*-dynamical system with G compact abelian, acting ergodically and faithfully on the C^*-algebra $\mathfrak{A}$. Assume $\mathfrak{A}$ is simple with unit. If every finite-rank subgroup of $\hat{G}$ is finitely generated, then we have the following Lie algebraic decomposition of the *-derivations of the ring $\mathfrak{A}_\infty$ of G-smooth elements: Define,*

$\mathscr{L}$: *all *-derivations*, $\delta: \mathfrak{U}_\infty \to \mathfrak{U}_\infty$.

$\mathscr{L}_0 := \{ \delta \in \mathscr{L}$: *$\delta$ generates some one-parameter group*, $t \to \tau(g(t)) \}$

$\mathscr{A} := \{ \delta \in \mathscr{L}$: *$\delta$ is approximately inner on* $\mathfrak{A}_\infty \}$.

Then

(*i*) *$\mathscr{L}_0$ is maximal abelian in $\mathscr{L}$ if $\hat{G}$ is torsion free.*

and

(*ii*) $[\mathscr{A}, \mathscr{L}] \subset \mathscr{A}$.

PROOF: It is shown in [1] (the Remark following Theorem 2.1) that every *-derivation which commutes with an ergodic G-action is the infinitesimal generator of some one-parameter group, $t \to \tau(g(t))$, where $(g(t))_{t \in \mathbb{R}} \subset G$ is a continuous one-parameter subgroup. But if $\mathfrak{A}$ is simple, $\mathscr{Z}$ is trivial, and G_0 is all of G. $\mathscr{Z}$ is spanned by the identity element I in $\mathfrak{A}$, and $\delta(I) = 0$ for every closed *-derivation. It follows that $\mathscr{L}_{00} = \mathscr{L}_0$. Hence, the corollary follows, as a special case, from Theorem 3.1.

REMARK:

(a) O. Bratteli and G.A. Elliot have kindly let me know that the implication in part (i) of the corollary goes both ways. Hence, if $\mathscr{L}_0$ is known to be maximal abelian, then we may deduce from this that $\hat{G}$ is torsion free.

(b) The following problem seems to be open: Let δ_1 and δ_2 be *-derivations in a C^*-algebra $\mathfrak{A}$, both defined on a dense invariant domain $\mathscr{D}$.
Assume that δ_1 is approximately inner on $\mathscr{D}$ with approximating elements $H_\nu \in \mathscr{D}$. Does it follow that the derivation $[\delta_1, \delta_1]$ is also approximately inner on $\mathscr{D}$?

(c) It is expected that the Lie theory in the present note may be used in the calculation of the spectrum of the derivations δ in $\mathscr{L}$. For derivations $\delta \in \mathscr{A}$ (approximately inner) we expect continuous spectrum.

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Both made several substantial improvements after reading a first draft of the manuscript.

The present paper rests in a very essential way on the earlier paper [1]. Part (b) in Theorem 3.1 is due to Bratteli (personal correspondence).

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Appendix

Problems and conjectures

The following questions for unbounded derivations seem to be natural. Interesting suggestions from O. Bratteli, G. Elliott, F. Goodman, R.T. Powers and G. Price are gratefully acknowledged.

Throughout, in the appendix, we consider *-derivations δ with dense domain $D(\delta)$ in a given C^*-algebra $\mathfrak{A}$.

A1. The normalizer to the approximately inner derivations

PROBLEM 1: Is the conclusion in Corollary 3.2 special to the "pseudo-torus" algebras $\mathfrak{A}(\rho)$? More specifically, it would be interesting to know if the implication in (a) of the above Remark holds true.

Note that we have, for any derivation δ and any H in $D(\delta)$,

$$[\delta, ad(H)] = ad(\delta(H))$$

by pure algebra. But passing to the limit, $n \to \infty$, creates several complications. These complications, presumably, take quite different forms in the different non-commutative C^*-algebras under considerations.

PROBLEM 2: Find general methods of constructing approximately inner derivations in C^*-algebras. The Fourier analysis technique from [1] appears to be the first such constructive approach to "approximate innerness".

PROBLEM 3: Does the reduced C^*-algebra over the free group G_2 on two generators, i.e., $C_r^*(G_2)$, have some unbounded closed *-derivation (with dense domain) which is approximately inner? (The bounded (inner) ones are excluded!)

Lance and Niknam constructed in [[4]] a family of non-approximately inner derivations in $C_r^*(G_2)$ associated to elements in $\mathrm{Hom}(G_2, \mathbb{R})$.

Let G_2 be the free group on two generators, and let

$$(u_g f)(g') = f(g^{-1}g') \qquad \text{for } g, g' \in G_2,$$

be the regular representation. Then the linear span of the elements $\{u_g \colon g \in G_2\}$ forms a dense *-subalgebra $\mathscr{D}$ in $C_r^*(G_2)$.

PROBLEM 4: Classify all the closable *-derivations δ in $C_r^*(G_2)$ such that $\mathscr{D} \subset D(\delta)$.

(The derivations δ considered in [[4]] satisfy this. They are given by $\delta(u_g) = ih(g)u_g$, $g \in G_2$, when $h \in \mathrm{Hom}(G_2, \mathbb{R})$.)

REMARK (added after the completion of the paper): Using central sequences in joint work with Dr. G.A. Elliott we showed recently that $C_r^*(G_2)$ has unbounded approximately inner *-derivations.

A2. The CAR-algebra

Let $\mathscr{H}$ be a separable complex ∞-dimensional Hilbert space, and let $\mathfrak{A} = \mathfrak{A}(\mathscr{H}) = \mathrm{CAR}(\mathscr{H})$ be the C^*-algebra which is generated by the canonical anticommutation relations over $\mathscr{H}$. Let $\mathscr{F} = \mathscr{F}(\mathscr{H})$ be the anti-symmetric Fock space over $\mathscr{H}$, and let N be the number operator (regarded as a self-adjoint unbounded operators in $\mathscr{F}$). The gauge-action α on $\mathfrak{A}$ is then given by

$$\alpha_\theta(A) = e^{i\theta N} A e^{-i\theta N}, \qquad \theta \in \mathbf{R}, A \in \mathfrak{A}.$$

Let δ_0 be the corresponding infinitesimal generator. Formally,

$$\delta_0 = iad(N).$$

Let $\mathfrak{A}^\alpha$ be the "current algebra" (alias, the GICAR) which is defined as

$$\mathfrak{A}^\alpha = \{ A \in \mathfrak{A} : \alpha_\theta(A) = A, \text{all } \theta \}.$$

PROBLEM 5: Let δ be a closed *-derivation in $\mathfrak{A}$ which commutes with the gauge-action α. Assume that the restriction of δ to $\mathfrak{A}^\alpha$ is the infinitesimal generator of a strongly continuous one-parameter group of *-automorphisms on $\mathfrak{A}^\alpha$ (i.e., "is a generator on $\mathfrak{A}^\alpha$" for short). Then does it follow that δ is itself a generator on $\mathfrak{A}$?

Examples due to Bratteli and the author [[2]] are known of closed derivations δ which commute with a compact abelian action such that δ is a generator on the fixed-point subalgebra of the action, but is *not* a generator on the original algebra.

Let δ_1 and δ_2 be two generators on the CAR-algebra $\mathfrak{A}$, and let $\tau_i(t) = e^{t\delta_i}$ be the corresponding strongly continuous one-parameter groups of *-automorphisms. Suppose that

$$\tau_1(t)(A) = \tau_2(t)(A) \qquad \text{for all} \quad t \in \mathbf{R}, \qquad \text{and} \quad A \in \mathfrak{A}^\alpha;$$

i.e., that the two one-parameter groups coincide on the gauge-invariant elements.

PROBLEM 6: Is the chemical potential accountable for the different extensions from $\mathfrak{A}^\alpha$? More precisely, is there a real number μ such that

$$\tau_2(t)(A) = \lim_{n \to \infty} \left(\tau_1\left(\frac{t}{n}\right) \alpha\left(\mu \frac{t}{n}\right)\right)^n (A) \qquad \text{for} \quad t \in \mathbf{R}, \qquad \text{and} \quad A \in \mathfrak{A}? \qquad (*)$$

(the number μ has the interpretation of "the chemical potential". The convergence is in the sense of Trotter). Formally (but *only* formally) we might think of the derivations δ_i, $i = 1, 2$, as being given by hamiltonians H_i. Then $(*)$ reflects the formal (unprecise) relations, $H_2 = H_1 + \mu N$, or $ad(H_2) = ad(H_1) + \mu ad(N)$, or

$$\delta_2 = \delta_1 + \mu \delta_0. \qquad (**)$$

The following special case of the conjecture may be derived from [[2]], [[3]], or [[5]].

It it is assumed, in addition to the conditions in Pr.6, that there is some dense *-subalgebra $\mathscr{D} \subset \mathfrak{A}$ such that

$$\mathscr{D} \subset D(\delta_1) \cap D(\delta_2),$$

then it follows that $(**)$ is satisfies *on* $\mathscr{D}$ for some $\mu \in \mathbf{R}$.

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REMARK: In joint work with Dr. O. Bratteli we showed recently that Problem 5 is affirmative. (Paper in *Commun. Math. Phys. 87* (1982) 353–364.)

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