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EXPECTATION AND VARIANCE OF THE VOLUME COVERED BY A LARGE NUMBER OF INDEPENDENT RANDOM SETS

A.J. Stam

1. Introduction

Throughout this paper the following assumptions and notations apply. The set $A \in \mathbb{R}^m$ is bounded and measurable with Lebesgue measure $|A| > 0$, but sometimes more stringent assumptions on A will be made. The random m-vectors $u_1, u_2, \ldots$ are independent with common probability density $p(u) = f(\|u\|)$, where f is nonincreasing and positive. Let N be either a nonrandom positive integer or a random variable having Poisson distribution with parameter λ and independent of the u_i. The volume in $\mathbb{R}^m$ covered by at least k of the random sets $A + u_i$, $i = 1, \ldots, N$, is denoted V_k. Here $A + y = \{x + y | x \in A\}$. The volume covered by exactly k of the random sets is denoted W_k. We take $k \geqq 1$. If $N = 0$ we define $V_k = W_k = 0$. Replacing A by $A + a$ does not change W_k or V_k, so it is no restriction to assume $0 \in A$ or $0 \in \operatorname{Int}(A)$ if A has nonempty interior.

Under the above conditions we have $V_k \to \infty$, a.s. if $N \to \infty$ and in probability if $\lambda \to \infty$, so that $EV_k \to \infty$. Moran in his 1974 paper [6] took $m = 3$, assumed that the u_i have standard normal distribution and derived the principal term in the asymptotic expansion of EV_1 as $\lambda \to \infty$ or $N \to \infty$. This term does not depend on A. He also proved the curious result that if A is a ball the variance $\sigma^2(V_1)$ tends to zero as $\lambda \to \infty$, or $N \to \infty$, viz.

$$0 < K_1 \leqq (\log EN)^{1/2} \sigma^2(V_1) \leqq K_2 < \infty, \quad EN \geqq 2, \tag{1.1}$$

and he showed that V_1 when centered and normalized in asymptotically normal. We will extend the results on expectation and variance, with special attention to the dependence on A, to densities of the form

$$p(u) = f(\|u\|) = c \exp(-\alpha^{-2\theta}\|u\|^{2\theta}), \tag{1.2}$$

$$p(u) = f(\|u\|) = c \exp\{-g(\|u\|)\}, \tag{1.3}$$

$$p(u) = f(\|u\|) = 1/h(\|u\|), \tag{1.4}$$

where $\theta > 0$ and g and h vary regularly at ∞. We need some results on regularly varying functions, to be found in De Haan [1], Feller [3] and Seneta [7]. The function $f:(0, \infty) \to (0, \infty)$ varies regularly of order ρ at ∞ or at 0, respectively, if

$$f(tx)/f(t) \to x^{\rho}, \quad t \to \infty (t \to 0), \quad x > 0. \tag{1.5}$$

If $\rho = 0$ we say that f varies slowly. If (1.5) holds and $\rho > -1$, then

$$\int_0^x f(s) \mathrm{d}s \sim (\rho + 1)^{-1} x f(x), \quad x \to \infty (x \to 0). \tag{1.6}$$

If f varies regularly of order ρ at ∞, it has the Karamata representation

$$f(x) = c(x) x^{\rho} \left\{ \exp \int_1^x s^{-1} \epsilon(s) \, \mathrm{d}s \right\}, \quad x \geqq 1, \tag{1.7}$$

with $c(x) > 0$, $x \geqq 1$, $c(x) \to c > 0$, $\epsilon(x) \to 0$, $x \to \infty$. If f is nondecreasing, it follows from (1.7) with $f(\lambda x) \leqq f(\lambda)$, $x \leqq 1$, that

$$f(\lambda x)/f(\lambda) \leqq \max\{1, C x^{\rho + \delta}\}, \tag{1.8}$$

with $C = C(\delta)$, for $x > 0$, $\lambda \geqq \lambda(\delta)$, for any $\delta > 0$. Let $g:(0, \infty) \to (0, \infty)$ be nondecreasing with $g(x) \to \infty$, $x \to \infty$. The generalized inverse g^{-1} of g is defined by

$$g^{-1}(y) = \sup\{x | g(x) \leqq y\}, \quad g(0^+) < y < \infty, \tag{1.9}$$

If $\rho > 0$, then g varies regularly at ∞ of order ρ if and only if g^{-1} varies regularly at ∞ of order ρ^{-1}. But g^{-1} may vary slowly without g varying regularly and then $x^n/g(x) \to 0$, $x \to \infty$, $n > 0$.

In Section 3 the asymptotic behaviour of EV_k and EW_k as $\lambda \to \infty$ for Poissonian N will be studied. Under (1.4) they tend to ∞ of the same order depending on h and m with coefficients involving k and $|A|$. Under (1.3) there is no dependence on A in the principal term and we may have $EW_k \to 0$.

Estimating the variance is far more difficult and we have to restrict our attention to V_1 since in contrast to (2.12) below the expressions for the other variances contain terms with opposite signs. In Section 4 we consider Poissonian N. Our results suggest that $\sigma^2(V_1)$ may tend to zero only if $p(u)$ decreases faster than exponentially. If the u_t have probability density (1.2) with $\theta > \frac{1}{2}$ and A is convex and bounded by a regular surface ∂A with curvatures bounded below and above,

$$0 < K_1 \leqq (\log EN)^{-b_1} \sigma^2(V_1) \leqq K_1 < \infty, \quad EN \geqq 2, \tag{1.10}$$

$$b_1 = (2\theta)^{-1} \{ m - 1 + (m+3)(\tfrac{1}{2} - \theta) \}, \tag{1.11}$$

in accordance with (1.1). So $\sigma^2(V_1) \to 0$, $\sigma^2(V_1)$ is bounded and $\sigma^2(V_1) \to \infty$, if $m - 1 + (m+3)(\frac{1}{2} - \theta)$ is negative, zero and positive, respectively. However, the order of decrease or increase of $\sigma^2(V_1)$ does not only depend on m and θ, but, strongly, on the shape of A. It was shown in Stam [8] that if $\theta = 1$ and A is a parallelopipedum, we have for Poissonian N

$$\sigma^2(V_1) \sim B(\log\log\lambda)^{m-1}/\log\lambda, \quad \lambda \to \infty.$$

For subexponentially decreasing p of the form (1.3) or (1.4) the exact first-order term for $\sigma^2(V_1)$ is derived. We see that then $\sigma^2(V_1) \to \infty$.

In Section 5 the results for Poissonian N are extended to nonrandom N by comparing the corresponding expectations and variances. Section 2 contains the general relations from which our proofs will start. We will take $m \geqq 2$. Results for $m = 1$ will be stated in remarks. The proofs that may different from those for $m \geqq 2$ are given in Stam [9], as are the derivations of the second-order terms that in some cases may be found.

We write $\int g(x)\mathrm{d}x$ for the Lebesgue integral of g over $\mathbb{R}^m$,

$$d_A = \operatorname{diam}(A) = \max\{\|x - y\| : x, y \in A\}, \tag{1.12}$$

$A - B = \{x - y | x \in A, y \in B\}$, $-A = \{-x | x \in A\}$ and I_A for the indicator function of A. The inner product of m-vectors is written (x, y). The notation (r, ω) will always stand for the spherical coordinates of $x \in \mathbb{R}^m$, so that $r = \|x\|$, $\omega = x/r$ and ω is the generic point of the unit sphere Ω. The area element of Ω will be written $\mathrm{d}\sigma(\omega)$ or $\mathrm{d}\sigma$, so that

$$S_m = \int_\Omega \mathrm{d}\sigma(\omega) = 2\pi^{m/2}/\Gamma(m/2), \quad m \geqq 2, \tag{1.13}$$

is the area of the unit sphere in $\mathbb{R}^m$. If $m = 1$ a good definition turns out to be $S_1 = 2$. By the identification $x = (r, \omega)$ we will sometimes write $g(r, \omega)$ for $g(x)$. The argument ω often will be suppressed.

In estimates and remainder terms we will often have to do with functions of r, ω, λ or N that are bounded by some constant. These functions all will be denoted by the same symbol η or by η_+ if they are nonnegative. The argument usually will be suppressed.

2. General formulae

The probability that the point $x \in \mathbb{R}^m$ is covered by the set $A + u_i$ is, since $p(u) = f(\|u\|)$,

$$P(x) = P(r, \omega) = P(u_i \in -A + x) = \int_{-A+x} p(u)\mathrm{d}u$$

$$= \int_A p(x - v)\mathrm{d}v = \int_A f\left(\{r^2 - 2r(v, \omega) + \|v\|^2\}^{1/2}\right)\mathrm{d}v. \tag{2.1}$$

The probability that x and y are covered by $A + u_i$ is

$$P(x, y) = P(u_i \in (-A + x) \cap (-A + y))$$

$$= \int I_{-A+x}(u) I_{-A+y}(u) p(u) \mathrm{d}u = \int_{A(y-x)} p(x - v) \mathrm{d}v, \tag{2.2}$$

$$A(z) = A_z = A \cap (A - z). \tag{2.3}$$

By interchanging intergrations and using the fifth term in (2.1) and the third term in (2.2) we find

$$\int P(x) \mathrm{d}x = |A|, \quad \int P(x, y) \mathrm{d}y = |A| P(x). \tag{2.4}$$

From (2.2) and (2.3)

$$P(x, y) = 0, \quad y - x \notin A - A. \tag{2.5}$$

Let $\xi(x)$ be the number of random sets that cover x. Then

$$EW_k = \int P(\xi(x) = k) \mathrm{d}x, \tag{2.6}$$

$$\sigma^2(V_1) = \int\int \{ P(\xi(x) \geqq 1, \xi(y) \geqq 1)$$

$$- P(\xi(x) \geqq 1) P(\xi(y) \geqq 1) \} \mathrm{d}x \mathrm{d}y$$

$$= \int\int \{ P(\xi(x) = \xi(y) = 0)$$

$$- P(\xi(x) = 0) P(\xi(y) = 0) \} \mathrm{d}x \mathrm{d}y. \tag{2.7}$$

From (2.6), the independence of the u_i and (2.1) we find for nonrandom N

$$EW_k = \binom{N}{k} \int P^k(x)(1 - P(x))^{N-k} \mathrm{d}x, \tag{2.8}$$

and for Poissonian N by conditioning with respect to N

$$EW_k = \int (\lambda P(x))^k \exp(-\lambda P(x)) \mathrm{d}x / k!. \tag{2.9}$$

In a similar way, for nonrandom N

$$P(\xi(x)=0)=(1-P(x))^{N}, \tag{2.10}$$

$$P(\xi(x)=\xi(y)=0)=(1-P(x)-P(y)+P(x,y))^{N}, \tag{2.11}$$

and for Poissonian N

$$\sigma^2(V_1)=\int\int \exp(-\lambda P(x)-\lambda P(y))\{\exp(\lambda P(x,y))-1\}\mathrm{d}x\mathrm{d}y, \tag{2.12}$$

Since $V_k = W_k + W_{k+1} + \dots$, we have for Poissonian N by (2.9) and the relation

$$\sum_{j=k}^{\infty} \alpha^j \,\mathrm{e}^{-\alpha}/j! = \int_0^{\alpha} y^{k-1}\mathrm{e}^{-y}/(k-1)!, \quad k \geqq 1,$$

$$EV_k = \int\left\{\int_0^{P(x)} y^{k-1}\,\mathrm{e}^{-y}\mathrm{d}y\right\}\mathrm{d}x/(k-1)!, \quad k\geqq 1. \tag{2.13}$$

Let $w_k(R)$ and $v_k(R)$ be the contribution to (2.9) and (2.13) from the domain $D_R=\{x|r\leqq R\}$.

Since $p(u)\geqq c(R)>0$ on D_R we have for some $\gamma=\gamma(R)>0$

$$w_k(R)=\mathcal{O}(\mathrm{e}^{-\gamma\lambda}), \quad v_k(R)-|D_R|=\mathcal{O}(\mathrm{e}^{-\gamma\lambda}), \quad \lambda\to\infty. \tag{2.14}$$

Now let μ be the measure on $\mathbb{R}$ induced from Lebesgue measure on D_R^c by the restriction to D_R^c of P given by (2.1), i.e.

$$\mu(A)=|\{x\in D_R^c: P(x)\in A\}|. \tag{2.15}$$

Then from (2.13), with $H=\sup\{P(x)|r>R\}$,

$$\begin{aligned} EV_k &= v_k(R)+\int_0^H\left(\int_0^{\lambda\xi} y^{k-1}\,\mathrm{e}^{-y}\mathrm{d}y\right)\mathrm{d}\mu(\xi)/(k-1)! \\ &= v_k(R)+\int_0^{\lambda H} y^{k-1}\,\mathrm{e}^{-y}\mu[\lambda^{-1}y,H]\mathrm{d}y/(k-1)! \end{aligned} \tag{2.16}$$

We will always take R so large that $P(r,\omega)$ is nonincreasing in r for $r\geqq R$. Let

$$r_u=r_u(\omega)=r(u,\omega)=\sup\{r|P(r,\omega)\geqq u\}. \tag{2.17}$$

Then, since for $0 < u \leqq H$ and $m \geqq 2$

$$\mu[u,H] = m^{-1}\int_{\Omega} r^m(u,\omega)\mathrm{d}\sigma(\omega) - |D_R|, \tag{2.18}$$

$$EV_k = v_k'(R) + m^{-1}\int_{\Omega}\mathrm{d}\sigma(\omega)\int_0^{\lambda H} y^{k-1} \times \mathrm{e}^{-y} r^m(\lambda^{-1}y,\omega)\mathrm{d}y/(k-1)! \tag{2.19}$$

for $m \geqq 2$ where we have by (2.14) for some $\gamma = \gamma(R) > 0$

$$v_k'(R) = \mathcal{O}(\exp(-\gamma\lambda)), \quad \lambda \to \infty. \tag{2.20}$$

If $P(r,\omega)$ for $r \geqq R$ has a strictly negative continuous partial derivative

$$P'(r) = P'(r,\omega) = \partial/\partial r P(r,\omega), \tag{2.21}$$

so that $r(\cdot,\omega)$ defined by (2.17) is the inverse of $P(\cdot,\omega)$, we have from (2.9) for Poissonian N and $m \geqq 2$

$$EW_k = w_k(R) + \int_{\Omega}\mathrm{d}\sigma(\omega)\int_0^{P(R,\omega)}(\lambda u)^k\,\mathrm{e}^{-\lambda u} r_u^{m-1}(\omega) \times |P'(r_u)|^{-1}\mathrm{d}u/k! \tag{2.22}$$

3. Expectation for Poissonian N

THEOREM 3.1.: *Let the u_i have probability density (1.4), with h nondecreasing and h^{-1}, defined by (1.9), varying regularly at ∞ of order $\rho \in [0, m^{-1})$. Then as $\lambda \to \infty$*

$$EV_k \sim m^{-1}S_m|A|^{m\rho}\Gamma(k-m\rho)\left(h^{-1}(\lambda)\right)^m/(k-1)!, \tag{3.1}$$

$$EW_k \sim \rho S_m|A|^{m\rho}\Gamma(k-m\rho)\left(h^{-1}(\lambda)\right)^m/k!. \tag{3.2}$$

REMARK: if $m = 1$ these relations hold with $S_1 = 2$, see Stam [9]. That $\rho \geqq 0$ follows from the fact that h is nondecreasing. From (1.7) we see that $\rho m \leqq 1$ is necessary in order that $\int p(x)\mathrm{d}x < \infty$, since h varies regularly of order ρ^{-1} if $\rho > 0$.

PROOF: To prove (3.1) we start from (2.19) where we take $R > d_A = \operatorname{diam}(A)$. From (2.1), for $r > R$,

$$|A|/h(r+d_A) \leqq P(r,\omega) \leqq |A|/h(r-d_A).$$

Since $P(r, \omega)$ is nonincreasing in r for $r > R$ we have by (1.9) and (2.17), for $0 < u < P(R, \omega)$

$$h^{-1}(u^{-1}|A|) - d_A \leqq r(u, \omega) \leqq h^{-1}(u^{-1}|A|) + d_A. \tag{3.3}$$

By the regular variation of h^{-1} and by (1.8), since $(a+b)^m < 2^m(a^m + b^m)$, $a \geqq 0$, $b \geqq 0$,

$$\{h^{-1}(\lambda y^{-1}|A|) + d_A\}^m / \{h^{-1}(\lambda)\}^m \to y^{-m\rho}|A|^{m\rho}, \quad \lambda \to \infty,$$

$$\{h^{-1}(\lambda y^{-1}|A|) + d_A\}^m / \{h^{-1}(\lambda)\}^m \leqq \max\{1, Cy^{-m\rho-\delta}\},$$

with $m\rho + \delta < 1$ for $\lambda > \lambda(\delta)$. It follows from (2.19), (2.20) and (3.3) by dominated convergence that

$$\limsup_{\lambda \to \infty} EV_k / \{h^{-1}(\lambda)\}^m \leqq m^{-1} S_m |A|^{m\rho} \Gamma(k - m\rho)/(k-1)!$$

From the other inequality in (3.3) the reversed inequality for liminf follows and this proves (3.1).

The relation (3.2) follows from (3.1) since $W_k = V_k - V_{k+1}$.

Theorem 3.1 applies to densities like (1.2) and (1.3) with $g(x)/\log x \to \infty$. Then we have $\rho = 0$ so that (3.2) gives no more than $EW_k/(H^{-1}(\lambda))^m \to 0$. The principal term for W_k then is given by

THEOREM 3.2: *Let the u_i have probability density (1.3) where g has a continuous positive derivative g' that varies regularly at ∞ of order $\tau \geqq -1$, where if $\tau = -1$ we assume that $g^{-1}(\log x)$ varies slowly at ∞. Then, as $\lambda \to \infty$,*

$$EW_k \sim (mk)^{-1} S_m \xi_0(\log \lambda), \quad \xi_0(x) = \frac{d}{dx}(g^{-1}(x))^m. \tag{3.4}$$

PROOF: We start from (2.22) with R sufficiently large. From (2.1) by applying (1.7) to g'

$$|P'(r, \omega)| = \int_A p(x\text{-}v) g'(\|x - v\|) \|x - v\|^{-1}(r - (v, \omega)) dv$$

$$= \int_A p(x - v) g'(r)(1 + \epsilon(r, v, \omega)) dv$$

$$= g'(r) P(r, \omega)(1 + \epsilon(r, \omega)), \tag{3.5}$$

where $\epsilon(r, v, \omega) \to 0$, $\epsilon(r, \omega) \to 0$, $r \to \infty$, uniformly in (v, ω) and ω. For

$r > d_A$ with (2.17), noting that $P(r, \omega)$ decrease with r for $r > d_A$,

$$c|A| \exp(-g(r+d_A)) \leqq P(r, \omega) \leqq c|A| \exp(-g(r-d_A)),$$

$$-d_A + g^{-1}(\log c|A|u^{-1}) \leqq r_u(\omega) \leqq g^{-1}(\log c|A|u^{-1}) + d_A \qquad (3.6)$$

Since $P(r_u, \omega) = u$ we have with (2.14) and with (3.5), noting that as $u \to \infty$ $r(u, \omega) \to \infty$ uniformly in ω by (3.5), for $R = R_1(\delta)$,

$$EW_k \leqq (1+\delta) \int_\Omega d\sigma(\omega) \int_0^{P(R,\omega)} \lambda^k u^{k-1} r_u^{m-1} e^{-\lambda u} \{k! g'(r_u)\}^{-1} du. \qquad (3.7)$$

If $\tau > -1$ we see from (1.6) that g varies regularly at ∞ of order $\tau + 1 > 0$, so that g^{-1} varies regularly of order $(\tau+1)^{-1}$. By (3.6) and (1.7) for g

$$r(u, \omega) \sim \psi(u) \stackrel{\text{def}}{=} g^{-1}(\log u^{-1}), \quad u \downarrow 0, \qquad (3.8)$$

uniformly in ω. If $\tau = 1$, this follows from the slow variation of $g^{-1}(\log x)$. From (3.8) and (1.7) for g', noting that $\psi(u) \to \infty$ as $u \downarrow 0$,

$$g'(r(u, \omega)) \sim g'(\psi(u)), \quad u \downarrow 0, \qquad (3.9)$$

uniformly in ω. From (3.7), (3.8) and (3.9) for $R = R_2(\delta)$ and some $a > 0$

$$EW_k \leqq (1+\delta)^2 S_m \lambda^k \int_0^a \psi^{m-1}(u) u^{k-1} e^{-\lambda u} \{k! g' \circ \psi(u)\}^{-1} du. \qquad (3.10)$$

From (3.8) and the regular variation of g^{-1} if $\tau < 1$, or by assumption if $\tau = 1$, it follows that ψ varies slowly at 0 and then, e.g. by (1.7), we see that $g' o \psi$ varies slowly at 0. So the integrand in (3.10) varies regularly of order k at 0. By applying the Abelian theorem given in Feller [3], Ch. XIII.5, we see that the right-hand side of (3.10) is asymptotically equal to

$$(1+\delta)^2 k^{-1} \psi^{m-1}(\lambda^{-1}) / g' \circ \psi(\lambda^{-1}) = (1+\delta)^2 (mk)^{-1} S_m \xi_0(\log \lambda).$$

So, since this holds for all $\delta > 0$,

$$\limsup_{\lambda \to \infty} EW_k / \xi_0(\log \lambda) \leqq (mk)^{-1} S_m.$$

From (3.5) and (3.6) we find in the same way the reversed inequality for liminf and this proves (3.4).

If the density p has the special form (1.2) a second-order term in the expansion of EV_k and EW_k may be found. The proofs start from (2.19) and (2.22) and proceed by expanding (2.1) and (3.5) into a suitable number of powers of r and a remainder term. From the expansion of $P(r, \omega)$ or from an inequality like (3.6) an expansion for $r(u, \omega)$ with remainder term uniform in ω is derived. Substitution into (2.19) and (2.22) then gives the asymptotic relations. The proofs are given in Stam [9]. Writing

$$R_k(\lambda) = EV_k - m^{-1}S_m\alpha^m(\log\lambda)^p, \quad p = m/2\theta,$$

$$T_k(\lambda) = EW_k - S_m\alpha^m(2\theta k)^{-1}(\log\lambda)^{p-1},$$

we find for $m \geqq 2$, if $p(u)$ is given by (1.2): If $\theta > \frac{1}{2}$ and A is a convex body and $0 \in \mathrm{Int}(A)$,

$$R_k(\lambda) \sim M_A\alpha^{m-1}(\log\lambda)^q, \quad q = (m-1)/2\theta,$$

$$T_k(\lambda) \sim (m-1)M_A\alpha^{m-1}(2\theta k)^{-1}(\log\lambda)^{q-1},$$

$$M_A = \int_\Omega \max\{(\omega, x) | x \in A\} d\sigma(\omega).$$

If $m = 2$, then M_A is equal to the length of the perimeter of A and if $m = 3$ it is equal to the integral of the mean curvature of ∂A. See Valentine [10], Ch. XII, and Hadwiger [5], §18. If $\theta = \frac{1}{2}$,

$$R_k(\lambda) = b_2(\log\lambda)^{m-1} + \mathcal{O}(\log\lambda)^{m-2},$$

$$T_k(\lambda) = (m-1)k^{-1}b_2(\log\lambda)^{m-2} + \mathcal{O}(\log\lambda)^{m-3},$$

$$b_2 = \alpha^m\left\{\int_\Omega \log\varphi(\omega) d\sigma(\omega) - S_m\Gamma'(k)/\Gamma(k)\right\},$$

$$\varphi(\omega) = c\int_A \exp(-\alpha^{-1}(v, \omega)) dv.$$

If $0 < \theta < \frac{1}{2}$, with $a = \min\{2,(2\theta)^{-1}\}$,

$$R_k(\lambda) = b_3(\log\lambda)^{p-1} + \mathcal{O}(\log\lambda)^{p-a},$$

$$T_k(\lambda) = (p-1)k^{-1}b_3(\log\lambda)^{p-2} + \mathcal{O}(\log\lambda)^{p-1-a},$$

$$b_3 = \alpha^m S_m(2\theta)^{-1}\{\log c|A| - \Gamma'(k)/\Gamma(k)\}.$$

4. Variance for Poissonian N

THEOREM 4.1: *Let $m \geqq 2$ and let the u_i have probability density (1.2) with $2\theta > 1$. If A is convex and its boundary ∂A is a twice continuously* *differentiable surface in* $\mathbb{R}^m$ *whose principal curvatures* $k_1, \ldots, k_{m-1}$ *satisfy* $0 < c_1 \leqq k_i(x) \leqq c_2 < \infty$, $i = 1, \ldots, m-1$, $x \in \partial A$, *then, with* b_1 *given by (1.11),*

$$\tfrac{1}{2} B_1 \leqq \limsup_{\lambda \to \infty} (\log \lambda)^{-b_1} \sigma^2(V_1) \leqq \limsup_{\lambda \to \infty} (\log \lambda)^{-b_1} \sigma^2(V_1) \leqq B_1, \tag{4.1}$$

$$B_1 = \alpha^{(3m+1)/2} \theta^{-(m+3)/2} S_{m-1} m! (m^2 - 1)^{-1} \left(\Gamma\left(\tfrac{1}{2} m + \tfrac{1}{2}\right)\right)^{-1} \times \int_{\partial A} K^{1/2}(\xi) \mathrm{d}\mathcal{O}(\xi) \tag{4.2}$$

Here S_m is given by (1.13), $\mathrm{d}\mathcal{O}(\xi)$ is the surface area element of ∂A at ξ and $K(\xi)$ the gaussian curvature of ∂A at ξ, i.e.

$$K(\xi) = k_1(\xi) \ldots k_{m-1}(\xi). \tag{4.3}$$

Remark: If $m = 2$ we will have to define $S_{m-1} = 2$, as will be seen from the proof. It is conjectured that the inequality (4.1) may be replaced by an actual limit. If $m = 1$ and $A = [-a, a]$ we have, see Stam [9],

$$(\log \lambda)^{2-\theta^{-1}} \sigma^2(V_1) \to (\pi \alpha \theta^{-1})^2/6, \quad \lambda \to \infty.$$

If $p(u)$ decreases with $\|u\|$ faster than exponentially, estimates for $\sigma^2(V_1)$ are difficult. This makes the proof of Theorem 4.1 very long. It is given in the appendix.

If $p(u)$ decreases with $\|u\|$ exponentially or slower, we may obtain first-order asymptotic terms for $\sigma^2(V_1)$ with the methods of Section 3. We use the notation $\omega = x/r$, $r = \|x\|$ defined in Section 1 and start from (2.12). Let U_1 be the contribution to (2.12) for $r < \gamma(\lambda)$ to be chosen later on. Putting $y - x = z$ and noting (2.4) and (2.5) we have, with $P(y) \leqq P(x, y)$,

$$\sigma^2(V_1) = U_1 + \int_{A-A} \mathrm{d}z \int_\Omega \mathrm{d}\sigma(\omega) \int_{\gamma(\lambda)}^\infty \mathrm{d}r\, r^{m-1} \cdot \cdot \exp\{-\lambda P(x) - \lambda P(x+z)\} \{\exp(\lambda P(x, x+z)) - 1\}, \tag{4.4}$$

$$U_1 \leqq \int_{r<\gamma(\lambda)} \mathrm{d}x \exp(-\lambda P(x)) \int \{1-\exp(-\lambda P(y))\mathrm{d}y$$

$$\leqq \lambda \int_{r<\gamma(\lambda)} \mathrm{d}x \exp(-\lambda P(x)) \int P(y)\mathrm{d}y$$

$$= \lambda |A| \int_{r<\gamma(\lambda)} \exp(-\lambda P(x))\mathrm{d}x. \tag{4.5}$$

THEOREM 4.2: *Let $m \geqq 2$ and let the u_i have probability density (1.2) with $\theta = \frac{1}{2}$. Then as $\lambda \to \infty$*

$$\sigma^2(V_1) \sim \alpha^m (\log \lambda)^{m-1} \int_{A-A} \mathrm{d}z \int_\Omega \mathrm{d}\sigma(\omega) \log W(z,\omega), \tag{4.6}$$

$$W(z,\omega) = \{\exp(-\alpha^{-1}(z,\omega)) + 1\}\{\exp(-\alpha^{-1}(z,\omega)) + 1 - \chi(A_z)/\chi(A)\}^{-1}, \tag{4.7}$$

where A_z is given by (2.3) and

$$\chi(E) = \chi(E,\omega) = \int_E \exp(\alpha^{-1}(v,\omega))\mathrm{d}v, \quad E \in \mathscr{B}(\mathbb{R}^m). \tag{4.8}$$

REMARK 1: The integral in (4.6) is finite since $0 \leqq \chi(A_z) \leqq \chi(A)$ and $\exp(-\alpha^{-1}(z,\omega)) \geqq c_0 > 0$.

REMARK 2: If A is a ball with radius a, then $\chi(A)$ does not depend on ω and, see Gröbner und Hofreiter [4], §313, 21^b, and (1.13),

$$\chi(A) = (m-1)^{-1} S_{m-1} \int_{-a}^{a} \exp(-x/\alpha)(a^2-x^2)^{(m-1)/2}\mathrm{d}x$$

$$= a^m (m-1)^{-1} S_{m-1} \Gamma(\tfrac{1}{2}) \Gamma(\tfrac{1}{2}m + \tfrac{1}{2})(2\alpha/ia)^{m/2} J_{m/2}(ia\alpha^{-1})$$

$$= a^m \pi^{m/2} (2\alpha/ia)^{m/2} J_{m/2}(ia\alpha^{-1}),$$

where J_β is the Bessel function of index β. Also $\chi(A_z,\omega) = \psi(\rho,\varphi)$ depends only on $\rho = \|z\|$ and the angle φ between z and ω. So, since $A-A$ is a ball with radius 2a and φ is distributed uniformly on $[0,2\pi]$,

$$\sigma^2(V_1) \sim \alpha^m (\log \lambda)^{m-1} (2\pi)^{-1} S_{m-1}^2 \int_0^{2a} \rho^{m-1}\mathrm{d}\rho \times \int_0^{2\pi} \log W^*(\rho,\varphi)\mathrm{d}\varphi,$$

$$W^*(\rho, \varphi) = \{\exp(-\alpha^{-1}\rho \cos \varphi) + 1\}$$
$$\times \{\exp(-\alpha^{-1}\rho \cos \varphi) + 1 - \psi(\rho, \varphi)/\chi(A)\}.$$

PROOF OF THEOREM 4.2: From (2.1), (2.2) and

$$\|x - v\| = r(1 - 2r^{-1}(v, \omega) + r^{-2}\|v\|^2)^{1/2} = r - (v, \omega) + \eta r^{-1},$$

$$P(x) = c \exp(-r/\alpha)\chi(A, \omega)(1 + \eta r^{-1}), \tag{4.9}$$

$$P(x + z) = c \exp(-r\alpha^{-1} - (z, \omega)\alpha^{-1})\chi(A, \omega)(1 + \eta r^{-1}) \tag{4.10}$$

$$P(x, x + z) = c \exp(-r/\alpha)\chi(A_z, \omega)(1 - \eta r^{-1}). \tag{4.11}$$

In (4.4) and (4.5) we take $\gamma(\lambda) = \frac{1}{2}\alpha \log \lambda$. From (4.9) for some $c_1 > 0$

$$U_1 = \mathcal{O}(\lambda(\log \lambda)^m) \exp(-c_1\lambda^{1/2}), \quad \lambda \to \infty. \tag{4.12}$$

Substituting (4.9)–(4.11) into (4.4) and putting $y = \lambda \exp(-r/\alpha)$ we find

$$\sigma^2(V_1) = U_1 + \alpha^m \int_{A-A} \mathrm{d}z \int_\Omega \mathrm{d}\sigma(\omega) \int_0^{\lambda^{1/2}} y^{-1}\mathrm{d}y(\log \lambda y^{-1})^{m-1}\cdot$$
$$\cdot \exp\{-cy\chi(A, \omega)(1 + \mathrm{e}^{-(z, \omega)/\alpha})(1 + \eta/\log \lambda y^{-1})\}\cdot$$
$$\cdot[-1 + \exp\{cy\chi(A_z, \omega)(1 + \eta/\log \lambda y^{-1})\}].$$

After dividing by $(\log \lambda)^{m-1}$ we may interchange limit and integral, since $\chi(A, \omega) \geqq b_0 > 0$ and $\lambda y^{-1} > \lambda^{1/2}$ in the integral. With (4.12) and Gröbner und Hofreiter [4], §313, 3^b, we find (4.6).

THEOREM 4.3: *Let the u_i have probability density (1.3), where g has a positive continuous derivative g' that varies regularly at ∞ of order $\tau \in [-1,0)$. Let $g^{-1}(\log x)$ vary slowly at ∞. Then as $\lambda \to \infty$, with A_z given by (2.3) and ξ_0 by (3.4),*

$$\sigma^2(V_1) \sim m^{-1}S_m\xi_0(\log \lambda)\int_{A-A} \mathrm{d}z \log(1 - \tfrac{1}{2}|A_z|/|A|)^{-1}. \tag{4.13}$$

REMARK 1: If $m = 1$ we have to take $S_m = 2$. If $\tau > -1$, the regular variation of g' implies the slow variation of $g^{-1}(\log x)$, see the proof of (3.8).

REMARK 2: If $g(x) = (x/\alpha)^{2\theta}$, with $0 < 2\theta < 1$, we have $\xi_0(\log \lambda) = m\alpha^m(\log \lambda)^{p-1}/2\theta$, $p = m/2\theta$. Comparison with (4.1) and (4.6) shows that the order of the asymptotic behaviour of $\sigma^2(V_1)$ depends continuously on θ if $p(u)$ is given by (1.2), but the coefficient of the leading term does not.

PROOF OF THEOREM 4.3: Since $g'(x) \to 0$, $x \to \infty$, we have $f(x+b)/f(x) \to 1$, $x \to \infty$, uniformly with respect to b in compacta. So from (2.1)-(2.3)

$$P(x) = |A| f(r)(1+\epsilon_1(x)), \quad P(x+z) = |A| f(r)(1+\epsilon_2(x,z)),$$

$$P(x, x+z) = |A_z| f(r)(1+\epsilon_3(x,z), \tag{4.14}$$

where $\epsilon_i \to 0$, $r \to \infty$, $i = 1,2,3$, uniformly with respect to ω and $z \in A - A$. In (4.4) we take $\gamma(\lambda) = \gamma$ fixed and so large that $\epsilon_i \geqq -\delta$, $i = 1, 2$, $\epsilon_3 \leqq \delta$ for $r \geqq \gamma$. Then

$$\sigma^2(V_1) \leqq U_1 + \int_{A-A} \mathrm{d}z \int_\Omega \mathrm{d}\sigma(\omega) \int_\gamma^\infty r^{m-1} \mathrm{d}r.$$

$$\cdot \exp(-\lambda\nu f(r))\{-1+\exp(\lambda\beta(z) f(r))\}, \tag{4.15}$$

$$\nu = 2|A|(1-\delta), \quad \beta(z) = |A_z|(1+\delta). \tag{4.16}$$

The integrand in (4.15) does not depend on ω. We replace the integration variable r by $y = \lambda f(r)$ and find

$$\sigma^2(V_1) \leqq U_1 + m^{-1} S_m \int_{A-A} \mathrm{d}z \int_0^{\lambda f(\gamma)} \xi_0(\log \lambda c y^{-1})$$

$$\times \mathrm{e}^{-\nu y}(\mathrm{e}^{y\beta(z)} - 1) y^{-1} \mathrm{d}y, \tag{4.17}$$

with ξ_0 given by (3.4). Here $U_1 \to 0$, exponentially as $\lambda \to \infty$, by (4.5). Since $g^{-1}(\log x)$ varies slowly and $g'(x)$ varies regularly at ∞, also $\xi_0(\log x)$ varies slowly. We want to divide (4.17) by $\xi_0(\log \lambda)$ and interchange limit and integral in the right-hand side. In the domain of integration we have $c\lambda y^{-1} > \exp(g(\gamma))$. Since $g(x) \to \infty$, $x \to \infty$, we may take γ so large that by applying (1.7) with $\rho = 0$ to $\xi_0(\log x)$ we find, with $|\epsilon(s)| < \delta_1$, $s \geqq s_1$,

$$\xi_0(\log \lambda c y^{-1}) / \xi_0(\log \lambda) \leqq c_1 \max\{y^{-\delta_1}, y^{\delta_1}\},$$

So, taking into account (4.16), we may apply dominated convergence.

This gives, with Gröbner und Hofreiter [4], §313,3b,

$$\limsup_{\lambda\to\infty} \sigma^2(V_1)/\xi_0(\log\lambda)$$

$$\leqq m^{-1}S_m\int_{A-A}\mathrm{d}z\int_0^\infty \exp(-\nu y)\{\exp(y\beta(z))-1\}y^{-1}\mathrm{d}y$$

$$= -m^{-1}S_m\int_{A-A}\mathrm{d}z\,\log\left\{1-\tfrac{1}{2}(1+\delta)|A_z|(1-\delta)^{-1}|A|^{-1}\right\}.$$

A reversed inequality for liminf may be obtained from (4.4) with Fatou's lemma in a similar way by taking $\gamma(\lambda)=\gamma$ so large that in (4.14) we have $\epsilon_i \leqq \delta$, $i=1,2$, $\epsilon_3 \geqq -\delta$. Since these inequalities hold for all $\delta > 0$, the relation (4.13) follows.

THEOREM 4.4: *Let the u_i have probability density (1.4) where h is nondecreasing and varies regularly at ∞ of order $\rho^{-1} > m$. Then, with h^{-1} defined by (1.9) and S_m by (1.13) and with $S_1 = 2$,*

$$\sigma^2(V_1) \sim M_4\left(h^{-1}(\lambda)\right)^m, \quad \lambda\to\infty, \tag{4.18}$$

$$M_4 = m^{-1}S_m\Gamma(1-m\rho)\int_{A-A}\left\{2|A|^{m\rho}-(2|A|-|A_z|)^{m\rho}\right\}\mathrm{d}z. \tag{4.19}$$

PROOF: With $\gamma=\gamma(\lambda)$ sufficiently large in (4.4), the relations (4.14)–(4.16) continue to hold since $f(x+b)/f(x)\to 1$, $x\to\infty$, uniformly in b, by (1.7) applied to h. Define the measure μ on the Borel sets of $\mathbb{R}_+$ by

$$\mu(E)=\int_B r^{m-1}\mathrm{d}r, \quad B=\{x|r\geqq\gamma,\ f(x)\in E\}.$$

Then (4.15) becomes

$$\sigma^2(V_1)\leqq U_1+S_m\int_{A-A}G(z,\lambda)\mathrm{d}z, \tag{4.19}$$

$$G(z,\lambda)=\int_0^\infty \mathrm{d}\mu(u)\,\mathrm{e}^{-\lambda\nu u}\{\exp(\lambda u\beta(z))-1\}$$

$$=\int_0^\infty \mathrm{d}\mu(u)\int_{\nu u-\beta(z)u}^{\nu u}\lambda\exp(-\lambda t)\mathrm{d}t$$

$$=\int_0^\infty \lambda\,\mathrm{e}^{-\lambda t}\left\{\mu\left[\nu^{-1}t,(\nu-\beta(z))^{-1}t\right)\right\}\mathrm{d}t. \tag{4.20}$$

For $0 < \xi < f(\gamma)$ we have

$$\mu[\xi, \infty) = m^{-1}\left(h^{-1}(\xi^{-1})\right)^m - m^{-1}\gamma^m,$$

where $h^{-1}(\xi^{-1})$ varies regularly at 0 of order $-\rho$. So as $t \downarrow 0$

$$\mu[\nu^{-1}t,(\nu-\beta(z))^{-1}t) \sim m^{-1}\left\{\nu^{m\rho} - (\nu-\beta(z))^{m\rho}\right\}\left(h^{-1}(t^{-1})\right)^m,$$

where the right-hand side varies regularly at 0 of order $-m\rho$ with $-1 < -m\rho < 0$. By applying the Abelian theorem in Feller [3], Ch. XIII. 5 to (4.20) we see that as $\lambda \to \infty$,

$$G(z,\lambda)\left(h^{-1}(\lambda)\right)^{-m} \to m^{-1}\Gamma(1-m\rho)\left\{\nu^{m\rho} - (\nu-\beta(z))^{m\rho}\right\}.$$

This limit may be interchanged with the integration in (4.19) since with (4.16) and $|A_z| \leqq |A|$

$$\mu[\nu^{-1}t,(\nu-\beta(z))^{-1}t) \leqq \mu\left[\nu^{-1}t,|A|^{-1}(1-3\delta)^{-1}t\right),$$

where the right-hand side in (4.20) would make applicable the same Abelian theorem, showing that $G(z,\lambda)(h^{-1}(\lambda))^m$ is bounded by a constant for $\lambda \geqq \lambda_0$, so that there is dominated convergence. So from (4.19)

$$\limsup_{\lambda\to\infty} \sigma^2(V_1)\left(h^{-1}(\lambda)\right)^{-m}$$

$$\leqq m^{-1}S_m\Gamma(1-m\rho)\int_{A-A}\left(\nu^{m\rho} - (\nu-\beta(z))^{m\rho}\right)\mathrm{d}z.$$

Since this inequality, and a similar reversed one for liminf to be derived by Fatou's lemma, hold for any $\delta > 0$, the relation (4.18) follows with (4.16).

5. A nonrandom number of sets

We compare corresponding expectations and corresponding variances for a nonrandom number $N = M$ of sets and for a number of sets that is Poissonian with parameter $\lambda = M$. Expectations and variances for the Poissonian number of sets will be denoted by E_1, σ_1^2 and for the nonrandom number of sets by E_2, σ_2^2.

THEOREM 5.1: *We have, for $k = 1, 2, \ldots,$*

$$\begin{aligned}|E_1W_k - E_2W_k| \leqq\ & \tfrac{1}{2}k(k-1)M^{-1}E_1W_k \\ & + \tfrac{1}{2}\mathrm{e}(k+1)(k+2)M^{-1}E_1W_{k+2} \\ & + k(k+1)\mathrm{e}^kM^{-1}E_1W_{k+1},\end{aligned} \tag{5.1}$$

PROOF: We write, also for $k=0$,

$$Q_k(x) = (MP(x))^k \exp(-MP(x))/k!, \tag{5.2}$$

$$R_k(x) = \binom{M}{k} P^k(x)(1-P(x))^{M-k}, \tag{5.3}$$

$$Q_k(x) - R_k(x) = T_1(x) + T_2(x) + T_3(x), \tag{5.4}$$

$$T_1(x) = Q_k(x) - \binom{M}{k} P^k(x) \exp(-MP(x)),$$

$$T_2(x) = \binom{M}{k} P^k(x)\left\{\exp(-MP(x)) - (1-P(x))^M\right\},$$

$$T_3(x) = \binom{M}{k} P^k(x)(1-P(x))^M - R_k(x).$$

We use the inequalities for $x \geqq 0$, $y \geqq 0$:

$$1 - y \leqq \mathrm{e}^{-y}, \quad 0 \leqq \mathrm{e}^{-y} - 1 + y \leqq \tfrac{1}{2}y^2, \tag{5.5}$$

$$\mathrm{e}^{-My} - (1-y)^M = (\mathrm{e}^{-y} - 1 + y) \sum_{j=0}^{M-1} \mathrm{e}^{-jy}(1-y)^{M-1-j}$$

$$\leqq \tfrac{1}{2}My^2 \exp\{-(M-1)y\}, \tag{5.6}$$

$$(1-x)(1-y) \geqq 1 - x - y. \tag{5.7}$$

With (5.7) and (2.9)

$$0 \leqq T_1(x) \leqq \tfrac{1}{2}k(k-1)M^{k-1}P^k(x) \exp(-MP(x))/k!,$$

$$0 \leqq \int T_1(x)\mathrm{d}x \leqq \tfrac{1}{2}k(k-1)M^{-1}E_1W_k, \quad k \geqq 1. \tag{5.8}$$

With (5.6) and (2.9), since $P(x) \leqq 1$,

$$0 \leqq T_2(x) \leqq \tfrac{1}{2}\mathrm{e}M^{k+1}P^{k+2}(x) \exp(-MP(x))/k!,$$

$$0 \leqq \int T_2(x)\mathrm{d}x \leqq \tfrac{1}{2}\mathrm{e}(k+1)(k+2)M^{-1}E_1W_{k+2}, \quad k \geqq 0. \tag{5.9}$$

With (5.5), (5.7) and (2.9), since $P(x) \leqq 1$,

$$0 \leqq -T_3(x) \leqq k\mathrm{e}^k M^k P^{k+1}(x) \exp(-MP(x))/k!$$

$$0 \leqq -\int T_3(x)\mathrm{d}x \leqq k(k+1)\mathrm{e}^k M^{-1}E_1W_{k+1}, \quad k \geqq 0. \tag{5.10}$$

The relation (5.1) follows from (2.8), (2.9), (5.2), (5.3), (5.4), (5.8), (5.9), and (5.10). For $k=0$ we have $T_1(x)=T_3(x)=0$, so with (5.9)

$$0 \leqq \int (Q_0(x)-R_0(x))\mathrm{d}x \leqq \mathrm{e}M^{-1}E_1W_2. \tag{5.11}$$

COROLLARY 5.1: *For nonrandom $N=M$ theorems 3.1 and 3.2 with $\lambda = M$ continue to hold.*

PROOF: Since

$$\sum_{j=0}^{\infty} Q_k(x) = \sum_{j=1}^{\infty} R_k(x) = 1,$$

we have from (2.8), (2.9) and $V_k = \Sigma_{j \geqq k} W_k$,

$$E_1V_k - E_2V_k = \sum_{j=0}^{k-1} \int (R_j(x)-Q_j(x))\mathrm{d}x$$

$$= \int (R_0(x)-Q_0(x))\mathrm{d}x + \sum_{j=1}^{k-1} (E_2W_j - E_1W_j). \tag{5.12}$$

The corollary follows from (5.1), (5.11) and (5.12) since in theorem 3.1 we have $EW_k = \mathcal{O}(EV_j)$ and in Theorem 3.1 and 3.2 all EW_k are of the same order.

REMARK: The asymptotic relations stated at the end of Section 3 also hold for nonrandom N, since they only contain logarithmic terms.

THEOREM 5.2: *with $\sigma_i^2(V_1)$, $i=1,2$ as defined above,*

$$|\sigma_1^2(V_1)-\sigma_2^2(V_1)| \leqq e^2M^{-1}\{4|A-A|E_1W_2 + (E_1W_1)^2\}. \tag{5.13}$$

PROOF: From (2.7), (2.10), (2.11), (2.12) and (2.5) we have, writing $U(x,y)=P(x)+P(y)-P(x,y)$ and $L=\{(x,y)|y-x \in A-A\}$,

$$\sigma_1^2(V_1)-\sigma_2^2(V_1) = K_1+K_2+K_3, \tag{5.14}$$

$$K_1 = \int\int_L \{\exp(-MU(x,y)) - (1-U(x,y))^M\}\mathrm{d}x\mathrm{d}y,$$

$$K_2 = \int\int_L \{(1-P(x))^M(1-P(y))^M$$

$$-\exp(-MP(x)-MP(y))\}\mathrm{d}x\mathrm{d}y,$$

$$K_3 = \int\int_{L^c} \{(1-P(x))^M (1-P(y))^M - (1-P(x)-P(y))^M\} \mathrm{d}x\mathrm{d}y.$$

From (5.6) by symmetry, with $U(x, y) \geqq P(x)$ and by (2.9)

$$0 \leqq K_1 \leqq \tfrac{1}{2}M\int\int_L U^2(x,y) \exp(-(M-1)U(x,y))\mathrm{d}x\mathrm{d}y$$
$$\leqq M\int\int_{L\cap\{P(y)<P(x)\}} (2P(x))^2 \exp(-(M-1)P(x))\mathrm{d}x\mathrm{d}y$$
$$\leqq 4M|A-A|e\int P^2(x) \exp(-MP(x))\mathrm{d}x = 8M^{-1}\mathrm{e}|A-A|E_1W_2. \tag{5.15}$$

In a similar way

$$0 \leqq -K_2 \leqq M\int\int_L \{\exp(-P(x)-P(y)) - 1 + P(x) + P(y)\}$$
$$\cdot \exp\{-(M-1)(P(x)+P(y))\}\mathrm{d}x\mathrm{d}y$$
$$\leqq \tfrac{1}{2}Me^2\int\int_L (P(x)+P(y))^2 \exp(-MP(x)-MP(y))\mathrm{d}x\mathrm{d}y$$
$$\leqq 4M^{-1}\mathrm{e}^2|A-A|E_1W_2, \tag{5.16}$$

$$0 \leqq K_3 \leqq M\int\int P(x)P(y)(1-P(x))^{M-1}(1-P(y))^{M-1}\mathrm{d}x\mathrm{d}y$$
$$\leqq Me^2\left\{\int P(x) \exp(-MP(x))\right\}^2 = M^{-1}\mathrm{e}^2(E_1W_2)^2. \tag{5.17}$$

The relation (5.13) follows from (5.14)–(5.17).

COROLLARY 5.2: *For nonrandom $N = M$ Theorems 4.1, 4.2, 4.3 and 4.4 with $\lambda = M$ continue to hold.*

PROOF: From (5.13). In Theorem 4.1 and 4.2 the behaviour of $\sigma_1^2(V_1)$ is logarithmic, whereas $E_1W_k \to 0$, $\lambda \to \infty$, by Theorem 3.1. In Theorem 4.3 there is slow variation of $\sigma_1^2(V_1)$ as $\lambda \to \infty$ and the same holds for E_1W_k by Theorem 3.2. In Theorem 4.4 the order of $\sigma_1^2(V_1)$ and E_1W_k is the same by Theorem 3.1 and we have $M^{-1}E_1W_1 \to 0$ since $(h^{-1}(x))^m$ varies regularly at ∞ of order $m\rho < 1$.

Appendix

PROOF OF THEOREM 4.1: Define $d = d(x)$ and $T = T(x)$ by $d(x) = \|x\|$, $x \in A$, and

$$d(x) = \inf\{\|x-u\| : u \in A\}, \quad x \notin A, \tag{1}$$

$$T(x) \in \partial A, \quad \|x - T(x)\| = d(x), \quad x \notin A. \tag{2}$$

So $d(x)$, if $x \notin A$, is the distance of x to the foot $T(x)$ of the unique normal to ∂A, that passes through x. For fixed $x \notin A$ we define a Cartesian coordinate system with coordinates $(w_1, \dots, w_m) = w$ and origin $T(x)$. The w_m-axis is the inner normal to ∂A at $T(x)$ and the other axes are along the principal directions in the tangent plane to ∂A at $T(x)$. See Do Carmo [2], §3.2 and §3.3. In a neighbourhood of $T(x)$ the surface ∂A is described by

$$2w_m = \sum_{i=1}^{m-1} k_i w_i^2 + \varphi(w_1, \dots, w_{m-1}), \tag{3}$$

$$t^{-2}\varphi(w_1 \dots w_{m-1}) \to 0, \quad t \to 0, \quad t = \left(w_1^2 + \dots w_{m-1}^2\right)^{1/2}, \tag{4}$$

where $k_i = k_i(T(x))$.

In (2.1) we take w as new integration variable. Note that the v- and w-systems have different origins. We have

$$\|x - v\|^2 = \left(d(x) + w_m\right)^2 + t^2, \tag{5}$$

$$P(x) = c\int_B \exp\left\{-\alpha^{-2\theta}\left(\left(d(x) + w_m\right)^2 + t^2\right)^{\theta}\right\} \mathrm{d}w, \tag{6}$$

where $B = B(x)$ is the set A "as seen from $T(x)$ in the w-coordinates". For $d = d(x) \geqq d_1$ sufficiently large, with the convention about bounded functions stated in section 1, we have

$$\left((d + w_m)^2 + t^2\right)^{\theta} = d^{2\theta} + 2\theta w_m\left(1 + \eta d^{-1}\right)d^{2\theta-1} + \eta_+ t^2 d^{2\theta-2}, \tag{7}$$

$$P(x) = c\exp\left(-\alpha^{-2\theta}d^{2\theta}(x)\right)\int_0^{b(x)} \mathrm{d}w_m \times \int_{S(w_m)} \mathrm{d}w_1 \dots \mathrm{d}w_{m-1}\psi_1(x, w), \tag{8}$$

$$\psi_1(x, w) = \exp\left\{-2\theta\alpha^{-2\theta}w_m\left(1 + \eta d^{-1}\right)d^{2\theta-1} - \eta_+ t^2 d^{2\theta-2}\right\}, \tag{9}$$

where $b(x) = \max\{w_m | w \in B(x)\}$ and $S(y)$ is the intersection of $B(x)$ with the plane $w_m = y$. Let $P_\epsilon(x)$ be the contribution to (8) for $0 \leqq w_m \leqq \epsilon$. Then $P_\epsilon(x)/P(x) \to 1$ as $d(x) \to \infty$, uniformly in $T(x)$. In a neighbourhood of $T(x)$ the boundary ∂A lies between the paraboloids

$$2w_m = (1 \pm \delta) \sum_{i=1}^{m-1} k_i(x) w_i^2, \tag{10}$$

so we may take ϵ so small that for $0 \leqq w_m \leqq \epsilon$

$$S(w_m) \subset \left\{ (w_1, \ldots, w_{m-1}) | \sum_{i=1}^{m-1} k_i w_i^2 < 2(1-\delta)^{-1} w_m \right\}. \tag{11}$$

In (8) this gives, by putting $u_i = w_i d^{\theta - 1}(x)$, $i = 1, \ldots, m-1$, $u_m = w_m d^{2\theta - 1}(x)$,

$$P_\epsilon(x) \leqq P_\epsilon^*(x) = c \exp(-\alpha^{-2\theta} d^{2\theta}(x)) (d(x))^{m-(m+1)\theta}$$

$$\cdot \int_0^{\epsilon d^{2\theta - 1}} \mathrm{d}u_m \exp\{-2\theta \alpha^{-2\theta} u_m (1 + \eta d^{-1})\} \varphi_1(x, u), \tag{12}$$

$$\varphi_1(x, u) = \int_H \exp\{-\eta_+ (u_1^2 + \ldots u_{m-1}^2)\} \mathrm{d}u_1 \ldots \mathrm{d}u_{m-1}, \tag{13}$$

$$H = \left\{ (u_1, \ldots, u_{m-1}) | \sum_{i=1}^{m-1} k_i u_i^2 \leqq 2(1-\delta)^{-1} u_m / d(x) \right\}.$$

Since ∂A has uniformly continuous derivatives by the compactness of $\bar{A}$ and $0 < c_1 \leqq k_i(x) \leqq c_2 < \infty$, $i = 1, \ldots, m-1$, $x \in \partial A$, we have, uniformly in $T(x)$, as $d(x) \to \infty$,

$$\varphi_1(x, u) \sim (m-1)^{-1} S_{m-1} K^{-1/2}(T(x))$$

$$\times \left(2(1-\delta)^{-1} u_m / d(x)\right)^{(m-1)/2}.$$

This gives in (12)

$$P_\epsilon^*(x) \sim (1-\delta)^{(1-m)/2} M_1 K^{-1/2}(T(x)) (d(x))^{(m+1)(1/2 - \theta)}$$

$$\times \exp(-\alpha^{-2\theta} d^{2\theta}(x)),$$

$$M_1 = c 2^{(m-1)/2} (m-1)^{-1} S_{m-1} \Gamma\left(\tfrac{1}{2} m + \tfrac{1}{2}\right) \left(\alpha^{2\theta} / 2\theta\right)^{(m+1)/2}. \tag{14}$$

By considering the reversed inclusion with $1 + \delta$, analogous to (11), following from (10), and noting that $P_\epsilon(x)/P(x) \to 1$ we find that as $d(x) \to \infty$, uniformly in $T(x)$,

$$P(x) \sim M_1 K^{-1/2}(T(x))(d(x))^{(m+1)(1/2-\theta)} \exp(-\alpha^{-2\theta} d^{2\theta}(x)). \tag{15}$$

We now return to (2.12). By symmetry and with the inequality $\exp(-a) - \exp(-b) \leqq b - a$, $0 \leqq a \leqq b$, noting that $P(x, y) \leqq P(y)$, we have, with

$$J(x) = \{ y | d(y) > d(x) \}, \tag{16}$$

$$\sigma^2(V_1) = 2 \int \mathrm{d}x \int_{J(x)} dy\, \mathrm{e}^{-\lambda P(x)} (\mathrm{e}^{-\lambda P(y) + \lambda P(x,y)} - \mathrm{e}^{-\lambda P(y)})$$

$$\leqq 2\lambda \int dx \exp(-\lambda P(x)) \int_{J(x)} dy P(x, y) = H_1 + H_2,$$

where H_1 and H_2 are the contributions to the integral for $x \in D^c$ and $x \in D$, respectively, with

$$D = D(\lambda) = \{ x | d(x) \geqq q(\lambda) \}. \tag{17}$$

By (15), since $0 < a_1 \leqq K(\xi) \leqq a_2$, $\xi \in \partial A$, we may choose $q(\lambda)$ so that for some $c_1 > 0$ if $\lambda \geqq \lambda_1$,

$$P(x) \leqq \lambda^{-1/2}, \quad x \in D(\lambda), \quad P(x) \geqq c_1 \lambda^{-1/2}, \quad x \in D^c(\lambda). \tag{18}$$

With (2.4)

$$H_1 \leqq 2\lambda |A| \int_{D^c} P(x) \exp(-\lambda P(x)) \mathrm{d}x$$

$$\leqq 2\lambda |A| \exp(-c_1 \lambda^{1/2}) \int_{D^c} P(x) \mathrm{d}x \leqq 2\lambda |A|^2 \exp(-c_1 \lambda^{1/2}). \tag{19}$$

We proceed with

$$\sigma^2(V_1) \leqq H_1 + 2\lambda \int_D \exp(-\lambda P(x)) Q(x) \mathrm{d}x, \tag{20}$$

$$Q(x) = \int_{J(x)} P(x, y) \mathrm{d}y. \tag{21}$$

Also from (2.12)

$$\sigma^2(V_1) \geqq 2\lambda \int_D \mathrm{d}x \int_{J(x)} \mathrm{d}y \exp(-\lambda P(x) - \lambda P(y)) P(x, y). \qquad (22)$$

From (2.5) we see that in (22) we may assume $\|y - x\| < 2d_A$ so that $T(x) - T(y) \to 0$ and $K(T(x)) - K(T(y)) \to 0$ as $d(x) \to \infty$. Since $q(\lambda) \to \infty$ as $\lambda \to \infty$ it follows from (17) that in (22)

$$P(y) \leqq (1 + \epsilon_2(\lambda)) P(x), \quad \epsilon_2(\lambda) \to 0, \quad \lambda \to \infty, \qquad (23)$$

So that with (21)

$$\sigma^2(V_1) \geqq 2\lambda \int_D \exp\{-\lambda(2 + \epsilon_2(\lambda)) P(x)\} Q(x) \mathrm{d}x. \qquad (24)$$

We now have to estimate $Q(x)$. From (21), (2.2) and (2.3)

$$Q(x) = \int_{J(x)} \mathrm{d}y \int_A I_{A-y+x}(v) p(x - v) \mathrm{d}v$$

$$= \int_A p(x - v) \mathrm{d}v \int I_{J(x)}(y) I_A(v + y - x) \mathrm{d}y$$

$$= \int_A p(x - v) \mathrm{d}v \int I_{J(x)}(w + x - v) I_A(w) \mathrm{d}w$$

$$= \int p(x - v) |A \cap (J(x) - x + v)| \mathrm{d}v.$$

With the same coordinates and notations as in (4), (5) and (6)

$$Q(x) = c \int_{B(x)} \exp\left\{ -\alpha^{-2\theta} \left((d(x) + w_m)^2 + t^2 \right)^{\theta} \right\} |F(x, w)| \mathrm{d}w, \qquad (25)$$

where $F(x, w)$ for fixed w is the set $A \cap (J(x) - x + v)$ "as seen in the new coordinates" (see Fig. 1), where $F(x, w) = F_1 \cup F_2$, with $F_1 = F(x, w) \cap \{y | y_m \geq w_m\}$ and $F_2 = F(x, w) \cap \{y | y_m < w_m\}$.

With $S(y)$ defined as following (8) and (9) and with the same

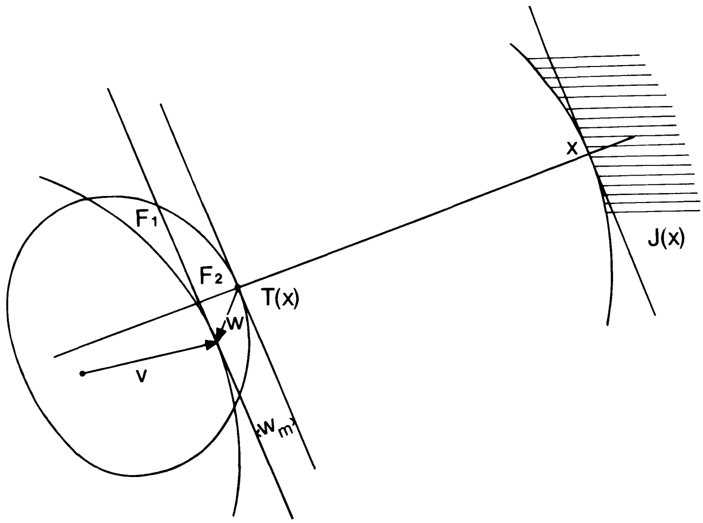

Figure 1. $F(x,w)=F_1 \cup F_2$

inclusions as in (10) and (11), denoting by $|\ |_{m-1}$ volume in $\mathbb{R}^{m-1}$, we have

$$|F_2| = \int_0^{w_m} |S(y)|_{m-1} \mathrm{d}y \sim \int_0^{w_m} (m-1)^{-1} S_{m-1} \prod_{i=1}^{m-1} \left(2yk_i^{-1}\right)^{1/2} \mathrm{d}y$$

$$= (m^2-1)^{-1} S_{m-1} K^{-1/2}(T(x))(2w_m)^{(m+1)/2}, \quad w_m \to 0, \quad (26)$$

uniformly in $T(x)$. For $d(x) \to \infty$ the set $J(x)$ approximates to a ball with radius $d(x)$, uniformly in $T(x)$. The projection of F_1 on the plane $w_m = 0$ has diameter $\mathcal{O}(w_m^{1/2})$, uniformly in x and we have

$$|F_1| \leqq c_3 \int_0^{c_4 w_m^{1/2}} t^m \mathrm{d}t/\mathrm{d}(x) = c_5 w_m^{(m+1)/2}/\mathrm{d}(x), \quad 0 \leqq w_m \leqq \epsilon.$$

With (26) this gives

$$|F(x,w)| = \left(1 + \epsilon(w_m) + \eta d^{-1}\right)(m^2-1)^{-1} S_{m-1} K^{-1/2}(T(x))$$

$$\times (2w_m)^{(m+1)/2}, \quad (27)$$

with $\epsilon(w_m) \to 0$, $w_m \to 0$, uniformly in x. We also may prove (27) by considering the paraboloids approximating ∂A in $T(x)$, see (10), and to $\partial F(x, w)$ in w and integrate the volume between them.

We now have to substitute (7) and (27) into (25), write (25) in the same form as (8) and apply the same technique of two-sided inclusion that derived (15) from (8). We find

$$Q(x) \sim M_2(d(x))^{(m+1)(1-2\theta)} \exp(-\alpha^{-2\theta} d^{2\theta}(x))/K(T(x)),$$

$$M_2 = c(m-1)^{-1}(m^2-1)^{-1} 2^m m! S_{m-1}^2 (\alpha^{2\theta}/2\theta)^{m+1}, \tag{28}$$

as $\mathrm{d}(x) \to \infty$, uniformly in $T(x)$. With (15) this gives, uniformly in $T(x)$,

$$Q(x)/P(x) \sim M_3 K^{-1/2}(T(x))(\mathrm{d}(x))^{(m+1)(1/2-\theta)}, \quad \mathrm{d}(x) \to \infty, \tag{29}$$

$$M_3 = S_{m-1} m! (\theta^{-1}\alpha^2\theta)^{(m+1)/2} / \{(m^2-1)\Gamma(\tfrac{1}{2}m + \tfrac{1}{2})\}. \tag{30}$$

In order to apply (15) and (30) to (20) we define a third, curvilinear, coordinate system on A^c. The new coordinates (R, ξ) and the Cartesian coordinates $x = (x_1, \ldots, x_m)$ of a point in A^c are connected by the relations

$$\xi = T(x), \quad R = \mathrm{d}(x), \tag{31}$$

$$x = \xi - RN(\xi), \tag{32}$$

with $T(x)$ and $\mathrm{d}(x)$ defined by (1), (2), so that $\xi \in \partial A$, and $N(\xi)$ the unit inward normal to ∂A at ξ. So $x(R, \xi)$ is the endpoint of the outer normal to ∂A at ξ with length R. To derive the Jacobian of (32) we replace ξ by a set of coordinates $u_1, \ldots, u_m$ in ∂A in a neighbourhood of ξ, i.e. a diffeomorphism $f: U \subset \mathbb{R}^{m-1} \to V \cap \partial A$, where V is a neighbourhood in $\mathbb{R}^m$ of ξ. Then (32) becomes

$$x = f(u_1, \ldots, u_{m-1}) - RN(u_1, \ldots, u_{m-1}), \tag{33}$$

and the Jacobian J of (33) is the determinant of the matrix consisting of the column vectors $-N$, $f_{u_1} - RN_{u_1}, \ldots, f_{u_{m-1}} - RN_{u_{m-1}}$, where f_{u_i}, etc., denotes partial differentiation. So

$$J = (-1)^m R^{m-1}(1 + \eta R^{-1}) J_1,$$

where J_1 is the determinant of the matrix consisting of the column vectors $N, N_{u_1}, \ldots, N_{u_{m-1}}$. Now let $\{a_{ij} = a_{ij}(u),\ i, j = 1, \ldots, m-1\}$ be the matrix describing the differential of the Gauss map at ξ with respect to the basis of the tangent vectors f_{u_i}, $i = 1, \ldots, m-1$. Then we have, see Do Carmo

[2], §3.3,

$$N_{u_i} = \sum_{r=1}^{m-1} a_{ri} f_{u_r}, \quad i,\ldots,m-1.$$

Applying this to J_1 we find

$$J = (-1)^m R^{m-1}(1+\eta R^{-1})\mathrm{Det}\{a_{ij}\}J_2,$$

where J_2 is the determinant of the matrix consisting of the column vectors $N, f_{u_1},\ldots,f_{u_{m-1}}$.

Now N is a unit vector orthogonal to the f_{u_i}. So J_2 is the $(m-1)$-dimensional volume of the parallelopiped spanned by the f_{u_i} and J_2 $\mathrm{d}u_1\ldots\mathrm{d}u_{m-1}$ is equal to the surface area element $\mathrm{d}\mathcal{O}(\xi)$ at ξ of ∂A. We have $\mathrm{Det}\{a_{ij}\} = K(\xi)$, see Do Carmo [2], §3.3. So

$$\int_{A^c} h(x)\mathrm{d}x = \int_{\partial A} \mathrm{d}\mathcal{O}(\xi)K(\xi)\int_0^\infty R^{m-1}(1+\eta R^{-1})h(R,\xi)\mathrm{d}R.$$

Writing $P(R,\xi)$ for $P(x)$ we see from (20), (17) and (29)

$$\sigma^2(V_1) \leqq H_1 + 2\lambda M_3 \int_{\partial A} \mathrm{d}\mathcal{O}(\xi)K^{1/2}(\xi)Z(\lambda,\xi), \tag{34}$$

$$Z(\lambda,\xi) = \int_{q(\lambda)}^\infty R^{m-1+(m+1)(1/2-\theta)}(1+\epsilon_3)P(R,\xi)$$

$$\times \exp(-\lambda P(R,\xi))\mathrm{d}R, \tag{35}$$

with $\epsilon_3 = \epsilon_3(R,\xi) \to 0$ as $R \to \infty$, uniformly in ξ. In (35) we take $P(R,\xi)$ as new integration variable. From (6)

$$|\partial P(R,\xi)/\partial R| = 2\theta\alpha^{-2\theta}R^{2\theta-1}(1+\eta R^{-1})P(R,\xi).$$

So with (18) and (1.11)

$$Z(\lambda,\xi) \leqq \alpha^{2\theta}(2\theta)^{-1}\int_0^{\lambda^{-1/2}} R_u^{2\theta b_1}(1+\epsilon_4)\,\mathrm{e}^{-\lambda u}\mathrm{d}u, \tag{36}$$

where $\epsilon_4 = \epsilon_4(u,\epsilon) \to 0$, $u \to 0$, uniformly in ξ and $R_u = R(u,\xi)$ is defined by

$$P(R_u,\xi) = u. \tag{37}$$

Note that $P(R, \xi)$ decreases strictly and continuously with R for $R > 2d_A$. From (6)

$$c|A| \exp\left\{-\alpha^{-2\theta}\left((R+d_A)^2 + d_A^2\right)^{\theta}\right\} \leqq P(R, \xi)$$

$$\leqq c|A| \exp\left\{-\alpha^{-2\theta}(R-d_A)^{2\theta}\right\}.$$

So from (37)

$$-d_A + \alpha\left\{\left(\log c|A|u^{-1}\right)^{\frac{1}{\theta}} - \alpha^{-2}d_A^2\right\}^{1/2} \leqq R(u, \xi)$$

$$\leqq d_A + \alpha\left(\log c|A|u^{-1}\right)^{1/2\theta},$$

$$R(u, \xi) \sim \alpha\left(\log u^{-1}\right)^{1/2\theta}, \quad u \downarrow 0,$$

uniformly in ξ. Substitution into (36) and putting $\lambda u = y$ gives

$$Z(\lambda, \xi) \leqq \alpha^{2\theta(1+b_1)}(2\theta\lambda)^{-1}\int_0^{\lambda^{1/2}}\left(\log \lambda y^{-1}\right)^{b_1}(1+\epsilon_5)\, e^{-y} dy,$$

where $\epsilon_5 = \epsilon_5(\lambda y^{-1}, \xi)$ and $\epsilon_5(t, \epsilon) \to 0$ as $t \to \infty$, uniformly in ξ. Substituting into (34) and dividing by $(\log \lambda)^{b_1}$ we find the right-hand inequality of (4.1) by (19) and dominated convergence. Note that $\lambda y^{-1} \geqq \lambda^{1/2}$ in the domain of integration. This disposes of the case $b_1 < 0$. If $b_1 > 0$ we apply the inequality

$$|\log \lambda y^{-1}/\log \lambda| \leqq 1 + c_3|\log y|, \quad \lambda \geqq \lambda_2.$$

From (24) we derive the left-hand inequality of (4.1) in a similar way, using Fatou's lemma, the factor $\frac{1}{2}$ arising from the exponent $-\lambda(2 + \epsilon_2(\lambda))P(x)$ instead of $-\lambda P(x)$.

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