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THE STABILITY OF CERTAIN CLASSES OF UNI-RULED VARIETIES

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Introduction

The first results on the deformations of ruled varieties were obtained by Kodaira and Spencer [8], where they show that all small deformations of a ruled surface are also ruled. Their argument uses the Castelnuovo-Enriques criterion for ruledness in an essential way, and thus, in view of the absence of a similar criterion for varieties of higher dimension, their method cannot be applied to the study of the deformations of a general ruled variety. In fact, our paper [9] shows that the straightforward generalization of the Kodaira-Spencer theorem is false, in that there is a ruled threefold that can be deformed into a threefold that is not ruled. Here we will study the deformations of ruled varieties, and other related varieties, and we will give some natural conditions which allow one to recover the stability results.

Section 1 is concerned with a technique of an approximate resolution of singularities in a family of varieties. In Section 2, we define the various classes of varieties that we will study, and we prove our main results in Section 3 (see especially Theorem 3.11).

We should mention that the smooth deformations of uni-ruled varieties have been studied by ourselves in [10] and in a work of Fujiki [4]. Both of these papers use somewhat different techniques from those employed in this work; the main result in both papers is that all smooth deformations of a uni-ruled variety are uni-ruled in characteristic zero. In characteristic $p > 0$ a similar result holds after imposing a separability hypothesis.

If X and Y are cycles, intersecting properly on a smooth variety U, we let $X \cdot Y$ denote the cycle intersection; if X and Y intersect in a zero cycle, we denote the degree of $X \cdot Y$ by $I(X \cdot Y;\ U)$. Subvarieties of U will be identified with cycles, without additional notation, as the context requires; if Z is a subscheme of U, we let $|Z|$ denote the associated cycle. We fix an algebraically closed base field k and a discrete valuation ring $\mathcal{O}$ with residue field k and quotient field K. Unless otherwise specified, all schemes, morphisms, and rational maps will be over k. If X and Y are schemes, with morphism $f: X \to Y$, we call X a fiber variety over Y if X is of finite type over Y, and f is faithfully flat with connected and reduced fibers.

If $f: V \to M$ is a flat, projective morphism, we denote the Hilbert scheme of V over M by $\mathrm{Hilb}(V/M)$, the universal bundle of subschemes of V, flat over M, by $h: H(V/M) \to \mathrm{Hilb}(V/M)$, and the Hilbert point of a subscheme X of V by $h(X)$.

This paper is a modification of the author's Brandeis doctoral thesis. I am greatly indebted to my thesis advisor, Teruhisa Matsusaka, for his guidance and encouragement.

§1. Resolution of singularities in a family

In this section we consider the following problem: suppose we are given a smooth projective fiber variety p: $V \to \mathrm{Spec}(\mathcal{O})$ over a DVR $\mathcal{O}$, and a rational map ϕ from the closed fiber V_0 to a variety Y. How can we resolve the indeterminacy of ϕ by replacing V with a birationally equivalent family p^*: $V^* \to \mathrm{Spec}(\mathcal{O})$, and keep the new family as smooth as possible?

We will use without comment the elementary properties of monoidal transformations; for definitions and proofs of these basic results we refer the reader to [5].

If X is a noetherian scheme and W a closed subscheme, we let

$$u_W: X_W \to X$$

denote the monoidal transformation with center W. Let Y be a closed subscheme of X. We define $\mathrm{Sing}_X(Y)$ to be the closed subset of Y,

$$\mathrm{Sing}_X(Y) = \left\{ \begin{array}{l} y \in Y \mid y \text{ is not smooth on } X, \\ \text{or } y \text{ is not smooth on } Y \end{array} \right\}.$$

Let p: $X \to \mathrm{Spec}(\mathcal{O})$ be an $\mathcal{O}$-scheme with closed subscheme Y. We define $\mathrm{Sing}(Y)$ to be the closed subset of Y,

$$\begin{aligned} \mathrm{Sing}(Y) = {} & \mathrm{Sing}_{X \otimes K}(Y \otimes K) \cup \mathrm{Sing}_{X \otimes k}(Y \otimes k) \\ & \cup \{ y \mid X \otimes k \text{ and } Y \text{ do not intersect properly at } y \}. \end{aligned}$$

In general, if $u: Y \to X$ is a birational morphism of smooth varieties, then the exceptional locus of u is a pure codimension one subset of Y. If this exceptional locus has irreducible components $E_1, \ldots, E_s$, we call the divisor $E = \sum_1^s E_i$ the exceptional divisor of u.

The following result is an easy consequence of the basic properties of monoidal transformations, and we leave its proof to the reader.

LEMMA 1.1: *Let* p: $X \to \mathrm{Spec}(\mathcal{O})$ *be a (quasi-)projective fiber variety over* $\mathrm{Spec}(\mathcal{O})$ *and let* W *be a reduced closed subscheme of* X. *Let* $X' = X - \mathrm{Sing}(W)$, $W' = W - \mathrm{Sing}(W)$ *and let* u: $X_W \to X$, u' : $X'_{W'} \to X'$ *be the respective monoidal transformation. Then*

(i) $p \circ u$: $X_W \to \mathrm{Spec}(\mathcal{O})$ is a (quasi-)projective fiber variety over $\mathrm{Spec}(\mathcal{O})$.

Suppose further that p: $X \to \mathrm{Spec}(\mathcal{O})$ is a smooth quasi-projective fiber variety with geometrically irreducible fibers and that $X \otimes k$ is not contained in $\mathrm{Sing}(W)$. Then

(ii) $p \circ u'$: $X'_{W'} \to \mathrm{Spec}(\mathcal{O})$ is a smooth quasi-projective morphism with geometrically irreducible fibers

(iii) $X'_{W'}$ is an open subscheme of X_W

(iv) $X'_{W'} \otimes K \to X \otimes K$ and $X'_{W'} \otimes k \to X \otimes k$ are birational morphisms.

(v) Let $\mathcal{E}$ be the exceptional divisor of u', and E the exceptional divisor of $u' \otimes k$. Then

$$\mathcal{E} \cdot (X'_W \otimes k) = E.$$

DEFINITION 1.2: Let p: $V \to \mathrm{Spec}(\mathcal{O})$ be a smooth projective fiber variety of fiber dimension n. Suppose there is a smooth complete variety S and a dominant rational map ϕ: $V \otimes k \to S$. A good resolution of the singularities of ϕ for the family V is a pair (u: $V' \to V$, U') where u: $V' \to V$ is a projective birational morphism, V' is irreducible, and U' is an open subscheme of V' satisfying

(i) $p \circ u$: $U' \to \mathrm{Spec}(\mathcal{O})$ is a smooth quasi-projective morphism with geometrically irreducible fibers;

(ii) the morphism $u \otimes k$: $U' \otimes k \to V \otimes k$ is birational.

(iii) Let $\overline{U' \otimes k}$ denote the closure of $U' \otimes k$ in V'. Then $\overline{U' \otimes k}$ is smooth and the induced rational map ϕ': $\overline{U' \otimes k} \to S$, $\phi' = \phi \circ (u \otimes k)$, is a morphism. Furthermore

$$\dim\big(\phi'(\overline{U' \otimes k} - U' \otimes k)\big) \leqslant n - 2.$$

(iv) Let $\mathcal{E} \subseteq U'$ be the exceptional divisor of the morphism $u : U' \to V$. Write $\mathcal{E}$ as a sum of irreducible divisors

$$\mathcal{E} = \sum_{i=1}^{r} \mathcal{E}_i,$$

and denote the divisor $\mathcal{E}_i \otimes k$ by $\mathcal{E}_i^0$. If E is an irreducible component of $\mathcal{E} \cap U' \otimes k$, then we can write the divisor $1 \cdot E$ as

$$1 \cdot E = \sum_{i=1}^{r} n_i \mathcal{E}_i^0$$

for suitable integers n_i.

We note that condition (iv) is trivially satisfied if all the $\mathscr{E}_i^0$ are irreducible; in general, (iv) states that the subgroup of the group of divisors on $U' \otimes k$ generated by the $\mathscr{E}_i^0$ is the same as the subgroup generated by their irreducible components. We also note that (iv) refers to *divisors* on $U' \otimes k$, *not* divisors modulo linear equivalence.

EXAMPLE: Let $V = \mathbb{P}^2 \times \operatorname{Spec}(\mathcal{O})$. Blow up $V \otimes k$ at a point p, and then blow up this surface at a point q lying on the exceptional curve. Let S be the resulting surface, let v: $S \to V \otimes k$ be the blowing up morphism and let $E = E_1 + E_2$ (E_i-irreducible) be the exceptional divisor of v, where E_2 is the exceptional divisor of the blowup at q. Let ϕ: $V \otimes k \to S$ be the inverse to v. Now let $C_1 \subseteq V$ be the image of a section s_1: $\operatorname{Spec}(\mathcal{O}) \to V$ that passes through p, and let u_1: $V_1 \to V$ be the blow up of V along C_1. We identify $V_1 \otimes k$ with the blow up of $V \otimes k$ at p. Let $C_2 \subseteq V_1$ be the image of a section s_2: $\operatorname{Spec}(\mathcal{O}) \to V_1$ that passes through q, but is not contained in the exceptional divisor of u_1. Let u_2: $V_2 \to V_1$ be the blow up of V_1 along C_2 and let u: $V_2 \to V$ be the composition $u_1 \circ u_2$. Then (u: $V_2 \to V$, V_2) is a good resolution of the singularities of ϕ for the family V. In fact (i)–(iii) are obvious since V_2 is smooth over $\mathcal{O}$ and since $V_2 \otimes k$ is isomorphic to S. To check (iv), we note that the exceptional divisor $\mathscr{E}$ of u: $V_2 \to V$ consists of two irreducible components $\mathscr{E}_1$ and $\mathscr{E}_2$, where $\mathscr{E}_2 = u_2^{-1}(C_2)$ and $\mathscr{E}_1$ is the proper transform $u_2^{-1}[u_1^{-1}(C_1)]$. We have $\mathscr{E}_1 \otimes k = E_1 + E_2$ and $\mathscr{E}_2 \otimes k = E_2$, so (iv) is satisfied.

Our main object is to show that a good resolution exists for each family and each rational map. Our procedure will be inductive; to help the induction along we require a slightly different notion: that of the replica of a birational morphism.

DEFINITION 1.3: Let p: $V \to \operatorname{Spec}(\mathcal{O})$ be a smooth quasi-projective morphism with geometrically irreducible fibers. Let X and X' be smooth varieties and let g: $X' \to X$ be a projective birational morphism. Suppose we have a birational morphism f: $V \otimes k \to X$ with

$$\operatorname{codim}_X \overline{\left(X - f(V \otimes k)\right)} \geqslant 2.$$

A *replica of g for the family V* is a pair (u: $V' \to V$, U') where u: $V' \to V$ is a projective birational morphism, and U' is open subscheme of V' satisfying

(i) $p' = p \circ u$: $U' \to \operatorname{Spec}(\mathcal{O})$ is a smooth quasi-projective morphism with geometrically irreducible fibers,
(ii) the morphism $p' \otimes k$: $U' \otimes k \to V \otimes k$ is birational,
(iii) the induced rational map

$$f' : U' \otimes k \to X', \qquad f' = g^{-1} \circ f \circ (u \otimes k),$$

is a morphism and

$$\operatorname{codim}_{X'}\overline{(X' - f'(U' \otimes k))} \geqslant 2.$$

(iv) Let $\mathscr{E}$ be the exceptional divisor of u: $U' \to V$. Write $\mathscr{E}$ as a sum of irreducible divisors

$$\mathscr{E} = \sum_{i=1}^{r} \mathscr{E}_i$$

and let $\mathscr{E}_i^0$ denote the divisor $\mathscr{E}_i \otimes k$. If E is an irreducible component of $\mathscr{E} \otimes k$, then we can write the divisor $1 \cdot E$ as

$$1 \cdot E = \sum_{i=1}^{r} n_i \mathscr{E}_i^0$$

for suitable integers n_i.

LEMMA 1.4 (*Severi's method of the projecting cone*): *Let X be a smooth quasi-projective variety contained in a projective space $\mathbb{P}^N$. Let $Z, B_1, \ldots, B_s$ be subvarieties of X. Then there is a subvariety $\mathscr{Z}$ of $\mathbb{P}^N$ such that*

(i) *$\mathscr{Z}$ and X intersect properly in $\mathbb{P}^N$.*

(ii) *$\mathscr{Z} \cdot X = Z + \sum_{i=1}^{r} Z_i$ as a cycle, with $Z \neq Z_i$ for all $i = 1, \ldots, r$ and $Z_i \neq Z_j$ for all $i \neq j$; $i, j = 1, \ldots, r$.*

(*iii*) *$Z_i \cap B_j$ is of codimension at least one on both B_j and Z_i for all $i = 1, \ldots, r$; $j = 1, \ldots, s$.*

PROOF (see [3], [12], or [13]): In each of these works, $\mathscr{Z}$ is the cone over Z with vertex a suitably general linear subvariety of $\mathbb{P}^N$.

The next lemma provides the key step in the inductive argument.

LEMMA 1.5: *Suppose $\mathcal{O}$ is equi-characteristic. Let p: $V \to \operatorname{Spec}(\mathcal{O})$ be a smooth quasi-projective morphism with geometrically irreducible fibers.*

Let X be a smooth variety, C a smooth subvariety of X of codimension at least two, u_C: $X_C \to X$ the monoidal transformation with center C; let $B_1, \ldots, B_s$ be irreducible subvarieties of X_C and let B_i' denote the image $u_C(B_i)$ in X. Suppose there is a birational morphism f: $V \otimes k \to X$. Letting $P = \overline{X - f(V \otimes k)}$ and $X^0 = X - P$, we further suppose that

(*a*) *$\operatorname{codim}_X(P) \geqslant 2$,*

(*b*) *$C \cap X^0$ and $B_i' \cap X^0$, $i = 1, \ldots, s$, are nonempty,*

(*c*) *f^{-1} is a morphism in a neighborhood of $C \cap X^0$ and $B_i' \cap X^0$, $i = 1, \ldots, s$.*

Then there is an open subset U of V, and a subvariety $\overline{W}$ of V such that

(A) *$(u_{\overline{W}}\colon V_{\overline{W}} \to V, U_W)$ is a replica of u_C for the family V, where $W = \overline{W} \cap U$.*

(B) *Letting $f_W\colon U_W \otimes k \to X_C$ be the induced morphism, and letting $P_C = \overline{X_C - f_W(U_W \otimes k)}$ and $X_C^0 = X_C - P_C$, then $B_i \cap X_C^0$ is nonempty for $i = 1,\dots,s$, and f_W^{-1} is an isomorphism in a neighborhood of $B_i \cap X_C^0$.*

(C) *W is smooth over Spec($\mathcal{O}$).*

Proof: We first note that we may replace $V \otimes k$ with $V \otimes k - f^{-1}(P)$, and V with $V - f^{-1}(P)$. Having done so, and changing notation, we may assume that $f(V \otimes k)$ is contained in X^0. We may also assume $X = X^0$.

Noting that f^{-1} is an isomorphism in a neighborhood of C and the B_i', we let $\tilde{C} = f^{-1}(C)$, $\tilde{B}_i = f^{-1}(B_i')$.

We may assume that V is a Spec($\mathcal{O}$) subscheme of $\mathbb{P}_k^N \times_k \mathrm{Spec}(\mathcal{O})$. By Lemma 1.4, there is a subvariety Z of $\mathbb{P}_k^N$ such that

(1) $Z \cap V \otimes k$ is proper; $Z \cdot (V \otimes k) = \tilde{C} + \sum_{i=1}^r D_i$,

(2) $D_i \neq D_j$ for $i \neq j$ and $D_i \neq \tilde{C}$ for all i,

(3) $D_i \cap \tilde{B}_j$ is of codimension at least one on both D_i and $\tilde{B}_j$, for all $i = 1,\dots,r$ and $j = 1,\dots,s$.

Let $\overline{W}$ be the closed subscheme $(Z \times \mathrm{Spec}(\mathcal{O})) \cap V$ of V and let T be the closed subset of V

$$T = \mathrm{Sing}(\overline{W}) \cup \left(\bigcup_i D_i\right).$$

Note that (b) and (c), together with statements (1)–(3) imply that

(4) $\mathrm{codim}_V(T \otimes k) \geq 2$; neither C nor any B_i' is contained in $f(T \otimes k)$.

Let U be the open subscheme $V - T$ of V. Let $\overline{C}$ and $\overline{B}_j$ denote the intersection of U with $\tilde{C}$ and $\tilde{B}_j$ respectively. Then from (1)–(3) it follows that

(5) $\overline{W} \cap U \otimes k$ is proper; $\overline{W} \cdot (U \otimes k) = \overline{C}$; $\overline{B}_j \cap (U \otimes k) \neq \emptyset$.

If $\overline{W}$ is reducible, replace $\overline{W}$ with an irreducible component $\overline{W}^0$ such that $\overline{W}^0 \cap (U \otimes k) = \overline{C}$; changing notation, we may assume that $\overline{W}$ is irreducible.

Let $W = U \cap \overline{W}$ and let $u_W\colon U_W \to U$, $u_{\overline{W}}\colon V_{\overline{W}} \to V$ be the respective monoidal transformations. We claim that $(u_{\overline{W}}\colon V_{\overline{W}} \to V, U_W)$ is a replica of u_C.

Conditions (i) and (ii) of Definition 1.3 are immediate from our construction. For condition (iii) we note the following diagram of rational maps

$$\begin{array}{ccc} U_W \otimes k & \xrightarrow{u_W \otimes k} & U \otimes k \\ \downarrow f_W & & \downarrow f_{|U \otimes k} \\ X_C & \longrightarrow & X \end{array} \tag{6}$$

where f_W is defined to make the diagram commute.

Since f^{-1} is an isomorphism in a neighborhood of $C \cap X^0$ and since

$$f^{-1}_{|U \otimes k}(C \cap X^0) = \bar{C} = W \cap (U \otimes k),$$

it follows that $u_W \otimes k$: $U_W \otimes k \to U \otimes k$ is the monoidal transformation with center $\bar{C}$, and f_W is a birational morphism. Let P_C be the closed subset $\overline{X_C - f_W(U_W \otimes k)}$ of X_C. By (6), we see that

$$P_C \subseteq u_C^{-1}(P) \cup u_C^{-1}(f(T)).$$

Suppose there was an irreducible component A of P_C of codimension one on X_C. Then

$$u_C(A) \subseteq P \cup f(T),$$

and as $\operatorname{codim}_X(P \cup f(T)) \geqslant 2$, it follows that

$$u_C(A) \subseteq C \cap (P \cup f(T))$$

and hence

$$A \subseteq u_C^{-1}(C \cap (P \cup f(T))).$$

But by (b), (c), and (4), $P \cup f(T)$ does not contain C. Thus $\operatorname{codim}_{X_C}[u_C^{-1}(C \cap (P \cup f(T)))]$ is at least two, and no such A could exist. This completes the verification of (iii) in Definition 1.3. We now check (iv).

As W is smooth and irreducible, the exceptional divisor $\mathscr{E}$ of u_W is the irreducible subvariety $u_W^{-1}(W)$ taken with multiplicity one. From the diagram (6), we have

$$\mathscr{E} \otimes k = (u_W \otimes k)^{-1}(\bar{C}).$$

As the right hand side is irreducible, property (iv) is clear and the proof of (A) is complete.

(B) follows easily from (5) and diagram (6); to show (C), we note that W is flat over Spec($\mathcal{O}$) since $W \otimes K$ and $W \otimes k$ have the same dimension. As both $W \otimes K$ and $W \otimes k$ are smooth, W is smooth over Spec($\mathcal{O}$) by [6, exp. II, thm. 2.1]. This completes the proof of the lemma. Q.E.D.

PROPOSITION 1.6: *Assume $\mathcal{O}$ is equi-characteristic. Let p: $V \to$ Spec($\mathcal{O}$) be a smooth quasi-projective morphism with geometrically irreducible fibers. Let X be a smooth variety and let*

$$v_i = u_{C_i} \colon X_{i+1} \to X_i, \qquad i = 0, \dots, n-1$$

be a sequence of monoidal transformations with smooth center $C_i \subseteq X_i$. *Let* C_i' *denote the image of* C_i *in* X_0 *for* $i = 0, \dots, n-1$ *and let* $v \colon X_n \to X$ *denote the composition* $v = v_0 \circ \dots \circ v_{n-1}$. *Suppose there is a birational morphism* $f \colon V \otimes k \to X$. *Letting* $P = X - f(V)$, $X^0 = X - P$, *we further suppose that*
(a) $\operatorname{codim}_X(P) \geqslant 2$;
(b) $C_i' \cap X^0$ *is non-empty, for* $i = 0, \dots, n-1$;
(c) f^{-1} *is an isomorphism in a neighborhood of* $C_i' \cap X^0$, *for* $i = 0, \dots, n-1$.

Then there is a sequence of monoidal transformations

$$u_i = u_{W_i} \colon V_{i+1} \to V_i; \qquad i = 0, \dots, n-1; \quad V_0 = V,$$

with irreducible center $W_i \subseteq V_i$, and open subsets U_i of V_i such that U_i and $U_i \cap W_i$ are smooth over $\operatorname{Spec}(\mathcal{O})$, $u_i(U_{i+1})$ is contained in U_i, and

$$(u \colon V_n \to V;\ U_n)$$

is a replica of v for the family V, where u is the composition, $u = u_0 \circ \dots \circ u_{n-1}$.

PROOF: We proceed by induction on n, the case $n = 0$ being trivial. Let C_i'' denote the image of C_i in X_1, for $i \geqslant 1$. By Lemma 1.5, there is a subvariety W_0 of V, and an open subscheme U_0 of V, such that,
(1) $(u_{W_0} \colon V_1 \to V;\ U_1)$ is a replica of $v_0 \colon X_1 \to X$ for V, where $V_1 = V_{W_0}$, $U_1 = (U_0)_{(W_0 \cap U_0)}$.
(2) Let $f_1 \colon U_1 \otimes k \to X_1$ be the morphism, $f_1 = v_0^{-1} \circ f \circ (u_{W_0} \otimes k)$, let $P_1 = X_1 - f_1(U_1 \otimes k)$, and let $X_1^0 = X_1 - P_1$. Then $X_1^0 \cap C_i'' \neq \emptyset$ and f_1^{-1} is an isomorphism in a neighborhood of $X_1^0 \cap C_i''$, for each $i = 1, \dots, n-1$.
(3) $W_0 \cap U_0$ is smooth over $\mathcal{O}$.

From (1) and (2), we see that the family $p_1 \colon U_1 \to \operatorname{Spec}(\mathcal{O})$, $p_1 = p \circ u_{W_0}$, the sequence of monoidal transformations

$$v_i \colon X_{i+1} \to X_i, \qquad i = 1, \dots, n-1$$

and the birational morphism

$$f_1 \colon U_1 \otimes k \to X_1$$

satisfy the hypotheses of the proposition. Let v^1 be the composition $v^1 = v_1 \circ \dots \circ v_n$. By induction, there is a sequence of monoidal transformations

$$u_i^1 = u_{W_i^1} \colon V_{i+1}^1 \to V_i^1; \qquad i = 1, \dots, n-1; \quad V_i^1 = U_1$$

with irreducible center $W_i^1 \subseteq V_i^1$, and open subsets U_i^1 of V_i^1 such that U_i^1 and $U_i^1 \cap W_i^1$ are smooth over Spec($\mathcal{O}$), and

$$\left(u^1 \colon V_n^1 \to U_1;\, U_n^1\right); \qquad u^1 = u_{n-1}^1 \circ \ldots \circ u_1^1,$$

is replica of v^1 for the family U_1. Define $\mathcal{O}$-varieties V_i, $i = 1,\ldots,n$, and subvarieties W_i, $i = 1,\ldots,n-1$, of V_i inductively by letting W_i be the closure of W_i^1 in V_i and letting V_{i+1} be the monoidal transform of V_i with center W_i. Clearly V_i^1 is a subscheme of V_i; let U_i be the open subscheme U_i^1 of V_i, and let u: $V_n \to V$ be the composition of the monoidal transformations

$$u_i = u_{W_i} : V_{i+1} \to V_i$$

We claim that (u: $V_n \to V$; U_n) is a replica of v for the family V. Indeed, properties (i)–(iii) of Definition 1.3 follow immediately from the fact that (u^1: $V_n^1 \to U_1$; U_n) and (u_{W_0}: $V_1 \to V$; U_1) are replicas for v^1 and v_0 respectively. It remains to check condition (iv).

Let $\mathcal{E}$ be the exceptional divisor of u: $U_n \to V$, and write $\mathcal{E}$ as a sum of irreducible divisors

$$\mathcal{E} = \sum_{i=1}^{r} \mathcal{E}_i .$$

Let $\mathcal{E}^1$ be the exceptional divisor of u^1: $U_n \to U_1$. From the factorization of u,

$$U_n \underset{u^1}{\to} U_1 \underset{u_{W_0}}{\to} V$$

we see that we may take one of the irreducible components of $\mathcal{E}$, say $\mathcal{E}_r$, to be the proper transform of $u_{W_0}^{-1}(W_0)$ under $(u^1)^{-1}$. In addition, we can write $\mathcal{E}^1$ as

$$\mathcal{E}^1 = \sum_{i=1}^{r-1} \mathcal{E}_i .$$

Furthermore, since U_i and $U_i \cap W_i$ are smooth over Spec($\mathcal{O}$), the exceptional locus of $u^1 \otimes k$: $U_n \otimes k \to U_1 \otimes k$ is $\mathcal{E}^1 \cap (U_n \otimes k)$, as one sees by an easy induction.

Let E be an irreducible component of $\mathcal{E} \otimes k$. If E is an exceptional subvariety for $u^1 \otimes k$, then E is a component of $\mathcal{E}^1 \otimes k$. As (u^1: $V_n^1 \to U_1$; U_n) is a replica of v^1, we can write $1 \cdot E$ as a sum

$$1 \cdot E = \sum_{i=1}^{r-1} n_i (\mathcal{E}_i \otimes k)$$

for suitable integers n_i. In case $u^1 \otimes k$ does not collapse E, we must have

$$\overline{(u^1 \otimes k)(E)} = u_{W_0}^{-1}(W_0) \cap (U_1 \otimes k).$$

Thus each irreducible component of $1 \cdot E - \mathscr{E}_r \otimes k$ is exceptional for $u^1 \otimes k$, and we are reduced to the preceeding case. This completes the verification of (iv), and the proof of the proposition. Q.E.D.

We now prove the main result of this section.

THEOREM 1.7: *Let p: $V \to \mathrm{Spec}(\mathscr{O})$ be a smooth projective fiber variety of fiber dimension n. Suppose $\mathscr{O}$ is equi-characteristic and that either* $\mathrm{char}(\mathscr{O}) > 5$ *and* $n \leqslant 3$, *or* $\mathrm{char}(\mathscr{O}) = 0$. *Suppose there is a smooth complete variety S and a dominant rational map ϕ: $V \otimes k \to S$. Then there is a good resolution of the singularities of ϕ for the family V.*

PROOF: By [2] in case $\mathrm{char}(\mathscr{O}) > 5$ and $n \leqslant 3$, or by [7] in case $\mathrm{char}(\mathscr{O}) = 0$, there is a birational morphism v: $X \to V \otimes k$ that can be factored into a sequence of monoidal transformations with smooth center, such that the induced rational map $\tilde{\phi}$: $X \to S$ is a morphism. Let $(u_1$: $V_1 \to V$, $U_1)$ be a replica of v for the family V. Let $\overline{U_1 \otimes k}$ be the closure of $U_1 \otimes k$ in V_1, and let

$$\bar{\phi}_1 : \overline{U_1 \otimes k} \to S$$

be the induced rational map $\bar{\phi}_1 = \phi \circ v^{-1} \circ (u_1 \otimes k)$. U_1 and $U_1 \otimes k$ are smooth and ϕ_1: $U_1 \otimes k \to S$ is a morphism. Applying [2] or [7] again, there is a projective birational morphism u_2: $V' \to V_1$ that resolves the singularities of the variety $\overline{U_1 \otimes k}$ and the indeterminacy of the map $\bar{\phi}_1$. In addition, u_2^{-1} is an isomorphism when restricted to U_1. Let $U' = u_2^{-1}(U_1)$, and let u': $V' \to V$ be the composition $u' = u_1 \circ u_2$. We claim that $(u'$: $V' \to V$, $U')$ is a good resolution of the singularities of ϕ for the family V.

Let ϕ': $U' \otimes k \to S$ be the induced morphism $\phi' = \bar{\phi}_1 \circ (u_2 \otimes k)$.

Let f_1: $U_1 \otimes k \to X$ be the birational morphism, $f_1 = v^{-1} \circ (u_1 \otimes k)$. Let $F \subseteq X$ be the closure of the fundamental locus of f_1^{-1}. Then

$$\phi'\left(\overline{U' \otimes k} - U' \otimes k\right) \subseteq \tilde{\phi}\left[\left(X - f_1(U_1 \otimes k)\right) \cup F\right].$$

But $\mathrm{codim}_X(X - f_1(U_1 \otimes k))$ is at least two, since $(u_1$: $V_1 \to V$; $U_1)$ is a replica of v. Also $\mathrm{codim}_X(F)$ is at least two. Thus

$$\dim\left(\phi'\left(\overline{U' \otimes k} - U' \otimes k\right)\right) \leqslant n - 2.$$

This verifies condition (iii) in definition 1.2. Conditions (i), (ii), and (iv) follow from the isomorphism u_2: $U' \to U_1$ and the fact that (u_1: $V_1 \to V$, U_1) is a replica of v. Q.E.D.

§2. Ruled, quasi-ruled, and strongly uni-ruled varieties

In this section we define our notion of stability, we introduce the classes of varieties which are natural higher dimensional generalizations of the class of ruled surfaces, and we verify the easy half of a criterion for stability of these classes.

DEFINITION 2.1: Let X be a variety of dimension n, and let Y be a variety of dimension $n-1$.

(i) X is said to be ruled over Y if X is birationally isomorphic to $Y \times \mathbb{P}^1$. The induced rational map ϕ: $X \to Y$ is called a ruling of X.

(ii) X is said to be quasi-ruled over Y if there is a dominant separable rational map ϕ: $X \to Y$ such that $\phi^{-1}(y)$ is an irreducible rational curve for all y in a Zariski open subset of Y.

(iii) X is called uni-ruled if there is a ruled variety W of dimension n, and a dominant rational map ϕ: $W \to X$.

(iv) X is called strongly uni-ruled if there is a variety W, birationally isomorphic to X, a variety Z of dimension $n-1$, and a subvariety $\mathscr{X}$ of $Z \times W$ such that

(a) p_1: $\mathscr{X} \to Z$ is smooth and proper,

(b) $p_1^{-1}(z)$ is a rational curve for all z in Z,

(c) p_2: $\mathscr{X} \to W$ is etale.

The notions of ruledness, quasi-ruledness, uni-ruledness and strong uni-ruledness are birational in nature.

Before discussing these varieties further, we will prove a lemma that will aid in their description.

LEMMA 2.2: *Let ϕ: $X \to Y$ be a dominant rational map of varieties with X, Y, and ϕ defined over a field k. Let y be a generic point of Y over k. Suppose that $\phi^{-1}(y)$ is a complete non-singular rational curve, say C_y. If there is a divisor $D \subseteq X$ such that $C_y \cap D$ is contained in the smooth locus of X and* $\deg(C_y \cdot D) = 1$, *then X is ruled over Y via ϕ.*

PROOF: Let $K \supseteq k$ be a field of definition for D and let y' be a generic point of Y over K. By our assumptions, $C_{y'} = \phi^{-1}(y')$ is defined over the field $K(y')$. Let x be a generic point of $C_{y'}$ over $K(y')$, hence x is also a generic point of X over K. The divisor $\mathscr{A} = D \cdot C_{y'}$ is rational over $K(y')$, hence by Riemann-Roch, there is a function $t_1 \in K(y')(x)$ such that

$$(t_1) + \mathscr{A} = 1 \cdot x_0, \qquad x_0 \in C_{y'}.$$

Thus x_0 is a $K(y')$ rational point of $C_{y'}$. Again by Riemann-Roch, there is a function $t \in K(y')(x)$ with

$$(t) = x_0 - x_\infty, \qquad x_\infty \in C_{y'}.$$

Thus t: $C_{y'} \to \mathbb{P}^1$ is an isomorphism and hence

$$K(x) = K(y')(x) = K(y')(t)$$

which proves the lemma. Q.E.D.

REMARK: Suppose X is a variety defined over k. If $\mathrm{char}(k) = 0$, or if $\mathrm{char}(k) > 5$ and $\dim(X) \leqslant 3$, then

(i) if ϕ: $X \to Y$ is a quasi-ruling then ϕ is a ruling if and only if ϕ admits a rational section s: $Y \to X$. This follows from the lemma once we replace X with a smooth projective variety X^*, birationally isomorphic to X, such that the induced rational map ϕ^*: $X^* \to Y$ is a morphism;

(ii) if X is a ruled variety when X is a quasi-ruled variety;

(iii) if X is a quasi-ruled variety, then X is a strongly uni-ruled variety, for suppose X is quasi-ruled over Y via ϕ: $X \to Y$. As above there is a smooth projective X^* birationally ismorphic to X such that the induced rational map ϕ^*: $X^* \to Y$ is a morphism. If Y_0 is an open subset of Y such that ϕ^* is smooth over Y_0, then the graph of ϕ^*, restricted to X^*, Y_0, exhibits X as a strongly uni-ruled variety.

(iv) If X is a strongly uni-ruled variety then X is a uni-ruled variety, for suppose X is strongly uni-ruled via a family $\mathscr{X}$ of curves on W parametrized by Z,

$$\mathscr{X} \subseteq Z \times W$$

where W is a variety birationally isomorphic to X. Let p_1: $\mathscr{X} \to Z$, p_2: $\mathscr{X} \to W$ be the restrictions to $\mathscr{X}$ of the projections on the first and second factor, respectively. Let q: $Z' \to Z$ be a quasi-finite morphism such that the pullback $\mathscr{X}'$,

$$\mathscr{X}' = \mathscr{X} \times_Z Z'$$

admits a rational section to the morphism $p_{Z'}$: $\mathscr{X}' \to Z'$. By (i) $\mathscr{X}'$ is ruled over Z', furthermore the morphism $p_2 \circ p$: $\mathscr{X}' \to W$ exhibits W, and hence X, as a uni-ruled variety.

(v) Suppose that X is a projective variety defined over k, and let x be a generic point of X over k. Then X is uni-ruled if and only if there is a rational curve on X containing x. See Lemma 2 of [10] for proof.

The celebrated example of the cubic threefold was shown by Clemens-Griffiths to be uni-rational but not rational; one easily deduces from this that the cubic threefold is not ruled. It is quasi-ruled (in fact a conic bundle) by the conics cut out by the $\mathbf{P}^2$ of planes passing through a line contained in the threefold, which gives an example of a quasi-ruled variety that is not ruled. More generally, it is easy to show that any non-trivial conic bundle over a base S, where S is not uni-ruled, is quasi-ruled but not ruled. See the discussion following Theorem 3.4 for an example of a strongly uni-ruled variety that is not quasi-ruled.

We do not know of an example of a uni-ruled variety that is not strongly uni-ruled. It is not difficult, however, to construct a family of rational curves on a variety V which exhibit V as a uni-ruled variety, but not as a strongly uni-ruled variety. For example, if x is a general point of a general quartic threefold V, then the intersection of V, the tangent hyperplane to V at x, and the tangent cone to V at x gives a degree eight rational curve C_x with an eight-tuple point at x. If S is a general surface section of V, then the curves C_x, as x runs over S, gives a uni-ruling of V. This is not a strong uni-ruling, since in a strong uni-ruling the k-closure of the singular points of the generic curve in the uni-ruling must have codimension at least two in the ambient variety.

DEFINITION 2.3: Let $\mathscr{P}$ be a class of nonsingular projective varieties. We say that $\mathscr{P}$ is stable under small deformations (in the algebraic sense) if, given a variety X_0 in $\mathscr{P}$ and smooth projective fiber variety

$$p: X \to M$$

over a variety M, with

$$X_0 = p^{-1}(m_0), \qquad m_0 \in M,$$

then there is a Zariski neighborhood U of m_0 in M such that the variety $p^{-1}(m)$ is in $\mathscr{P}$ for all m in U.

LEMMA 2.4: *Let $\mathscr{P}$ be a class of smooth projective varieties such that*

(*i*) *If p: $X \to M$ is a smooth projective fiber variety over a variety M, and if $X \otimes k(M)$ is in $\mathscr{P}$, then there is an open subset U of M such that $p^{-1}(u)$ is in $\mathscr{P}$ for all u in U;*

(*ii*) *if p: $X \to Spec(\mathscr{D})$ is a smooth projective fiber variety over an equicharacteristic $DVR\,\mathscr{D}$, with quotient field L and algebraically closed residue field K_0, and if $X \otimes K_0$ is in $\mathscr{P}$, then $X \otimes L$ is in $\mathscr{P}$.*

Then $\mathscr{P}$ is stable under small deformations.

PROOF: This follows by a simple noetherian induction.

LEMMA 2.5: *Let p: $X \to M$ be a proper fiber variety over a variety M. Suppose there is a variety Y_M, defined and proper over $k(M)$, and a* *dominant rational (resp. birational) map ψ_M: $X \otimes k(M) \to Y_M$, also defined over $k(M)$. Then there is an open subset U of M, a flat and proper morphism q: $Y \to U$ and a subvariety Γ of $p^{-1}(U) \times_U Y$ such that*

(*i*) *$Y \otimes k(U)$ is $k(U) = k(M)$ isomorphic to Y_M; $\Gamma \otimes k(U)$ is the graph of ψ_M;*

(*ii*) *$Y \otimes k(u)$ is reduced and geometrically irreducible for each u in U;*

(*iii*) *$\Gamma \otimes k(u)$ is the graph of a dominant rational (resp. birational) map ψ_u: $X \otimes k(u) \to Y \otimes k(u)$, for each u in U.*

PROOF: Y_M defines by specialization a proper fiber-variety $\overline{Y}_M$ over an open subset U_1 of M. Let U_2 be an open subset of M_1 over which $\overline{Y}_M$ is flat, and such that $\overline{Y}_M \otimes k(u)$ is reduced and geometrically irreducible for each u of U_1. Let Γ_M denote the graph of ψ_M, and let $\overline{\Gamma}_M$ denote the k-closure of Γ_M in $X \times_M \overline{Y}_M$. As Γ_M is $k(M)$-closed, $\overline{\Gamma}_M \otimes k(M) = \Gamma_M$. If ψ_M is a birational map, we take U to be an open subset of U_2 such that $\overline{\Gamma}_M$ is flat over U, $\overline{\Gamma}_M \otimes k(u)$ is reduced and geometrically irreducible for each u of U, and such that

$$\deg\left(\overline{\Gamma}_M \otimes k(u)/X \otimes k(u)\right) = \deg\left(\overline{\Gamma}_M \otimes k(u)/\overline{Y}_M \otimes k(u)\right) = 1$$

for each u in U. If ψ_M is merely a dominant rational map, we take U to be an open subset as above, only we require that

$$\deg\left(\overline{\Gamma}_M \otimes k(u)/X \otimes k(u)\right) = 1,$$

and that p_2: $\overline{\Gamma}_M \otimes k(u) \to \overline{Y}_M \otimes k(u)$ is dominant for each in U. Letting $\Gamma = \overline{\Gamma}_M \times_M U$ and $Y = \overline{Y}_M \times_M U$ completes the proof. Q.E.D.

PROPOSITION 2.6: *Let $\mathscr{P}_1$ be the class of strongly uni-ruled varieties, $\mathscr{P}_2$ the class of quasi-ruled varieties, and $\mathscr{P}_3$ the class of ruled varieties. Then $\mathscr{P}_1$, $\mathscr{P}_2$, and $\mathscr{P}_3$ satisfy condition (i) of Lemma* 2.4.

PROOF: Let p: $X \to M$ be a smooth projective fiber variety over a variety M. Let X_M denote the generic fiber $X \otimes k(M)$

(1) Suppose X_M is strongly uni-ruled. Then there is a variety W_M, a variety Z_M and a subvariety $\mathscr{Z}_M$ of $Z_M \times W_M$ satisfying conditions (a)–(c) of Definition 2.1 (iv). We may assume that W_M, Z_M and $\mathscr{Z}_M$ are defined over a finite field extension L of $k(M)$, and that $X_M \otimes L$ and W_M are birational over L. Replacing M with its normalization in L, and changing notation, we may assume that $L = k(M)$. Let $\overline{W}_M$ and $\overline{Z}_M$ be varieties proper over $k(M)$, containing W_M and Z_M respectively, as $k(M)$-open subsets. By Lemma 2.5, there is an open subset U of M, and proper morphisms q: $\overline{Z} \to U$, r: $\overline{W} \to U$ such that

(i) $\overline{Z} \otimes k(U) = \overline{Z}_M$; $\overline{W} \otimes k(U) = \overline{W}_M$;

(ii) $\overline{Z} \otimes k(u)$ and $\overline{W} \otimes k(u)$ are reduced and geometrically irreducible for each u in U;

(iii) $\overline{W} \otimes k(u)$ is birational to $X \otimes K(u)$ for each u in U.

Let $\bar{\mathscr{Z}}$ be the k-closure of $\mathscr{Z}_M$ in $\bar{Z} \times_U \bar{W}$. As $\mathscr{Z}_M$ is smooth and proper over Z_M, and etale over W_M, there is an open subset Z of $\bar{Z}$ such that

(iv) (a) $\bar{\mathscr{Z}} \cap p_1^{-1}(Z)$ is smooth and proper over Z;

(b) $\bar{\mathscr{Z}} \cap p_1^{-1}(Z)$ is etale over $\bar{W}$.

By Lemma 5 of [13], we have

(v) $p_1^{-1}(z)$ is a rational curve, for each z in Z.

Let $\mathscr{Z} = \bar{\mathscr{Z}} \cap p_1^{-1}(Z)$.

As smooth, proper, and etale morphisms are stable under base change, we have

(vi) (a) $\mathscr{Z} \otimes k(u)$ is smooth and proper over $Z \otimes k(u)$;

(b) $\mathscr{Z} \otimes k(u)$ is etale over $\bar{W} \otimes k(u)$.

By (iii), (v), and (vi), $X \otimes k(u)$ is strongly uni-ruled.

(2) Suppose X_M is quasi ruled via ϕ_M: $X_M \to Y_M$. Arguing as in (1), we may assume that Y_M and ϕ_M are defined over $k(M)$, and that Y_M is proper over $k(M)$. Apply Lemma 2.5 to Y_M and ϕ_M, and let p: $Y \to U$, $\Gamma \subseteq p^{-1}(U) \times_U Y$ be as given by that lemma. By assumption, the generic fiber of ϕ_M is an irreducible rational curve, and ϕ_M is separable. In particular, there is an open subset of X_M that is smooth over Y_M; thus there is an open subset Γ_0 of Γ, smooth over Y. Shrinking U if necessary, we may assume that the rational map ϕ_u: $X \otimes k(u) \to Y \otimes k(u)$ defined by the graph $\Gamma \otimes k(u)$ is separable, and that the generic fiber of ϕ_u is irreducible. By Lemma 5 of [13], the generic fiber of ϕ_u is a rational curve, hence $X \otimes k(u)$ is quasi-ruled for each u in U.

(3) If X_M is a ruled variety, then our result follows from Theorem 1.1 of [11]. Q.E.D.

A similar result also holds for the class of uni-ruled varieties. As the uni-ruled varieties (in characteristic zero) have been shown to be stable under smooth deformation (see [4] and [10]), we omit the proof.

§3. Stability

In this section we will prove our main results on the stability of ruled, quasi-ruled and strongly uni-ruled varieties. We first prove a result about the deformations of rational curves.

DEFINITION 3.1: Let p: $V \to M$ be a flat projective morphism of schemes. Let O be a point of U, and let Z be a subscheme of $p^{-1}(0)$. A locally closed irreducible subset Y of $\mathrm{Hilb}(V/M)$ is called a maximal algebraic family of deformations of Z in V if

(i) $h(Z) \in Y$;

(ii) if $T \to M$ is an M-scheme, O_T a point of T over O, $Z \subset T \times_M V$ a subscheme flat over T such that $k(O_T) \otimes \mathscr{Z} = Z$, and f: $T \to \mathrm{Hilb}(V/M)$ the morphism induced by $\mathscr{Z}$, then there is an open neighborhood U of O_T in T such that $f(U)$ is contained in Y.

PROPOSITION 3.2: *Let $p\colon V \to \mathrm{Spec}(\mathcal{O})$ be a projective morphism and let U be an open subset of V, smooth over $\mathcal{O}$ with geometrically irreducible fibers of dimension n. Suppose $U \otimes k$ contains a complete smooth rational curve X with*

$$N_{X/U\otimes k} \cong (\mathcal{O}_X)^{n-1}.$$

Then there is a subvariety Y of $Hilb(V/\mathcal{O})$ such that, letting $\mathcal{Y}$ be the pullback $Y \times_{Hilb} H(V/\mathcal{O})$, we have

(*i*) *Y is a maximal algebraic family of deformations of X in V;*

(*ii*) *the morphism $g\colon Y \to Spec(\mathcal{O})$ induced by the inclusion of Y into $Hilb(V/\mathcal{O})$ is smooth, and the fibers of g have dimension $n-1$.*

(*iii*) *$\mathcal{Y}$ is a subscheme of $Y \times_{Spec(\mathcal{O})} U$;*

(*iv*) *$\mathcal{Y}$ is smooth and proper over Y and etale over U;*

(*v*) *each fiber of $p_1\colon \mathcal{Y} \to Y$ is a rational curve.*

In particular, $U \otimes k$, and $U \otimes K$ are strongly uni-ruled.

PROOF: Let Y' be an irreducible component of $\mathrm{Hilb}(V/\mathcal{O})$ containing $h(X)$. Since

$$h^1(X; N_{X/U\otimes k}) = h^1\left(\mathbb{P}^1; (\mathcal{O}_{\mathbb{P}^1})^{n-1}\right) = 0$$

and since U is smooth over $\mathrm{Spec}(\mathcal{O})$, we have

(a) Y' is the only component of $\mathrm{Hilb}(V/\mathcal{O})$ passing through $h(X)$;

(b) $h(X)$ is smooth on Y';

(c) $T_{h(X)}(Y')$ is isomorphic to $H^0(X, N_{X/U})$,

by corollary 5.2, exp. III of [6]. We now give a description of the isomorphism in (c).

Let $\mathcal{Y}' \subseteq Y' \times_{\mathrm{Spec}(\mathcal{O})} V$ be the subscheme induced by the inclusion of Y' in $\mathrm{Hilb}(V/\mathcal{O})$.

Denoting $h(X)$ by 0, we note that $\mathcal{Y}'_0 = \mathcal{Y}' \otimes k(0)$ is isomorphic to X, hence is a smooth variety. As $\mathcal{Y}'$ is flat over Y', this implies that there is a neighborhood of $\mathcal{Y}'_0$ in $\mathcal{Y}'$, smooth over Y'. In particular, the normal sheaf of $\mathcal{Y}'_0$ in $\mathcal{Y}'$ is the trivial sheaf,

$$N_{\mathcal{Y}'_0/\mathcal{Y}'} \cong \mathcal{O}_{\mathcal{Y}'_0} \otimes_k T_0(Y').$$

Thus, if v is a tangent vector, $v \in T_0(Y')$, there is a unique section S_v in $H^0(\mathcal{Y}'_0; N_{\mathcal{Y}'_0/\mathcal{Y}'})$ such that, for each x in $\mathcal{Y}'_0$,

$$\mathrm{d}\bar{p}_1(x)(S_v(x)) = v$$

where $\mathrm{d}\bar{p}_1(x)\colon N_{\mathcal{Y}'_0/\mathcal{Y}'} \otimes k(x) \to T_0(Y)$ is the homomorphism induced by the tangent map $\mathrm{d}p_1$.

The tangent map $\mathrm{d}p_2$: $T(\mathscr{Y}_0') \to T(X)$ induces a map $\mathrm{d}\bar{p}_2$: $H^0(\mathscr{Y}_0'; N_{\mathscr{Y}_0'/\mathscr{Y}'}) \to H^0(X; N_{X/U})$.

We define the map ρ: $T_0(Y') \to H^0(X; N_{X/U})$ by

$$\rho(v) = \mathrm{d}\bar{p}_2(S_v).$$

This described the isomorphism of (c).

We have the exact sheaf sequence on X:

$$0 \to N_{X/U \otimes k} \to N_{X/U} \to N_{U \otimes k_{/U}} \otimes \mathscr{O}_X \to 0$$

As $N_{X/U \otimes k} = (\mathscr{O}_X)^{n-1}$, and $N_{U \otimes k_{/U}} \otimes \mathscr{O}_X = \mathscr{O}_X$, and since $\mathrm{Ext}^1_X(\mathscr{O}_X, (\mathscr{O}_X)^{n-1}) = 0$, the above sequence splits, and

$$N_{X/U} \cong (\mathscr{O}_X)^n.$$

Thus, by (c), $\dim(Y') = n$.

Next, let u be a non-zero vector in the Zariski tangent space of $\mathrm{Spec}(\mathscr{O})$ at the closed point, and let S_u be a global section of $N_{X/U}$ such that

$$\mathrm{d}\bar{p}(x)(S_u(x)) = u$$

for each x in X. Let $v \in T_0(Y')$ be such that

$$\rho(v) = S_u.$$

Let g: $Y \to \mathrm{Spec}(\mathscr{O})$ be the morphism induced by the inclusion of Y' into $\mathrm{Hilb}(V/\mathscr{O})$. We note that g is given by the map of $\mathscr{O}$ into $\mathscr{O}_Y$ defined by

$$\mathscr{O} \underset{p^*}{\to} H^0(V, \mathscr{O}_V) \underset{p_2^*}{\to} H^0(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}) \underset{p_1^*}{\leftarrow} \mathscr{O}_{Y'}.$$

Thus, we see that

$$\mathrm{d}g(v) = \mathrm{d}\bar{p}\,(\rho(v)) = u,$$

and hence the map g is smooth at $h(X)$.

Furthermore, we claim that $\mathscr{Y}'$ is etale over V in a neighborhood of $\mathscr{Y}_0'$. As p_2: $\mathscr{Y}_0' \to X$ is an isomorphism, it is enough to show that

$$\mathrm{d}\bar{p}_2(x)\colon N_{\mathscr{Y}_0'/\mathscr{Y}'} \otimes k(x) \to N_{X/V} \otimes k(p_2(x))$$

is an isomorphism for each x in $\mathscr{Y}_0'$. Let v be a non-zero element of $N_{\mathscr{Y}_0'/\mathscr{Y}'} \otimes k(x)$. As $N_{\mathscr{Y}_0'/\mathscr{Y}'}$ is a free sheaf, there is a unique global section S_v of $N_{\mathscr{Y}_0'/\mathscr{Y}'}$ such that $S_v(x) = v$. Clearly we have

$$\mathrm{d}\bar{p}_2(S_v) = \rho(\mathrm{d}\bar{p}_1(v)) \neq 0,$$

since $d\bar{p}_1(v) \neq 0$. As $N_{X/U}$ is the trivial sheaf $(\mathcal{O}_X)^n$, this implies that

$$0 \neq d\bar{p}_2(S_v)(p_2(x)) = d\bar{p}_2(x)(v).$$

Thus $dp_2(x)$ is an injection, hence an isomorphism as desired.

Thus there is an open neighborhood Y of $h(X)$ in Y' such that Y is smooth over Spec($\mathcal{O}$), $\mathcal{Y}' \cap p_1^{-1}(Y)$ is smooth over Y and $\mathcal{Y}' \cap p_1^{-1}(Y)$ is etale over V.

As X is contained in U, we may take Y so that $p_2(\mathcal{Y}' \cap p_1^{-1}(Y))$ is contained in U. Let $\mathcal{Y} = \mathcal{Y}' \cap p_1^{-1}(Y)$. Since Y is open in Y', (i) is clear, as is (ii) and (iii), noting that $\dim(Y) = \dim(Y') = n$. The previous paragraph proves (iv). As for (v), the fiber $\mathcal{Y} \otimes k(0)$ is isomorphic to the rational curve X. Since the genus of a smooth complete curve is a deformation invariant all smooth deformations of $\mathcal{Y} \otimes k(0)$ are also rational, proving (v). Our final assertion follows from (iv), (v), and the base change theorems for smooth, proper, and etale morphisms. Q.E.D.

We also require the following basic result on extensions of invertible sheaves. As the result is well known, we merely sketch the proof.

LEMMA 3.3: *Let f: $V \to M$ be a smooth, projective morphism of integral schemes with geometrically irreducible fibers, O a point of M, and L_0 an invertible sheaf on $f^{-1}(O) = V_0$. Suppose that $h^2(V_0, \mathcal{O}_{V_0}) = 0$. Then there is an etale neighborhood r: $M' \to M$ of O, an invertible sheaf L' on $V' = V \times_M M'$, and a point O' of $r^{-1}(O)$ such that $L' \otimes \mathcal{O}_{V'_{0'}}$ is isomorphic to $p_1^*(L_0)(V'_{0'} = p_2^{-1}(O'))$.*

PROOF: Let $\hat{M}$ denote the formal completion of M at m_0 and let $\hat{V}$ denote the formal scheme $V \times_M \hat{M}$. By proposition 7.1 of [5], there is an invertible sheaf $\hat{L}$ on $\hat{V}$ with $\hat{L} \otimes \mathcal{O}_{V_0}$ isomorphic to L_0. Let $\hat{\mathcal{O}}$ denote the completion of the local ring of O in M at its maximal ideal, $\bar{M}$ the scheme Spec($\hat{\mathcal{O}}$). By Grothendieck's existence theorem [EGA III, 5.4.5], L extends to an invertible sheaf $\bar{L}$ on $V \times_M \bar{M}$. Finally, Artin's algebraization theorem [2] applied to the functor F,

$$F(T) = \mathrm{Pic}(T \times_M V); \qquad T \text{ an } M\text{-scheme},$$

yields the desired sheaf L' and etale neighborhood r: $M' \to M$. Q.E.D.

THEOREM 3.4: *Let $\mathcal{U}_k(n)$ denote the class of smooth projective strongly uni-ruled varieties of dimension n, defined over a field containing k. If* $\mathrm{char}(k) = 0$, *or if* $\mathrm{char}(k) > 5$ *and $n \leqslant 3$, then $\mathcal{U}_k(n)$ is stable under small deformations.*

PROOF: By Proposition 2.6, we need only verify (ii) in Lemma 2.4. Let $\mathcal{O}$ be an equi-characteristic DVR with quotient field K and algebraically closed residue field K_0, and let p: $V \to \operatorname{Spec}(\mathcal{O})$ be a smooth projective morphism with geometrically irreducible fibers of dimension n. Changing notation, we may assume $K_0 = k$. Suppose $V \otimes k$ is strongly uni-ruled. Then there is a variety W, birational to $V \otimes k$, a variety Y and a family of curves $\mathscr{Y} \subseteq Y \times W$ such that

(a) p_1: $\mathscr{Y} \to Y$ is smooth and proper;

(b) p_2: $\mathscr{Y} \to W$ is etale;

(c) each fiber of p_1: $\mathscr{Y} \to Y$ is a rational curve.

Let v: $W' \to V \otimes k$ be a projective birational morphism such that W' is smooth, and the induced map v: $W' \to W$ is a morphism. Shrinking Y if necessary, we may replace W with W'; changing notation we may assume $W = W'$. Let (u': $V' \to V$; U') be a good resolution for the singularities of the map v^{-1} for the family V. Let ψ: $U' \otimes k \to W$ denote the induced morphism, $\psi = v^{-1} \circ (u' \otimes k)$. Let $P = \overline{W - \psi(U' \otimes k)}$ and let F be the closure of the fundamental locus of ψ^{-1}. Since (u': $V' \to V$; U') is a good resolution, we have

$$\operatorname{codim}_W(P \cup F) \geqslant 2.$$

Let O be a point of Y such that

$$p_2\big(p_1^{-1}(O)\big) \cap (P \cup F) = \phi$$

and let $X = p_2(p_1^{-1}(O))$. Since $\mathscr{Y}$ is etale over W, the formal neighborhood of $p_1^{-1}(O)$ in $\mathscr{Y}$ is isomorphic to the formal neighborhood of X in W; thus

$$N_{X/W} \cong (\mathcal{O}_X)^{n-1}.$$

Furthermore, since $X \cap (P \cup F) = \phi$, X is contained in $\psi(U' \otimes k)$ and ψ^{-1} is an isomorphism in a neighborhood of X. Letting $X' = \psi^{-1}(X)$, we have

$$N_{X'/U \otimes k} \cong (\mathcal{O}_{X'})^{n-1}$$

and X' is a smooth complete rational curve. Applying Proposition 3.2, we see that $V' \otimes K$ and hence $V \otimes K$, is strongly uni-ruled, which completes the verification of (ii) and the proof of the theorem. Q.E.D.

We now turn to stability results for quasi-ruled and ruled varieties. The reader will note that we require additional hypotheses in Theorems 3.8, 3.9, and 3.10 to yield stability. These hypotheses are actually quite necessary. Regarding Theorem 3.8, we give in [9] an example of a family $p: V \to \operatorname{Spec}(\mathcal{O})$, with $V \otimes k$ quasi-ruled over a ruled threefold, such that $V \otimes K$ is not quasi-ruled.

We also give an example of a family $q: W \to \mathrm{Spec}(\mathcal{O})$, with $W \otimes k$ ruled over, a base Y via a morphism, but with $h^2(Y, \mathcal{O}_Y) \neq 0$, such that $W \otimes K$ is not ruled, which illuminates the final hypothesis of theorem 3.10. The variety $V \otimes K$ above, being the deformation of the quasi-ruled (hence strongly uni-ruled) $V \otimes k$ is, in virtue of the above theorem, a strongly uni-ruled variety. This gives an example of a variety which is strongly uni-ruled, but not quasi-ruled, as mentioned in Section 2.

We first prove some basic results about extending rational maps to a family of varieties.

DEFINITION 3.5: Let p: $U \to \mathrm{Spec}(\mathcal{O})$ be a smooth morphism with geometrically irreducible fibers. Suppose there is a variety Y_0 and a dominant rational map f: $U \otimes k \to Y_0$. An extension of f to the family U is a pair (g: $Y \to \mathrm{Spec}(\mathcal{O})$; F) where g: $Y \to \mathrm{Spec}(\mathcal{O})$ is a smooth morphism with geometrically irreducible fibers, and F: $U \to Y$ is a dominant rational map over $\mathcal{O}$, defined along $U \otimes k$, such that

(i) Y_0 is birational to $Y \otimes k$, by some ξ: $Y_0 \to Y \otimes k$;

(ii) $F_{|U \otimes k} = \xi \circ f$.

The following lemma gives a criterion for a subvariety $\mathcal{Y}$ of $U \times_{\mathrm{Spec}(\mathcal{O})} Y$ to be the graph of the extension of a rational map.

LEMMA 3.6: *Let p: $U \to \mathrm{Spec}(\mathcal{O})$ be a smooth quasi-projective morphism with geometrically irreducible fibers, let Y_0 be a variety and let f: $U \otimes k \to Y_0$ be a dominant rational map. Suppose there is a smooth morphism g: $Y \to \mathrm{Spec}(\mathcal{O})$, with geometrically irreducible fibers, a subvariety $\mathcal{Y}$ of $U \times_{\mathcal{O}} Y$, a birational map ξ: $Y_0 \to Y \otimes k$, and a point x_0 of $U \otimes k$ such that*

(i) $\mathcal{Y} \otimes K$ and $\mathcal{Y} \otimes k$ are irreducible and reduced;

(ii) $\mathcal{Y} \otimes k$ is the graph of $\xi \circ f$: $U \otimes k \to Y \otimes k$;

(iii) $\xi \circ f$ is a morphism at x_0.

(iv) Let x_t be a point of U and $x_t \to x_0$ a specialization. Let y_t be a point of $p_2((x_t \times Y) \cap \mathcal{Y})$. If we extend $x_t \to x_0$ to a specialization $y_t \to \bar{y}$, then $\bar{y} = \xi \circ f(x_0)$.

Then $\mathcal{Y}$ is the graph of a dominant rational map F: $U \to Y$ over $\mathcal{O}$, and (g: $Y \to \mathrm{Spec}(\mathcal{O})$; F) is an extension of f to the family U.

PROOF: Since p_2: $\mathcal{Y} \otimes k \to Y \otimes k$ is dominant, and $\mathcal{Y}$ and Y are irreducible, it follows that p_2: $\mathcal{Y} \otimes K \to Y \otimes K$ is also dominant. To complete the proof, we need only show that there is an open subset W of $\mathcal{Y}$, such that W is isomorphic to $p_1(W)$ via p_1.

Since $\mathcal{Y}$ is integral and dominates $\mathrm{Spec}(\mathcal{O})$, $\mathcal{Y}$ is flat over $\mathcal{O}$. Furthermore, since $\xi \circ f$ is morphism at x_0, the graph $\mathcal{Y} \otimes k$ of $\xi \circ f$ is smooth at $(x_0, \xi \circ f(x_0))$. Thus $\mathcal{Y}$ is smooth over $\mathcal{O}$ in a neighborhood of $(x_0, \xi \circ f(x_0))$, hence there is a neighborhood W of $(x_0, \xi \circ f(x_0))$ on $\mathcal{Y}$ such that

(a) $g \circ p_2$: $W \to \mathrm{Spec}(\mathcal{O})$ is a smooth morphism;

(b) p_1: $W \to U$ is quasi-finite and dominant.

Our assumption (iv), together with the valuative criterion for properness, shows that the map p_1: $W \to U$ is proper over a neighborhood of x_0 in U. Shrinking W if necessary, we may assume that W is proper over its image $p_1(W)$. Thus, since p_1: $W \to p_1(W)$ is both quasi-finite and proper,

$$p_1\colon W \to p_1(W)$$

is a finite morphism.

In particular, if x is a point of $p_1(W)$, specializing to x_0, then

$$\begin{aligned}\dim_{k(x)}\left(\mathcal{O}_W \otimes_{\mathcal{O}_U} k(x)\right) &\leqq \dim_{k(x_0)}\left(\mathcal{O}_W \otimes_{\mathcal{O}_U} k(x_0)\right)\\ &= \dim_{k(x_0)}\left(\mathcal{O}_{Y\otimes k} \otimes_{\mathcal{O}_{U\otimes k}} k(x_0)\right)\\ &= 1\end{aligned}$$

where "dim" refers to the vector space dimension. Thus p_1: $W \to p_1(W)$ is an isomorphism, as desired Q.E.D.

Most of the work in verifying the conditions of Lemma 3.6 for the situation at hand has already been done, as is shown below. We recall that the Hilbert point of a subvariety X of a projective variety V is denoted $h(X)$.

LEMMA 3.7: *Let p: $V \to \operatorname{Spec}(\mathcal{O})$ be a smooth projective fiber space of fiber dimension n. Suppose* $\operatorname{char}(k) = 0$, *or that* $\operatorname{char}(k) > 5$ *and* $n \leqslant 3$. *Suppose that $V \otimes k$ is quasi-ruled over a variety Y_0 via ϕ: $V \otimes k \to Y_0$. Then there is a local extension $\mathcal{O}'$ of $\mathcal{O}$, with quotient field K' and residue field k,*

$$\begin{array}{ccc} V' = V \otimes_{\mathcal{O}} \mathcal{O}' & \xrightarrow{p_1} & V \\ p_2 \downarrow & & \downarrow \\ \operatorname{Spec}(\mathcal{O}') & \to & \operatorname{Spec}(\mathcal{O}), \end{array}$$

a good resolution (u^*: $V^* \to V'$; U^*) of the singularities of $\phi \circ p_1$: $V' \otimes k \to Y_0$ for the family V', a smooth quasi-projective fiber variety g: $Y \to Spec(\mathcal{O}')$, and a subvariety $\mathcal{Y}$ of $U^* \times_{Spec(\mathcal{O}')} Y$ such that

- (i) *$\mathcal{Y}$ is smooth over $\mathcal{O}'$; $\mathcal{Y} \otimes K'$ and $\mathcal{Y} \otimes k$ are geometrically irreducible;*
- (ii) *there is a birational map ξ: $Y_0 \to Y \otimes k$ such that $\mathcal{Y} \otimes k$ is the proper transform under $id \times \xi$ of the graph of the morphism ϕ^*: $U^* \otimes k \to Y_0$ induced by $\phi \circ p_2$;*
- (iii) *$\mathcal{Y}$ is smooth and proper over y, and each fiber of p_2: $\mathcal{Y} \to Y$ is a rational curve;*
- (iv) *Y is a locally closed subset of $Hilb(V^*|\mathcal{O}')$ and $\mathcal{Y} = Y \times_{Hilb} H(V^*/\mathcal{O}')$. Furthermore, if ϕ is a morphism, we may take $U^* = V'$.*

PROOF: Let (u_1: $V_1 \to V$; U_1) be a good resolution of the singularities of ϕ: $V \otimes k \to Y_0$. Let $\overline{U_1 \otimes k}$ be the closure of $U_1 \otimes k$ in V_1 and let ϕ_1: $\overline{U_1 \otimes k} \to Y_0$ be the induced morphism. Since (u_1: $V_1 \to V$; U_1) is a good resolution, we have

(a) $\overline{U_1 \otimes k}$ is smooth;

(b) $\operatorname{codim}_{Y_0}[\phi_1(\overline{U_1 \otimes k} - \mathrm{U}_1 \otimes \mathrm{k})] \geqslant 1$.

Thus there is a smooth point y_0 of Y_0 such that $\phi_1^{-1}(y_0)$ is a smooth complete rational curve contained in $U_1 \otimes k$, with trivial normal bundle. Denote $\phi_1^{-1}(y_0)$ by X.

We apply Proposition 3.2, to give a maximal irreducible algebraic family, Y_1, of deformations of X in U_1. Let g_1: $Y_1 \to \mathrm{Spec}(\mathcal{O})$ and $\mathcal{Y}_1 \subseteq Y_1 \times_{\mathrm{Spec}(\mathcal{O})} U_1$ be as given by that proposition.

Let K' be a finite extension of K such that $Y_1 \otimes K'$ is a union of geometrically irreducible components defined over K'. Let $\mathcal{O}'$ be a local extension of $\mathcal{O}$ with quotient field K'. Take the various pull-backs

$$\begin{tikzcd}
U_1' \arrow[r, phantom, "\subseteq"] \arrow[d] & V_1' \arrow[r] \arrow[d] & V' \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathrm{Spec}(\mathcal{O}') \arrow[d] & Y_1' \arrow[l] \arrow[d] \\
U_1 \arrow[r, phantom, "\subseteq"] & V_1 \arrow[r] & V \arrow[r, "p"'] & \mathrm{Spec}(\mathcal{O}). & Y_1 \arrow[l]
\end{tikzcd}$$

Both U_1' and V' are smooth over $\mathcal{O}'$. Let ϕ_1': $\overline{U_1' \otimes k} \to Y_0$ denote the rational map induced by ϕ_1. We note that ϕ_1' restricted to $U_1' \otimes k$ is a morphism.

Just as in Theorem 1.7, we resolve the singularities of $\overline{U_1' \otimes k}$ and ϕ_1' to yield a good resolution (u^*: $V^* \to V'$; U^*) of the singularities of $\phi \circ p_2$: $V' \otimes k \to Y_0$.

Let ϕ^*: $\overline{U^* \otimes k} \to Y_0$ be the morphism induced by $\phi \circ p_2$, and let X^* be the curve $\phi^{*-1}(y_0)$. By construction, $U^* \otimes k$ is isomorphic to $U_1 \otimes k$, and hence X^* is a smooth rational curve, with trivial normal bundle in $U^* \otimes k$.

We identify Y_1' with a locally closed subset of $\mathrm{Hilb}(V^*/\mathcal{O}')$ in the obvious fashion, so that $\mathcal{Y}_1' = \mathcal{Y}_1 \otimes_{\mathcal{O}} \mathcal{O}'$ is identified with the restriction to Y_1' of the universal family $H(V^*/\mathcal{O}')$. One of the irreducible components of Y_1', say Y' contains $h(X^*)$.

Let g: $Y \to \mathrm{Spec}(\mathcal{O}')$ be a maximal algebraic family of deformations of X^* in U^* as given by Proposition 3.2. Counting dimensions, we find that Y contains an open subset of Y', and vice versa. As $Y' \otimes K'$ is geometrically irreducible so is $Y \otimes K'$. Shrinking Y if necessary we may assume that $Y \otimes k$ is irreducible.

Let $\mathcal{Y} = Y \times_{\mathrm{Hilb}(V^*/\mathcal{O}')} H(V^*/\mathcal{O}')$. $\mathcal{Y}$ satisfies (iii) by Proposition 3.2; as Y is smooth over $\mathcal{O}$ with irreducible fibers, so is $\mathcal{Y}$, which proves (i). We note that $Y \otimes k$ is a maximal algebraic family of deformations of X^* in $U^* \otimes k$. Thus the rational map

$$\xi\colon Y_0 \to \mathrm{Hilb}(V^* \otimes k/k); \qquad \xi(y) = h(\phi^{*-1}(y))$$

factors through $\overline{Y \otimes k}$, and is easily seen to be birational. Clearly $id \times \xi$: $(U^* \otimes k) \times Y_0 \to (U^* \otimes k) \times (Y \otimes k)$ transforms the graph of ϕ^* to $\mathscr{Y} \otimes k$. This verifies (ii) and completes the proof. Q.E.D.

THEOREM 3.8: *Let p: $V \to \mathrm{Spec}(\mathcal{O})$ be a smooth projective fiber variety, of fiber dimension n. Suppose* $\mathrm{char}(k) = 0$, *or that* $\mathrm{char}(k) > 5$ *and* $n \leqslant 3$. *Suppose that $V \otimes k$ is quasi-ruled via ϕ: $V \otimes k \to Y_0$. Suppose further that Y_0 is not uni-ruled. Then $V \otimes K$ is a quasi-ruled variety.*

PROOF: Let $\mathcal{O}'$, p': $V' \to \mathrm{Spec}(\mathcal{O}')$, ($u^*$: $V^* \to V'$; U^*), ξ: $Y_0 \to Y \otimes k$, g: $Y \to \mathrm{Spec}(\mathcal{O}')$, and $\mathscr{Y} \subseteq U^* \times_{\mathcal{O}'} Y$ be as given by Lemma 3.7. We need only show that $\mathscr{Y} \otimes K' \subseteq (U^* \otimes K') \times (Y \otimes K')$ is the graph of a rational map $\phi_{K'}$: $U^* \otimes K' \to Y \otimes K'$. By Lemma 3.6 and 3.7, we need only verify (iv) of Lemma 3.6.

Since $Y \otimes k$ is not uni-ruled, there is a point y_0 of $Y \otimes k$ that is not contained in any rational curve lying in $Y \otimes k$. We may choose y_0 so that the rational map $\tilde{\phi} = \xi \circ \phi^*$: $U^* \otimes k \to Y \otimes k$ is a morphism in a neighborhood of $\tilde{\phi}^{-1}(y_0)$. Let X denote the smooth rational curve $\tilde{\phi}^{-1}(y_0)$ and let x_0 be a point of X. Let $x_t \to x_0$ be a specialization, let y_t be a point of $p_2(\mathscr{Y} \cap (x_t \times Y))$, and let $y_t \to \bar{y}$ be a specialization extending $x_t \to x_0$. We must show $\bar{y} = \tilde{\phi}(x_0) = y_0$.

Let X_t be the smooth rational curve $p_1(\mathscr{Y} \cap (U^* \times y_t))$. Clearly x_t is in X_t, and $h(X_t) = y_t$. As y_t is in $\mathrm{Hilb}(V^*/\mathcal{O}')$, which is a union of components projective over $\mathcal{O}'$, $\bar{y}$ is also in $\mathrm{Hilb}(V^*/\mathcal{O}')$. Thus the specialization $y_t \to \bar{y}$ defines a specialization of subschemes of V^*, $X_t \to \bar{X}$ with

$$h(\bar{X}) = \bar{y}.$$

As x_t is a point of X_t, x_0 is a point of $\bar{X}$; furthermore, by Lemma 5 of [13] each irreducible component of $\bar{X}$ is a rational curve. Let $\bar{X}_0$ be a component of $\bar{X}$ containing x_0. Then either $\bar{X}_0 = X$, or $\tilde{\phi}(\bar{X}_0)$ is a rational curve on $Y \otimes k$ containing y_0. Since there are no such curves on $Y \otimes k$, we have $\bar{X}_0 = X$. Also, X, X_t, and $\bar{X}$ have the same Hilbert polynomials, which forces $X = \bar{X}$. Thus $\bar{y} = h(\bar{X}) = h(X) = y_0$, and (iv) is verified. This completes the proof of the theorem. Q.E.D.

THEOREM 3.9: *Let p: $V \to \mathrm{Spec}(\mathcal{O})$ be a smooth projective fiber variety. Suppose that $V \otimes k$ is quasi-ruled over a projective variety $Y_0 \subseteq \mathbb{P}^m$ via a morphism ϕ: $V \otimes k \to Y_0$. Then $V \otimes K$ is quasi-ruled.*

PROOF: Let $\mathcal{O}'$, p': $V' \to \mathrm{Spec}(\mathcal{O}')$, g: $Y \to \mathrm{Spec}(\mathcal{O}')$, ξ: $Y_0 \to Y \otimes k$, and $\mathcal{Y} \subseteq V' \times_{\mathcal{O}'} Y$ be a given by Lemma 3.7 for the morphism ϕ. As above, we need only verify (iv) in Lemma 3.6.

Let y_0 be a point of Y such that $\phi' = \xi \circ \phi$: $V' \otimes k \to Y \otimes k$ is a morphism in a neighborhood of $\phi'^{-1}(y_0)$. Let x_0 be a point of $X = \phi'^{-1}(y_0)$, let $x_t \to x_0$ be a specialization, let y_t be a point of $p_2(\mathcal{Y} \cap (x_t \times Y))$, and let $y_t \to \bar{y}$ be a specialization extending $x_t \to x_0$. Let $\bar{Y}$ denote the closure of Y in $\mathrm{Hilb}(V'/\mathcal{O}')$.

As y_t is in $\bar{Y}$, $\bar{y}$ is in $\bar{Y} \otimes k$, which is connected by Zariski's connectedness theorem.

Let X_t be the subscheme of V' with $h(X_t) = y_t$, let $\bar{X}$ be the subscheme with $h(\bar{X}) = \bar{y}$ and let $X_t \to \bar{X}$ be the specialization defined by $y_t \to \bar{y}$. $X_t \to \bar{X}$ defines a specialization of positive cycles $|X_t| \to |\bar{X}|$. Since $\bar{Y} \otimes k$ is connected, we have

$$|\bar{X}| \sim_{alg} |X| \qquad \text{on} \quad V' \otimes k.$$

Thus $\phi_*(|\bar{X}|) \sim_{\mathrm{alg}} \phi_*(X) \sim_{\mathrm{alg}} 0$ on $\mathbb{P}^M$. As $\mathrm{supp}|\bar{X}|$ contains x_0, and is connected, this forces

$$\mathrm{supp}(\bar{X}) = X.$$

Since $\bar{X}$ and X have the same Hilbert polynomial, we have

$$\bar{X} = X$$

and

$$\bar{y} = y_0,$$

which completes the proof. Q.E.D.

We now consider the case of ruled varieties

THEOREM 3.10: *Let p: $V \to \mathrm{Spec}(\mathcal{O})$ be a smooth projective fiber variety of fiber dimension n. Suppose that $V \otimes k$ is ruled over a variety Y_0 via ϕ: $V \otimes k \to Y_0$. Suppose that either*

(a) ϕ is a morphism and Y_0 is projective,

or

(b) Y_0 is not uni-ruled and char$(k) = 0$,

or

(c) Y_0 is not uni-ruled and char$(k) > 5$ and $n \leqslant 3$.

Finally suppose that $h^2(V \otimes k, \mathcal{O}_{V \otimes k}) = 0$. Then $V \otimes K$ is ruled.

PROOF: Let $\mathcal{O}'$, p: $V' \to \mathrm{Spec}(\mathcal{O}')$, $(u^*$: $V^* \to V', U^*)$, g: $Y \to \mathrm{Spec}(\mathcal{O}')$, ξ: $Y_0 \to Y \otimes k$, and $\mathcal{Y} \subseteq U^* \times_{\mathcal{O}'} Y$ be as given by Lemma 3.7 for the map ϕ.

Let $\phi^*\colon U^* \otimes k \to Y_0$ be the map induced by ϕ. Just as in Theorems 3.8 and 3.9, we have that

$$(g\colon Y \to \operatorname{Spec}(\mathcal{O}'), F)$$

is an extension of $\phi^*\colon U^* \otimes k \to Y_0$ to the family U^*, where $F\colon U^* \to Y$ is the map with graph $\mathcal{Y}$.

Let $\sigma\colon Y_0 \to \overline{U^* \otimes k}$ be a rational section to $\overline{\phi^*\colon U^* \otimes k} \to Y_0$. Let D be the divisor $1 \cdot \overline{\sigma(Y_0)}$ on $\overline{U^* \otimes k}$, and let D' be the divisor $(u^*)_*(D)$ on $V' \otimes k$.

Since $h^2(V' \otimes k, \mathcal{O}_{V' \otimes k}) = 0$, Lemma 3.3 implies there is a local extension $\mathcal{O}''$ of $\mathcal{O}'$ and an invertible sheaf L on $V' \otimes_{\mathcal{O}'} \mathcal{O}''$ such that $L \otimes k \cong \mathcal{O}_{V' \otimes k}(D')$. Changing notation, we may assume $\mathcal{O}' = \mathcal{O}''$.

Let f be a rational section of L over V' and let Z be the divisor of f. Choosing f appropriately, we may assume that $\operatorname{supp}(Z)$ does not contain $V' \otimes k$. Then $Z \otimes k$ is linearly equivalent to D' on $V' \otimes k$. Let Z^* denote the divisor $(u^*)^{-1}(Z)$ on U^*.

Let $\mathcal{E} \subseteq U^*$ be the exceptional divisor of $u^*\colon U^* \to V'$. Write $\mathcal{E}$ as a sum of irreducible divisors

$$\mathcal{E} = \sum_{i=1}^{r} \mathcal{E}_i,$$

and write $\operatorname{supp}(\mathcal{E} \otimes k)$ as a union of irreducible components

$$\operatorname{supp}(\mathcal{E} \otimes k) = \bigcup_{j=1}^{e} E_j.$$

Then

$$\begin{aligned}
(1) \qquad Z^* \otimes k &= (u^*)^{-1}(Z \otimes k) \\
&\sim_l (u^*)^{-1}(D') \qquad \text{on} \quad U^* \otimes k \\
&= (u^*)^{-1}((u^*)(D)) \\
&= D + \sum_{j=1}^{s} r_j E_j \qquad \text{for suitable integers } r_j \\
&= D + \sum_{i=1}^{r} n_i (\mathcal{E}_i \otimes k) \qquad \text{for suitable integers } n_i,
\end{aligned}$$

the last line following from Definition 1.2 (iv).

Let y_t be a generic point of $Y \otimes \overline{K}'$ over $\overline{K}'$. Let X_t be the fiber

$$(F \otimes K')^{-1}(y_t) = p_1(\mathcal{Y} \cap (U^* \times y_t)).$$

Let y_0 be a generic point of $Y \otimes k$ over k, and let X_0 be the fiber

$$(\xi \circ \phi)^{-1}(y_0) = (F \otimes k)^{-1}(y_0) = p_1(\mathscr{Y} \cap (U^* \times y_0))$$

X_0 and X_t are smooth complete rational curves on $U^* \otimes k$ and $U^* \otimes K'$ respectively, and X_t is algebraically equivalent to X_0 on U^*. Thus

$$\begin{aligned} &I(X_t \cdot (Z^* \otimes K' - \Sigma n_t(\mathscr{E}_t \otimes K')); U^* \otimes K') \\ &= I(X_t \cdot (Z^* - \Sigma n_t \mathscr{E}_t); U^*) \\ &= I(X_0 \cdot (Z^* - \Sigma n_t \mathscr{E}_t); U^*) \\ &= I(X_0 \cdot (Z^* \otimes k - \Sigma n_i \mathscr{E}_i \otimes k); U^* \otimes k) \\ &= I(X_0 \cdot D; U^* \otimes k), \text{ by } (1) \\ &= 1. \end{aligned}$$

(The intersection numbers are preserved since X_0 and X_t are both members of a family of *complete* curves on U^*.)

By Lemma 2.2, $F \otimes \bar{K}$: $U^* \otimes \bar{K} \to Y \otimes \bar{K}$ is a ruling, which completes the proof. Q.E.D.

We summarize our results in the following theorem.

THEOREM 3.11: *Let p: $V \to M$ be a smooth projective fiber variety of fiber dimension n over a variety M. Let O be a point of M. Suppose $V \otimes k$ is quasi-ruled over a variety Y_0 via ϕ: $V \otimes k \to Y_0$. Suppose that either*

(a) ϕ is a morphism and Y_0 is projective,

or

(b) Y_0 is not uni-ruled and char$(k) = 0$,

or

(c) Y_0 is not uni-ruled and char$(k) > 5$ and $n \leqslant 3$.

Then there is an open neighborhood U of O in M such that $V \otimes k(u)$ is quasi-ruled for each u in U.

If in addition ϕ is a ruling and $h^2(V \otimes k, \mathcal{O}_{V \otimes k}) = 0$, then $V \otimes k(u)$ is ruled for each u in M.

PROOF: Our first conclusion follows from Lemma 2.5, Theorems 3.8 and 3.9 and a simple noetherian induction. The second conclusion follows from Lemma 2.5, Theorem 3.8, and the following result of Matsusaka (Theorem 1.1 [11]):

Let V be a smooth ruled variety in a projective space V' a variety in a projective space, and $\mathcal{O}$ a DVR such that V' is a specialization of V over $\mathcal{O}$. Then V' is also a ruled variety. Q.E.D.

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