S. RALLIS

On the howe duality conjecture

Compositio Mathematica, tome 51, nº 3 (1984), p. 333-399
<http://www.numdam.org/item?id=CM_1984__51_3_333_0>

Compositio Mathematica 51 (1984) 333–399

ON THE HOWE DUALITY CONJECTURE

S. Rallis

Table of contents

Introduction

The purpose of this paper is to establish certain results in the theory of cusp forms for the classical groups of the type Sp_n or $\mathrm{O}(Q)$ (symplectic and orthogonal). The theory for $G\ell_n$ has been initiated and is continuing to be developed, thanks mainly to the efforts of Hecke, Godement, Jacquet, Langlands, Piatetski-Shapiro, and Shalika. There are essentially two main problems in the $G\ell_n$ case. First, it is necessary to develop a Hecke theory for automorphic cuspidal representations of $G\ell_n$. That is, using the L-function theory of admissible $G\ell_n(\mathbb{A})$ representations given in [J-Ps-Sh] and [G-J], one wants to characterize the automorphic cuspidal representation in terms of the holomorphicity of the associated L-function and the family of functional equations that the "twists" of the L-functions must satisfy. The second question is to follow the general Langlands philosophy and determine whether such L-functions can be interpreted as Artin-type L-functions, i.e., L-functions associated to finite dimensional representations of certain Weil-Deligne groups. On the other hand, to answer similar questions for Sp_n or $\mathrm{O}(Q)$ seems to be at a very rudimentary stage. Indeed, very recent progress has been given in [Ps-1] to develop a systematic theory of L-functions for Sp_2. What is evident

This research was supported, in part, by NSF Grant MCS 78-02414.

from the various empirical observations in [H-Ps-1] and [Ps-2] is several phenomena that do not occur in the $G\ell_n$ case. Namely (1) L-functions of cuspidal representations may have poles and (2) "Ramanujan" type conjectures fail; that is, eigenvalues of Hecke operators operating on cusp forms fail to be "unitary" or tempered in certain directions. These specific phenomena are, in fact, very intimately connected; they come from using lifting theory associated to Weil representations of dual reductive pairs.

Thus our purpose here is to use lifting theory to give a decomposition of the space of cusp forms on these groups into orthogonal "pieces" R_i, where all the irreducible components occurring in R_i (for certain i) exhibit the specific phenomena indicated above. The key point here is to show that such a lifting theory can actually be formulated in precise terms (see [H]). That is, for each subspace R_i, we require that there is a well defined mapping from automorphic representations occurring in R_i to cuspidal automorphic representations on some other group G_i. Moreover we want that such a mapping be *injective* and preserve the multiplicities of the representations.

One additional point that inevitably comes with such an investigation is how to use the Selberg Trace Formula to get an effective "comparison of traces" statement between Hecke operators on the various groups involved in the lifting. Specifically we present below preliminary evidence that such a comparison of trace is possible; the main departure from the classical cases is that we require comparison of traces for not just one fixed dual pair but for a whole family of dual pairs!

Indeed we start with a fixed orthogonal group $O(Q)$ (where Q is defined on an m dimensional space) and consider the space of cusp forms $L^2_{\text{cusp}}(O(Q)(\mathbb{A}))$ on $O(Q)(K)\backslash O(Q)(\mathbb{A})$. Then we define for each f in L^2_{cusp} the "lift" of f as the span of functions

$$G \rightsquigarrow \langle \theta_\varphi(G, g) | f(g) \rangle_{O(Q)(K)\backslash O(Q)(\mathbb{A})}$$

where θ_φ is a θ-series on the group $\mathrm{Sp}_n(\mathbb{A}) \times O(Q)(\mathbb{A})$ constructed from the associated Weil representation of the dual reductive pair $(\mathrm{Sp}_n, O(Q))$ (where φ varies over all Schwartz-Bruhat functions). Then for each irreducible component (π) in L^2_{cusp}, we let $\mathscr{A}_n(\pi) =$ the space of the "lift" of f as f varies in (π). Then the first point we show is that

(1) The space $\mathscr{A}_n(\pi) \subseteq L^2_{\text{cusp}}(\mathrm{Sp}_n(\mathbb{A})) \Leftrightarrow$ the lift $\mathscr{A}_r(\pi)$ of π is identically zero for all $1 \leqslant r < n-1$.

This rather elementary criterion then allows us to determine a *natural decomposition* (Theorem I.2.1) of the space L^2_{cusp} into an orthogonal direct sum of $O(Q)(\mathbb{A})$ invariant subspaces

$$L^2_{\text{cusp}} = R_1 \oplus R_2 \oplus \ldots \oplus R_m \qquad (\text{with } m = \dim Q),$$

where a representation π occurring in R_i has the important property that the representation will die under "lifting" to $\mathrm{Sp}_r(\mathbb{A})$ when $r < t$, and $r = t$ is the first positive number where π lifts to a *nonzero* cuspidal representation in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_t(\mathbb{A}))$. It may happen that π will lift to a nonzero cuspidal representation in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_t(\mathbb{A}))$ for $t > r$. This then leads naturally to the *global Howe duality conjecture* (see [H]) which states that

(∗) given a cuspidal representation π occurring in R_t, then there is a unique cuspidal representation $\beta_t(\pi)$ occurring in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_t(\mathbb{A}))$ such that $\mathscr{A}_t(\pi) \cap \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_t(\mathbb{A}))$ is a nonzero multiple of $\beta_t(\pi)$. Moreover the induced mapping

$$\pi \rightsquigarrow \beta_t(\pi)$$

is an *injective* mapping from the set of representations of $\mathrm{O}(Q)(\mathbb{A})$ occurring in R_t into the set of cuspidal representations of $\mathrm{Sp}_t(\mathbb{A})$.

The import of the conjecture is that the lifting defined between R_t and $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_t(\mathbb{A}))$ gives, in fact, a *well-defined mapping* between representations which is *injective*!

We then show that the global duality conjecture is implied by the local duality conjecture (Theorem I.2.2) *for all primes* v in K. In particular, the local duality conjecture is a statement about local Weil representations of $\mathrm{O}(Q_v) \times \mathrm{Sp}_n(K_v)$; namely if $\mathrm{Hom}_{\mathrm{O}(Q_v)\times \mathrm{Sp}_n(K_v)}(S[M_{mn}(K_v)], {}^{\omega\otimes\sigma_1}_{\omega\otimes\sigma_2}) \neq 0$, then $\sigma_1 \cong \sigma_2$ ($\mathrm{Sp}_n(K_v)$ equivalent), and if $\mathrm{Hom}_{\mathrm{O}(Q_v)\times \mathrm{Sp}_n(K_v)}(S[M_{mn}(K_v)], {}^{\omega_1\otimes\sigma}_{\omega_2\otimes\sigma}) \neq 0$, then $\omega_1 \cong \omega_2$ ($\mathrm{O}(Q_v)$ equivalent). Here ω, ω_1, ω_2, and σ, σ_1, and σ_2 are all unitarizable representations. The main subject in this paper is then to indicate how the local duality conjecture can be proved in various cases; namely, if

(i) Q_v is anisotropic (already done in [As], [K-V], and [H]),
(ii) Q_v is an arbitrary form with v finite and $\dim Q_v > 4n + 2$,
(iii) Q_v is a nonquaternionic form with v finite and $\dim Q_v = 4n + 2$,
(iv) Q_v is a split form with v finite and $\dim Q_v = 4n$,
(v) Q_v is an unramified form with $\dim Q_v \leqslant n + 2$.

We note here the "almost" everywhere statement in [H]. We think that the methods given below will probably extend (with some effort) to all forms Q_v for each local prime v.

Then returning to the *global Howe duality conjecture*, it is possible to refine the conjecture by specifying the behavior of multiplicities under the map β_t given above. That is, we conjecture that for each π occurring in R_t, we have "multiplicity of π in R_t = multiplicity of $\beta_t(\pi)$ in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_t(\mathbb{A}))$". Indeed we can show that this multiplicity preserving conjecture is implied by a local multiplicity-one conjecture (Corollary to Theorem I.2.2) for *all primes* v in K. Again this is a statement about the local Weil representations of $\mathrm{O}(Q_v) \times \mathrm{Sp}_n(K_v)$; namely *for unitary representations* ω and σ, we have $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{O}(Q_v)\times \mathrm{Sp}_n(K_v)}(S[M_{mn}(K_v)], \omega \otimes \sigma)) \leqslant 1$. But then we show that this is true in the cases where we can prove the local duality conjecture.

Thus we can actually show the *global Howe duality conjecture and* the *multiplicity preserving statement* are valid in certain cases (see end of I.2).

We note that in [R-1] we have given the explicit relation between the action of the Hecke algebras of $\mathrm{O}(Q_v)$ and $\mathrm{Sp}_n(K_v)$ in the local Weil representation π_{Q_l} on $S[M_{mn}(K_v)]$. In particular, using the decomposition of $\mathrm{L}^2_{\mathrm{cusp}}$ above, we can reasonably expect that there exists a comparison of trace formula of the following form. If f_v is a given Hecke operator on $\mathrm{L}^2_{\mathrm{cusp}}$ (i.e., f_v belongs to the local Hecke algebra of $\mathrm{O}(Q_v)$ at the prime v), then $\mathrm{Tr}(f_v)$ on $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ has to be compared with the sum

$$\sum_{l=1}^{l=m} \mathrm{Tr}\left(f_v^l|_{\mathrm{Im}\,\beta_l}\right),$$

where f_v^l is the Hecke operator on $\mathrm{Sp}_l(K_v)$ paired to f_v (in the correspondence in [R-1]). The main difficulty here is to have an effective way to compute $\mathrm{Tr}(f_v^l|_{\mathrm{Im}\,\beta_l})$ using the Selberg trace formula.

We organize the paper in the following fashion.

In §0 we present preliminary definitions and notation. In Chapter I there are two sections. We are concerned with the global theory of cusp forms on $\mathrm{O}(Q)$ or Sp. First in §1 of Chapter I, we determine (Theorem I.1) the constant term of the "lift" of a cusp form f ($f \in \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$) along any maximal parabolic in $\mathrm{Sp}_r(\mathbb{A})$. We also determine in an analogous fashion the constant term of the "lift" of a cusp form f (for $f \in \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_r(\mathbb{A}))$) along any maximal parabolic in $\mathrm{O}(Q)$. Then in §2 of Chapter I, we deduce the simple criterion (Corollary to Theorem I.1) stated above for the "lift" to be a cusp form. Again we do the analogous statement for Sp_r. Then we define the spaces R_l and show in Theorem I.2.1 a "finiteness" statement about the R_l and deduce the orthogonal decomposition of $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A})) = \oplus_{l=1}^{l=m} R_l$. Again we deduce an analogous statement for $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$. Then we state the *global Howe duality conjecture* and show in Theorem I.2.2 that this conjecture is implied by the local duality conjecture for all primes v in K. In the proof we introduce a certain $\mathrm{O}(Q)(\mathbb{A}) \times \mathrm{Sp}_l(\mathbb{A})$ invariant, bilinear nonzero form on the space $S[M_{ml}(\mathbb{A})] \otimes \pi \otimes \sigma$ (where π, σ are cuspidal representations of $\mathrm{O}(Q)$ and Sp), which is factorizable into a product of local $\mathrm{O}(Q_v) \times \mathrm{Sp}_l(K_v)$ invariant sesquilinear forms; this allows us to reduce the global duality conjecture to the local duality conjecture stated above (for all primes). On the other hand, we again use this global distribution and its factorizable property to show that (Corollary to Theorem I.2.2) local multiplicity-one implies the global multiplicity preserving statement. Then we indicate at the end of I.2 for which cases we can prove the global duality conjecture.

In §3 of Chapter I, we consider the special example of the space $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_2(\mathbb{A}))$. In particular, there is an orthogonal decomposition of this space into three subspaces $I_i(H_{i-1})$ with $i = 3, 4, 5$, such that $I_i(H_{i-1})$ lifts to $L^2_{\text{cusp}}(O(H_{i-1})(\mathbb{A}))$ with H_{i-1} the unique split form in $2i$-2 variables. Then we give a simple criterion for functions to belong to the various $I_i(H_{i-1})$ in terms of the *vanishing* of certain "periods." We show that the space $I_5(H_4)$ contains all those automorphic cuspidal representations without standard Whittaker models. Thus the Saito-Kurokawa space of [Ps-2] lies in $I_5(H_4)$; we show that this latter space does not fill up $I_5(H_4)$ by giving examples of automorphic cuspidal representations of $\mathrm{Sp}_2(\mathbb{A})$ which are nonzero lifts from quaternion algebras (see [H-Ps-2]). The point here is that such examples satisfy the generalized Ramanujan conjecture whereas those representations lying in the Saito-Kurokawa space do not.

Chapter II is concerned with proving the local duality conjecture for the cases mentioned above.

Indeed here we follow a method of proof similar to that proposed in [H]. In particular, in II.1 we prove the "invariant distribution" Theorem (Theorem II.1). That is, for any local Weil representation π_{Q_v} of $O(Q_v) \times \mathrm{Sp}_n(K_v)$ on the space $S[M_{mn}(K_v)]$, we show that the Jacquet module $S[M_{mn}(K_v)]_{O(Q_v)}$ in the sense of [B-Z] as an $\mathrm{Sp}_n(K_v)$ module is isomorphic to the $\mathrm{Sp}_n(K_v)$ module determined as the range of the mapping $\varphi \to \{\pi_{Q_v}(G)(\varphi)[0]\}$ (with $\varphi \in S[M_{mn}(K_v)]$). Thus we have a structure Theorem about the space of $O(Q_v)$ invariant distributions on $S[M_{mn}(K_v)]$. In particular, this generalizes a similar structure Theorem for the case when $n = 1$ in [R-S-1]. We also indicate that an analogous statement is true for the symplectic case.

In II.2 we consider the restriction of $\mathrm{Sp}_{2n}(K_v)$ module $\rho_{Q_v} = S[M_{m2n}(K_v)]_{O(Q_v)}$ to the subgroup $\mathrm{Sp}_n(K_v) \times \mathrm{Sp}_n(K_v)$ (embedded in $\mathrm{Sp}_{2n}(K_v)$ via $(g_1, g_2) \rightsquigarrow \left[\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array}\right]$). We first consider ρ_{Q_v} embedded in a bigger Sp_{2n} module $V_{m/2}$, which is an Sp_{2n} induced module, coming from the quasi-character on the maximal parabolic P_{2n}

$$\left[\begin{array}{c|c} A & X \\ \hline 0 & (A')^{-1} \end{array}\right] \rightsquigarrow \det|A|^{m/2}\langle \det A | \Delta(Q_v)\rangle_{K_v},$$

$\Delta(Q_v) =$ the discriminant of Q_v. Then (using the analogue of the Frobenius Subgroup Theorem) it is easy to analyze $V_{m/2}$ when restricted to $\mathrm{Sp}_n \times \mathrm{Sp}_n$. Indeed we know that the coset space $\mathrm{Sp}_{2n}/\mathrm{Sp}_n \times \mathrm{Sp}_n$ is an example of an affine symmetric space, and hence the orbit space $P_{2n}\backslash \mathrm{Sp}_{2n}/\mathrm{Sp}_n \times \mathrm{Sp}_n$ is finite; thus we can determine (Proposition II.2.1) a finite $\mathrm{Sp}_n \times \mathrm{Sp}_n$ composition series for $V_{m/2}$, where the first component of the composition series is the $\mathrm{Sp}_n \times \mathrm{Sp}_n$ module given by the left and right action on the space $C_c^\infty(\mathrm{Sp}_n)$. The importance of this observation is that "generically" $V_{m/2}$ as an $\mathrm{Sp}_n \times \mathrm{Sp}_n$ module has a multiplicity-one property (Remarks II.2.2 and II.2.3).

In II.3 we use the doubling principle of the Weil representation to show (Proposition II.3.1) that there exists an injective mapping of the space

$$\bigoplus_{\sigma} \operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q_v)}(S[M_{mn}(K_v)], \pi \times \sigma)$$

(σ varies over all $\mathrm{O}(Q_v)$ inequivalent, admissible, irreducible representations) into $\operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_{Q_v}, \pi \times \pi)$. Thus we reduce questions about multiplicity of $\operatorname{Hom}_{\mathrm{Sp}_n(K_v) \times \mathrm{O}(Q_v)}(S[M_{mn}(K_v)], \pi \otimes \sigma)$ to studying the multiplicity of $\operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_{Q_v}, \pi \otimes \pi)$. However we note that, since ρ_{Q_v} does not in general equal $V_{m/2}$, we need finer information about the Sp_{2n} module $V_{m/2}$. This we give in Proposition II.3.2.

Finally in II.4, we give the strategy of proof of the local Howe duality conjecture and the local multiplicity-one conjecture for the cases discussed above. We introduce the notion of $\mathrm{Sp}_n \times \mathrm{Sp}_n$ boundary components on the space ρ_{Q_v} and prove the key technical step (Theorem II.4.1) that every $\mathrm{Sp}_n \times \mathrm{Sp}_n$ intertwining operator from ρ_{Q_v} to $\pi \otimes \pi$ which "lives" on a boundary component will, in fact, imply that π or the associated $\pi(\sigma)$ (representation of $\mathrm{O}(Q)$) is either nonunitary or trivial. Then we must handle computationally the case where π is trivial; here we have to analyze a specific Jacquet module on $S[M_{mn}(K_v)]$ relative to a parabolic P_t in Sp_n.

§0. Notation and preliminaries

(I) Let k be a local field of characteristic O. We fix a nontrivial additive character τ on k. Let $\langle\ ,\ \rangle_k$ be the usual Hilbert symbol on k. Let $\mathrm{d}x$ be a Haar measure on k which is self dual relative to τ. We let $|\ |_k$ be an absolute value of k.

If k is a nonarchimedean field, we let $\mathcal{O}_k =$ ring of integers of k, $\pi_k =$ the maximal ideal in $\mathcal{O}_k$, and $q =$ the cardinality of $\mathcal{O}_k/\pi_k$.

(II) Let K be a number field (i.e. finite degree extension of $\mathbb{Q}$, the rational numbers). Let $\mathbb{A}_K$ be the corresponding adelic group. Then embed K as a discrete subring in $\mathbb{A}_K$. Let K_v be the completion of K relative to a prime v in K. Let τ be a nontrivial character on $\mathbb{A}_K$ which equals 1 on K; then there exist compatible characters τ_v on K_v (for all primes v in K) such that $\tau(X) = \prod_v \tau_v(X_v)$. Let $\mathrm{d}X$ be the measure (Tamagawa measure) on $\mathbb{A}_K$ such that the group $\mathbb{A}_K/K$ is self dual relative to τ and $\mathbb{A}_K/K$ has mass 1. When the context is clear, we drop K in $\mathbb{A}_K$ and just use $\mathbb{A}$ for $\mathbb{A}_K$.

(III) Let Q be a nondegenerate quadratic form on K^m. Let Q_v be the corresponding local versions on K_v^m. If Q_v is a totally split form which is the direct sum of r hyperbolic planes, then we let $Q_v = H_r$. Let $\mathrm{O}(Q)$ be the orthogonal group of Q. Then we can form the corresponding adelic group $O(Q)(\mathbb{A})$ and the corresponding local orthogonal groups $O(Q_v)$ of Q_v at K_v. Let $O(Q)(K)$ = the K rational points in $O(Q)$ and embed $O(Q)(K)$ into $O(Q)(\mathbb{A})$ in the standard way. Choose a Tamagawa measure on the quotient $O(Q)(K)\backslash O(Q)(\mathbb{A})$ as given in [Ar].

Similarly let A be a nondegenerate alternating form on K^{2n}. Let Sp_n be the corresponding symplectic group and $\mathrm{Sp}_n(\mathbb{A})$, $\mathrm{Sp}_n(K_v)$ the associated adelic and local objects. Let $\mathrm{Sp}_n(K)$ = the K rational points in Sp_n and embed $\mathrm{Sp}_n(K)$ into $\mathrm{Sp}_n(\mathbb{A})$ again in the standard fashion and choose a Tamagawa measure on the quotient $\mathrm{Sp}_n(K)\backslash \mathrm{Sp}_n(\mathbb{A})$ as given in [Ar].

(IV) We consider the category of smooth representations for the local and global objects in question. That is, π_v is smooth locally for G_v (a local group) if (1) at the Archimedean primes, π_v is a differentiable module for G_v, i.e., $(\pi_v)_\infty = C^\infty$ vectors in $\pi_v = \pi_v$ and (2) at the finite non-Archimedean primes, π_v is a smooth module for G_v in the sense of [B-Z]. Then we consider also the category of admissible modules as given in [B-Z] and [B-J].

In the non-Archimedean case, we use the notion of Jacquet functor given in [B-Z]. That is, if $N_v \subset G_v$ is any closed subgroup, and if π_v is any smooth G_v module, we have a functor from π_v to $(\pi_v)_{N_v} = \pi_v/\pi_v(N_v)$, where $\pi_v(N_v) = \{$all linear combinations of the form $x - \pi_v(n)x$ as x varies in π_v and n varies in $N_v\}$. Moreover we consider the category of admissible $G_{\mathbb{A}}$ modules as given in [F]. In this context, we note the well known relation in [F] between unitary irreducible modules of $G_{\mathbb{A}}$ and admissible irreducible modules of $G_{\mathbb{A}}$. We also use the notion of automorphic irreducible representation of $G_{\mathbb{A}}$ as given in [B-J].

(V) Let X_v be an ℓ-space in the sense of [B-Z].

Then $S(X_v)$ is the space of locally constant and compact support functions on X_v. If X_∞ is a C^∞ manifold, then $S(X_\infty)$ is the space of C^∞ functions on X_∞ which have an appropriate rapid decrease property (relative to derivatives, etc.) Then if X is the restricted direct product $\prod_v X_v$, then $S(X)$ is the restricted tensor product $\prod_v S(X_v)$ in the sense of [F].

If S is any set of places of K, we let $C^\infty(X_S)$ be the space of smooth complex-valued functions on X_S, and $C_c^\infty(X_S)$, those elements in $C^\infty(X_S)$ having compact support. Note that f in $C_c^\infty(X_S)$ is smooth if f is continuous on X_S and as a function of 2 variables $X_{S,\infty} \times X_{S,\mathrm{fin}} \xrightarrow{f} \mathbb{C}$ with f in C^∞ in X_∞ (locally constant in X_{fin}, resp.) for fixed x_∞ (x_{fin} resp.). For instance, we know that if $X = M_{mn}(K)$, $m \times n$ matrices over K, then $S_{\mathrm{comp}}(X_{S_1}) \otimes S(X_{\mathrm{fin}})$ is *dense* in $C_c^\infty(X)$, where S_1 = set of Archimedean primes of K, $S_{\mathrm{comp}}(X_{S_1})$ = the compactly supported elements in $S(X_{S_1})$, and fin = all finite primes in K.

(VI) If $G_{\mathbb{A}}$ is a global group, then we denote the space of cusp forms on $G_K\backslash G_{\mathbb{A}}$ by $L^2_{\mathrm{cusp}}(G_{\mathbb{A}})$. We note (*by our convention*) that if $G_K\backslash G_{\mathbb{A}}$ is compact, then $L^2_{\mathrm{cusp}}(G_{\mathbb{A}}) = \{$all functions $f \perp$ constants$\}$. We know that

$L^2_{\mathrm{cusp}}(G_{\mathbb{A}})$ is discretely decomposable as a $G_{\mathbb{A}}$ module, and each unitary irreducible representation occurring in $L^2_{\mathrm{cusp}}(G_{\mathbb{A}})$ has a finite multiplicity. We denote

$$\langle f_1 | f_2 \rangle_{G(\mathbb{A})} = \int_{G(K)\backslash G(\mathbb{A})} f_1(g) f_2(g) \mathrm{d}g,$$

where $\mathrm{d}g$ is some Tamagawa measure on the quotient $G(K)\backslash G(\mathbb{A})$.

(VII) We consider the following Weil representation. Namely we fix a nondegenerate form Q on k^m (k, a local field) and a nondegenerate alternating form A on k^{2n}. (Here we have dropped the subscript v from Q to avoid excessive notation; throughout the paper it will be clear from the context when the symbol Q is meant to denote Q_v.) Then we consider the alternating form $Q \otimes A$ on $M_{m2n}(k)$. To this form we associate a Weil representation of the two-fold cover $\tilde{\mathrm{Sp}}_{m \cdot n}$ of Sp_{mn} given in [We]. Then we restrict the representation to a subgroup $\tilde{\mathrm{Sp}}_n \times \mathrm{O}(Q) =$ inverse image of $\mathrm{Sp}_n \times \mathrm{O}(Q)$ in $\tilde{\mathrm{Sp}}_{mn}$.

The representation π_Q of $\tilde{\mathrm{Sp}}_n \times \mathrm{O}(Q)$ is given by

$$\pi_Q\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0 & (A')^{-1} \end{array}\right], \varepsilon\right)\varphi[X] = \varepsilon |\det A|^{m/2} \frac{\gamma(1)}{\gamma((\det A)^m)} \varphi[X \cdot A],$$

$$\pi_Q\left(\left[\begin{array}{c|c} I & B \\ \hline 0 & I \end{array}\right], \varepsilon\right)\varphi[X] = \varepsilon \tau\left(\mathrm{Tr}\left(\tfrac{1}{2} B X' Q X\right)\right) \varphi[X],$$

$$\pi_Q\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \varepsilon\right)\varphi[Y] = \varepsilon \gamma(1)^{-nm} \int_{M_{nm}(k)} \tau(-\mathrm{Tr}(Y' Q X)) \times \varphi[X] \mathrm{d}X,$$

$$\pi_Q(g)\varphi[X] = \varphi[g^{-1} \cdot X], \qquad g \in \mathrm{O}(Q),$$

where $\gamma(\ )$ is given in [R-S-2]. Then, if m is even, we know that there exists a splitting map $\mathrm{Sp}_n \to \tilde{\mathrm{Sp}}_n(k)$, $G \rightsquigarrow (G, \mathscr{s}_2(G))$ such that $G \rightsquigarrow (G, \mathscr{s}_2(G)) \to \pi_Q(G, \mathscr{s}_2(G))$ gives a representation of Sp_n (see [R-1] for details).

In the case $Q = Q' \oplus (-Q')$, where Q' is any nondegenerate form, it is possible to linearize the Weil representation π_Q. That is, we define the partial Fourier transform

$$F_T(f)\begin{bmatrix} Z \\ W \end{bmatrix} = \int f\begin{bmatrix} (U+W)/2 \\ (U-W)/2 \end{bmatrix} \tau(\mathrm{Tr}(U' Q' Z)) \mathrm{d}U.$$

Here we note that the action of $\mathrm{Sp}_n(k)$ is given by

$$\pi_\ell(G)\varphi\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \varphi\left[\Omega^{-1}G^{\Delta}\Omega\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right]$$

where $\Omega\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [X_1|X_2]$ and $[X_1|X_2]G^{\Delta} = [X_1'|X_2']$ with

$$G^{\Delta} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{t^{-1}}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

and $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Here we have the identity

$$F_T \circ \left(\pi_{Q' + (-Q')}(G)\right)(\varphi)\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \pi_\ell(G)F_T \circ \varphi\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

We note that, via a change of basis, that if $Q = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$, then the corresponding linearization F_T is given by

$$F_T(f)\begin{bmatrix} Z \\ W \end{bmatrix} = \int f\begin{bmatrix} R \\ W \end{bmatrix}\tau(\mathrm{Tr}(R'Z))\mathrm{d}R.$$

(VIII) Using the local data in (VII), we can define a global Weil representation π_Q of $\dot{\mathrm{Sp}}_n(\mathbb{A}) \times \mathcal{O}(Q)(\mathbb{A})$ on the space $S[M_{mn}(\mathbb{A})]$ (for details, see [We]). For every $\varphi \in S[M_{mn}(\mathbb{A})]$, we can construct a function $((G, g) \in \mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A}))$:

$$\theta_\varphi(G, g) = \sum_{\xi \in M_{mn}(K)} \pi_Q(G, g)(\varphi)(\xi).$$

Then θ_φ is an automorphic function on $\mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A})$. That is, $\theta_\varphi(\gamma_1 G, \gamma_2 g) = \theta_\varphi(G, g)$ for all $\gamma_1 \in \mathrm{Sp}_n(K)$, $\gamma_2 \in \mathrm{O}(Q)(K)$. Moreover θ_φ is a slowly increasing function on $\mathrm{Sp}_n(K) \times \mathrm{O}(Q)(\mathrm{K})\backslash\mathrm{Sp}_n(\mathbb{A}) \times \mathrm{O}(Q)(\mathbb{A})$ in the sense of [B-J].

(IX) We construct here representatives for the *maximal parabolic subgroups* of $\mathrm{O}(Q)$ and Sp_n.

(A) Sp_n has parabolics given by P_{n-i}, $i = 0, \dots, n-1$, where

$$P_{n-i} \cong \mathrm{Sp}_i \times \mathrm{G}\ell_{n-i} \times U_i^n$$

with

$$\mathrm{Sp}_l = \left\{ \left[\begin{array}{c|c|c|c} A & 0 & B & 0 \\ \hline 0 & I & 0 & 0 \\ \hline C & 0 & D & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right] \,\Bigg|\, \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathrm{Sp}_l \right\}$$

and

$$\mathrm{G}\ell_{n-l} = \left\{ \left[\begin{array}{c|c} \begin{array}{c|c} I & 0 \\ \hline 0 & A \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & (A^t)^{-1} \end{array} \end{array}\right] \,\Bigg|\, A \in \mathrm{G}\ell_{n-l} \right\}$$

and

U_l^n is the semidirect product $U_l^{n,1} \times U_l^{n,2}$,

where

$$U_l^{n,1} = \left\{ \left[\begin{array}{c|c} \begin{array}{c|c} I & 0 \\ \hline T & I \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} I & -T^t \\ \hline 0 & I \end{array} \end{array}\right] \,\Bigg|\, T \in M_{n-l,l}(k) \right\}$$

and

$$U_l^{n,2} = \left\{ \left[\begin{array}{c|c} I & \begin{array}{c|c} 0 & T \\ \hline T^t & X \end{array} \\ \hline 0 & I \end{array}\right] \,\Bigg|\, T \in M_{l,n-l}(k), \qquad X \in \mathrm{Sym}_{n-l}(k) \right\}.$$

(B) $\mathcal{O}(Q)$ has parabolics given by $\tilde{P}_r$, $r = 1, \ldots,$ index Q, where

$$\tilde{P}_r \cong \mathcal{O}(Q|_{\Sigma_r}) \times \mathrm{G}\ell(\Sigma) \times \tilde{N}_r, \qquad r = \dim \Sigma,$$

where $\Sigma \oplus \Sigma^* \oplus \Sigma_r = k^m$ is a Q *orthogonal splitting* of k^m into spaces where Σ, Σ^* are Q isotropic subspaces which are paired nonsingularly by Q. Here we identify Σ, Σ^*, and Σ_r to

$$\left[\begin{array}{c} * \\ \hline 0 \\ \hline 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline * \\ \hline 0 \end{array}\right], \text{ and } \left[\begin{array}{c} 0 \\ \hline 0 \\ \hline * \end{array}\right], \text{ respectively,}$$

when we consider k^m as a space of column vectors. Also

$O(Q|_{\Sigma_r})$ = the orthogonal group of Q restricted to the subspace Σ_r,
$G\ell(\Sigma)$ = the general linear group $\subseteq O(Q)$ operating on Σ and by contragredience on Σ^*,
and

$$\tilde{N}_r = \text{the unipotent radical of } \tilde{P}_r \text{ which is given by}$$

$$\tilde{N}_r = \{((\mu_1|\mu_2|\ldots|\mu_r), \mathscr{s})|\mu_i \in \Sigma_r, \quad \mathscr{s} \in \mathrm{Skew}_r(k)\}$$

$$(\mathrm{skew}_r(\ ) = \text{set of } r \times r \text{ skew symmetric matrices})$$

where the action on an element

$$\tilde{N}_r((\mu_1|\ldots|\mu_r), \mathscr{s}) = \tilde{N}_r[(\mu, \mathscr{s})], \quad \text{with } \mu \in M_{\lambda_r}(k),$$

$\mathscr{s} \in \mathrm{Skew}_r(k)$, on k^m is given by
(1) *Identity* on Σ^*,

(2)

$$\left[\begin{array}{c} 0 \\ \hline 0 \\ \hline Z \end{array}\right] \rightsquigarrow \left[\begin{array}{c} -\mu' Q Z \\ \hline 0 \\ \hline Z \end{array}\right] \quad \text{on } \Sigma_r$$

(where we view Q as a form on Σ_r),

(3)

$$\left[\begin{array}{c} 0 \\ \hline W \\ \hline 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{c} (-\mu' Q\mu/2 + \mathscr{s})W \\ \hline W \\ \hline \mu \cdot W \end{array}\right] \quad \text{on } \Sigma.$$

(X) Given an automorphic form f on $\mathrm{Sp}_n(\mathbb{A})$ or φ on $\mathcal{O}(Q)(\mathbb{A})$, we know that the *constant term* relative to a maximal parabolic is given by

$$f^{(P_{n-i})}(G) = \int_{U_i''(K)\backslash U_i''(\mathbb{A})} f(uG)\mathrm{d}u$$

on $\mathrm{Sp}_n(\mathbb{A})$ and

$$\varphi^{(\tilde{P}_r)}(g) = \int_{\tilde{N}_r(K)\backslash \tilde{N}_r(\mathbb{A})} \varphi(u^* g)\mathrm{d}u^*$$

on $O(Q)(\mathbb{A})$, where $\mathrm{d}u$ and $\mathrm{d}u^*$ are induced measures on $U_i''(K)\backslash U_i''(\mathbb{A})$ and $\tilde{N}_r(K)\backslash \tilde{N}_r(\mathbb{A})$ having normalized mass 1 for these compact quotient spaces.

(XI) The standard maximal compact subgroups of Sp_n, $G\ell_m$, and $O(Q)$ are determined (up to conjugacy) as follows:

(i) $\mathrm{Sp}_n(\mathcal{O}_k) \subset \mathrm{Sp}_n(k)$,

(ii) $\mathrm{G}\ell_m(\mathcal{O}_k) \subset \mathrm{G}\ell_m(k)$,

(iii) the subgroup of $\mathrm{O}(Q)$ stabilizing a lattice in k^m with a "Witt basis", i.e., a basis $\{e_i\}$ of k^m such that

(a) $e_1, \dots, e_r$ ($e_{r+1}, \dots, e_{2r}$, resp.) generate over k maximal Q-isotropic subspaces of k^m,

(b) $Q(e_i, e_{r+j}) = \delta_{ij}$, $1 \leqslant i \leqslant r$,

(c) $e_{2r+1}, \dots, e_m$ generate over k the Q-orthogonal of the span $\{e_1, \dots, e_r, e_{r+1}, \dots, e_{2r}\}$ and generate the unique maximal $\mathcal{O}_k$ integral lattice of this space.

(XII) Given an arbitrary, connected, reductive group G, we let P' be a minimal parabolic subgroup of G, and P, a parabolic of G containing P'. Let $A' \subset P'$ be a maximal split torus of G, Φ a root system of G with respect to A', and Π a set of simple roots of Φ. Then we say that the parabolic pair (P, A) of G dominates (P', A') if there exists a subset Π' of Π such that $A =$ largest torus contained in the intersection of the kernels of the elements of Π'. For any $\delta > 0$, we let $A^-(\delta) = \{a \in A \| \alpha(a)| \leqslant \delta$ for all roots α_1 in $\Pi - \Pi'\}$. Let $\tilde{P}$ be the opposed parabolic to P in G.

We let N and $\tilde{N}$ be the unipotent radicals of P and $\tilde{P}$, respectively.

Let π be an admissible, finitely generated representation of G. Let $\check{\pi}$ be the contragredient representation of G. We apply the Jacquet functors to π and $\check{\pi}$ relative to N and $\tilde{N}$.

Then we know from [C] that $(\check{\pi})_{\tilde{N}}$ is the contragredient representation to π_N (relative to the connected reductive subgroup M of G which satisfies $P = M \cdot N$ and $\tilde{P} = M \cdot \tilde{N}$, semidirect). Indeed we have the following precise version of this duality. Namely there exists a unique M-invariant nondegenerate pairing $\langle \cdot, \cdot \rangle_N$ between π_N and $(\check{\pi})_{\tilde{N}}$ with the following properties: given v_1 and v_2 in π and $\check{\pi}$, respectively, and v_1' and v_2' the corresponding images in π_N and $(\check{\pi})_{\tilde{N}}$, there is a real number $\varepsilon_1 > 0$ such that

$$\langle v_2, \pi(a) v_1 \rangle = \langle v_2', \pi_N(a) v_1' \rangle_N$$

for all $a \in A^-(\varepsilon_1)$.

I. Global theory

§1. Constant term of θ-functions

We assume that m is even and Q an arbitrary nondegenerate form on k^m.

We fix $\varphi \in S(M_{mn}(\mathbb{A}))$ and we consider the associated θ-series

$$\theta_\varphi(G, g) = \sum_{\xi \in M_{mn}(K)} \pi_Q(G, g)(\varphi)[\xi].$$

Then we know from [20] that as a function on $\mathrm{Sp}_n(K)\times \mathrm{O}(Q)(K)\backslash \mathrm{Sp}_n(\mathbb{A})\times \mathrm{O}(Q)(\mathbb{A})$, θ_φ is a *slowly increasing function,* i.e.

$$|\theta_\varphi(G,g)|\leqslant \|(G,g)\|^m$$

for some positive integer m, $\|\ \|$ a norm on $\mathrm{Sp}_n(\mathbb{A})\times \mathrm{O}(Q)$ given in [4].

The purpose of this section is to compute for any maximal parabolic subgroup of Sp_n or $\mathrm{O}(Q)$, the constant term of θ_φ in the direction of that parabolic subgroup. First we consider the following maps:

(1) $\varphi \rightsquigarrow \pi_Q(G)(\varphi)[X|0]$, $G\in \mathrm{Sp}_n(\mathbb{A})$ and $X\in M_{ml}(\mathbb{A})$;

(2) $\varphi \rightsquigarrow \int_{R\in M_{ln}(\mathbb{A})}\pi_Q(g)(\varphi)\begin{bmatrix} R\\ \hline 0\\ \hline X\end{bmatrix}\mathrm{d}R$, $g\in \mathrm{O}(Q)(\mathbb{A})$ and $X\in M_{m-2l,n}(\mathbb{A})$.

Then we construct the associated θ functions, namely $\Theta'(\pi_Q(G,g)\varphi) = \sum_{\xi\in M_{ml}(K)}\pi_Q(G,g)(\varphi)[\xi|0]$ and

$$\tilde{\Theta}'(\pi_Q(G,g)\varphi)=\sum_{\mu\in M_{m-2l,n}(K)}\int \pi_Q(G,g)(\varphi)\begin{bmatrix} R\\ \hline 0\\ \hline \mu\end{bmatrix}\mathrm{d}R$$

(see §0(VII) and §0(IX)).

We than state the main Theorem which determines the constant terms.

We note that the following theorem is a generalization of the results of [R-2]. In qualitative terms, the content of this Theorem is essentially that the constant term of a lift along a given maximal parabolic is the lift associated to a θ series in a smaller number of variables. This, in fact, is very close to the use of the Φ operator in the theory of Siegel modular forms.

THEOREM I.1.1:
(1) *Let P_{n-l} be as in* §0(IX). *Then for f a cusp form on* $\mathrm{O}(Q)(\mathbb{A})$,

$$\langle \theta_\varphi^{(P_{n-l})}(G,g)|f(g)\rangle_{\mathrm{O}(Q)(\mathbb{A})} = \langle \Theta'(\pi_Q(G,g)\varphi)|f(g)\rangle_{\mathrm{O}(Q)(\mathbb{A})}.$$

(2) *Let $\tilde{P}_l$ be as in* §0(IX). *Then for f a cusp form on* $\mathrm{Sp}_n(\mathbb{A})$

$$\langle \theta_\varphi^{(\tilde{P}_l)}(G,g)|f^*(G)\rangle_{\mathrm{Sp}_n(\mathbb{A})} = \langle \tilde{\Theta}'(\pi_Q(G,g)\varphi)|f^*(G)\rangle_{\mathrm{Sp}_n(\mathbb{A})}.$$

PROOF: We first note that

$$\theta_\varphi^{(P_{n-l})}(G,g)=\int_{U_l^{n,1}(K)\backslash U_l^{n,1}(\mathbb{A})}\sum_{\tilde{\xi}\in W_l}\pi_Q(G,g)(\varphi)$$

$$[\tilde{\xi}_1+a_{l+1,1}\tilde{\xi}_{l+1}+\dots+a_{n1}\tilde{\xi}_n|\tilde{\xi}_2+a_{l+1,2}\tilde{\xi}_{l+1}+\dots+a_{n2}\tilde{\xi}_n|$$

$$\dots\tilde{\xi}_l+a_{l+1,l}\tilde{\xi}_{l+1}+\dots+a_{n,l}\tilde{\xi}_n|\tilde{\xi}_{l+1}|\dots|\tilde{\xi}_n]\,\Pi\,\mathrm{d}a_{ij},$$

where $W_i = \{ X \in M_{mn}(K) | X'QX = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & 0 \end{array}\right]$, $*$ arbitrary $i \times i$ matrix$\}$ (see §II.4).

Then we consider the orbits of $\mathrm{O}(Q)(K) \times Z_t(K)$ in W_t, where $Z_t(K) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline * & B \end{array} \right) | A \in G\ell_t(K), B \in G\ell_{n-t}(K), \textit{and } * \textit{ arbitrary} \right\}$ $\subseteq \mathrm{G}\ell_n(K)$. The set of equivalence classes of such orbits is parametrized by the following set of representatives:

$$[X|\xi_1|\xi_2 \ldots |\xi_t|0] = [X|\xi] \quad \text{with } \xi = [\xi_1|\ldots|\xi_t|0],$$

where $\xi_1, \ldots, \xi_t$ form a nonzero, linearly independent set of mutually Q isotropic vectors spanning the subspace Σ (see §0 (IX) (B)), and X runs through a set of representatives of $\mathrm{O}(Q|_{\Sigma_t}) \times \mathrm{G}\ell_t$ orbits in $M_{m-2t,t}(K)$ (with $m - 2t = \dim \Sigma_t$). We note that the number of such $\mathrm{O}(Q)(K) \times Z_t(K)$ orbits in W_t is finite.

Then we have that

$$\sum_{(\gamma_1, \gamma_2) \in \mathrm{Stab}[X|\xi] \backslash \mathrm{O}(Q)(K) \times Z_t(K)} \pi(\gamma_1 G_1, \gamma_2 g)(\varphi)[X|\xi]$$

$$= \sum_{\gamma_2^* \in (\tilde{P}_t \backslash \mathrm{O}(Q)(K)} \; \sum_{\gamma_1^* \in (Z_t(K)_{[X|\xi]} \backslash Z_t(K))} \pi_Q(\gamma_1^* G, \gamma_2^* g)(\varphi)[X|\xi],$$

where $Z_t(K)_{[X|\xi]}$ = stabilizer of $[X|\xi]$ in $Z_t(K)$ and $\tilde{P}_t$ = the maximal parabolic in $\mathrm{O}(Q)(K)$ given by $\mathrm{O}(Q|_{\Sigma_t})(K) \times \mathrm{G}\ell_{m-2t}(K) \times \tilde{N}_t(K)$ (which in $\mathrm{O}(Q)(K)$ is the stabilizer of the $Z_t(K)$ orbit of $[X|\xi]$). Thus we have that

$$\theta_\varphi^{(P_{n-t})} = \sum_\xi \sum_X \sum_{\gamma_2^* \in (\tilde{P}_t \backslash \mathrm{O}(Q)(K))} \; \sum_{\gamma_1^* \in Z_t(K)_{[X|\xi]} \backslash Z_t(K)}$$

$$\times \int_{U_t^{n,1}(K) \backslash U_t^{n,1}(\mathbf{A})} \pi_Q[\gamma_1^* X(S) G, \gamma_2^* g](\varphi)[X|\xi] \mathrm{d}S.$$

Here

$$X(S) = \left\{ \left[\begin{array}{cc|cc} I & 0 & \multicolumn{2}{c}{0} \\ S & I & & \\ \hline \multicolumn{2}{c|}{} & I & -S^t \\ \multicolumn{2}{c|}{0} & 0 & I \end{array} \right] \in Z_t \right\} \subseteq \mathrm{Sp}_n.$$

But then we note that $Z_t(K)_{[X|\xi]}$ splits as a semidirect product of 2 groups:

$$\left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \in \mathrm{G}\ell_t(K) \times \mathrm{G}\ell_{n-t}(K) \mid X \cdot A = X,\ \xi \cdot B = \xi\right\} \quad \text{and}$$

$$\left\{\left[\begin{array}{c|c} I & 0 \\ \hline Y & I \end{array}\right] \mid \xi \cdot Y \equiv 0\right\}.$$

Thus we have that

$$\sum_{\gamma_1^* \in Z_t(K)_{[X|\xi]} \backslash Z_t(K)} \int_{U_t^{n,1}(K) \backslash U_t^{n,1}(\mathbb{A})} \pi_Q\left[\gamma_1^* X(S) G,\ \gamma_2^* g\right](\varphi)[X|\xi] \mathrm{d}S$$

$$= \sum_{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \in (\mathrm{G}\ell_t(K) \times \mathrm{G}\ell_{n-t}(K))_{[Z|\xi]} \backslash \mathrm{G}\ell_t(K) \times \mathrm{G}\ell_{n-t}(K)}$$

$$\times \left\{ \int_{M_{tt}(\mathbb{A})} \pi_Q[G,\ \gamma_2^* g](\varphi) \left[\begin{array}{c|c} W & \xi \cdot B \\ \hline 0 & 0 \\ \hline X \cdot A & \end{array}\right] \mathrm{d}W \right\}.$$

Hence we have by (formally) integrating $\theta_\varphi^{(P_{n-t})}$ against a cusp form f on $\mathrm{O}(Q)(K) \backslash \mathrm{O}(Q)(\mathbb{A})$,

$$\int_{\mathrm{O}(Q)(K) \backslash \mathrm{O}(Q)(\mathbb{A})} f(g) \theta_\varphi^{(P_{n-t})}(G, g) \mathrm{d}g = \int_{\mathrm{O}(Q)(K) \backslash \mathrm{O}(Q)(\mathbb{A})} f(g)$$

$$\times \sum_\xi \sum_X \int_{(\tilde{P}_t)_K \backslash \mathrm{O}(Q)(\mathbb{A})} \left\{ \left\{ \sum_{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \ldots} \int_{M_{tt}(\mathbb{A})} \pi_Q(G, g)(\varphi) \right.\right.$$

$$\left.\left. \times \left[\begin{array}{c|c} W & \xi \cdot B \\ \hline 0 & 0 \\ \hline X \cdot A & \end{array}\right] \mathrm{d}W \right\}\right\} \mathrm{d}g$$

$$= \sum_\xi \sum_X \int_{L'_K \tilde{N}_t(\mathbb{A}) \backslash \mathrm{O}(Q)(\mathbb{A})}$$

$$\times \left\{ \left\{ \int_{\tilde{N}_t(K) \backslash \tilde{N}_t(\mathbb{A})} \left\{ \sum_{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \ldots} \int_{M_{tt}(A)} \pi_Q(G, ug)(\varphi) \right.\right.\right.$$

$$\left.\left.\left. \times \left[\begin{array}{c|c} W & \xi \cdot B \\ \hline 0 & 0 \\ \hline X \cdot A & \end{array}\right] \mathrm{d}W \right\} f(ug) \mathrm{d}u \right\}\right\} \mathrm{d}g,$$

where $L_K^t = \mathrm{O}(Q|_{\Sigma_t})(K) \times G\ell_{m-2t}(K)$. But we note that the function

$$u \rightsquigarrow \sum_{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)\ldots} \int_{M_{tt}(\mathbb{A})} \pi_Q(G, ug)(\varphi)\left[\begin{array}{c|c} W & \xi\cdot B \\ \hline 0 & 0 \\ \hline X\cdot A & \end{array}\right] \mathrm{d}W$$

is $\tilde{N}_t(A)$ *invariant*! Thus the inner integral above is merely $f^{(\tilde{P}_t)}(g) =$ the constant term of f in the direction of the parabolic $\tilde{P}_t$ in $\mathrm{O}(Q)$. But f is cuspidal implies that the above sum vanishes if ξ is not the *zero matrix*!

Thus we have that

$$\langle \theta_\varphi^{(P_{n-t})}(G, g) | f(g) \rangle_{\mathrm{O}(Q)(\mathbb{A})} = \langle \Theta'\big(\pi_Q(G, g)(\varphi)\big) | f(g) \rangle_{\mathrm{O}(Q)(\mathbb{A})}$$

for *all* G.

Although the proof of (2) is very similar to that of (1), we include the proof to emphasize the importance of the linearizing property of the Weil representation (see §0(VII)). In particular we have that

$$\theta_\varphi^{(\tilde{P}_r)}(G, g) = \int_{M_{m-2r,r}(K)\backslash M_{m-2r,r}(\mathbb{A})} \left\{ \sum_{\substack{\{M, Y\in M_{rn}(K) | YM' \text{ is symmetric}\} \\ X\in M_{m-2r,n}(K)}} F_{T_r}\big[\pi_Q(G, \tilde{N}_r(T)g)(\varphi)\big]\left[\begin{array}{c} M \\ \hline Y \\ \hline X \end{array}\right] \right\} \mathrm{d}T$$

where F_{T_r} is the partial Fourier transform "relative to Σ_r" given by

$$F_{T_r}(\varphi)\left[\begin{array}{c} A \\ \hline B \\ \hline C \end{array}\right] = \int_{M_{rn}(\mathbb{A})} \varphi\left[\begin{array}{c} X \\ \hline B \\ \hline C \end{array}\right] \tau\big[\mathrm{Tr}(X'A)\big]\,\mathrm{d}X$$

for $\varphi \in S[M_{mn}(\mathbb{A})]$ (see §0(VII)) and

$$\tilde{N}_r(T) = \{\tilde{N}_r((T, 0)) | T \in M_{m-2r,r}(\mathbb{A})\}$$

(see §0(IX)).

But then, similar to (1) above, we note that the $G\ell_r(K) \times \mathrm{Sp}_n(K)$ orbits in $\{[R|S] | RS'$ symmetric with $R, S \in M_{rn}(K)\} = \{X \in M_{r2n}(K) | X \left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right] X' = 0\}$ are given by representatives of the form $V_i = [I_i|0]$ with $i = 0, \ldots, \min(n, r)$ (with $I_i = i \times i$ identity matrix).

Thus we have (formally) that

$$\theta_{\varphi}^{(\tilde{P}_r)}(G,g)=\sum_{t}\ \sum_{\gamma_1\in G\ell_r(K)_{V_t}\backslash G\ell_r(K)}\ \sum_{\gamma_2\in \mathrm{Stab}(V_t)\backslash(\mathrm{Sp}_n)_K}\ \sum_{X\in M_{m-2r,n}(K)}$$

$$\int_{M_{m-2t,t}(K)\backslash M_{m-2t,t}(\mathbb{A})} F_{T_r}\left[\pi_Q\left(\gamma_2 G,\gamma_1\tilde{N}_r(T)g\right)\varphi\right]$$

$$\begin{bmatrix} I_t\\ \overline{0}\\ \overline{X}\end{bmatrix}\mathrm{d}T=\sum_{t}\ \sum_{\gamma_1\in G\ell_r(K)_{V_t}\backslash G\ell_r(K)}\ \sum_{\gamma_2\in \mathrm{Stab}(V_t)\backslash(\mathrm{Sp}_n)_K}\ \sum_{X\in M_{m-2t,n}(K)}$$

$$\int_{M_{m-2r,r}(K)\backslash M_{m-2t,t}(\mathbb{A})} F_{T_r}\left[\pi_Q\left(\gamma_2 G,\tilde{N}_r(T)\gamma_1 g\right)\varphi\right]$$

$$\begin{bmatrix} I_t\\ 0\\ X\end{bmatrix}\mathrm{d}T$$

where $G\ell_r(K)_{V_t}$ = Stabilizer of the $\mathrm{Sp}_r(K)$ orbit of V_t in $G\ell_r(K)$. Here we have used the fact that $\gamma_1\tilde{N}_r(T)=\tilde{N}_r(T\gamma_1)\gamma_1$ and $\mathrm{d}(T\gamma)=|\det\gamma|_K\,\mathrm{d}(T)=\mathrm{d}(T)$ on $M_{m-2r,r}(K)\backslash M_{m-2r,r}(\mathbb{A})$.

Then we have (using the definition of $\tilde{N}_r$ in §0(IX))

$$\int_{M_{m-2t,t}(K)\backslash M_{m-2r,t}(\mathbb{A})} F_{T_r}\left[\pi_Q\left(\gamma_2 G,\tilde{N}_r(T)\gamma_1 g\right)(\varphi)\right]\begin{bmatrix} I_t\\ \overline{0}\\ \overline{X}\end{bmatrix}\mathrm{d}T$$

$$=F_{T_r}\left[\pi_Q\left(\gamma_2 G,\gamma_1 g\right)(\varphi)\right]\begin{bmatrix} I_t\\ \overline{0}\\ \overline{X}\end{bmatrix}$$

$$\left\{\int_{M_{m-2r,r}(K)\backslash M_{m-2r,r}(\mathbb{A})}\tau\left(\mathrm{Tr}\left(T'QXI_t'\right)\right)\mathrm{d}T\right\}.$$

But then the latter integral vanishes if and only if $X\cdot I_t'=0$. That is, $X\cdot I_t'=[X_t|0]\equiv 0$ if and only if $X_t\equiv 0$.

Then we note that

$$\sum_{\{X\in M_{m-2t,n}(K)|X_t\equiv 0\}} F_{T_r}\left[\pi_Q(G,g)(\varphi)\right]\begin{bmatrix} I_t\\ \overline{0}\\ \overline{X}\end{bmatrix}$$

as a function of G is *invariant* on the left by the subgroup of $\mathrm{Stab}_{\mathbb{A}}(V_t)$ isomorphic to

$$\left\{\left[\begin{array}{cc|cc} I_t & * & * & A \\ \hline 0 & I_{n-t} & A' & 0 \\ \hline & & I_t & 0 \\ \cline{3-4} 0 & & *^t & I_{n-t} \end{array}\right] \;\Big|\; *\ \textit{arbitrary} \text{ and } A\ \textit{arbitrary}\right\}.$$

But this latter group is the unipotent radical of a fixed parabolic subgroup of $\mathrm{Sp}_n(K)$. Then we can apply the same reasoning as in (1) and deduce (2). Q.E.D.

Appendix to §1

We must prove the *absolute convergence* of the following:

$$(A)\int_{\tilde{P}_t\backslash \mathrm{O}(Q)(\mathbb{A})}|f(g)|$$

$$\times\left\{\sum_{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B\end{array}\right)\cdots}\int_{M_{t,t}(\mathbb{A})}\left|\pi_Q(G,g)(\varphi)\left[\begin{array}{c|c} W & \xi B \\ \hline 0 & 0 \\ \hline XA & \end{array}\right]\right|\mathrm{d}W\right\}\mathrm{d}g.$$

This allows us to switch orders of integration when we compute in I.1 the integral

$$\int_{\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A})}\theta_\varphi^{(P_{n-t})}(G,g)f(g)\mathrm{d}g.$$

But we note that $\mathrm{O}(Q)(\mathbb{A})=\tilde{P}(\mathbb{A})\cdot K$, where K is a compact subgroup of $\mathrm{O}(Q)(\mathbb{A})$. Then (A) is majorized by a finite number of terms of the form

$$\int_{(\mathrm{G}\ell_t(K)\backslash \mathrm{G}\ell_t(\mathbb{A}))\times((\mathrm{O}(Q|_{\Sigma_t})(K)\backslash \mathrm{O}(Q|_{\Sigma_t})(\mathbb{A})))\times(\tilde{N}_t(K)\backslash\tilde{N}_t(\mathbb{A}))}|f_t(g_1g_2g_3)|$$

$$\times\sum_{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B\end{array}\right)\cdots}\left(\int_{M_{t,t}(\mathbb{A})}\left|\pi_{Q_1}(G,g)(\varphi_t)\left[\begin{array}{c|c} W & g_1^{-1}\xi B \\ \hline 0 & 0\end{array}\right]\right|\mathrm{d}W\right)|\det g_1|^{-r}$$

$$|\pi_{Q_2}(G,g)(\tilde{\varphi}_t)\left[g_2^{-1}XA\right]|\mathrm{d}g_1\mathrm{d}g_2\mathrm{d}g_3,$$

where φ_t and $\tilde{\varphi}_t$ belong to $S[M_{2t,n}(\mathbb{A})]$ and $S[M_{m-2t,n}(\mathbb{A})]$, resp. and f_t is a cusp form on $\mathrm{O}(Q)(\mathbb{A})$.

Note π_{Q_1} and π_{Q_2} are the Weil representations of the pairs $O(Q_1)\times \mathrm{Sp}_n$ on $S[M_{2t,n}(\mathbb{A})]$ (where Q_1 = the restriction of Q to $\Sigma\oplus\Sigma^*$, i.e., see §0(IX)(B)) and $(O(Q|_{\Sigma_t})=O(Q_2))\times \mathrm{Sp}_n$ on $S[M_{m-2t,n}(\mathbb{A})]$. But then we use the rapid decreasing properties of f_t to deduce the absolute convergence of the above series!

§2. *Howe duality conjecture*

Thus as a consequence of Theorem I.1.1 we deduce the following corollary.

COROLLARY TO THEOREM I.1.1:

(a) *Let* $f\in L^2_{\mathrm{cusp}}(O(Q)(\mathbb{A}))$. *Then* $\langle\theta_\varphi(G,g)|f(g)\rangle_{O(Q)(\mathbb{A})}$ *is a cusp form on* $\mathrm{Sp}_n(K)\backslash \mathrm{Sp}_n(\mathbb{A})$ *if and only if*

$$\langle\Theta^i(\pi_Q(G,g)(\varphi))|f(g)\rangle_{O(Q)(\mathbb{A})}\equiv 0$$

for all $G\in \mathrm{Sp}_n(\mathbb{A})$ *and* $i=1,\dots,n$.

(b) *Let* $f^*\in L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$. *Then* $\langle\Theta_\varphi(G,g)|f^*(G)\rangle_{\mathrm{Sp}_n(\mathbb{A})}$ *is a cusp form on* $O(Q)(K)\backslash O(Q)(\mathbb{A})$ *if and only if*

$$\langle\tilde\Theta^i(\pi_Q(G,g)(\varphi))|f^*(G)\rangle_{\mathrm{Sp}_n(\mathbb{A})}\equiv 0$$

for all $g\in O(Q)(\mathbb{A})$ and $i=1,\dots,$ index(Q).

REMARK I.2.1: We note here that a simple exercise will verify that $\langle\Theta^i(\pi_Q(G,g)(\varphi))|f(g)\rangle_{O(Q)(\mathbb{A})}\equiv 0$ for *all* $G\in\mathrm{Sp}_n(A)$ and *all* $\varphi\in S[M_{mn}(\mathbb{A})]$ is equivalent to

$$\langle\theta_\varphi(G,g)|f(g)\rangle_{O(Q)(\mathbb{A})}\equiv 0$$

for *all* $G\in\mathrm{Sp}_i(\mathbb{A})$ and *all* $\varphi\in S[M_{mi}(\mathbb{A})]$. A similar statement holds for $\tilde\Theta^i(\pi_Q(G,g)(\varphi))$ given above.

Then we define the subspaces $L_r(Q)=\{f\in L^2_{\mathrm{cusp}}(O(Q)(\mathbb{A}))\,|\langle\theta_\varphi(G,g)|f(g)\rangle_{O(Q)(\mathbb{A})}\equiv 0$ for all $G\in\mathrm{Sp}_r(\mathbb{A})$ and all $\varphi\in S[M_{mr}(\mathbb{A})]\}$ (recall here that θ_φ is the θ-function associated to the Weil representation of $\mathrm{Sp}_r\times O(Q)$ on $S[M_{mr}(\mathbb{A})]$) and $I(Q)=\{f^*\in L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))|\langle\theta_\varphi(G,g)|f^*(G)\rangle_{\mathrm{Sp}_n(\mathbb{A})}\equiv 0$ for all $g\in O(Q)(\mathbb{A})$ and all $\varphi\in S[M_{mn}(k)]\}$, $m=\dim Q$.

Then we define inductively $R_1(Q)=L_1(Q)^\perp$ in $L^2_{\mathrm{cusp}}(O(Q)(\mathbb{A}))$, $R_2(Q)=(L_1(Q)\cap L_2(Q))^\perp$ in $L_1(Q),\dots,R_t(Q)=(L_1(Q)\cap\dots L_t(Q))^\perp$ in $L_1(Q)\cap L_2(Q)\cap\dots L_{t-1}(Q)$. Similarly let $I_1(Q)=I(Q)^\perp$ in $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$, $I_2(Q)=(I(Q)\cap I(Q\oplus H_1))^\perp$ in $I(Q),\dots,I_{t+1}(Q)=(I(Q)\cap I(Q\oplus H_1)\dots\cap I(Q\oplus H_t))^\perp$ in $I(Q)\cap I(Q\oplus H_1)\cap\dots I(Q\oplus H_{t-1})$. Here we adopt the convention that if Q is the *zero form*, then $I_1(Q)=\{0\}$.

Then we have the following Structure Theorem about the cusp forms on Sp_n or $\mathrm{O}(Q)$.

THEOREM I.2.1:

(1) $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ *is an orthogonal direct sum of*

$$R_1(Q) \oplus R_2(Q) \oplus \ldots \oplus R_m(Q)$$

where $m = \dim Q$.

(2) *Let Q be an anisotropic form over K. Then* $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$ *is an orthogonal direct sum of*

$$I_1(Q) \oplus I_2(Q \oplus H_1) \oplus \ldots \oplus I_{r+1}(Q \oplus H_r)$$

where $r = 2n$.

PROOF: (1) By employing Corollary to Theorem I.1.1 and Remark I.2.1, we have that $\mathrm{L}^2_{\mathrm{cusp}} = R_1(Q) \oplus R_2(Q) \oplus \ldots R_l(Q) \oplus \mathrm{L}_1(Q) \cap \mathrm{L}_2(Q) \cap \ldots \cap L_l(Q)$. Then it suffices to show that $\mathrm{L}_1(Q) \cap \ldots \cap \mathrm{L}_m(Q) = (0)$. Indeed we show that $\mathrm{L}_m(Q) = (0)$. First we note that $\mathrm{O}(Q)(\mathbb{A})$ embeds as a closed subset of $M_{mn}(\mathbb{A})$ (for $n \geqslant m$) via $\mathrm{O}(Q)(\mathbb{A}) \cong \left\{ \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array} \right) | B \in O(Q)(\mathbb{A}) \right\} \subseteq M_{mn}(\mathbb{A})$. Moreover the restriction map $C_c^\infty(M_{mn}(\mathbb{A})) \to C_c^\infty(\mathrm{O}(Q)(\mathbb{A}))$ is surjective. Then we note that every function in $C_c^\infty(\mathrm{O}(Q)(K) \backslash \mathrm{O}(Q)(\mathbb{A}))$ is obtained by averaging a function in $C_c^\infty(\mathrm{O}(Q)(\mathbb{A}))$, i.e., $H \in C_c^\infty(\mathrm{O}(Q)(K) \backslash \mathrm{O}(Q)(\mathbb{A}))$ has the form $\sum_{\gamma \in \mathrm{O}(Q)(K)} \varphi(\gamma \cdot x)$, $x \in \mathrm{O}(Q)(\mathbb{A})$ and $\varphi \in C_c^\infty(\mathrm{O}(Q)(\mathbb{A}))$.

Thus, if f in $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ has the property that $\langle f | H \rangle_{\mathrm{O}(Q)(\mathbb{A})} \equiv 0$ for a *dense set* of H above, then $f \equiv 0$.

Then if $\varphi \in S[M_{mn}(\mathbb{A})]$, we deduce easily that

$$\theta_\varphi^{P_n, \chi_Q}(G, g) = \int_{\mathrm{Sym}_n(K) \backslash \mathrm{Sym}_n(\mathbb{A})} \theta_\varphi\left(\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} G, g \right) \tau\left(\mathrm{tr}\left(T \cdot \chi_Q \right) \right) \mathrm{d}T$$

$$= \sum_{\{\xi \in M_{mn}(K) | \xi' Q \xi = \chi_Q\}} \pi_Q(G, g)(\varphi)(\xi),$$

where $\chi_Q = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array} \right]$. But we note that $\{\xi \in M_{mn}(K) | \xi' Q \xi = \chi_Q\} = \{ \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 0 \end{array} \right] | R \in \mathrm{O}(Q)(K) \}$.

But we know that $S[M_{mn}(\mathbb{A})]$ contains $S_{\mathrm{comp}}(M_{mn}(K_\infty)) \otimes S[M_{mn}(\mathbb{A}_{\mathrm{fin}})]$ (see §0(V)), which is dense in $C_c^\infty(M_{mn}(\mathbb{A}))$, and the image of $S[M_{mn}(K_\infty)] \otimes S[M_{mn}(\mathbb{A}_{\mathrm{fin}})]$ (via restriction) in $C_c^\infty(\mathrm{O}(Q)(\mathbb{A}))$ is dense in $C_c^\infty(\mathrm{O}(Q)(\mathbb{A}))$. Hence if $f \in \mathrm{L}_m(Q)$, then $\langle f | \theta_\varphi^{P_n, \chi_Q}(G, \ ) \rangle_{\mathrm{O}(Q)(\mathbb{A})} \equiv 0$ for all $\varphi \in S[M_{mn}(\mathbb{A})]$; this implies by the comments above that $f \equiv 0$.

(2) We apply a similar argument as above. First we note that $\mathrm{Sp}_n(\mathbb{A})$ embeds as a closed subset of $M_{\lambda n}(\mathbb{A})$ (with $\lambda \geqslant 2\mu \geqslant 4n$, where $\mu =$ Witt index of Q) via $\mathrm{Sp}_n(\mathbb{A}) \cong \left\{ \left[\dfrac{\Omega^{-1} G \Omega(X_n)}{0} \right] | G \in Sp_n(\mathbb{A}) \right\} \subseteq M_{\lambda n}(\mathbb{A})$, with

$$X_n = \left[\begin{array}{c} I_n \\ \hline 0 \\ \hline 0 \\ \hline I_n \end{array} \right]$$

(see II.1). Then if $\varphi \in S[M_{\lambda n}(\mathbb{A})]$, we deduce that the Fourier coefficient of θ_φ in the direction of the central subgroup $Z_\mu(\mathscr{J}) = \{ \tilde{N}_\mu((0, \mathscr{J})) | \mathscr{J}$ a $\mu \times \mu$ skew symmetric matrix$\}$ of the unipotent radical $\tilde{P}_\mu$ in $\mathrm{O}(Q)(\mathbb{A})$ relative to the skew symmetric matrix

$$A = \left[\begin{array}{cc|c} 0 & I_n & \\ \hline -I_n & 0 & 0 \\ \hline \multicolumn{2}{c}{0} & 0 \end{array} \right]$$

is given by

$$\begin{aligned} \theta_\varphi^{Z_\mu, A}(G, g) &= \int_{Z_\mu(K) \backslash Z_\mu(\mathbb{A})} \theta_\varphi\big(G, Z_\mu(X) g\big) \tau(\mathrm{Tr}(X \cdot A)) \mathrm{d} X \\ &= \sum_{\gamma \in (\mathrm{Sp}_n(K))} F_{T_\mu}\big(\pi_Q(G, g)(\varphi)\big) \left[\frac{\Omega^{-1} \gamma \Omega(X_n)}{0} \right] \\ &= \sum_{\gamma \in (\mathrm{Sp}_n(K))} \pi_\ell(\gamma) F_{T_\mu}\big(\pi_Q(G, g)(\varphi)\big) \left[\frac{X_n}{0} \right], \end{aligned}$$

where $\pi_l(\ )$ operates linearly on $S[M_{2rn}(\mathbb{A})]$ (see §0(VII)).

Thus following the same reasoning as in case (1), we deduce that if $f^* \in I(Q \oplus H_{r-1})$ (where $r \geqslant 2n$), then $\langle f^* | \theta_\varphi^{Z_\mu, A}(G, g) \rangle_{\mathrm{Sp}_n(\mathbb{A})} \equiv 0$ for all $\varphi \in S[M_{\lambda n}(\mathbb{A})]$; thus $f^* \equiv 0$. Q.E.D.

REMARK I.2.2: We have shown that in (1) $\mathrm{L}_t(Q) = \{0\}$ if $t \geqslant m$ and in (2) $I(Q \oplus H_r) = \{0\}$ if $r \geqslant 2n$.

Then for a fixed cuspidal representation π occurring in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$, we consider the subspace

$$\mathscr{A}_n(\pi) = \left\{ \text{the complex linear span of } \int_{\mathrm{O}(Q)(K) \backslash \mathrm{O}(Q)(\mathbb{A})} \theta_\varphi(G, g) \right.$$

$$\left. f(g) \mathrm{d} g | f \in (\pi), \qquad \varphi \in S\big[M_{mn}(\mathbb{A})\big] \right\}.$$

Here (π) denotes the isotypic component in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ which transforms according to π (here the multiplicity of π in (π) is finite!).

Similarly for a cuspidal representation σ in $L^2_{\mathrm{cusp}}(\mathrm{Sp}_n(\mathbb{A}))$, we let

$$\mathscr{B}_{Q+r}(\sigma)=\left\{\text{the complex linear span of }\int_{\mathrm{Sp}_n(K)\backslash\mathrm{Sp}_n(\mathbb{A})}\theta_\varphi(G,g)\right.$$

$$\left. f^*(G)\mathrm{d}G \mid f^*\in(\sigma),\qquad \varphi\in S\left[M_{mn}(\mathbb{A})\right]\right\},$$

$$m=\dim Q+2r.$$

Then we have two elementary consequences of the above Theorem.

Corollary 1 to Theorem I.2.1:

(1) *If π occurs in $R_t(Q)$, then $\mathscr{A}_t(\pi)\cap L^2_{\mathrm{cusp}}(\mathrm{Sp}_t(\mathbb{A}))\neq(0)$.*

(2) *(Q anisotropic)*

If σ occurs in $I_r(Q\oplus H_{r-1})$, then $\mathscr{B}_{Q+r-1}(\sigma)\cap L^2_{\mathrm{cusp}}(\mathrm{O}(Q\oplus H_{r-1})(\mathbb{A}))\neq 0$. We note that if $r=1$, then $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathscr{A}))$ must be replaced by $L^2(\mathrm{O}(Q)(\mathbb{A}))$ except if $\dim Q>2n+2$. *In the latter case, we use Siegel's formula to deduce that*

$$\mathscr{B}_Q(\sigma)\cap L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))\neq 0.$$

Corollary 2 to Theorem I.2.1: *If Q is a split 2-dimensional form over K, then $I(Q)=\{0\}$.*

Proof: We use the same argument as in Theorem I.2.1 and see that

$$\theta_\varphi(G,g)=\sum_{\xi_1,\xi_2\in M_{1n}(K)}F_{T_1}\big(\pi_Q(G,g)\varphi\big)\left[\frac{\xi_1}{\xi_2}\right].$$

But using the linearizing property of F_T, we have

$$\theta_\varphi(G,g)=\sum_{\gamma\in\mathrm{Sp}_{n-1}(K)\times U^n_{n-1}(K)\backslash\mathrm{Sp}_n(K)}\pi_\ell(\gamma)F_{T_1}\big(\pi_Q(G,g)(\varphi)\big)\left[\frac{\xi_0}{0}\right]$$

$$+F_T\big(\pi_Q(G,g)(\varphi)\big)[0],$$

where $\mathrm{Sp}_{n-1}(K)\times U^n_{n-1}(K)$ stabilizes $\xi_0=(1\ \ 0\ldots0)$ (see §0(IX)).

But then we note that if f is a cusp form on $\mathrm{Sp}_n(\mathbb{A})$, it is easy to see that f is perpendicular to both terms in the series above for θ_φ. Q.E.D.

Then we can state the global Howe duality conjecture.

Conjecture:

(1) *There is a unique cuspidal representation $\beta_t(\pi)$ occurring in*

$L^2_{\text{cusp}}(\mathrm{Sp}_l(\mathbb{A}))$ *so that* $\mathscr{A}_l(\pi) \cap L^2_{\text{cusp}}(\mathrm{Sp}_l(\mathbb{A}))$ *is a nonzero multiple of* $\beta_l(\pi)$. *Then the map* $\pi \rightsquigarrow \beta_l(\pi)$ *defines an injective correspondence from cuspidal representations occurring in the space* $R_l(Q)$ *to cuspidal representations in* $L^2_{\text{cusp}}(\mathrm{Sp}_l(\mathbb{A}))$.

REMARK I.2.3: We can state a similar conjecture about the space $\mathscr{B}_{Q+r}(\sigma)$. On the other hand we can actually consider a more general (albeit less interesting) conjecture than the duality conjecture. Indeed we look at all π in $L^2_{\text{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ which satisfy $\mathscr{A}_r(\pi) \cap L^2_{\text{cusp}}(\mathrm{Sp}_r(\mathbb{A})) \neq 0$. Then there exists a unique $\beta_r^*(\pi)$, an irreducible component in $L^2_{\text{cusp}}(\mathrm{Sp}_r(\mathbb{A}))$ such that $\mathscr{A}_r(\pi) \cap L^2_{\text{cusp}}(\mathrm{Sp}_r(\mathbb{A}))$ is a nonzero multiple of $\beta_r^*(\pi)$. Moreover the map $\pi \rightsquigarrow \beta_r^*(\pi)$ is an *injective mapping* from the cuspidal representations π having the nonvanishing property above to cuspidal representations in $L^2_{\text{cusp}}(\mathrm{Sp}_r(\mathbb{A}))$.

Then we can reduce the global Howe conjecture to the similar problem for the local case. Indeed we have the following Theorem.

THEOREM I.2.2: *The global Howe duality conjecture is implied by the local Howe duality conjecture for every prime v in K.*

Fix a local Weil representation π_{Q_l} *of the dual pair* $\mathrm{Sp}_n(K_v) \times \mathrm{O}(Q_v)$ *on* $S[M_{mn}(K_v)]$. *Then if* ω, σ_1, *and* σ_2 *are unitary irreducible representations of* $\mathrm{O}(Q_v)$ *and* $\mathrm{Sp}_n(K_v)$, *we have that*

$$\mathrm{Hom}_{\mathrm{Sp}_n(K_v)\times \mathrm{O}(Q_v)}\left(S[M_{mn}(K_v)], \begin{matrix}\omega\otimes\sigma_1\\ \\ \omega\otimes\sigma_2\end{matrix}\right) \neq 0$$

implies $\sigma_1 \cong \sigma_2$. *Similarly if* μ_1 *and* μ_2 *and* ρ *are unitary irreducible representations of* $\mathrm{Sp}_n(K_v)$ *and* $\mathrm{O}(Q_v)$, *then*

$$\mathrm{Hom}_{\mathrm{Sp}_n(K_v)\times \mathrm{O}(Q_v)}\left(S[M_{mn}(K_v)], \begin{matrix}\mu_1\otimes\rho\\ \\ \mu_2\otimes\rho\end{matrix}\right) \neq 0$$

implies that $\mu_1 \cong \mu_2$. (Here the intertwining space is all intertwining maps in the smooth category of representations of the associated groups (with continuity assumptions at the Archimedean primes) and equivalence of irreducibles is understood as equivalence in the admissible category (see §0).)

PROOF: We fix $\pi \in L^2_{\text{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ so that $\mathscr{A}_l(\pi) \cap L^2_{\text{cusp}}(\mathrm{Sp}_l(\mathbb{A})) \neq 0$. Then there exists a *cuspidal* representation ρ in $L^2_{\text{cusp}}(\mathrm{Sp}_l(\mathbb{A}))$ such that the $\mathrm{O}(Q)(\mathbb{A}) \times \mathrm{Sp}_l(\mathbb{A})$ *invariant bilinear form* on $S[M_{ml}(K)] \otimes (\pi \otimes \bar{\rho})$ defined by

$$(\varphi, f_1, \bar{f}_2) \rightsquigarrow \int_{(\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A}) \times (\mathrm{Sp}_l(K)\backslash \mathrm{Sp}_l(\mathbb{A}))} \theta_\varphi(G, g)$$

$$f_1(g)\bar{f}_2(G)\mathrm{d}g\mathrm{d}G$$

($\bar{\rho} = \{\bar{f} \mid f \in \rho\} = \check{\rho}$, the contragredient module to ρ) is *nonzero*. This then determines for each v, a prime in K, a *nonzero* $\mathrm{O}(Q_v) \times \mathrm{Sp}_l(K_v)$ *invariant bilinear form* on the local space $S[M_{ml}(K_v)] \otimes \pi_v \otimes \check{\rho}_v$. That is, we can embed $S[M_{ml}(K_v)] \otimes \pi_v \otimes \check{\rho}_v$ into $S[M_{ml}(\mathbb{A})] \otimes \pi \otimes \check{\rho}$ in an $\mathrm{O}(Q_v) \times \mathrm{Sp}_l(K_v)$ equivariant manner by sending $\varphi_v \otimes e_v \otimes \bar{h}_v \rightsquigarrow (\varphi_v \otimes \psi) \otimes (e_v \otimes e) \otimes (\bar{h}_v \otimes \bar{h})$, where ψ, e, h are certain fixed elements in the restricted products $\prod_{v' \neq v} S[M_{ml}(K'_v)]$, $\otimes_{v' \neq v} \pi_{v'}$, $\otimes_{v' \neq v} \check{\rho}_{v'}$. We then choose, ψ, e, h such that the bilinear form

$$(\varphi_v, e_v \otimes \bar{h}_v) \rightsquigarrow \int \theta_{\varphi_v \otimes \psi}(G, g)(e_v \otimes e)(g)\overline{(h_v \otimes h)(G)}\,\mathrm{d}G\,\mathrm{d}g$$

is a *nonzero* form. In particular, this form then gives a nonzero bilinear form on the associated local smooth modules.

On the other hand, such a form then gives rise to a *nonzero element* in $\mathrm{Hom}_{\mathrm{O}(Q_v) \times \mathrm{Sp}_l(K_v)}(S[M_{ml}(K_v)], \check{\pi}_v \otimes \rho_v)$.

In particular if $\mathscr{A}_l(\pi) \cap \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_l(\mathbb{A}))$ contains 2 inequivalent cuspidal representations (ρ_1) and (ρ_2), then for *some prime* v in K we have $(\rho_1)_v \not\equiv (\rho_2)_v$ (relative to $\mathrm{Sp}_l(K_v)$) and $\mathrm{Hom}_{\mathrm{O}(Q_v) \times \mathrm{Sp}_l(K_v)}(S[M_{ml}(K_v)], \left\{ {(\check{\pi})_v \otimes (\rho_1)_v \atop (\check{\pi})_v \otimes (\rho_2)_v} \right) \neq 0$.

On the other hand, if π_1 and π_2 are inequivalent cuspidal representations in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ such that $\mathscr{A}(\pi_l) \cap \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_l(\mathbb{A}))$ are nonzero multiples of σ, then there exists a prime v on K so that $(\pi_1)_v \neq (\pi_2)_v$ and $\mathrm{Hom}_{\mathrm{O}(Q_v) \times \mathrm{Sp}_l(K_v)}(S[M_{ml}(K_v)], {(\check{\pi}_1)_v \otimes \sigma_v \atop (\check{\pi}_2)_v \otimes \sigma_v}) \neq 0$.

Thus the local Howe duality conjecture for every prime v in K implies the global Howe duality conjecture. Q.E.D.

It is also possible to refine the global Howe duality conjecture. Indeed we have the following Corollary.

COROLLARY TO THEOREM II.2.2 (*Multiplicity Preserving*):
If, for every v in K, the local Howe conjecture holds and

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{O}(Q_v) \times \mathrm{Sp}_l(K_v)}\left(S\left[M_{ml}(K_v)\right], \omega_1 \otimes \sigma_1\right) \leqslant 1$$

is true for all irreducible unitary representations ω_1 and σ_1 of $\mathrm{O}(Q_v)$ and $\mathrm{Sp}_l(K_v)$, then both the global Howe duality conjecture and the relation

$$\begin{aligned} &\text{multiplicity of } \pi \text{ in } R_l \\ &\quad = \text{multiplicity of } \beta_l(\pi) \text{ in } \mathscr{A}(\pi) \cap \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_l(\mathbb{A})) \end{aligned}$$

are true.

PROOF: We fix an irreducible component $[\pi]$ in (π). Then we claim that the image of $[\pi]$ in $\mathscr{A}(\pi) \cap \mathrm{L}^2_{\mathrm{cusp}}$ is $\mathrm{Sp}_l(\mathbb{A})$ irreducible. If not, then there exist, via the construction of Theorem 2.2, at least 2 linearly independent $\mathrm{O}(Q)(\mathbb{A})\times \mathrm{Sp}_l(\mathbb{A})$ invariant bilinear forms on $S[M_{ml}(\mathbb{A})]\otimes(\pi\otimes\beta_l(\pi))$. We sketch the argument for this fact. Indeed, if the image of $[\pi]$ in $\mathscr{A}(\pi)\cap \mathrm{L}^2_{\mathrm{cusp}}$ contains 2 irreducible factors V and W, then we may assume that

$$\int\theta_\varphi(G,g)\rho(g)\overline{(\psi-\psi^*)}(G)\mathrm{d}g\mathrm{d}G\equiv 0$$

for *all* $\varphi\in S[M_{ml}(\mathbb{A})]$, $\rho\in[\pi]$, $\psi\in V$, and $\psi^*\in W$ (where under an $\mathrm{Sp}_l(\mathbb{A})$ equivariant isomorphism between V and W, ψ and ψ^* are equivalent). But this is a contradiction since the vectors in $V\oplus W$ of the form $\alpha-\alpha^*$ (as α varies in $\beta_l(\pi)$) lies in the image of $[\pi]$.

But then we apply the arguments of §4 of [Sh] and deduce that for any *finite set* S of primes in K, where $S\supseteq S_1=\{$the set of v in K when either one of the representations π_v and ρ_v does not have a fixed vector under the standard maximal compact subgroup of $\mathrm{O}(Q_v)\times\mathrm{Sp}_n(K_v)\}\cup\{$Archimedean primes in $K\}$, the map

$$\Big(\bigotimes_{v\in S}\varphi_v\Big)\otimes\Big(\bigotimes_{v\in S}e_v\Big)\otimes\Big(\bigotimes_{v\in S}\bar{h}_v\Big)\rightsquigarrow\psi\otimes e\otimes\bar{h}$$

$$\rightsquigarrow\int\theta_\psi(G,g)e(g)\overline{h(G)}\mathrm{d}G\mathrm{d}g$$

is factorizable. Here $\psi=\otimes_{v\in S}\varphi_v\otimes(\otimes_{v\notin S}\chi_v)$ (where $\chi_v=$ characteristic function of a fixed lattice in $M_{mn}(K_v)$), $e=(\otimes_{v\in S}e_v)\otimes(\otimes_{v\notin S}e_v^*)$ (where e_v^* is a fixed vector in π_v under the standard maximal compact subgroup of $\mathrm{Sp}_n(K_v)$), and $h=\otimes_{v\in S}h_v\otimes(\otimes_{v\notin S}h_v^*)$ (where h_v^* is a fixed vector under the standard maximal compact subgroup of $\mathrm{O}(Q_v)$). The factorizable property follows from the *local multiplicity one* statement in the hypothesis above.

But locally this implies that for at least one prime v in K that $S[M_{ml}(K_v)]\otimes\pi_v\otimes\beta_l(\pi)_v$ has 2 linearly independent $\mathrm{O}(Q_v)\times\mathrm{Sp}_l(K_v)$ invariant bilinear forms; this contradicts the hypothesis given above.

On the other hand, if $[\pi]_1$ and $[\pi]_2$ are disjoint irreducible components in (π), then the images $[\pi]_1^*$ and $[\pi]_2^*$ are disjoint in $\mathscr{A}(\pi)\cap\mathrm{L}^2_{\mathrm{cusp}}$ by the same reasoning as above.

We sketch the argument for this fact. Indeed the images of $[\pi]_1$ and $[\pi]_2$ in $\mathscr{A}(\pi)\cap\mathrm{L}^2_{\mathrm{cusp}}$ determine the *same* irreducible subspace. Now we assume that

$$\int\theta_\varphi(G,g)\{\rho_1(g)-\rho_1^*(g)\}\overline{\beta(G)}\mathrm{d}g\mathrm{d}G=0$$

for *all* $\varphi\in S[M_{ml}(\mathbb{A})]$, $\beta\in$ image of $[\pi]_l$ in $\mathscr{A}(\pi)\cap\mathrm{L}^2_{\mathrm{cusp}}$, and $\rho_l\in[\pi]_l$

(where under an $\mathrm{O}(Q)(\mathbb{A})$ equivariant isomorphism between $[\pi]_1$ and $[\pi]_2$, ρ_1 and ρ_1^* are equivalent). But this is a contradiction to the fact that the space spanned by the $\rho_1 - \rho_1^*$ must lift *nonzero* to the image of $[\pi]_i$!

Thus we have shown that multiplicity is preserved under the mapping $\pi \rightsquigarrow \beta_i(\pi)$, i.e., multiplicity of π in R_i = multiplicity of $\beta_i(\pi)$ in $\mathscr{A}(\pi) \cap \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_i(\mathbb{A}))$! Q.E.D.

REMARK I.2.4: Assuming the analogue of the global Howe duality conjecture and the multiplicity one statement for the pair $(I_r(H_{r-1}), \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(H_{r-1})(\mathbb{A})))$, we see that if $n = 1$ and Q is the zero form, then the space $I_3(H_2) = \mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$ and the lifting between $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$ and $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(H_2)(\mathbb{A}))$ determines an injective mapping from cuspidal representations of $\mathrm{Sp}_1(\mathbb{A})$ to cuspidal representations of $\mathrm{O}(H_2)(\mathbb{A})$. Moreover we have that multiplicity of a cuspidal representation π in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_1(\mathbb{A}))$ is preserved under the lifting!

Thus the main problem of this paper is to verify the *local Howe duality conjecture* and the *local multiplicity one* statement of the above Corollary. We verify this conjecture in certain cases stated precisely below.

In particular, using Theorem I.2.2, we thus can safely define a family of correspondences β_i $(i = 1, \ldots, \dim Q)$ between cuspidal representations appearing in $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ and the various $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{Sp}_i(\mathbb{A}))$. In essence, the importance of this partition of the cuspidal representations occurring in $\mathrm{O}(Q)(K)\backslash\mathrm{O}(Q)(\mathbb{A})$ is that for each subspace R_i, the correspondence given by the Weil representation for the dual pair $(\mathrm{O}(Q), \mathrm{Sp}_i)$ is, in fact, well defined and determines an injective mapping. Moreover, using Corollary to Theorem 2.2, the multiplicities are preserved under the mappings.

We, in fact, shall verify the *local Howe duality conjecture* and the *multiplicity one* statements in the following cases.

(1) Q_v is *compact*. This has been done in [K-V] for $K_v = \mathbb{R}$ using infinitesimal arguments. Moreover, if K_v is non-Archimedean, then Q_v is anisotropic in the cases (i) when $m = 2$ and Q_v is a multiple of a norm form of a quadratic extension of K_v and (ii) when $m = 4$ and Q_v is the norm form of the unique quaternion algebra over K_v. We note that the local duality conjecture and multiplicity one have been demonstrated in case (i) for $m = 2$ and all n in [As] and [H] and in case (ii) for $m = 4$ and $n \geqslant 4$ in [As]. We give a proof below for the case $m = 4$ and all n (see Corollary to Theorem II.4.1). We note that these cases are fairly well known to the experts in the field, but we give this proof because we do not find complete proofs (in full generality) in the literature.

(2) Q_v an arbitrary nondegenerate form with v finite and $\dim Q > 4n + 2$.

(3) Q_v a nonquaternionic form with v finite and $\dim Q = 4n + 2$. (i.e., Q_v *cannot* be of the form $Q_v = H_{2n-1} \oplus \mathrm{L}$ where L is an anisotropic form equivalent to the norm form of a quaternion algebra over $\mathbb{Q}_v$).

(4) Q_v a split form with v finite and $\dim Q = 4n$.
(5) Q_v an unramified form with v finite and $\dim Q \leqslant n + 2$.

The rest of this paper is devoted to working out Cases (2) through (5) above. The same arguments will probably extend to the real and complex cases with some effort. Thus, for instance, we can prove the *global Howe duality conjecture* and the *multiplicity preserving* statement in the case where

(1) Q is the *norm form* of an *imaginary quadratic field* K over $\mathbb{Q}$, the rationals, and $i = 1, 2$ above.
(2) Q is the *norm form* of a *definite quaterion algebra over* $\mathbb{Q}$ and $i = 1$, 2, 3, and 4 above.
(3) Q *any anisotropic* form and $1 \leqslant i \leqslant (m-2)/4$ above.
(4) Q a sum of $m = 8t$ squares over $\mathbb{Q}$ and $1 \leqslant i \leqslant m/4$ and $i \geqslant m - 2$ above.

§3. *An example:* $L^2_{\mathrm{cusp}}(\mathrm{Sp}_2(\mathbb{A}))$

We consider the case $\mathrm{Sp}_2(\mathbb{A})$ and the associated decomposition of

$$L^2_{\mathrm{cusp}}(\mathrm{Sp}_2(\mathbb{A})) = I_3(H_2) \oplus I_4(H_3) \oplus I_5(H_4).$$

In this low dimensional case, it is easy to give other characterizations of the spaces $I_i(H_{i-1})$ ($i = 3, 4, 5$) (see [H-Ps-2] for a characterization similar to that below). That is

(A) $f \in I_4(H_3) \oplus I_5(H_4)$ if and only if

$$\oint_{\mathrm{Sp}_1(K)\backslash \mathrm{Sp}_1(\mathbb{A})} f(\dot{G}'G)\mathrm{d}\mu(G') \equiv 0$$

for all $G \in \mathrm{Sp}_2(\mathbb{A})$. Here $\mathrm{Sp}_1(\mathbb{A})$ embeds into $\mathrm{Sp}_2(\mathbb{A})$ via the map $x \to (x, 1) \in \mathrm{Sp}_1(\mathbb{A}) \times \mathrm{Sp}_1(\mathbb{A}) \subset \mathrm{Sp}_2(\mathbb{A})$, and $\mathrm{d}\mu$ is an $\mathrm{Sp}_1(\mathbb{A})$ invariant measure on $\mathrm{Sp}_1(K)\backslash \mathrm{Sp}_1(\mathbb{A})$.

(B) $f \in I_5(H_4)$ if and only if $\oint_{Z(K)\backslash Z(\mathbb{A})} f(ZG)\mathrm{d}\mu(Z) \equiv 0$ for all $G \in \mathrm{Sp}_2(\mathbb{A})$. Here Z is the unipotent group given by

$$\left\{ \left[\begin{array}{cc|cc} 1 & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 1 & 0 \\ & & 0 & 1 \end{array} \right] \,\middle|\, * \in \mathbb{A} \right\}$$

and $\mathrm{d}\mu$ is again a $Z(\mathbb{A})$ invariant measure on $Z(K)\backslash Z(\mathbb{A})$.

First, before proving (A) and (B), some comments are in order. We note from [Ps-2] that if the function $f \in L^2_{\mathrm{cusp}}(\mathrm{Sp}_2(\mathbb{A}))$ does not possess a standard Whittaker model in the sense that

$$\int_{U(K)\backslash U(\mathbb{A})} f\left\{\left[\begin{array}{cc|cc} 1 & \alpha & \beta & \gamma \\ 0 & 1 & \gamma & \delta \\ \hline \multicolumn{2}{c|}{0} & 1 & 0 \\ & & -\alpha & 1 \end{array}\right] G\right\} \tau(\alpha\lambda_1 + \delta\lambda_2)$$

$$\times \mathrm{d}U(\alpha, \beta, \gamma, \delta) \equiv 0$$

for all $G \in \mathrm{Sp}_2(\mathbb{A})$, all $\lambda_i \in K^{\times}$, and U is the maximal unipotent subgroup

$$\left\{\left[\begin{array}{cc|cc} 1 & * & * & * \\ 0 & 1 & * & * \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -* & 1 \end{array}\right] \Big| * \in \mathbb{A}\right\},$$

then

$$\int_{Z(K)\backslash Z(\mathbb{A})} f(ZG)\mathrm{d}\mu(Z) \equiv 0$$

for all G.

Thus, as a conclusion, all those automorphic cuspidal representations of $\mathrm{Sp}_2(\mathbb{A})$ not possessing standard Whittaker models lie in the space $I_5(H_4)$. In particular, this means that the *Saito-Kurokawa* space defined in [Ps-2] lies in $I_5(H_4)$. However it is also evident that the space $I_5(H_4)$ is not completely filled up by Saito-Kurokawa. Namely we recall that if Q is any non-degenerate anisotropic form over K and if we use the decomposition of $L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A})) = R_1(Q) \oplus R_2(Q) \oplus \ldots$, then it is clear that the image of the space $R_2(Q)$ lies in $I_5(H_4)$. That is, we observe that for $\varphi \in S[M_{m2}(\mathbb{A})]$ (relative to the dual pair $\mathrm{O}(Q)(\mathbb{A})$ and $\mathrm{Sp}_2(\mathbb{A})$)

$$\int_{Z(K)\backslash Z(\mathbb{A})} \theta_\varphi(ZG, g)\mathrm{d}Z = \sum_{\substack{\xi = (\xi_1|\xi_2) \in M_{m2}(K) \\ Q(\xi_1) = 0}} \pi_Q(G, g)(\varphi)(\xi_1|\xi_2).$$

But since Q is anisotropic, the sum above is simply over the set $\{(0|\xi_2)|\xi_2 \in M_{m1}(K)\}$. Thus the lift of an element $\psi \in L^2_{\mathrm{cusp}}(\mathrm{O}(Q)(\mathbb{A}))$ (more precisely we mean the space of the functions

$$\int_{\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A})} \psi(g)\theta_\varphi(G, g)\mathrm{d}g$$

as φ varies in $S[M_{m2}(\mathbb{A})]$ and G in $\mathrm{Sp}_2(\mathbb{A})$) lies in $I_5(\mathrm{H}_4)$ (by using (B) above) if and only if

$$\int_{\mathrm{O}(Q)(K)\backslash \mathrm{O}(Q)(\mathbb{A})} \psi(g)\theta_\varphi(G, g)\mathrm{d}g \equiv 0$$

for all $G \in \mathrm{Sp}_1(\mathbb{A})$ and all $\varphi \in S[M_{m1}(\mathbb{A})]$ (relative to the dual pair $\mathrm{O}(Q)(\mathbb{A})$ and $\mathrm{Sp}_1(\mathbb{A})$). Thus, in particular, we have that

$$\sum_Q \text{“the lift of } R_2(Q)\text{”} \subseteq I_5(H_4)$$

where Q varies over all inequivalent classes of quadratic forms anisotropic over K. We recall from [H-Ps-2] that (where $K = \mathbb{Q}$, the rationals) if Q is the norm form of a quaternion algebra over $\mathbb{Q}$, then the lift of $R_2(Q) \neq \{0\}$. Hence we have a subspace (i.e. $R_2(Q)$) in $I_5(H_4)$ which cannot overlap the Saito-Kurokawa space. Indeed it is easy to see that the lift of $R_2(Q)$ will satisfy the generalized Ramanujan conjecture for the eigenvalues of Hecke operators, whereas the Saito-Kurokawa space clearly cannot satisfy such conditions (see [R-1]).

Proof of (A) and (B): We recall that in general for $f \in I_i(H_{i-1}) \oplus \ldots \oplus I_{2n}(H_{2n-1})$, it is necessary and sufficient that

$$\int f(G)\theta_\varphi(G, g)\mathrm{d}G \equiv 0$$

for all $\varphi \in S[M_{2\nu,n}(\mathbb{A})]$ and $g \in \mathrm{O}(Q)(\mathbb{A})$ (as ν varies from 2 to $i-2$). But unwinding the sum for θ_φ and using the linearization of the Weil representation for $\mathrm{Sp}(\mathbb{A})$ and $\mathrm{O}(Q)(\mathbb{A})$ on $S[M_{2\nu,n}(\mathbb{A})]$ (see §0(VII)), we have that

$$\theta_\varphi(G, g) = \sum_{\xi \in M_{2\nu,n}(K)} \pi_\ell(G)F_T\big(\pi_Q(g)\varphi\big)(\xi).$$

But then since π_ℓ linearizes the Sp_n action, we can write the above sum over the Sp_n orbits in the space $M_{2\nu,n}(K)$ (where via π_ℓ, Sp_n is acting on $k^{2n} \oplus \ldots \oplus k^{2n}$, taken ν times, with k^{2n} the standard Sp_n module). Then it is a matter of classifying the orbits in these cases. In particular, we let $n = 2$ and $\nu = 2$ or $\nu = 3$. In these cases we have

(i) $\nu = 2$. Then the orbits have representatives which are of the following types:
  (a) $(0, 0)$
  (b) $(X, \lambda X)$ with $X \neq 0$ and $\lambda \in K$
  (c) (X, Y) with X and Y perpendicular to each other (relative to the alternating form $\langle\ ,\ \rangle$ defining Sp_2) and linearly indepen-

dent

(d) (X, Y) with X and Y having nonzero inner product and linearly independent.

In the cases above, the isotropy groups are (a) Sp_2, (b) $\mathrm{Sp}_1 \times U_1^2$, (c) U_2^2, and (d) $\mathrm{Sp}_1 \to \mathrm{Sp}_1 \times \mathrm{Sp}_1$ via $X \to (X, 1)$.

(ii) $\nu = 3$. Then the orbits have representative which are of the following types:

(a) $(0, 0, 0)$

(b) $(X, 0, 0)$ with $X \neq 0$ and a possible $\mathrm{G}\ell_3$ translate of this Sp orbit (here $\mathrm{G}\ell_3$ is acting on the left relative to $k^4 \oplus k^4 \oplus k^4$)

(c) $(X, Y, 0)$ with X and Y as in case (i, c) above and a possible $\mathrm{G}\ell_3$ translate of this orbit

(d) $(X, Y, 0)$ with X and Y as in case (i, d) above and a possible $\mathrm{G}\ell_3$ translate of this orbit

(e) (X, Y, Z) with X, Y, and Z spanning a three dimensional space with $\langle X, Y \rangle = 1$ and $\langle X, Z \rangle = \langle Y, Z \rangle = 0$. We also allow a possible $\mathrm{G}\ell_3$ translate of this orbit.

In these cases the isotropy groups are (a) Sp_2, (b) $\mathrm{Sp}_1 \times U_1^2$, (c) U_2^2, (d) Sp_1, and (e) Z.

Then when we take the inner product of θ_φ against a cusp form ψ on $\mathrm{Sp}_2(K)\backslash\mathrm{Sp}_2(\mathbb{A})$, we can first integrate ψ, in particular, against

$$\sum_{\gamma \in (X_\xi)(K)\backslash(\mathrm{Sp}_2)(K)} \pi_\ell(\gamma G) F_T\big(\pi_Q(g)\varphi\big)(\xi_*),$$

where X_ξ is the isotropy group of ξ, and have

$$\int_{X_\xi(\mathbb{A})\backslash \mathrm{Sp}_2(\mathbb{A})} \pi_\ell(G) F_T\left(\pi_Q(g)(\varphi)(\xi_*)\left(\int_{X_\xi(K)\backslash X_\xi(\mathbb{A})} \psi(hG)\mathrm{d}h\right)\mathrm{d}G.\right.$$

Then when we consider points ξ where the isotropy group X_ξ contains a unipotent radical (i.e., cases (a), (b), and (c) above for $\nu = 2$ and 3), we see that since ψ is a cusp form,

$$\int_{X_\xi(K)\backslash X_\xi(\mathbb{A})} \psi(hG)\mathrm{d}h \equiv 0.$$

Thus since the function $G \rightsquigarrow \pi_\ell(G) F_T(\psi)(\xi_*)$ is sufficiently generic on the coset space $X_\xi(\mathbb{A})\backslash\mathrm{Sp}_2(\mathbb{A})$ (i.e., $X_\xi(\mathbb{A})\backslash\mathrm{Sp}_2(\mathbb{A})$ is an $\mathrm{Sp}_2(\mathbb{A})$ orbit and we use simple approximation arguments given in [H-Ps-2] noting that φ is an arbitrary Schwartz-Bruhat function on the ambient space), we deduce (A) and (B) above. We note here that

$$\oint_{Z(K)\backslash Z(\mathbb{A})} \psi(G'G)\mathrm{d}\mu(Z) \equiv 0 \qquad \text{for all } G$$

implies that

$$\oint_{\mathrm{Sp}_1(K)\backslash \mathrm{Sp}_1(\mathbf{A})} \psi(G'G)\mathrm{d}\mu(G') \equiv 0 \qquad \text{for all } G.$$

II. The local theory

§1. The local structure of invariant distributions associated to the Weil representation

We shall be concerned in (II) with local fields, which we denote by k.

We recall here the definition of the Weil representation π_Q of $\mathrm{Sp}_n \times \mathrm{O}(Q)$ given in §0(VII).

We consider the following spaces. Let $\rho_Q^* = \{T \in S[M_{mn}(k)]^* | \pi_Q(g)T = T \text{ for all } g \in \mathrm{O}(Q)\}$, and let $\omega_Q^* = \{T \in S[M_{mn}(k)]^* | \pi_\ell(g)T = T \text{ for all } g \in \mathrm{Sp}_n\}$. Here $S(\ )^*$ = the space of all linear functionals on $S(\ )$. The space ρ_Q^* is a $\pi_Q(\mathrm{Sp}_n)$ module. Then we can also regard ω_Q^* as an $\mathrm{O}(Q' \oplus (-Q'))$ module (where the action of $\mathrm{O}(Q' \oplus (-Q'))$ is given via

$$\varphi \rightsquigarrow F_T\big(\pi_{Q'\oplus(-Q')}(g)\varphi\big)$$

(F_T defined in §0(IX)). The problem is then to determine the structure of ρ_Q^* ($\omega_{Q'}^*$, resp.) as an $\mathrm{Sp}_n(\mathrm{O}(Q' \oplus (-Q'))$, resp.) module. We note that ρ_Q^* is the contragredient of ρ_Q, which is defined in the next remark.

REMARK II.1.1: In the non-Archimedean local field case, it suffices to determine the Jacquet modules $\rho_Q = S[M_{mn}(k)]_{\mathrm{O}(Q)}$, $\omega_{Q'} = S[M_{mn}(k)]_{\mathrm{Sp}_n}$ as Sp_n or $\mathrm{O}(Q' \oplus (-Q'))$ modules.

First we must define 2 modules which are naturally associated to the problem above.

Let V_Q = range of the map $\varphi \to \pi_Q(G)(\varphi)[0]$ as φ varies over all Schwartz-Bruhat functions in $S[M_{mn}(k)]$. Then we deduce that

$$V_Q \subset \mathrm{Ind}\left[P_n \nearrow \mathrm{Sp}_n, \left[\begin{array}{c|c} A & X \\ \hline 0 & (A^t)^{-1} \end{array}\right] \rightsquigarrow |\det A|^{m/2} \frac{\gamma(1)}{\gamma((\det A)^m)}\right].$$

Let $W_{Q'}$ = range of the map $\varphi \to F_T \circ \pi_{Q'+(-Q')}(g)(\varphi)\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\varphi \in S[M_{mn}(k)]$. Then let $\mathbb{P}_Q$ = the parabolic of $\mathrm{O}(Q' \oplus (-Q'))$ given by

$$\mathcal{M}\left\{\left[\begin{array}{c|c} g & * \\ \hline 0 & (g^t)^{-1} \end{array}\right]\right\}\mathcal{M}^{-1}$$

where $\mathscr{M} = \left[\begin{array}{c|c} I_n & I_n \\ \hline I_n & -I_n \end{array}\right]$. Here $W_{Q'}$ is a submodule of

$$\operatorname{Ind}\left(\mathbb{P}_{Q'} \nearrow \mathrm{O}(Q' \oplus (-Q')) \,|\, \mathscr{M}\left[\begin{array}{c|c} g & * \\ \hline 0 & (g^t)^{-1} \end{array}\right]\mathscr{M}^{-1} \to |\det g|^n\right).$$

Then we prove the following theorem.

THEOREM II.1.1: *Let k be a non-Archimedean local field ($ch(k) = 0$). Then ρ_Q is Sp_n isomorphic to V_Q and $\omega_{Q'}$ is $\mathrm{O}(Q' \oplus (-Q'))$ isomorphic to $W_{Q'}$.*

The proof of this Theorem will be given in several steps. First we define the generalized moment maps. That is, we let

$$\begin{array}{ccc} & M_{mn}(k) & \\ {\scriptstyle A}\swarrow & & \searrow{\scriptstyle Q} \\ \mathrm{Alt}_m(k) & & \mathrm{Sym}_n(k) \end{array}$$

where

$$Q(X) = X^tQX, \qquad X \in M_{mn}(k)$$

and

$$A(Y) = Y\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}Y^t, \qquad Y \in M_{mn}(k) \text{ (with } m \text{ even).}$$

Then we note for $g \in \mathrm{O}(Q)$ and $\mathrm{G} \in \mathrm{Sp}_n$, we have

$$Q(g \cdot X) = Q(X) \qquad \text{and} \quad A(Y \cdot G) = A(Y).$$

Also we have for $H \in \mathrm{G}\ell_n(k)$ and $H \in \mathrm{G}\ell_m(k)$ that

$$Q(X \cdot H) = H^tQ(X)H \qquad \text{and} \quad A(H \cdot Y) = H^tA(Y)H.$$

Thus in any case, we have for any $\mathrm{G}\ell_n(k)$ ($\mathrm{G}\ell_m(k)$ resp.) *orbit* $\mathscr{O}$ in $\mathrm{Sym}_n(k)$ or $\mathrm{Alt}_m(k)$

$$Q^{-1}(\mathscr{O}) \qquad \text{or} \quad A^{-1}(\mathscr{O})$$

contains a finite number of $\mathrm{O}(Q) \times \mathrm{G}\ell_n(k)$ orbits ($\mathrm{G}\ell_m(k) \times \mathrm{Sp}_n$ orbits). In particular, this implies that the sets $X_i(Q) = \{\xi \in M_{mn}(k) | \operatorname{rank}(Q(\xi)) \leqslant i\}$ ($X_i(A)$ resp.) consists of a *finite number* of $\mathrm{O}(Q) \times \mathrm{G}\ell_n(k)$ ($\mathrm{G}\ell_m(k)$

$\times \mathrm{Sp}_n(k)$ resp.) orbits. Also we have a stratification of the space $M_{mn}(k)$ as follows:

$$X_0(Q) \subseteq X_1(Q) \subseteq \ldots \subseteq X_r(Q) \subseteq \ldots$$

Then the scheme of the proof goes as follows. We must show that Kernel $\{\varphi \to \pi_Q(G)\varphi[0]\} = S[M_{mn}(k)][\mathrm{O}(Q)]$ (see §0(IV)). Then we note $M_{mn}(k) - X_i(Q)$ is an open subset of $M_{mn}(k)$. We are going to show that given $f \in S[M_{mn}(k) - X_i(Q)]$ and $f \in$ Kernel$(\ldots)$, then there exists φ_f in $S[M_{mn}(k)][\mathrm{O}(Q)]$ such that $f - \varphi_f \in$ Kernel$(\ldots)$ and support $(f - \varphi_f) \subseteq M_{mn}(k) - X_{i+1}(Q)$. We note here that trivially $S[M_{mn}(k)][\mathrm{O}(Q)] \subseteq$ Kernel$(\ldots)$.

Then we have the exact sequence of spaces

$$0 \to S[M_{mn}(k) - X_{i+1}(Q)] \to S[M_{mn}(k) - X_i(Q)]$$
$$\to S[X_{i+1}(Q) - X_i(Q)] \to 0.$$

Then we can relativize this sequence by the group $\mathrm{O}(Q)$ and obtain

$$S[M_{mn}(k) - X_{i+1}(Q)]_{O(Q)} \to S[M_{mn}(k) - X_i(Q)]_{O(Q)}$$
$$\to S[X_{i+1}(Q) - X_i(Q)]_{O(Q)} \to 0.$$

Then the method of proof is to show that if $f \in S[M_{mn}(k) - X_i(Q)]$ and $f \in$ Kernel$(\ldots)$, then $T(f) \equiv 0$ for all $\mathrm{O}(Q)$ invariant distributions in $M_{mn}(k) - X_i(Q)$ supported on $X_{i+1}(Q) - X_i(Q)$. In particular, this implies that f (modulo relativizing above) vanishes in the last step of the exact sequence above; hence we can find a $\varphi_f \in S[M_{mn}(k) - X_i(Q)][\mathrm{O}(Q)]$ such that $f - \varphi_f$ has support in $M_{mn}(k) - X_{i+1}(Q)$. It is automatic that $f - \varphi_f \in$ Kernel$(\ldots)$.

Then we consider another (and finer) stratification of the space $M_{mn}(k) - X_i(Q)$ into sets $Y_{i,t}(Q)$ where

$$Y_{i,0} = M_{mn}(k) - X_i(Q) \subset Y_{i,1} \subset \ldots \subset Y_{i,r} = M_{mn}(k) - X_{i-1}(Q)$$

and $Y_{i,t} = (M_{mn}(k) - X_i(Q)) \cup \{\xi \in X_i(Q) - X_{i-1}(Q) | \mathrm{rank}(\xi) \leqslant i + t - 1\}$. Then using similar reasoning as above, we must show that if $f \in S[Y_{i,r} - (Y_{i,t} - Y_{i,0})]$ and $f \in$ Kernel$(\ldots)$, then $T(f) = 0$ for all $\mathrm{O}(Q)$ invariant distributions T supported in $Y_{i,t+1} - Y_{i,t}$. In particular, this again implies that there is a function $\varphi_f \in S[Y_{i,r} - (Y_{i,t} - Y_{i,0})][\mathrm{O}(Q)]$ such that $f - \varphi_f$ has support in $Y_{i,r} - (Y_{i,t+1} - Y_{i,0})$.

Hence the problem reduces to showing that if $f \in S[Y_{i,r} - (Y_{i,t} - Y_{i,0})] \cap$ Kernel$(\ldots)$, then $T(f) \equiv 0$ for all $\mathrm{O}(Q)$ invariant distributions supported on each of the $\mathrm{O}(Q) \times G\ell_n(k)$ orbits $\mathcal{O}_\alpha$ in $Y_{i,t+1} - Y_{i,t}$. We note that there are a finite number of such orbits; hence each such orbit $\mathcal{O}_\alpha$ is a closed subset of $Y_{i,r} - (Y_{i,t} - Y_{i,0})$ and the dimension of each such orbit $\mathcal{O}_\alpha$ is the *same*.

The proof then reduces to the following series of steps.

First we characterize the orbits in $Y_{i,t+1} - Y_{i,t}$. Indeed an orbit $\mathcal{O}_\alpha$ in $Y_{i,t+1} - Y_{i,t}$ will contain a representative X_α of the form $[\xi|0]$, where ξ is a $m \times (i+t)$ matrix of $i+t$ linearly independent columns such that $Q(\xi_\ell, \xi_\eta) = 0$ if $\ell \neq \eta$ ($\xi_\ell = \ell$th column of ξ) and $Q(\xi_\ell, \xi_\ell) = \alpha_\ell$, with $\alpha_\ell \neq 0$ for $1 \leqslant \ell \leqslant i$ and $\alpha_\ell = 0$ for $i+1 \leqslant \ell \leqslant i+t$. Then if M_α is the subspace spanned by the $i+t$ columns $\{\xi_i\}$ of ξ, we let $\mathrm{O}(Q)_{M_\alpha} = \{g \in \mathrm{O}(Q) | g$ stabilizes the subspace $M_\alpha\}$. Then we obtain a linear representation ν_α: $\mathrm{O}(Q)_{M_\alpha} \to \mathrm{Aut}_k(M_\alpha)$ (relative to the choice of basis ξ_i above). Then the isotropy group of X_α equals

$$\left\{\left(g, \left[\begin{array}{c|c} \nu_\alpha(g) & 0 \\ \hline X & B \end{array}\right]\right) | g \in \mathrm{O}(Q)_{M_\alpha},\right.$$

$$B \in \mathrm{G}\ell_{n-(i+t)}(k) \text{ and } X \text{ an arbitrary } (n-(i+t))$$

$$\left. \times (i+t) \text{ matrix}\right\}.$$

Then we consider $W(M_\alpha) =$ the $\mathrm{O}(Q)$ orbit of ξ in $M_{m,i+t}(k)$, and we define a map

$$W(M_\alpha) \times \mathrm{G}\ell_n(k) \to \mathcal{O}_\alpha$$

$$(X, G) \overset{\psi_\alpha}{\to} [X|0]G.$$

Then we observe that ψ_α is a surjective morphism (in the category of ℓ-spaces of [B-Z]). With this map, it is possible to describe all $\mathrm{O}(Q)$ invariant distributions on the orbit $\mathcal{O}_\alpha$. That is, we have the following Lemma.

LEMMA: *Choose an* $\mathrm{O}(Q)$ *invariant measure* $d\mu_{M_\alpha}$ *on* $W(M_\alpha)$, *a right invariant Haar measure du on* $\mathrm{G}\ell_n(k)$, *and an* $\mathrm{O}(Q) \times \mathrm{G}\ell_n(k)$ *relatively invariant measure on* $\mathcal{O}_\alpha$. *Then relative to this data, there exists a surjective map*

$$S[W(M_\alpha) \times \mathrm{G}\ell_n(k)] \to S[\mathcal{O}_\alpha]$$

$$\beta \to f_\beta$$

such that every O(Q) *invariant distribution* S *on* $S[\mathcal{O}_\alpha]$ *has the form*

$$S(f_\beta)=\oint\left(\int_{(\mathrm{O}(Q)/\mathrm{O}(Q)^{M_\alpha})\times Z_\alpha}\beta(g\cdot\xi, uG)\left(\Delta_{Z_\alpha}(u)\right)^{-1}\right.$$

$$\left.\times\,\mathrm{d}\mu_{M_\alpha}(g)d_r(u)\right)\mathrm{d}R(G)$$

where $\oint\ldots\mathrm{d}R$ *is an arbitrary distribution on* $S[Z_\alpha\backslash \mathrm{G}\ell_n(k)]$.

Here $Z_\alpha=\left\{\left[\begin{array}{c|c}A&0\\\hline X&B\end{array}\right]\middle| A\in\nu_\alpha(\mathrm{O}(Q)_{M_\alpha}),\ B\in\mathrm{G}\ell_{n-(i+t)}(k),\ X\right.$ an arbitrary $(n-(i+t))\times(i+t)$ matrix, and $d_r(\ )$ is a right invariant measure on Z_α and Δ_{Z_α}, the associated delta function (defined to satisfy $d_r(\xi\cdot x)=\Delta_{Z_\alpha}(x)d_r(\xi)$ for all $x,\xi\in Z_\alpha$).

PROOF: Each distribution on $S[\mathcal{O}_\alpha]$ gives rise to a distribution on $S[W(M_\alpha)\times\mathrm{G}\ell_n(k)]$ via the map $\beta\to f_\beta$. (Note the map $\beta\to f_\beta$ is constructed using [HC] and the fact that ψ_α is a surjective mapping.) Thus it suffices to determine the O(Q) invariant distribution S on $S[W(M_\alpha)\times\mathrm{G}\ell_n(k)]$. However we observe that such an S must also satisfy

$$S\left(f_{G_1\cdot\beta}\right)=S\left(f_\beta\right),\qquad \text{where } G_1=\left[\begin{array}{c|c}\nu_\alpha(g)&0\\\hline X&B\end{array}\right]$$

and $g\in\mathrm{O}(Q)_{M_\alpha}$, with G_1 acting on the second factor in $W(M_\alpha)\times\mathrm{G}\ell_n(k)$ by left multiplication. Thus we deduce the statement of the Lemma.

Q.E.D.

Then we let $f\in S[Y_{i,r}-(Y_{i,t}-Y_{i,0})]\cap\mathrm{Kernel}(\ldots)$. We want to study the behavior of $T(f)$ for all O(Q) invariant distributions T supported on an O$(Q)\times\mathrm{G}\ell_n(k)$ orbit $\mathcal{O}_\alpha$ in $Y_{i,t+1}-Y_{i,t}$.

We let $A\in\mathrm{G}\ell_n(k)$ and we define the map

$$f\rightsquigarrow\pi_Q\left[\begin{array}{c|c}A&0\\\hline 0&(A^t)^{-1}\end{array}\right](f)[X|0],$$

where X is an arbitrary matrix in $M_{m,i+t}(k)$. Then we have that $\tilde{f}_A(X)=\pi_Q\left[\begin{array}{c|c}A&0\\\hline 0&(A^t)^{-1}\end{array}\right](f)[X|0]$ is a function in $S[M_{m,i+t}(k)]$ and $\tilde{f}_A$ has support in the set

$$M'_{m,i+t}=\left\{\xi\in M_{m,i+t}(k)\,|\,\mathrm{rank}(Q(\xi))\geqslant i \text{ and } \mathrm{rank}(\xi)=i+t\right\}.$$

Then we consider the *moment mapping* $Q\colon M_{m,i+t}(k)\to\mathrm{Sym}_{i+t}(k)$, and we note that the differential $\mathrm{d}Q$ of this map is submersive at all

$\xi \in M_{m,i+t}(k)$, where $\operatorname{rank}(\xi) = i + t$ (we recall that $m \geqslant i + t$). Thus we can apply the regularity Theorem of [HC] and deduce that there exists an orbital integral map of

$$S\left[M'_{m,i+t}\right] \to S\left[\operatorname{Sym}_{i+t}(k)\right]$$

$$\varphi \to M_\varphi$$

which satisfies

$$\int_{M'_{m,i+t}} \varphi[T]\psi[Q(T)]\,\mathrm{d}T = \int_{\operatorname{Sym}_{i+t}(k)} M_\varphi[X]\psi(X)\,\mathrm{d}X$$

for all ψ, a locally constant function on $\operatorname{Sym}_{i+t}(k)$. Here $\mathrm{d}T$ and $\mathrm{d}X$ are Haar measures on $M'_{m,i+t}$ and $\operatorname{Sym}_{i+t}(k)$, resp.

Then we consider the Bruhat decomposition of Sp_n relative to P_n, i.e. a decomposition of the form $\mathrm{Sp}_n = \bigcup_{i=0}^{n} P_n w_i P_n$ where

$$w_i = \left[\begin{array}{c|c|c|c} I_i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i} \\ \hline 0 & 0 & I_i & 0 \\ \hline 0 & -I_{n-i} & 0 & 0 \end{array}\right].$$

Then we assume $f \in \operatorname{Ker}(\dots)$. Hence we have $\pi_Q(pw_i p')(f)[0] \equiv 0$ for all $p, p' \in P_n$. But this implies that

$$\int \pi_Q \left[\begin{array}{c|c} A & 0 \\ \hline 0 & (A')^{-1} \end{array}\right](f)[X_i|0]\,\tau\left(\operatorname{Tr}\left(SX_i'QX_i\right)\mathrm{d}X_i \equiv 0,\right.$$

where $X_i \in M_{mi}(k)(1 \leqslant i \leqslant n)$ and $S \in \operatorname{Sym}_i(k)$. Then if $f \in S[Y_{i,r} - (Y_{i,t} - Y_{i,0})] \cap \operatorname{Kernel}(\dots)$, we deduce from the above comments that, for *all* $S \in \operatorname{Sym}_{i+t}(k)$ and *all* $A \in \mathrm{G}\ell_n(k)$,

$$\int_{\operatorname{Sym}_{i+t}(k)} M_\Psi[X]\tau(\operatorname{Tr}(S \cdot X))\,\mathrm{d}X \equiv 0,$$

where $\Psi = \pi_Q \left[\begin{array}{c|c} A & 0 \\ \hline 0 & (A')^{-1} \end{array}\right] f$. Hence $M_\Psi[X] \equiv 0$ for all $X \in \operatorname{Sym}_{i+t}(k)$. In particular, this function vanishes on the Q moment of the $\mathrm{O}(Q)$ orbit $W(M_\alpha)$ in $M_{m,i+t}(k)$.

Then we claim that the distribution (for *fixed* $A \in \mathrm{G}\ell_n(k)$)

$$f \rightsquigarrow M_{\tilde{f}_A}[M_\alpha] \qquad \text{for } f \in S\left[Y_{i,r} - (Y_{i,t} - Y_{i,0})\right] \tag{$*$}$$

is a nonzero multiple of

$$\int \beta(g\cdot M_\alpha, u\cdot A)\left(\Delta_{Z_\alpha}(u)\right)^{-1}\mathrm{d}\mu_{M_\alpha}d_r(u),$$

where $f_\beta = f$ restricted to $\mathcal{O}_\alpha$. (See Lemma above.)

First we note that the distribution given by $(*)$ has support in the closure of the set $\{g[M_\alpha|0]A \mid g\in \mathrm{O}(Q)\}$ in $Y_{t,r}-(Y_{t,t}-Y_{t,0})$. However we know that $\mathcal{O}_\alpha$ is a closed set in $Y_{t,r}-(Y_{t,t}-Y_{t,0})$. Then $\mathcal{O}_\alpha\cap\{[X|0] \mid X\in M_{m,t+t}(k)\}$ is a closed subset of $Y_{t,r}-(Y_{t,t}-Y_{t,0})$. On the other hand, we know that $\mathcal{O}_\alpha\cap\{[X|0] \mid X\in M_{m,t+t}(k)\}$ is an orbit of $\mathrm{O}(Q)\times \mathrm{G}\ell_{t+t}(k)$ given by $\{[gM_\alpha A|0] \mid g\in\mathrm{O}(Q),\ A\in \mathrm{G}\ell_{t+t}(k)\}$. Then we apply the Q moment map to this set and observe that the set $Q^{-1}(Q(M_\alpha))$ is a closed subset in $\mathcal{O}_\alpha\cap\{[X|0] \mid X\in M_{m,t+t}(k)\}$; hence $\{[gM_\alpha|0]\cdot A \mid g\in\mathrm{O}(Q)\}$ is closed in $Y_{t,r}-(Y_{t,t}-Y_{t,0})$ (A, a fixed matrix in $\mathrm{G}\ell_n(k)$). Thus the distribution $(*)$ has support in $\{[gM_\alpha|0]\cdot A \mid g\in\mathrm{O}(Q)\}$. Then we apply the *Lemma* and deduce the above statement!

Thus we have shown that if $f\in S[Y_{t,r}-(Y_{t,t}-Y_{t,0})]\cap \mathrm{Kernel}(\ldots)$, then $T(f)\equiv 0$ for all $\mathrm{O}(Q)$ invariant distributions in $Y_{t,r}-(Y_{t,t}-Y_{t,0})$ supported on an $\mathrm{O}(Q)\times\mathrm{G}\ell_n(k)$ orbit in $Y_{t,t+1}-Y_{t,t}$. In particular, this implies Theorem II.1.1.

We note that the proof for $W_{Q'}$ proceeds in the same manner as above. Q.E.D.

REMARK II.1.1: If $m\geqslant 2n+1$, then it is possible (from Proposition 6 of [We]) to describe

$$\pi_Q\left(\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}I & T\\ 0 & I\end{bmatrix}\right)(\varphi)[0]=\theta_\varphi(T)$$

in terms of the moment mappings. Indeed for each $Z\in\mathrm{Sym}_n(k)$, there exists an $\mathrm{O}(Q)$ invariant measure $\mathrm{d}\mu_Z$ in $Q^{-1}(Z)=\{X\in M_{mn}(k) \mid X'QX = Z\}\cap\{X\in M_{mn}(k)|\mathrm{rank}(X)=n\}$ such that the functional

$$\varphi\to M_\varphi(Z)=\int\varphi(X)\mathrm{d}\mu_Z(X)$$

is a tempered measure on $S[M_{mn}(k)]$. Moreover we have that the function M_φ is a continuous and integrable function on $\mathrm{Sym}_n(k)$ and satisfies

$$\hat{M}_\varphi(T)=\int_{M_{m,n}(k)}\varphi[X]\tau\left(\mathrm{Tr}\left(X'QX\cdot T\right)\right)\mathrm{d}X$$

(where now $\hat{M}_\varphi$ is continuous and integrable on $\mathrm{Sym}_n(k)$). Thus we have

$$\theta_\varphi[T]=\hat{M}_\varphi[T].$$

We note here that the set $Q^{-1}(Z) \cap \{X|\mathrm{rank}(X) = n\}$ is an $O(Q)$ orbit, and the orbit carries an $O(Q)$ invariant measure $d\mu_Z$. Thus if $Q^{-1}(Z) \cap \{X|\mathrm{rank}(X) = n\} = \emptyset$, then we have $M_\varphi[Z] \equiv 0$.

§2. *Decomposition of* ρ_Q *as a* $\mathrm{Sp}_n \times \mathrm{Sp}_n$ *module*

The first problem is to study ρ_Q as a module when restricted to the subgroup $\mathrm{Sp}_n \times \mathrm{Sp}_n \subseteq \mathrm{Sp}_{2n}$ embedded as follows:

$$(g_1, g_2) \to \left[\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array}\right].$$

The problem first becomes a standard one in the theory of induced representations (i.e. generalizing the Frobenius Subgroup Theorem of finite group representation theory). Indeed we first must analyze the double coset structure of the space

$$\mathrm{Sp}_n \times \mathrm{Sp}_n \backslash \mathrm{Sp}_{2n}/P_{2n}.$$

We consider first the problem over *finite fields*. We find a set of representatives for the $\mathrm{Sp}_n \times \mathrm{Sp}_n$ orbits in Sp_{2n}/P_{2n}; moreover we find the isotropy group of a fixed point in each such orbit! The arguments here are purely of a counting nature.

Then we consider the problem over general local fields ($Ch(k) = 0$ and k non-Archimedean). We show that a similar parametrization as in the finite field case is valid. Indeed we show that the problem reduces to the finite field case.

(i) Finite field case (k = a finite field with $\#(k) = q$)

We first comment that every $\mathrm{Sp}_n \times \mathrm{Sp}_n$ orbit in Sp_{2n}/P_{2n} intersects the open Bruhat cell $P_{2n}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}P_{2n}$. In fact we take as candidates for representatives of the $\mathrm{Sp}_n \times \mathrm{Sp}_n$ orbits the elements

$$V_i = \begin{bmatrix} I & T_i \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

where

$$T_i = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & I_i & 0 \\ & & 0 & 0 \\ \hline I_i & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \end{array}\right], \qquad I_i = i \times i \textit{ identity matrix}.$$

Then a straightforward argument using the Bruhat decomposition shows that

$$(\mathrm{Sp}_n \times \mathrm{Sp}_n)(V_i \cdot P_{2n}) \cap P_{2n}\left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]P_{2n}$$

$$= \left\{ \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P_{2n} \middle| Z = \begin{bmatrix} R & S \\ S^t & T \end{bmatrix},\right.$$

$$\left. R \text{ and } T\ n \times n \text{ symmetric, and } \operatorname{rank}(S) = i \right\}.$$

In particular, this fact implies that $(\mathrm{Sp}_n \times \mathrm{Sp}_n)(V_i \cdot P_{2n}) \cap (\mathrm{Sp}_n \times \mathrm{Sp}_n)(V_j \cdot P_{2n}) = \emptyset$ iff $i \neq j$. Then to show that the V_i form a complete set of representatives, it suffices to determine $\Omega_i = \#((\mathrm{Sp}_n \times \mathrm{Sp}_n)(V_i \cdot P_{2n}))$ and prove that

$$\sum_{i=0}^{n} \Omega_i = \#(\mathrm{Sp}_{2n}/P_{2n}).$$

Thus it suffices to determine the isotropy group of $V_i \cdot P_{2n}$ in $\mathrm{Sp}_n \times \mathrm{Sp}_n$. But a straightforward and tedious argument shows for

$$R = \left[\begin{array}{c|c|c|c} A & & B & \\ \hline & X & & 0 \\ \hline C & & D & \\ \hline & 0 & & (X^t)^{-1} \end{array}\right], \qquad S = \left[\begin{array}{c|c|c|c} D & & C & \\ \hline & Y & & 0 \\ \hline B & & A & \\ \hline & 0 & & (Y^t)^{-1} \end{array}\right]$$

that

$$(\mathrm{Sp}_n \times \mathrm{Sp}_n)^{V_i \cdot P_{2n}} = \left\{ (R, S) \middle| \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathrm{Sp}_i(k),\ X, Y \in \mathrm{G}\ell_{n-i}(k) \right\}$$

$$\times\left({}^tU_i^n \times {}^tU_i^n\right)$$

where U_i^n = a unipotent radical of the parabolic in Sp_n having $\mathrm{Sp}_i(k) \times \mathrm{G}\ell_{n-i}(k)$ as its *Levi factor* (see §0(IX)).

Then we compute the orders of the various groups in question.

(1) $\#(\mathrm{Sp}_r) = q^{2r-1}(q^{2r} - 1)q^{2r-3}(q^{2r-2} - 1)\ldots q(q^2 - 1)$

(2) $\#(\mathrm{G}\ell_r) = (q^r - 1)(q^r - q^1)\ldots(q^r - q^{r-1})$

(3) $\#(U_i^n) = q^{2i(n-i)+((n-i)(n-i+1)/2)}$

(4) $\#(P_r) = \#(\mathrm{G}\ell_r)q^{r(r+1)/2}$.

Thus the identity we must show is that

$$\frac{\#(\mathrm{Sp}_{2n})}{\#(P_{2n})\#(\mathrm{Sp}_n \times \mathrm{Sp}_n)} = \sum_{i=0}^{i=n} \frac{1}{\#(\mathrm{Sp}_i)\#(G\ell_{n-i})^2 \#(U_i^n)^2}$$

However, after substituting the above information into both sides, we see that this identity is equivalent to

$$(q^{2n}+1)\ldots(q^{n+1}+1) = \sum_{k=0}^{k=n} q^{k^2}\binom{n}{k}_q (q^n+1)\ldots$$

$$(q^{k+1}+1)(q^n-1)\ldots(q^{n-k+1}-1),$$

where $\binom{n}{k}_q = [(q^n-1)\ldots(q^{n-k+1}-1)]/(q^k-1)\ldots(q-1)$ (the binomial coefficient). But this identity is a special case of the *q-binomial theorem*: if $P_j(r,s) = (r-s)\ldots(r-q^{j-1}s)$, then

$$P_n(r,s) = \sum_{k=0}^{k=n} \binom{n}{k}_q P_k(r,y)P_{n-k}(y,s).$$

Then we let $r = q^{2n}$, $s = -1$, and $y = q^n$ in this identity to deduce the above equality!

(ii) Local field case ($Ch(k) = 0$ and n non-Archimedean)

We assert that the same parametrization as given in (i) works in this case. Namely we claim that the elements

$$V_i \cdot P_{2n}$$

form a complete set of representatives of the $\mathrm{Sp}_n \times \mathrm{Sp}_n$ orbits in Sp_{2n}/P_{2n}. Indeed it is clear that the orbits of these elements are disjoint. What we must show is that every element in Sp_{2n}/P_{2n} is $\mathrm{Sp}_n \times \mathrm{Sp}_n$ conjugate to an element in the open cell

$$P_{2n}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}P_{2n}.$$

For this we note that $\mathrm{Sp}_{2n}(k) = \mathrm{Sp}_{2n}(\mathcal{O}_k)P_{2n}$; hence it suffices to prove the statement for an element $z \in \mathrm{Sp}_{2n}(\mathcal{O}_k)$. That is, there exists $\omega \in \mathrm{Sp}_n \times \mathrm{Sp}_n$ such that

$$\omega \cdot z \in \text{the open cell}.$$

First we consider the *surjective* homomorphism $\mathrm{Sp}_{2n}(\mathcal{O}_k) \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$ and let $B =$ the inverse image of the group $P_{2n}(\mathbb{F}_q)$ in $\mathrm{Sp}_{2n}(\mathcal{O}_k)$. Then we apply case (i) and find an element ω' in $\mathrm{Sp}_n(\mathcal{O}_k)\times \mathrm{Sp}_n(\mathcal{O}_k)$ such that $\omega'\cdot z\in B\begin{bmatrix} 0 & I\\ -I & 0\end{bmatrix}B$. Moreover, again using the Bruhat decomposition of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ relative to $P_{2n}(\mathbb{F}_q)$, we have $\omega'\cdot z = b_1\begin{bmatrix} 0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix} I & T_1\\ 0 & I\end{bmatrix}$, where $T_1\in \mathrm{Sym}_{2n}(\mathcal{O}_k)$.

Then we recall the decomposition (as groups)

$$B = B\cap\left\{\begin{bmatrix} I & T\\ 0 & I\end{bmatrix} \mid T\in \mathrm{Sym}_{2n}(\mathcal{O}_k)\right\}\cdot B\cap \mathrm{G}\ell_{2n}(k)$$

$$\cdot B\cap\left\{\begin{bmatrix} I & 0\\ T & I\end{bmatrix} \mid T\in \mathrm{Sym}_{2n}(\mathcal{O}_k), \qquad T\equiv 0 \bmod \pi\right\}.$$

Thus

$$b_1=\begin{bmatrix} I & T'\\ 0 & I\end{bmatrix}\left[\begin{array}{c|c} A & 0\\ \hline 0 & (A')^{-1}\end{array}\right]\begin{bmatrix} I & 0\\ T'' & I\end{bmatrix}.$$

Hence we have

$$\omega'\cdot z=\begin{bmatrix} I & T'\\ 0 & I\end{bmatrix}\begin{bmatrix} 0 & I\\ -I & 0\end{bmatrix}\left[\begin{array}{c|c} (A')^{-1} & 0\\ \hline 0 & A\end{array}\right]\begin{bmatrix} I & T'''\\ 0 & I\end{bmatrix};$$

thus $\omega'\cdot z$ lies in the open cell.

Thus we can now describe the restriction of ρ_Q to $\mathrm{Sp}_n\times \mathrm{Sp}_n$. But, in fact, we shall do it more generally. Namely we consider the Sp_{2n} module $V_\chi=\{\varphi:\mathrm{Sp}_{2n}\to\mathbb{C}\mid\varphi$ locally constant and $\varphi[P\cdot G]=\chi(P)\varphi(G)$ for $P\in P_{2n}$ and $G\in \mathrm{Sp}_n\}$ where

$$\chi\left(\left[\begin{array}{c|c} A & X\\ \hline 0 & (A')^{-1}\end{array}\right]\right)=\chi(\det A),\ \chi \text{ a quasicharacter on } k^\times.$$

We note at this point we consider $\mathrm{Sp}_n\times\mathrm{Sp}_n$ orbits in $P_{2n}\backslash \mathrm{Sp}_{2n}$. It is straightforward to verify that $P_{2n}\cdot\tilde V_t$ (with $\tilde V_t = V_t^{-1}(w_t, w_t)$) is a typical representative of an $\mathrm{Sp}_n\times\mathrm{Sp}_n$ orbit (w_t given in the proof of Theorem II.1.1). The isotropy group of this element in $\mathrm{Sp}_n\times\mathrm{Sp}_n$ is the *transpose* of $(\mathrm{Sp}_n\times\mathrm{Sp}_n)^{V_t\cdot P_{2n}}$ given above $(=\{(R,S)\mid A\in \mathrm{Sp}_t(k),\ X,\ Y\in G\ell_{n-t}(k)\}\times(U_t^n\times U_t^n))$.

Then we have the decomposition of V_χ as a $\mathrm{Sp}_n\times\mathrm{Sp}_n$ module.

PROPOSITION II.2.1: *As a* $\mathrm{Sp}_n\times\mathrm{Sp}_n$ *module,* V_χ *has a finite composition series*

$$V_\chi = V_{\chi,0}\supset V_{\chi,1}\supset V_{\chi,2}\supset\ldots\supset V_{\chi,n}$$

where $V_{\chi,\iota}/V_{\chi,\iota+1}$ *is* $\mathrm{Sp}_n \times \mathrm{Sp}_n$ *equivalent to the compactly induced module*

$$\operatorname{ind}\left[(\mathrm{Sp}_n \times \mathrm{Sp}_n)^{P_{2n}\cdot \tilde{V}_\iota} \nearrow \mathrm{Sp}_n \times \mathrm{Sp}_n, (A, X)\cdot(A^\Delta, Y)\cdot(g_1, g_2) \rightsquigarrow \chi(X\cdot Y)\right],$$

where (A, X) *corresponds to the matrix R given above*, (A^Δ, Y) *corresponds to the matrix S given above, and* $g_1, g_2 \in U_\iota^n$.

REMARK II.2.1: We note that $V_{\chi,n}$ is the $\mathrm{Sp}_n \times \mathrm{Sp}_n$ representation on the space $S[D\backslash \mathrm{Sp}_n \times \mathrm{Sp}_n]$ where $D = \{(A, A^\Delta) | A \in \mathrm{Sp}_n\}$. But this latter module is $\mathrm{Sp}_n \times \mathrm{Sp}_n$ equivalent to the representation of $\mathrm{Sp}_n \times \mathrm{Sp}_n$ on the space $S[\mathrm{Sp}_n]$ via the action

$$\text{(2-A)}\quad \varphi \underset{(g_1,g_2)}{\rightarrow} \varphi\left[g_1^{-1} X g_2^\Delta\right], \qquad X \in \mathrm{Sp}_n.$$

REMARK II.2.2: We note that the module V_χ sometimes carries a Sp_{2n} unitary structure. In particular, this is the case when $\chi(X)|X|^{-(n+1/2)}$ is a *unitary character* on k^x. (Note $A \rightsquigarrow |\det A|^{2n+1}$ is the module of P_{2n}.) Thus we deduce from Remark II.2.1 that the module $\overline{V}_\chi =$ the Hilbert space completion of V_χ is $\mathrm{Sp}_n \times \mathrm{Sp}_n$ unitarily equivalent to the representation on $\mathrm{L}^2(\mathrm{Sp}_n)$ given by (2-A). Hence by Schur's Lemma, we have that the space of $\mathrm{Sp}_n \times \mathrm{Sp}_n$ smooth vectors $\mathrm{L}^2(\mathrm{Sp}_n)_\infty$ has the *multiplicity one property*:

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}\left(\mathrm{L}^2(\mathrm{Sp}_n)_\infty, \alpha\otimes\beta\right)\right) \cong \begin{cases} 1 \text{ if } \beta = \check{\alpha}^\Delta \\ 0 \text{ otherwise} \end{cases},$$

where $\check{\alpha} =$ the contragredient of α and $\alpha^\Delta =$ the twisted representation of α given by $\alpha^\Delta(g) = \alpha(g^\Delta)$ for $g \in \mathrm{Sp}_n$.

We note that the space $V_\chi \subset \mathrm{L}^2(\mathrm{Sp}_n)_\infty$, and we cannot infer from the above that

$$\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}\left(V_\chi, \alpha\otimes\beta\right)$$

has a similar multiplicity one property!

The main problem that we consider is the determination of $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(\rho_Q, \alpha\otimes\beta)$ or, more generally, $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(V_\chi, \alpha\otimes\beta)$. The main difficulty is first to find $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(V_{\chi,\iota}/V_{\chi,\iota+1}, \alpha\otimes\beta)$ for *all i* and then to patch together this information to find $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(V_\chi, \alpha\otimes\beta)$. We note here (of course) by Schur's Lemma that

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}\left(V_{\chi,n}, \alpha\otimes\beta\right)\right) = \begin{cases} 1 \text{ if } \beta \cong \check{\alpha}^\Delta \\ 0 \text{ otherwise} \end{cases}.$$

REMARK II.2.3: For *supercuspidal* π we have that dim $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(V_\chi, \pi \otimes \check{\pi}^\Delta)$ (dim $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(\rho_Q, \pi\otimes\check{\pi}^\Delta)$) equals one (equals zero or one resp.). Indeed we note that the Jacquet functor $\pi_U = 0$ for *all* unipotent radicals U of a parabolic in Sp_n. Thus $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(V_{\chi,i}/V_{\chi,i+1}, \pi\otimes\check{\pi}^\Delta) = 0$ if $i < n$. Then applying Proposition II.2.1 and the above comments, we deduce that $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(V_\chi, \pi\otimes\check{\pi}^\Delta)$ is exactly a one dimensional space. On the other hand, if there exists a *nonzero* $\mathrm{Sp}_n \times \mathrm{Sp}_n$ intertwining map of ρ_Q to $\pi\otimes\check{\pi}^\Delta$, we deduce that $\pi\otimes\check{\pi}^\Delta$ is, in fact, a summand of ρ_Q; this implies that $\pi\otimes\check{\pi}^\Delta$ is a summand of V_χ (for $\chi(x) = |x|^{m/2}\langle\Delta(Q)|x\rangle$). But then we apply the above statement and deduce the above multiplicity one result.

§3. *Local duality*

The problem of local duality of the Weil representation π_Q of $\mathrm{Sp}_n \times \mathrm{O}(Q)$ on $S[M_{mn}(k)]$ is to find the relationship between α and β when we have the condition that

$$\mathrm{Hom}_{\mathrm{Sp}_n\times \mathrm{O}(Q)}\big(S\big[M_{mn}(k)\big], \alpha\otimes\beta\big) \neq 0.$$

We restate the unitary Howe duality conjecture as follows.

If $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{O}(Q)}(S[M_{mn}(k)], \alpha\otimes\beta_i) \neq 0$ *for* $i = 1$ *and* 2, *then* $\beta_1 \cong \beta_2$ *as* $\mathrm{O}(Q)$ *modules and if* $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{O}(Q)}(S[M_{mn}(k)], \alpha_i\otimes\gamma) \neq 0$ *for* $i = 1$ *and* 2, *then* $\alpha_1 \cong \alpha_2$ *as* Sp_n *modules. Here* α, β_i *and* α_i, γ *are unitary representations.*

We showed in §2 that the validity of this conjecture for all local primes implies the *global Howe duality conjecture.*

We are going to sketch a method of proof for this conjecture. Indeed we shall use the methods of II.1 and II.2 to deduce this conjecture.

The first step is to relate the spaces $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{O}(Q)}(S[M_{mn}(k)], \alpha\otimes\beta)$ (which are nonzero) to the space $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(\rho_Q, \alpha\otimes\alpha)$. Indeed we have the following Proposition.

PROPOSITION II.3.1: *There exists an injection of linear spaces*

$$\bigoplus_\sigma \mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{O}(Q)}\big(S\big[M_{mn}(k)\big], \pi\otimes\sigma\big) \hookrightarrow \mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}\big(\rho_Q, \pi\otimes\pi\big).$$

REMARK II.3.1: We note that if dim $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(\rho_Q, \pi\otimes\pi) \leqslant 1$, then we have that the first part of the *unitary Howe conjecture* is valid for such π. This is the case (using Remark II.2.3) when π is a supercuspidal representation of Sp_n. Moreover we have that such a π is Sp_n equivalent to $\check{\pi}^\Delta$.

PROOF OF PROPOSITION II.3.1:

We define a linear mapping from

$$\bigoplus_{\sigma} \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{mn}(k)], \pi \otimes \sigma) \overset{\Sigma_\pi}{\rightsquigarrow} \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \pi \otimes \pi)$$

as follows.

First we have that $S[M_{mn}(k)] \otimes S[M_{mn}(k)] = S[M_{m2n}(k)]$. But then we consider the $(\mathrm{Sp}_n \times \mathrm{Sp}_n) \times (\mathrm{O}(Q) \times \mathrm{O}(Q))$ intertwinning map

$$S[M_{mn}(k)] \otimes S[M_{mn}(k)] \xrightarrow{T \otimes T} (\pi \otimes \sigma) \otimes (\pi \otimes \sigma)$$

$$= (\pi \otimes \pi) \otimes (\sigma \otimes \sigma),$$

where T is a nonzero element in $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{mn}(k)], \pi \otimes \sigma)$. Then we take the diagonal embedding of $\mathrm{O}(Q) \hookrightarrow \mathrm{O}(Q) \times \mathrm{O}(Q)$ and restrict the representation $\sigma \otimes \sigma$ to this group. But we recall that σ is $\mathrm{O}(Q)$ equivalent to its contragredient $\check{\sigma}$ (see [N]); hence there exists a unique (up to scalars) $\mathrm{O}(Q)$ invariant functional λ_σ on $\sigma \otimes \sigma$ (i.e., a $\mathrm{O}(Q)$ invariant linear map λ_σ of $\sigma \otimes \sigma \to \mathbb{C}$). Then we consider the composite map $\lambda_\sigma \circ (T \otimes T)$; this map is an $(\mathrm{Sp}_n \times \mathrm{Sp}_n) \times \mathrm{O}(Q)$ intertwining map of $S[M_{m2n}(k)] \to (\pi \otimes \pi) \otimes \mathbf{1}\!\mathbf{1}$, $\mathbf{1}\!\mathbf{1}$ = the trivial representation of $\mathrm{O}(Q)$. (We note that the $\mathrm{O}(Q)$ action on $S[M_{m2n}(k)]$ is merely the action induced from left multiplication on $M_{m2n}(k)$.) Then we construct the Jacquet module $S[M_{m2n}(k)]_{\mathrm{O}(Q)}$ and hence $\lambda_\sigma \circ T \otimes T$ induces an intertwining map T^σ of ρ_Q to $\pi \otimes \pi$. (Here we use Theorem II.1.1 relative to the Weil representation of the pair $\mathrm{Sp}_{2n} \times \mathrm{O}(Q)$ on the space $M_{m2n}(k)$.)

Thus we let Σ_π be the map $\Sigma_\sigma T^\sigma$ as σ varies over all irreducible, admissible representations of $\mathrm{O}(Q)$ for which $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{m2n}(k)], \pi \otimes \sigma) \neq 0$. Thus we must show that Σ_π is a linear injection!

We consider the adjoint of $\Sigma_\sigma T^\sigma : (\pi \otimes \pi)^* \to S'[M_{m2n}(k)]$ as factoring through

$$\begin{tikzcd}
(\pi \otimes \pi)^* \arrow[r, "\bigoplus_\sigma I \otimes \lambda_\sigma^*"] \arrow[dr] & \bigoplus_\sigma (\pi \otimes \pi)^* \otimes (\sigma \otimes \sigma)^* \arrow[d, "\bigoplus_\sigma T^* \otimes T^*"] \\
 & S'[M_{m2n}(k)]
\end{tikzcd}$$

We note here that π^*, σ^*, and $S[M_{m2n}(k)]^*$ denote the spaces of *all* linear functionals on π, σ, and $S[M_{m2n}(k)]$, respectively, whereas $\check{\pi}$ and $\check{\sigma}$ will be the spaces of smooth functionals in π^* and σ^* (i.e., those functionals invariant under a suitable compact open subgroup).

First it is clear that $\oplus_\sigma I \otimes \lambda_\sigma^*$ is injective. On the other hand, we suppose that $\Sigma(T_i^* \otimes T_i^*) \equiv 0$, where $T_i \in \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{mn}(k)], \pi \otimes$

σ). This implies there exists $w_1 \in \check{\pi} \otimes \check{\pi}$ such that

$$(T_1^* \otimes T_1^*)(w_1 \otimes \lambda_\sigma^*) = \sum_{i \geqslant 2} T_i^* \otimes T_i^*(w_1 \otimes \lambda_\sigma^*)$$

and the term on the left is nonzero in $S[M_{m2n}(k)]^*$. But then it is possible to find a compact open subgroup $\tilde{K}$ in $\mathrm{O}(Q) \times \mathrm{O}(Q)$ such that

$$\int_{\tilde{K}} (T_1^* \otimes T_1^*)(w_1 \otimes k \cdot \lambda_\sigma^*) \mathrm{d}k \neq 0.$$

However it is easy to see that the projection operator $\int_{\tilde{K}} \mathrm{d}k$ commutes with $T_1^* \otimes T_1^*$, and we have

$$\int_{\tilde{K}} (T_1^* \otimes T_1^*)(w_1 \otimes k \cdot \lambda_\sigma^*) \mathrm{d}k = (T_1^* \otimes T_1^*)\left(w_1 \otimes \int_{\tilde{K}} k \cdot \lambda_\sigma^* \mathrm{d}k \right).$$

But $\int_{\tilde{K}} k \cdot \lambda_\sigma^* \mathrm{d}k$ is the Trace form on the finite dimensional space $(\sigma \otimes \sigma)^{\tilde{K}}$. (Note $\sigma \otimes \sigma$ is admissible!) Thus we have that $\check{T}_1 \otimes \check{T}_1((\check{\pi} \otimes \check{\pi}) \otimes (\check{\sigma} \otimes \check{\sigma}))$ lies in $\sum_{i \geqslant 2} \check{T}_i \otimes \check{T}_i(\check{\pi} \otimes \check{\pi} \otimes \check{\sigma} \otimes \check{\sigma})$, where $\check{T}_i =$ the transpose of T_i on the smooth contragredient $\check{\pi} \otimes \check{\sigma}$. But the representations $\check{\pi} \otimes \check{\pi} \otimes \check{\sigma} \otimes \check{\sigma}$ are *inequivalent* (as representations of $\mathrm{Sp}_n \times \mathrm{Sp}_n \times \mathrm{O}(Q) \times \mathrm{O}(Q)$). Hence $T_1^* \otimes T_1^*$ must be the zero map, which is impossible! Thus we have that Σ_π is *injective*. Q.E.D.

Thus the strategy of proving the *unitary Howe conjecture* is to study the space $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \pi \otimes \pi)$ for all π. We must determine the behavior of the possible intertwining operators arising from the "boundary components" $V_{\chi,i}/V_{\lambda,i+1}$ $(i < n)$ to $\pi \otimes \pi$. The idea is to determine the π which satisfy $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(V_{\chi,i}/V_{\chi,i+1}, \pi \otimes \pi) \neq 0$ and then to determine the matrix coefficient behavior of π.

The technical problem that must be settled first is to see how ρ_Q embeds in V_χ and relate the "boundary components" $V_{\chi,i}/V_{\chi,i+1}$ to ρ_Q.

This point can be solved, in part, from the recent work of [Gu]. Indeed we restrict to the family of representations of the form V_χ where $\chi(x) = |x|^{\ell/2}$ or $|x|^{\ell/2}\langle x|\varepsilon\rangle$, ℓ an integer and ε a *unit* in $k^\times/(k^\times)^2$ which satisfies $\langle x|\varepsilon\rangle = (-1)^{\mathrm{ord}(x)}$. (Thus we have that $\langle x|\varepsilon\rangle = |x|^{\pi\sqrt{-1}/(\log q)}$.) We note that χ in these instances is an *unramified* character in $k^\times$, and the associated series V_χ is also *unramified*. In particular, this restricts the type of Q we can consider:

(1) If $Q =$ *direct sum of hyperbolic planes* or a *direct sum of hyperbolic planes and the unique anisotropic form of degree* 4, then $\Delta_Q \in (k^\times)^2$ and $\rho_Q \in V_\chi$ with $\chi(x) = |x|^{m/2}$ ($m =$ dimension(Q)). (We call the former type a "split" form and the latter type a "quaternionic" form.)

(2) If $Q =$ direct sum of hyperbolic planes and a multiple of the norm form of a quadratic extension of k which, via class field theory, corresponds to the element ε above, then $\Delta_Q \in \varepsilon(k^\times)^2$ and $\rho_Q \subseteq V_\chi$, with $\chi(x) = |x|^{m/2}\langle x|\varepsilon\rangle$. We call these forms Q the *unramified forms.*

Then we recall the following facts about the $\mathrm{Sp}_{2n}(k)$ module structure of the V_χ.

(1) There exists an $\mathrm{Sp}_{2n}(k)$ invariant bilinear form on the space $V_{\chi_1} \times V_{\chi_2}$ where $(\chi_1 \cdot \chi_2)(x) = |x|^{2n+1}$. In explicit terms, the pairing between V_{χ_1} and V_{χ_2} is given as follows:

$$(f_1|f_2) = \int_{\mathrm{Sym}_{2n}(k)} f_1\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\right)$$

$$f_2\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\right)\mathrm{d}X,$$

where $f_1 \in V_{\chi_1}$ and $f_2 \in V_{\chi_2}$.

(2) Let $\chi(x) = |x|^{\ell/2}$ with ℓ a real number. Then V_χ is irreducible if and only if $\ell \notin [0,2,4,\ldots,2(2n+1)]$. If $\ell = 0$, 2, $4n$, or $2(2n+1)$, then V_χ has 2 irreducible components; one component is the unique irreducible subrepresentation and the other is the unique irreducible quotient of V_χ. If $\ell = 4,\ldots,2(2n-1)$, then V_χ has 3 irreducible components. If $\ell > 2n+1$, then V_χ has a unique irreducible subrepresentation and the quotient of V_χ by this subrepresentation splits into a direct sum of 2 irreducible components. If $\ell < 2n+1$, then V_χ has 2 distinct irreducible subrepresentations and the quotient of V_χ by the sum is the unique irreducible quotient of V_χ.

(3) Let $\chi(x) = |x|^{\ell/2 + \pi\sqrt{-1}/\log(q)}$ (ℓ real). Then V_χ is irreducible if and only if $\ell \notin [2,4,\ldots,2(2n)]$. If $\ell \in [2,\ldots,2(2n)]$, then V_χ has 3 irreducible components with the same conditions holding as in (2)!

(4) Let $\chi(x) = |x|^{\ell/2}$ and $\chi^*(x) = |x|^{-\ell/2+(2n+1)}$. Then $\mathrm{Hom}_{\mathrm{Sp}_{2n}}(V_\chi, V_{\chi^*}) \neq 0$. If either $\ell < 2n+1$ or $\ell \notin [0,\ldots,2(2n-1)]$, then dim $\mathrm{Hom}_{\mathrm{Sp}_{2n}}(V_\chi, V_{\chi^*}) = 1$. On the other hand, if $\ell > 2n+1$ and $\ell \in [0,\ldots,2(2n-1)]$, then dim $\mathrm{Hom}_{\mathrm{Sp}_{2n}}(V_\chi, V_{\chi^*}) = 2$. (We note that the same comments hold if χ and χ^* are both multiplied by $|x|^{\pi\sqrt{-1}/\log(q)}$ with ℓ restricted to the range in (3) above).

With these structure Theorems concerning V_χ, we can deduce certain information about the embedding of ρ_Q in V_χ.

We deduce immediately the following Proposition.

Proposition II.3.2: *Q is an unramified form of type* (1) *above.*

(I) *If $\ell > 2(2n+1)$, then $\rho_Q = V_\chi$ is irreducible and $\rho_Q \cong V_{\chi^*}$.*

(II) (i) *If* $\ell = 2(2n+1)$, *then* $\rho_{Q_1} = V_\chi(Q_1 =$ *unique split form of dimension* $2(2n+1))$ *and* ρ_{Q_2} *is the unique irreducible submodule of* V_χ $(Q_2 =$ *"quaternionic" form of dimension* $2(2n+1))$.

(ii) *If* $\ell = 0$, *then the trivial representation of* Sp_{2n} *is the unique subrepresentation of* V_χ.

(iii) *If* $\chi(x) = |x|^{2n+1}$ *and* $\chi^*(x) =$ *trivial, then* $V_\chi/\rho_{Q_2} \cong$ *trivial representation and* $\rho_{Q_2} \cong V_{\chi^*}/($*trivial rep.*$)$.

(III) (i) *If* $\ell = 2(2n)$, *then* $V_\chi = \rho_{Q_1}$ $(Q_1 =$ *unique split form of dimension* $2(2n))$ *and* ρ_{Q_2} *is the unique irreducible subrepresentation of* V_χ $(Q_2 =$ *"quaternionic" form*$)$.

(ii) *If* $\ell = 2$, *then* $\rho_{``Q_1"}$ *is the unique irreducible subrepresentation of* $V_\chi$ $(``Q_1" =$ *"split" form*$)$.

(iii) *If* $\chi(x) = |x|^{2n}$ *and* $\chi^*(x) = |x|$, *then* $V_\chi/\rho_{Q_2} \cong \rho_{``Q_1"}$ *and* $V_{\chi^*}/\rho_{``Q_1"} \cong \rho_{Q_2}$.

(IV) *Let* $2n+1 < \ell < 2(2n)$ *and* $\chi(x) = |x|^{\ell/2}$, $\chi^*(x) = |x|^{-\ell/2+2n+1}$. *Let* Q_1 *and* Q_2 $(``Q_1"$ *and* $``Q_2")$ *be the split and quaterionic forms of dimension* $\ell(2(2n+1) - \ell$ *resp.*$)$. *Then*

(i) $\rho_{``Q_1"}$ *and* $\rho_{``Q_2"}$ *are irreducible submodules of* V_{χ^*}. *Moreover* $\rho_{``Q_1"} \oplus \rho_{``Q_2"}$ *is a maximal submodule of* V_{χ^*}. *There is a nonzero intertwining map from* V_χ *onto* $\rho_{``Q_1"}$ $(\rho_{``Q_2"}$ *resp.*$)$.

(ii) ρ_{Q_1} *and* ρ_{Q_2} *are maximal submodules of* V_χ. *Moreover* ρ_{Q_1} *has the same composition factors as either* $V_{\chi^*}/\rho_{``Q_1"}$ *or* $V_{\chi^*}/\rho_{``Q_2"}$ $(\rho_{Q_2}$ *has a similar property*$)$.

PROOF: (I) is evident from fact (2) stated above.

For (II) we consider the inner product (from (1) above) of $\pi_Q(G)(\varphi)[0]$ with the constant function 1 in V_{χ^*}. In particular we have

$$\pi_Q(G)(\varphi)[0]|\, 1\rangle = \int_{\mathrm{Sym}_{2n}(k)} \pi_Q\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}\right)(\varphi)[0]\mathrm{d}T.$$

But we know from Remark II.1.1 that the last integral equals

$$M_\varphi[0].$$

However we also know from Remark II.1.1 that $M_\varphi[0]$ is a *nonzero linear form* if Q is split and is zero in the case that Q is quaternionic. Hence

$$\langle f|1\rangle \equiv 0$$

for all $f \in \rho_Q$ (Q quaterionic) and $\langle f|1\rangle \neq 0$ for *some* $f \in \rho_Q$ (Q split). Hence $\rho_{Q_1} = V_\chi$ (Q_1 split) and ρ_{Q_2} is the unique irreducible submodule of V_χ (Q_2 quaterionic). Thus (II) (i) follows!

We note that (II) (ii) and (iii) follow from facts (2) and (4) given above.

Let $\rho_Q \subset V_\chi$ and $\rho_{``Q"} \subset V_{\chi^*}$, where $\chi$ and $\chi^*$ are related by $\chi \cdot \chi^*(x) = |x|^{2n+1}$. Then we restrict the $\mathrm{Sp}_{2n}$ invariant bilinear form on $V_\chi \otimes V_{\chi^*}$ to $\rho_Q \otimes \rho_{``Q"}$. In particular, we have that

$$\langle \pi_Q(G)(\varphi_1)[0] | \pi_{``Q"}(G)(\varphi_2)[0] \rangle$$

$$= \int_{\mathrm{Sym}_{2n}(k)} \pi_{Q \oplus ``Q"}\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}\right)(\varphi_1 \otimes \varphi_2)[0] \mathrm{d}T$$

$$= M_{\varphi_1 \otimes \varphi_2}[0]$$

for $\varphi_1 \in S[M_{m\ell}(k)]$, $\varphi_2 \in S[M_{m,2(2n+1)-\ell}(k)]$, and $\varphi_1 \otimes \varphi_2[X|Y] = \varphi_1[X] \cdot \varphi_2[Y]$ with $X \in M_{m\ell}(k)$ and $Y \in M_{m,2(2n+1)-\ell}(k)$.

Using Weil's criterion again, we deduce that ρ_Q is perpendicular to $\rho_{``Q"}$ if and only if $Q \oplus ``Q"$ is not a split form.

Thus $(\rho_{Q_1})^\perp \supseteq \rho_{``Q_2"}$, $(\rho_{Q_2})^\perp \supseteq \rho_{``Q_1"}$ $((\rho_{``Q_1"})^\perp \supseteq \rho_{Q_2}$, $(\rho_{``Q_2"})^\perp \supseteq \rho_{Q_1}$ resp.). Here Q_1 and Q_2 ("Q_1" and "Q_2" resp.) are the split and quaterionic forms of dimension $\ell(2(2n+1) - \ell$ resp.). (here $\ell < 2n+1$).

Thus (III) (i), (ii), and (iii) follow (using fact(3) above).

Now we let $\chi(x) = |x|^{\ell/2}$ and $\chi^*(x) = |x|^{-\ell/2+2n+1}$ with $l > 2n+1$.

We first observe that $\rho_{``Q_1"} \neq \rho_{``Q_2"}$. For otherwise by the above comments $\rho^\perp_{``Q_1"} = \rho^\perp_{``Q_2"}$ which, in turn, implies that $\rho_{Q_1} \subseteq \rho^\perp_{``Q_1"}$ (which, from the comments above, is clearly not the case).

Then we consider the moment mappings given by "Q_1" and "Q_2". In particular, it is easy to see that (the closure of range "Q_1") $\cup$ (the closure of range "Q_2") $\neq \mathrm{Sym}_{2n}(k)$. Thus there exists an open $\mathrm{GL}_{2n}(k)$ orbit $\mathcal{O}$ in $\mathrm{Sym}_{2n}(k)$ – [(closure of range "Q_1") $\cup$ (closure of range "Q_2")]. Let $\varphi \in S[\mathrm{Sym}_{2n}(k)]$ such that $\hat{\varphi}$ has support in $S[\mathcal{O}]$. Then it is possible to find a function f_φ in V_χ such that $f_\varphi(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}) = \varphi(T)$ for all $T \in \mathrm{Sym}_{2n}(k)$.

Then we consider the inner product of f_φ with an arbitrary function in $\rho_{``Q_i"}$ $(i = 1, 2)$

$$\langle f_\varphi | \pi_{``Q_i"}(G)(\psi)[0] \rangle = \int_{\mathrm{Sym}_{2n}(k)} \varphi(X) \pi_{Q_i}\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right)$$

$$\times (\psi)[0] \mathrm{d}X = \int_{\mathrm{Sym}_{2n}(k)} \varphi(X)$$

$$\times \left(\int_{M_{m,\ell}(k)} \psi(T) \tau(\mathrm{Tr}(XT'QT)) \mathrm{d}T \right) \mathrm{d}X$$

$$= \int_{M_{m,\ell}(k)} \psi(T) \hat{\varphi}(T'QT) \mathrm{d}T.$$

Both φ and ψ have compact support; hence it is possible to switch orders of integration above. But we have that support($\hat{\varphi}$) does not intersect the closure of range "Q". Thus the last integral above vanishes for all $\psi \in S[M_{m\ell}(k)]$. Thus f_φ is perpendicular to $\rho_{``Q_1''} + \rho_{``Q_2''}$. Hence we have $\rho_{``Q_1''} + \rho_{``Q_2''} \neq V_{\chi^*}$; then, applying fact (2) above and the previous comments, we see that $\rho_{``Q_1''} + \rho_{``Q_2''}$ is the maximal submodule of V_{χ^*}. But then we also have that $\rho_{``Q_2''} \subseteq (\rho_{Q_1})^\perp \subseteq \rho_{``Q_1''} + \rho_{``Q_2''}$ and $\rho_{``Q_1''} \subseteq (\rho_{Q_2})^\perp \subseteq \rho_{``Q_1''} + \rho_{``Q_2''}$. This implies that $\rho_{``Q_1''} = (\rho_{Q_2})^\perp$, $\rho_{``Q_2''} = (\rho_{Q_1})^\perp$ and $\rho_{``Q_1''} \cap \rho_{``Q_2''} = \{0\}$.

Thus we have that $\rho_{``Q_1''}$, $\rho_{``Q_2''}$ are irreducible components of V_{χ^*} and disjoint. If now $\rho_{``Q_1''}$ and $\rho_{``Q_2''}$ are Sp_{2n} *equivalent*, it follows that any nonzero intertwining operator from V_χ to V_{χ^*} must carry V_χ either *onto* $\rho_{``Q_1''} \oplus \rho_{``Q_2''}$ or to an irreducible component of $\rho_{``Q_1''} \oplus \rho_{``Q_2''}$; but such a component is Sp_n equivalent to $\rho_{``Q_1''} \cong \rho_{``Q_2''}$. Hence if $\rho_{``Q_1''} \cong \rho_{``Q_2''}$, we have that there exists a nonzero intertwining operator from V_χ to $\rho_{``Q_1''}$ and $\rho_{``Q_2''}$ respectively! On the other hand, let $\rho_{``Q_1''}$ be not equivalent to $\rho_{``Q_2''}$. Then we know that the embedding of $\rho_{``Q_1''} \oplus \rho_{``Q_2''}$ into V_{χ^*} determines a surjection of V_χ to $\rho^\vee_{``Q_1''} \oplus \rho^\vee_{``Q_2''}$. But using fact (2), we have that V_χ/(unique irreducible representation in V_χ) maps surjectively to $\rho^\vee_{``Q_1''} \oplus \rho^\vee_{``Q_2''}$. But the latter quotient is also a direct sum $W_1 \oplus W_2$ of irreducibles W_i. Hence we have that $W_1^\vee \cong \rho_{``Q_1''}$ and $W_2^\vee = \rho_{``Q_2''}$. Then using fact (3), we know that there exist 2 linearly independent intertwining operators from $W_1 \oplus W_2$ to $W_1^\vee \oplus W_2^\vee$ (i.e. every such intertwining operator has a nonzero kernel in V_χ and hence its kernel contains the unique irreducible subrepresentation of V_χ!).

Thus, in any case, there exists a nonzero intertwining operator from V_χ to $\rho_{``Q_1''}$ and $\rho_{``Q_2''}$.

Then we also deduce that V_χ/(unique irreducible) $\cong \rho_{``Q_1''} \oplus \rho_{``Q_2''}$. Moreover ρ_{Q_1} and ρ_{Q_2} are clearly *maximal* submodules (since $\rho_{``Q_1''}$, $\rho_{``Q_2''}$ are irreducible). Thus ρ_{Q_1} has in its composition series terms $\rho_{``Q_1''}$ and the "unique irreducible" or $\rho_{``Q_2''}$, and the "unique irreducible". A similar statement is valid for ρ_{Q_2}! Then we recall from fact (4) that there exists a nonzero intertwining operator from V_{χ^*} into V_χ. In particular, by examining the various possibilities for the image of this operator, we deduce easily IV (ii)! Here we need the fact that there is only one composition factor in V_χ (V_{χ^*} resp.) with a nonzero fixed vector under the maximal compact subgroup of Sp_n. Q.E.D.

If Q is an unramified form (with a two-dimensional anisotropic component), then using the same reasoning as in Proposition II.3.2, we can prove the following.

PROPOSITION II.3.2′: *Let Q be an unramified form of type* (2) *above.*

(I) *If $\ell \geqslant 2(2n+1)$, then $\rho_Q = V_\chi$ is irreducible and $\rho_Q \cong V_{\chi^*}$.*

(II) *Let* $2n+1 < \ell \leqslant 2(2n)$ *and* $\chi(x) = |x|^{\ell/2}\langle x|\varepsilon\rangle$, $\chi^*(x) = |x|^{-\ell/2+2n+1}\langle x|\varepsilon\rangle$. *Let* Q_1 *and* Q_2 *("*Q_1*" and "*Q_2*") be the unramified forms of dimension* ℓ *(dimension* $2(2n+1)-\ell$*, resp.) with anisotropic factors the norm form* $N_{E/k}$ *and* $\lambda N_{E/k}$*, respectively (with* $\langle\lambda|\varepsilon\rangle = -1$*). Then*

(i) $\rho_{``Q_1"}$ *and* $\rho_{``Q_2"}$ *are irreducible submodules of* V_{χ^*}*. Moreover* $\rho_{``Q_1"} \oplus \rho_{``Q_2"}$ *is a maximal submodule of* V_{χ^*}*. There is a nonzero intertwining map of* V_χ *onto* $\rho_{``Q_1"}$ *(*$\rho_{``Q_2"}$ *resp.).*

(ii) ρ_{Q_1} *and* ρ_{Q_2} *are maximal submodules of* V_χ*. Moreover* ρ_{Q_1} *has the same composition factors as either* $V_{\chi^*}/\rho_{``Q_1"}$ *or* $V_{\chi^*}/\rho_{``Q_2"}$ *(*ρ_{Q_2} *has a similar property).*

Proof: We recall that there exist 2 inequivalent *unramified* forms of a fixed dimension where the two-dimensional anisotropic factor is a multiple of the norm form $N_{E/k}$. Fixing these forms as Q_1 and Q_2, each of dimension ℓ, ("Q_1" and "Q_2", each of dimension $2(2n+1)-\ell$, resp.), then we have, following the same idea as in the proof of Proposition II.3.2, that ρ_{Q_1} is perpendicular to $\rho_{``Q_2"}$ and $\rho_{Q_2}$ is perpendicular to $\rho_{``Q_1"}$ relative to the pairing between V_χ and V_{χ^*}. Then we follow the same argument as in the proof of Proposition II.3.2 to complete the demonstration Q.E.D.

Remark II.3.2: More generally, if we assume that Q is a quadratic form of the type $H_r \oplus$ L (with L, two-dimensional anisotropic), then we can, in fact, show that if $\dim(Q) \geqslant 4n+2$, then $\rho_Q = V_\chi$. Indeed we note that if $\rho_Q \neq V_\chi$, then $(\rho_Q)^\perp \neq \{0\}$ in V_{χ^*}. In particular, this implies that there exists a function λ in V_{χ^*} such that

$$\int_{\mathrm{Sym}_n(k)} \lambda\left(w\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix}\right)\theta_\varphi(T)\,\mathrm{d}T \equiv 0$$

for all $\varphi \in S[M_{mn}(k)]$ (here $\theta_\varphi(T) = \pi_Q(w\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix})(\varphi)(0))$. Then we consider the splitting $S[M_{mn}(k)] = S[M_{2r,n}(k)] \otimes S[M_{2,n}(k)]$ and recall that π_Q has a tensor splitting $\pi_{H_r} \otimes \pi_L$, i.e., $\pi_Q(G)(\varphi_1 \otimes \varphi_2)\left[\dfrac{X}{Y}\right] = \pi_{H_r}(G)\varphi_1(X)\cdot\pi_L(G)\varphi_2(Y)$ for $\varphi_1 \in S[M_{2r,n}(k)]$, $\varphi_2 \in S[M_{2,n}(k)]$, $X \in M_{2r,n}(k)$, and $Y \in M_{2,n}(k)$. Thus we have that

$$\theta_{\varphi_1 \otimes \varphi_2}(T) = \theta_{\varphi_1}(T)\theta_{\varphi_2}(T).$$

But this implies that

$$\int_{\mathrm{Sym}_n(k)} \theta_{\varphi_1}(T)\left\{\theta_{\varphi_2}(T)\lambda\left(w\begin{pmatrix}I & T\\ 0 & I\end{pmatrix}\right)\right\}\mathrm{d}T \equiv 0$$

for *all* $\varphi_1 \in S[M_{2r,n}(k)]$ and $\varphi_2 \in S[M_{2,n}(k)]$. But it is straightforward to show that we can find φ_2 such that $\theta_{\varphi_2} \cdot \lambda \not\equiv 0$. Hence $(\rho_{H_r})^{\perp} \neq \{0\}$, which is a contradiction.

§4. *Local duality and multiplicity one*

We introduce here the notion of a representation $\omega \otimes \omega'$ of $\mathrm{Sp}_n \times \mathrm{Sp}_n$ occurring as a boundary component of ρ_Q. The starting point is Propositions II.3.2 and II.3.2′ and Remark II.3.2. Indeed we consider the following cases.

(i) Q is an *arbitrary form* with $\dim Q \geqslant 4n+2$. Then we say $\omega \otimes \omega'$ is a boundary component if

$$\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}\left(V_{\chi,i}/V_{\chi,i+1},\ \omega \otimes \omega'\right) \neq 0$$

for some $i < n$ (where $\chi(x) = |x|^{m/2}\langle \Delta(Q)|x\rangle$).

(ii) Q is an *unramified form* with $\dim Q \leqslant 4n$. Then we say $\omega \otimes \omega'$ is a boundary component if

(a) $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(V_{\chi^*,i}/V_{\chi^*,i+1},\ \omega \otimes \omega') \neq 0$ for some $i < n$ (where $\chi^*(x) = |x|^{-(m/2)+2n+1}\langle \Delta(Q)|x\rangle$) when $\dim Q < 2n+1$;

(b) either $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(V_{\chi,i}/V_{\chi,i+1},\ \omega \otimes \omega') \neq 0$ or $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(V_{\chi^*,i'}/V_{\chi^*,i'+1},\ \omega \otimes \omega') \neq 0$ for some i or i' both less than n and $2n+1 < \dim Q \leqslant 4n$. (Here $\chi(x) = |x|^{m/2}\langle \Delta(Q)|x\rangle$ and $\chi^*(x) = |x|^{-(m/2)+2n+1}\langle \Delta(Q)|x\rangle$.) Moreover if Q is split and $\dim Q = 4n$, then we require only the first condition in (b) (see Proposition II.3.2).

REMARK II.4.1: We note that if $\omega_1 \otimes \omega_2$ does not occur as a boundary component of ρ_Q, then $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \omega_1 \otimes \omega_2)$ is at most a two-dimensional space. Indeed, in case (i) and in case (ii, a) above, we deduce by using Proposition II.3.1 that $\dim \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \omega_1 \otimes \omega_2) \leqslant \dim \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(V_{\chi,n}, \omega_1 \otimes \omega_2)$ in case (i) or $\dim \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \omega_1 \otimes \omega_2) \leqslant \dim \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(V_{\chi^*,n}, \omega_1 \otimes \omega_2)$ in case (ii, a). But the latter spaces (in both cases) are at most one dimensional. In case (ii, b), ρ_Q fits as the center term of an exact sequence of Sp_{2n} modules:

$$0 \to E \to \rho_Q \to F \to 0,$$

where E is contained in the image of an Sp_{2n} intertwining map from V_{χ^*} and F is contained in the image of an Sp_{2n} intertwining map from V_χ. If $T \in \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \omega_1 \otimes \omega_2)$ and T does not give rise to a boundary component for $\omega_1 \otimes \omega_2$, we have that T induces an $\mathrm{Sp}_n \times \mathrm{Sp}_n$ intertwining map from $V_{\chi,n}$ or $V_{\chi^*,n}$ onto $\omega_1 \otimes \omega_2$. Thus in case (ii, b),

$$\dim \mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}\left(\rho_Q, \omega_1 \otimes \omega_2\right) \leqslant 2.$$

Again we note that if Q is split and $\dim Q = 4n$, then we have $\dim \operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \omega_1 \otimes \omega_2) \leqslant 1$.

Then we state the main technical Theorem which is the key step in the local Howe duality conjecture.

THEOREM II.4.1: *Let ω_1 and ω_2 be irreducible representations of* Sp_n. *Let*

$$\operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}\big(\rho_Q, \omega_1 \otimes \omega_2\big) \neq 0$$

and suppose $\omega_1 \otimes \omega_2$ is a boundary component of ρ_Q.

(1) *If Q is any form and* $\dim Q > 4n + 2$, *then ω_1 (ω_2 resp.) is nonunitarizable. If* $\dim Q = 4n + 2$ *and Q is not "quaternionic", then ω_1 (ω_2 resp.) is nonunitarizable.*

(2) *If Q is a split form and* $\dim Q = 2$ *or* $4n$, *then ω_1 (ω_2 resp.) is either nonunitarizable or the trivial representation.*

(3) *Let $n \geqslant 2$. If Q is an unramified form and* $\dim Q < n + 2$, *then if* $\operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{mn}(k)], \omega_i \otimes \sigma_i) \neq 0$, *either ω_i or σ_i is a nonunitarizable representation.*

(4) *Let $n \geqslant 2$. If Q is an unramified form and* $\dim Q = n + 2$, *then if* $\operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{mn}(k)], \omega_i \otimes \sigma_i) \neq 0$, *either ω_i is nonunitarizable or σ_i is nonunitarizable or trivial.*

COROLLARY TO THEOREM II.4.1:

The Howe duality conjecture (stated in §3) *and the statement that*

$$\dim\Big(\operatorname{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}\big(S\big[M_{mn}(k)\big], \pi \otimes \omega\big)\Big) \leqslant 1$$

for all π and ω unitary irreducibles of Sp_n *and* $\mathrm{O}(Q)$, *respectively, hold in the following cases:*

(1) *Q any form with* $\dim Q > 4n + 2$ *or with* $\dim Q = 4n + 2$ *provided Q is not quaternionic,*

(2) *Q a split form with* $\dim Q = 4n$ *or* 2,

(3) *if $n \geqslant 2$ and Q is an unramified form with* $\dim Q \leqslant n + 2$.

Before starting the proof of Theorem II.4.1, we show how to prove the above Corollary.

First by the above Theorem and Proposition II.3.1, we see that the first part of the Howe duality conjecture and local multiplicity one are valid in the cases

(i) Q any form with $\dim Q \geqslant 4n + 2$ (provided that Q is not "quaternionic" with $\dim Q = 4n + 2$) and

(ii) if $n \geqslant 2$ and Q is an unramified form with $\dim Q < n + 2$.

The remaining cases (2) and (3) in the Corollary above will be considered separately below.

Now we consider (i) and (ii) above and suppose that $\omega_i \otimes \sigma$ ($i = 1, 2$)

occurs in $S[M_{mn}(k)]$ (ω_i and σ are unitary). Then using the same arguments as in Proposition II.3.1, there exist nonzero $\mathrm{Sp}_n \times \mathrm{Sp}_n$ intertwining operators from ρ_Q to $\omega_1 \otimes \omega_1$, $\omega_1 \otimes \omega_2$ and $\omega_2 \otimes \omega_2$. Now if ω_i are inequivalent, then either $\omega_1 \otimes \omega_1$ or $\omega_2 \otimes \omega_2$ (but not both) may occur as a boundary component of ρ_Q. In any case, we have that $\omega_1 \otimes \omega_2$ and, say, $\omega_1 \otimes \omega_1$ are not boundary components; hence there exist nonzero $\mathrm{Sp}_n \times \mathrm{Sp}_n$ intertwining operators from $V_{\chi^*,n}$ into $\omega_1 \otimes \omega_2$ and $\omega_1 \otimes \omega_1$ respectively. This implies by Schur's Lemma that $\omega_1 \cong \omega_2$ (Sp_n equivalent).

We now digress and handle the separate cases mentioned above.

(A) The first extraordinary case for the Howe duality conjecture is when $m = 4n$ or 2 and Q is a split form. We note that if π is unitary and not the trivial representation, then $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{Sp}_n}(\rho_Q, \pi \otimes \pi)$ is at most one-dimensional. On the other hand, if $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(\pi_Q, \mathbf{1} \otimes \sigma) \neq 0$, then σ is a uniquely determined representation of $\mathrm{O}(Q)$ with a nonzero fixed vector under the standard maximal compact subgroup of $\mathrm{O}(Q)$ by Theorem 7.1 of [H]. Moreover we note from Theorem II.1.1 that σ can be represented as a quotient of $W_{Q'}$ (here Q' a split form of dimension $2n$ so that $Q = Q' \oplus (-Q')$). But $W_{Q'}$ admits *at most* one vector invariant under the standard maximal compact subgroup of $\mathrm{O}(Q)$; hence $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(\pi_Q, \mathbf{1} \otimes \sigma)$ is at most one-dimensional! On the other hand, using the same arguments as above, we deduce that if

$$\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}\left(\pi_Q, \begin{Bmatrix} \pi_1 \\ \pi_2 \end{Bmatrix} \otimes \omega\right) \neq 0,$$

then $\pi_1 \cong \pi_2$ (π_1, π_2, and ω are unitary).

(B) We assume that Q is unramified and that $\dim Q = n + 2$. Then from Theorem II.4.1, if π gives rise to a unitary boundary component of ρ_Q, we see that the only unitary representation $\sigma(\pi)$ of $\mathrm{O}(Q)$ such that $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{mn}(k)], \pi \otimes \sigma(\pi)) \neq 0$ must be the trivial representation. Moreover we know that $\mathrm{Hom}_{\mathrm{Sp}_n \times \mathrm{O}(Q)}(S[M_{n+2,n}(k)], \pi \otimes \mathbf{1}) \cong \mathrm{Hom}_{\mathrm{Sp}_n}(S[M_{n+2,n}(k)]_{\mathrm{O}(Q)}, \pi)$. But from Proposition II.3.2, we know that $S[M_{n+2,n}(k)]_{\mathrm{O}(Q)}$ is an irreducible Sp_n module (here n is even). Thus, such a π is uniquely determined. Hence we deduce easily that local Howe duality and multiplicity one hold in this case.

Thus, having concluded the proof of the Corollary to Theorem II.4.1, we start the proof of Theorem II.4.1.

Proof of Theorem II.4.1:

The first point is to analyze the possible irreducible components of $\mathrm{Sp}_n \times \mathrm{Sp}_n$ that occur in $V_{\chi,i}/V_{\chi,i+1}$ (for general χ). The first obvious consequence is the following.

LEMMA II.4.1: *Assume that* $\mathrm{Hom}_{\mathrm{Sp}_n\times\mathrm{Sp}_n}(V_{\chi,\iota}/V_{\chi,\iota+1},\ \omega_1\otimes\omega_2)\neq 0$. *Then there exists an irreducible representation* $\beta = A\otimes B$ *of* $\mathrm{Sp}_\iota\times\mathrm{G}\ell_{n-\iota}$ *with the central character of* B *given by*

$$\lambda\cdot I_{n-\iota}\rightsquigarrow\chi^{-1}(\lambda^{n-\iota})|\lambda|^{(n-\iota)(n+\iota+1)}$$

such that ω_1 (ω_2 *resp.*) *occurs as a subrepresentation of*

$$\mathrm{ind}(\mathrm{Sp}_\iota\times\mathrm{G}\ell_{n-\iota}\times U_\iota^n\nearrow\mathrm{Sp}_n|A\otimes B).$$

PROOF: From Proposition II.2.1, we have that there exists a nonzero $\mathrm{Sp}_n\times\mathrm{Sp}_n$ intertwining map:

$$\check{\omega}_1\otimes\check{\omega}_2\mapsto\mathrm{Ind}\Big((\mathrm{Sp}_n\times\mathrm{Sp}_n)^{P_{2n}\cdot\tilde{V}_\iota}$$

$$\nearrow\mathrm{Sp}_n\times\mathrm{Sp}_n|(A,X)\cdot(A^{\Delta},Y)\cdot(g_1,g_2)$$

$$\rightsquigarrow\chi^{-1}(\det X\det Y)|\det X\det Y|^{(n+\iota+1)}\Big).$$

Then applying the Jacquet functor relative to $U_\iota^n\times U_\iota^n$, we get a nonzero $(\mathrm{Sp}_\iota\times\mathrm{G}\ell_{n-\iota})\times(\mathrm{Sp}_\iota\times\mathrm{G}\ell_{n-\iota})$ intertwining map of

$$(\check{\omega}_1)_{U_\iota^n}\otimes(\check{\omega}_2)_{U_\iota^n}\mapsto S_{\iota w}[\mathrm{Sp}_\iota]\otimes\{(A,B)$$

$$\rightsquigarrow\chi^{-1}(\det A\det B)|\det A\det B|^{(n+\iota+1)}\}.$$

Here $S_{\iota w}[\mathrm{Sp}_\iota]$ is the twisted $\mathrm{Sp}_\iota\times\mathrm{Sp}_\iota$ representation given in Remark II.2.1 (see (2-A)). Thus it follows that there exists a $\mathrm{Sp}_\iota\times\mathrm{G}\ell_{n-\iota}$ irreducible representation ρ occurring in $(\check{\omega}_1)_{U_\iota^n}$ which has central character on $\mathrm{G}\ell_{n-i}$ given by $\lambda\cdot I_{n-\iota}\rightsquigarrow\chi^{-1}(\lambda^{n-\iota})|\lambda|^{(n-\iota)(n+\iota+1)}$. Then we apply the contragredient and induction functors (see [B-Z]); we deduce that ω_1 embeds in $\mathrm{ind}(\mathrm{Sp}_\iota\times\mathrm{G}\ell_{n-\iota}\times U_\iota^n\nearrow\mathrm{Sp}_n|\tilde{\rho})$, where $\tilde{\rho}$ is the representation of $\mathrm{Sp}_\iota\times\mathrm{G}\ell_{n-\iota}$ obtained by applying

$$\left[\begin{array}{cc|cc} I_\iota & 0 & 0 & 0\\ 0 & 0 & 0 & I_{n-\iota}\\ \hline 0 & 0 & I_\iota & 0\\ 0 & -I_{n-\iota} & 0 & 0\end{array}\right]$$

to the representation $\check{\rho}$. Hence the central character of $\tilde{\rho}$ on $\mathrm{G}\ell_{n-\iota}$ is given by

$$\lambda\cdot I_{n-\iota}\rightsquigarrow\chi^{-1}(\lambda^{n-\iota})|\lambda|^{(n-\iota)(n+\iota+1)}.\qquad\text{Q.E.D.}$$

REMARK II.4.2: The proof of Lemma II.4.1 implies that $(\check{\omega}_1)_{U_i''}$ has an eigenspace under Center($G\ell_{n-i}$) which transforms according to the character $\lambda \cdot I_{n-i} \rightsquigarrow \chi^{-1}(\lambda^{n-i})|\lambda|^{(n-i)(n+i+1)}$. We recall that the simple root system Π defining Sp_n is given by the set $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{i-1} - \varepsilon_i, 2\varepsilon_i, \varepsilon_{i+1} - \varepsilon_{i+2}, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n - \varepsilon_1\}$. Moreover the root subsystem Π' defining $\mathrm{Sp}_i \times G\ell_{n-i}$ is given by $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{i-1} - \varepsilon_i, 2\varepsilon_i, \varepsilon_{i+1} - \varepsilon_{i+2}, \ldots, \varepsilon_{n-1} - \varepsilon_n\}$. (see §0-XII). Then using §0, we deduce that for the sequence $a_m = \pi^m \cdot I_{n-i} \in$ Center $G\ell_{n-i}$ (m, a large positive integer), there exist vectors $v_1 \in (\check{\omega}_1)$ and $v_2 \in \omega_1$ such that

$$\langle v_2, \check{\omega}_1(a_m)v_1\rangle = \chi^{-1}(\pi^{m(n-i)})q^{-m(n-i)(n+i+1)}\langle v_2, v_1\rangle$$

with $\langle v_2, v_1\rangle \neq 0$. In particular, if $\mathrm{Re}(\chi) > n + i + 1$, then $\lim_{m\to+\infty}|\langle v_2, \check{\omega}_1(a_m)v_1\rangle| = +\infty$. Furthermore if $\mathrm{Re}(\chi) = n + i + 1$, then $|\langle v_2, \check{\omega}_1(a_m)v_1\rangle| = |\langle v_2, v_1\rangle| \neq 0$, and hence $(v_2, \check{\omega}_1(a_m)v_1\rangle$ does not go to zero as $m \to +\infty$. In any case (with $\mathrm{Re}(\chi) > n + i + 1$) by using the Howe unitary criterion, we see that $\check{\omega}_1$ (and hence ω_1) is either non-unitary or finite dimensional (and hence trivial, see Appendix) representation of Sp_n.

Then using Remark II.4.2, we deduce (1) and (2) of Theorem II.4.1.

Thus the remainder of this section is devoted to cases (3) and (4) of Theorem II.4.1.

We assume Q is unramified and $\dim Q \leqslant 4n$. Then

(a) if $m < 2n + 1$, we see that the possible values of i which may contribute *unitary* boundary components to ρ_Q satisfy $i \geqslant n - m/2$ (i.e. if $m = 2j$, then i varies from $n - j$ to $n - 1$),

(b) if $2n + 1 < m$, then the possible values of i which may contribute *unitary* boundary components to ρ_Q vary from 0 to $n - 1$.

We examine the Howe duality conjecture in a more computational manner than we have done above. Indeed, assuming that ω_1 embeds into $\mathrm{ind}(\mathrm{Sp}_i \times G\ell_{n-i} \times U_i'' \nearrow \mathrm{Sp}_n | A \otimes B)$, we try to examine the duality conjecture when ω_1 is replaced by $\mathrm{ind}(\mathrm{Sp}_i \times G\ell_{n-i} \times U_i'' \nearrow \mathrm{Sp}_n | A \otimes B)$.

We apply the Jacquet functor to get the equivalence of $\mathrm{Hom}_{\mathrm{Sp}_n \times O(Q)}(S[M_{mn}(k)], \mathrm{ind}(\mathrm{Sp}_i \times G\ell_{n-i} \times U_i'' | A \otimes B) \otimes \sigma)$ with $\mathrm{Hom}_{\mathrm{Sp}_i \times G\ell_{n-i} \times O(Q)}(S[M_{mn}(k)]_{U_i''}, A \otimes B \otimes \sigma)$.

Thus the next problem is to get explicit control of $S[M_{mn}(k)]_{U_i''}$ as an $\mathrm{Sp}_i \times G\ell_{n-i} \times O(Q)$ module. Indeed we are going to construct a certain resolution of the space $S[M_{mn}(k)]_{U_i''}$.

For this computation we now assume no restriction on Q.

We consider the following induced module:

$$V_\ell = \mathrm{Ind}\Bigg[\{\mathrm{Sp}_i \times O(Q|_{\Sigma_\ell})\} \times \left\{\left(R, \left(\begin{array}{c|c} R & 0 \\ \hline X & S \end{array}\right)\right) | R \in G\ell(\Sigma)\right\} \times \tilde{N}_\ell \nearrow$$

$$\mathrm{Sp}_i \times G\ell_{n-i} \times O(Q) | \pi_{Q_{\Sigma(\ell)}} \otimes \{(R, S) \to |\det R|^{(m/2)+i} |\det S|^{m/2}$$

$$\times \langle \Delta(Q) | \det R \det S\rangle\} \otimes \mathbf{1}_{\tilde{N}_\ell}\Bigg]$$

where $\Sigma(\ell)$ denotes Σ_ℓ. Let Z_ℓ be the inducing subgroup above. Here $\pi_{Q_{\Sigma(\ell)}}$ is the Weil representation of $\mathrm{Sp}_i \times \mathrm{O}(Q|_{\Sigma_\ell})$ on the space $S[M_{m-2\ell,i}(k)]$.

Then we consider the linear map from $S[M_{mn}(k)]$ to V_ℓ given by

$$\varphi \overset{T_\ell}{\rightsquigarrow} \int_{M_{\ell,i}(k)} \pi_Q(G,g)(\varphi)\left[\begin{array}{c|c} X & I_\ell \\ T_1 & 0 \\ 0 & \\ T_2 & \end{array}\right] \mathrm{d}X.$$

Here we use the decomposition of $M_{m,i}(k)$ such that

$$\left[\begin{array}{c} X \\ 0 \\ \hline 0 \\ \end{array}\right] \text{ spans } \Sigma, \quad \left[\begin{array}{c} 0 \\ \hline Y \\ 0 \\ \end{array}\right] \text{ spans } \Sigma^*, \text{ and } \left[\begin{array}{c} 0 \\ T_1 \\ \hline 0 \\ T_2 \end{array}\right] \text{ spans } \Sigma_\ell. \text{ (See §0.)}$$

Here $G \in \mathrm{Sp}_i \times \mathrm{G}\ell_{n-i}$ and $g \in \mathrm{O}(Q)$. Then we note that the above map is invariant under U_i^n; that is $\varphi - \pi_Q(u)\cdot\varphi$ lies in the kernel of the above map for all $u \in U_i^n$ and all $\varphi \in S[M_{mn}(k)]$.

We consider the sequence of subspaces (with $\mathscr{S}_0 = S[M_{mn}(k)]$)

$$\mathscr{S}_\ell = \bigcap_{j=0}^{j=\ell-1} \mathrm{Kernel}(T_j)$$

for $\ell = 0,\ldots,\min(n-i, m/2-t)$ where $t = \dim L/2$ ($Q = H_r \oplus L$, L either a 0, 2, or 4 dimensional anisotropic form). Then $\mathscr{S}_\ell$ is invariant under $\mathrm{Sp}_i \times \mathrm{G}\ell_{n-i} \times \mathrm{O}(Q)$. The structure of $\mathscr{S}_\ell$ is clarified by the following Lemma.

Lemma II.4.2: *T_ℓ maps $\mathscr{S}_\ell$ surjectively to v_ℓ = the subspace of V_ℓ of compactly supported* (mod Z_ℓ) *functions.*

Proof: It suffices to show for $f \in \mathscr{S}_\ell$ that

$$(*) \int_{M_{\ell,i}(k)} f\left[\begin{array}{c|c} U & \underline{S} \\ \hline T_1 & 0 \\ \hline 0 & 0 \\ \hline T_2 & \end{array}\right] \mathrm{d}U$$

as a function of $\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, S\right)$ belongs to

$$S[M_{m-2\ell,i}(k)] \otimes S[\mathrm{G}\ell(\Sigma)].$$

Indeed we know that $T_\ell(f)$ is determined by its values on the set $G\ell(\Sigma)\cdot K_1 \times K_2$ (where K_1 is the standard maximal compact of $O(Q)$ and K_2 is the standard maximal compact of $G\ell_{n-t}$). We let X_f be the finite dimensional span of the functions $\pi_Q(K_1 \times K_2)(f)$. Then we apply the above criterion to a basis of X_f and thus deduce that there exist at most a finite number of cosets of $K_1 \times K_2$ in $G\ell(\Sigma)\cdot(K_1 \times K_2)$ where $T_\ell(f)$ is nonvanishing. Thus $T_\ell(f)$ has support in a finite number of double cosets of $Z_\ell \backslash \mathrm{Sp}_t \times G\ell_{n-t} \times O(Q)/K_1 \times K_2$.

On the other hand, we have (by hypothesis) that for $f \in \mathscr{S}_\ell$,

$$\int \pi_Q(G, g) f \left[\begin{array}{c|c} \begin{array}{c} U_{\ell-\nu} \\ A_\nu \\ T_1' \\ \hline 0 \\ B_\nu \\ T_2' \end{array} & \begin{array}{c} \boxed{S} \\ \\ \\ 0 \\ \\ \\ \end{array} \end{array}\right] \mathrm{d}U_{\ell-\nu} \equiv 0$$

with A_ν and $B_\nu \in M_{\nu,t}(k)$, $\begin{bmatrix} T_1' \\ T_2' \end{bmatrix} \in M_{m-2\ell,t}(k)$ and S any invertible $(\ell-\nu)\times(\ell-\nu)$ matrix. If S' is any $\ell \times \ell$ matrix having rank $\ell-\nu$, then we know that it is possible to express S' as $k_1^{-1}\left[\begin{array}{c|c} S & 0 \\ \hline 0 & 0 \end{array}\right]k_2$, where $k_t \in G\ell_\ell(\mathcal{O}_k)$ and S is some invertible $(\ell-\nu)\times(\ell-\nu)$ matrix. Then in the above identity, we let $G = k_2$ and $g = g_1 \cdot k_1$ such that $g_1^{-1}\cdot I_{\ell-\nu} = S$ and $k_1 \in G\ell(\Sigma)$. Then we let $B_\nu = 0$, and we integrate against A_ν and deduce that the function in $(*)$ vanishes for all $\left(\begin{bmatrix} T_1 \\ \hline T_2 \end{bmatrix}, S\right)$ where $\mathrm{rank}(S) = \ell - \nu$. Hence if we vary ν, we deduce that the function in $(*)$ vanishes for all $\left(\begin{bmatrix} T_1 \\ \hline T_2 \end{bmatrix}, S\right)$ where $\mathrm{rank}(S) < \ell$. But since this function has compact support and is locally constant, we deduce the desired property!

Thus we have shown that the map T_ℓ restricted to $\mathscr{S}_\ell$ maps $\mathscr{S}_\ell$ into v_ℓ. If F is any function in v_ℓ, we know that F is determined by its values on a finite number of double cosets of the form $Z_\ell \backslash \mathrm{Sp}_t \times G\ell_{n-t} \times O(Q)/\tilde{K}_1 \times \tilde{K}_2$ where $\tilde{K}_1 \times \tilde{K}_2$ is an open normal subgroup of $K_1 \times K_2$ to be specified below.

Indeed we know that the span X_F of the functions in $S[M_{m-2\ell,t}(k)]$

$$\{F(\xi_\nu(K_1 \times K_2))(X) \mid X \in M_{m-2\ell,t}(k)\}$$

is a finite dimensional space, where ξ_ν runs through a set of representatives of those double cosets $Z_\ell \backslash \mathrm{Sp}_t \times G\ell_{n-t} \times O(Q)/K_1 \times K_2$ which carry the support of F (mod Z_ℓ) (here $\xi_\nu \in G\ell(\Sigma)$).

Then we choose m an integer which satisfies the following conditions:

(i) $X_F \subseteq S[M_{m-2\ell,\iota}(k)/\pi^m M_{m-2\ell,\iota}(\mathcal{O}_k)]$ (= the subspace of functions in $S[M_{m-2\ell,\iota}(k)]$ which are invariant under $\pi^m M_{m-2\ell,\iota}(\mathcal{O}_k)$ by translation),

(ii) $(\pi^m \cdot Q \cdot W) \equiv 0 \bmod \pi$ for all $W \in \text{Support}(\psi)$ as ψ varies in X_F,

(iii) $(1/2)\pi^{2m}Q \equiv 0 \bmod \pi$,

(iv) X_F is *pointwise invariant* by $\{\gamma \in \mathrm{O}(Q|_{\Sigma_\ell}) | \gamma \equiv 1 \bmod \pi^m\}$,

(v) F is *invariant* by $\tilde{K}_1 \times \tilde{K}_2 = \{(\gamma_1, \gamma_2) \in K_1 \times K_2 | \gamma_i \equiv I \bmod \pi^m\}$.

Then the function F is determined by knowing the values of $F(\xi_\nu \rho_j)$ where ρ_j forms a set of coset representatives of $K_1 \times K_2/\tilde{K}_1 \times \tilde{K}_2$.

We now choose a function $f \in \mathcal{S}_\ell$ such that f is invariant by $\tilde{K}_1 \times \tilde{K}_2$ and $T_\ell(f)(\xi_\nu \rho_j) = F(\xi_\nu \rho_j)$. First we let $\beta_{\nu j} \in S[Z_\ell \backslash \mathrm{Sp}_\iota \times \mathrm{G}\ell_{n-\iota} \times \mathrm{O}(Q)]$ such that $\beta_{\nu j}(Z_\ell \xi_{\nu'} \rho_{j'}(\tilde{K}_1 \times \tilde{K}_2)) = \delta_{\nu j, \nu' j'}$ (Kronecker delta function) and $\beta_{\nu j}$ vanishes on the remaining $Z_\ell \backslash \mathrm{Sp}_\iota \times \mathrm{G}\ell_{n-\iota} \times \mathrm{O}(Q)/\tilde{K}_1 \times \tilde{K}_2$ cosets. Then we let $\beta^*_{\nu j} \in S[M_{m,n-\iota}(k)]^{\tilde{K}_1 \times \tilde{K}_2}$ such that $\beta^*_{\nu j}$ vanishes on $\{\xi | \xi' Q \xi = 0$ and $\text{rank}(\xi) \leqslant \ell - 1\}$ and $\beta^*_{\nu j}$ restricted to the $\mathrm{O}(Q) \times \mathrm{G}\ell_{n-\iota}$ orbit of

$$\begin{bmatrix} \underline{I_\ell|} & \\ & 0 \end{bmatrix}$$

in $M_{m,n-\iota}(k)$ is $\beta_{\nu j}$. (We note that this orbit is closed in $M_{m,n-\iota}(k) - \{\xi | \xi' Q \xi = 0$ and $\text{rank}(\xi) \leqslant \ell - 1\}$. That is, we choose $\beta^*_{\nu j}$ such that

$$\pi_Q(g)\beta^*_{\nu j}\begin{bmatrix} \underline{I_\ell|} & \\ & 0 \end{bmatrix} = \beta_{\nu j}(g)$$

for all $g \in \mathrm{O}(Q) \times \mathrm{G}\ell_{n-\iota}$.

Next we choose $\varphi_{\nu j} \in S[M_{m,\iota}(k)]$ such that

$$\pi_Q(\rho_j)(\varphi_{\nu j})\begin{bmatrix} \frac{X}{T_1} \\ \frac{Y}{T_2} \end{bmatrix} = |\det \xi_\nu|^{-\iota} F(\xi_\nu \rho_j)\begin{bmatrix} \frac{T_1}{T_2} \end{bmatrix}\chi_{\mathcal{O}}(X)\chi_{\mathcal{O}}(Y)$$

(where $\chi_{\mathcal{O}}$ is the characteristic function of $M_{\ell,\iota}(\mathcal{O}_k)$ in $M_{\ell,\iota}(k)$).

Then we let

$$f(X) = \sum \varphi_{\nu j} \otimes \beta^*_{\nu j}.$$

We observe that

$$T_\ell(f)(\xi_{\nu'}\rho_{j'}) = \sum \left(\int \pi_Q(\rho_{j'})(\varphi_{\nu_j}) \begin{bmatrix} X \\ T_1 \\ \hline 0 \\ T_2 \end{bmatrix} \mathrm{d}X \right)$$

$$\times \left(\pi_Q(\xi_{\nu'}\rho_{j'})(\beta^*_{\nu j}) \begin{bmatrix} I_\ell & \\ \hline & 0 \end{bmatrix} \right) = F(\xi_{\nu'}\rho_{j'}) \begin{bmatrix} T_1 \\ \hline T_2 \end{bmatrix}.$$

To complete the proof, it suffices to show that f is $\tilde{K}_1 \times \tilde{K}_2$ invariant. We know that $\beta^*_{\nu j}$ is $\tilde{K}_1 \times \tilde{K}_2$ invariant by construction; hence it suffices to show that each φ_{ν_j} is $\tilde{K}_1 \times \tilde{K}_2$ invariant. Indeed we shall show that φ_{ν_j} is invariant by s_ℓ (the element of order two in $\mathrm{O}(Q)$ which is trivial on Σ_ℓ, and $s_\ell(\Sigma) = \Sigma^*$ and $s_\ell(\Sigma^*) = \Sigma$) and the subgroup $\{\gamma \in \tilde{P}_\ell | \gamma \equiv I \bmod \pi^m\}$.

By the choice of s_ℓ, we have that s_ℓ maps the integral elements in Σ bijectively to the integral elements in Σ^*. Thus

$$\chi_\mathcal{O}(s_\ell(Y))\chi_\mathcal{O}(s_\ell(X)) = \chi_\mathcal{O}(X)\chi_\mathcal{O}(Y).$$

Similarly if $\gamma \in \mathrm{G}\ell(\Sigma)$ satisfies $\gamma \equiv I \bmod \pi^m$, then

$$\chi_\mathcal{O}(\gamma(X))\chi_\mathcal{O}\big((\gamma')^{-1}(Y)\big) = \chi_\mathcal{O}(X)\chi_\mathcal{O}(Y).$$

Moreover by the choice of m above (condition (iv) above), we have that all the functions $F(\xi_\nu\rho_j)\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ are invariant by $\{\gamma \in \mathrm{O}(Q|_{\Sigma_\ell}) | \gamma \equiv I \bmod \pi^m\}$.

Thus we must study the invariance of φ_{ν_j} under the group $\{\gamma \in \tilde{N}_\ell | \gamma \equiv I \bmod \pi^m\}$. Then applying §0,

$$\pi_Q(N(\mu, \jmath))\pi_Q(\rho_j)\varphi_{\nu_j} \begin{bmatrix} X \\ T_1 \\ \hline Y \\ T_2 \end{bmatrix} = |\det \xi_\nu|^{-t} F(\xi_\nu\rho_j)\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \mu \cdot Y \right)$$

$$\chi_\mathcal{O}\left(X + ((-\mu' Q\mu/2) + \jmath)Y - \mu' Q \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) \chi_\mathcal{O}(Y).$$

Using conditions (ii) and (iii) above, we see that if X and Y belong to $M_{\ell,t}(\mathcal{O}_k)$, then $X + ((-\mu' Q\mu/2) + \jmath) \cdot Y - \mu' Q \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in M_{\ell,t}(\mathcal{O}_k)$ provided that $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \text{support}(F(\xi_\nu \rho_j))$ and m is chosen as above. But then by condition (i), we have that

$$F(\xi_\nu \rho_j)\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \mu \cdot Y\right) = F(\xi_\nu \rho_j)\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right)$$

for all $\mu \equiv 0 \bmod \pi^m$.

We note that if $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ is *not* in the support of $F(\xi_\nu \rho_j)$, then $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \mu \cdot Y$ does not belong to its support either (i.e., $\mu \cdot Y$ lies in $\pi^m M_{m-2\ell,\iota}(\mathcal{O}_k)$ and use condition (i) above).

On the other hand, if $X \notin M_{\ell,\iota}(\mathcal{O}_k)$ and $Y \in M_{\ell,\iota}(\mathcal{O}_k)$, then again by conditions (ii) and (iii), $X + ((-\mu' Q\mu/2) + \sigma)Y - \mu' Q\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \notin M_{\ell,\iota}(\mathcal{O}_k)$ (with $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ in the support of $F(\xi_\nu \rho_j)$). Again we can apply the same reasoning as in the above paragraph to cover the case when $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ does not lie in the support of $F(\xi_\nu \rho_j)$.

Finally, for the case when $Y \notin M_{\ell,\iota}(\mathcal{O}_k)$, we can see trivially the invariance of $\pi_Q(\rho_j)\varphi_{\nu_j}$ under $N(\mu, \sigma)$.

Thus we have shown that $\pi_Q(\rho_j)\varphi_{\nu_j}$ is invariant under $\{\gamma \in \tilde{N}_\ell | \gamma \equiv I \bmod \pi^m\}$. But since $(s_\ell\{\gamma \in \tilde{N}_\ell | \gamma \equiv I \bmod \pi^m\} s_\ell) \cdot \{\gamma \in \tilde{P}_\ell | \gamma \equiv I \bmod \pi^m\} = \tilde{K}_1$ is a normal subgroup of K_1, we have that φ_{ν_j} is invariant by $\tilde{K}_1$.

Thus we have shown that

$$T_\ell(f)(\xi_\nu \rho_j k) = F(\xi_\nu \rho_j k)$$

for all $k \in \tilde{K}_1 \times \tilde{K}_2$. Q.E.D.

Thus this Lemma gives an effective resolution of $S[M_{mn}(k)]$ as an $\mathrm{Sp}_\iota \times \mathrm{G}\ell_{n-\iota} \times \mathrm{O}(Q)$ module. Indeed for each ℓ, we see that T_ℓ induces a surjective mapping of $(\mathcal{S}_\ell)_{U_\iota^n}$ to v_ℓ. But to complete this resolution we need the following.

We let $\rho_\iota = \min(n - i, m/2 - t)$.

LEMMA II.4.3: *$(\mathcal{S}_{\rho_\iota})_{U_\iota^n}$ is $\mathrm{Sp}_\iota \times \mathrm{G}\ell_{n-\iota}$ isomorphic to v_{ρ_ι}.*

PROOF: All that is necessary to prove is that

$$\mathrm{Kernel}(T_{\rho_\iota} : \mathcal{S}_{\rho_\iota} \to v_{\rho_\iota}) \subseteq \mathcal{S}_{\rho_\iota}[U_\iota^n].$$

This means that for each $f \in \operatorname{Ker}(T_{\rho_t} : \mathscr{S}_{\rho_t} \to v_{\rho_t})$ there exists an integer m_0 such that

$$\int_{\pi^{m_0} M_{n-t,t}(\mathcal{O}_k)} f[X + Y \cdot T | Y] \mathrm{d}T \equiv 0$$

for *all* $(X|Y) \in W_t = \{ X \in M_{mn}(k) | X'QX = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & 0 \end{array}\right] \}$. But since each

$$Y = g^{-1} \begin{bmatrix} \underline{I_\ell|} & \\ & 0 \end{bmatrix} z$$

for $g \in \mathrm{O}(Q)$, $z \in \mathrm{G}\ell_{n-t}$, we see that the above is equivalent to (for $\ell = 0, \dots, \rho_t$)

$$\int_{\{Z \in \pi^{m_0} M_{\ell,t}(\mathcal{O}_k)\}} \pi_Q(z, g) f \left[\begin{array}{c|c} Z + W & \underline{I_\ell|} \\ T_1 & \\ \hline 0 & 0 \\ T_2 & \end{array}\right] \mathrm{d}Z \equiv 0 \qquad (**)$$

for *all* $(z, g) \in \mathrm{G}\ell(\Sigma) \cdot K_1 \times K_2$ and *all* $W \in M_{\ell,t}(k)$ and *all* $\left[\frac{T_1}{T_2}\right] \in M_{m-2\ell,t}(k)$.

We let X_f be the finite dimensional space spanned by $\pi_Q(K_1 \times K_2)(f)$. Then we observe that it suffices to show

$$\int_{\{Z \in \pi^{m_0} M_{\ell,t}(\mathcal{O}_k)\}} \pi_Q(z)(\varphi) \left[\begin{array}{c|c} Z + W & \underline{I_\ell|} \\ T_1 & \\ \hline 0 & 0 \\ T_2 & \end{array}\right] \mathrm{d}Z \equiv 0 \qquad (***)$$

holds for all $\varphi \in X_f$ where $z \in \mathrm{G}\ell(\Sigma)$ can be restricted to the diagonal matrices of the form $\mathrm{d}(\nu_1, \dots, \nu_\ell) = D(\pi^{\nu_1}, \dots, \pi^{\nu_\ell})$, and $W \in M_{\ell,t}(k)$ and $\left[\frac{T_1}{T_2}\right] \in M_{m-2\ell,t}(k)$ are arbitrary.

We assume the validity of $(***)$ for integers ranging from 0 to ℓ. Then we know that

$$\pi_Q(z)(\varphi)\left[\begin{array}{c|c} \frac{X}{Y} & 0 \end{array}\right] \equiv 0$$

(for all $z \in \mathrm{G}\ell(\Sigma)$ and all X and Y) implies that

$$\varphi\left[\begin{array}{c|c} \frac{X}{Y} & \begin{array}{c} \underline{\mathrm{d}(\nu_1, \dots, \nu_{\ell+1})|} \\ 0 \end{array} \end{array}\right] \equiv 0$$

if $\nu_{\ell+1} \geqslant \ldots \geqslant \nu_1 > M_1$ for a positive large integer M_1 or $\nu_1 < M_1'$ for some integer M_1'. Then we know that for $M_1 \geqslant \nu_1 \geqslant M_1'$ (by induction hypothesis)

$$\int_{\{c\in\pi^{m_0'}M_{1,\iota}(\mathcal{O}_k)\}} \varphi\left[\begin{array}{c|cc} \pi^{\nu_1}(c)+w_1 & \pi^{\nu_1} & \\ T_1' & & \\ \hline 0 & & 0 \\ T_2' & & \end{array}\right] dc \equiv 0$$

(where $T_\iota' \in M_{m/2-1,\iota}(k)$ and $w_1 \in M_{1,\iota}(k)$) implies that

$$\int_{\{Z\in\pi^{m_0'}M_{\ell+1,\iota}(\mathcal{O}_k)\}} \pi_Q\Big(d(\nu_1,\ldots,\nu_{\ell+1})^{-1}\Big)$$

$$(\varphi)\left[\begin{array}{c|cc} Z+W_1 & I_{\ell+1} & \\ T_1 & & \\ \hline 0 & & 0 \\ T_2 & & \end{array}\right] dZ \equiv 0$$

for $\nu_{\ell+1} \geqslant \nu_\ell \geqslant \cdots \geqslant \nu_2 \geqslant M_2$ (with M_2 a large positive integer). Then we repeat the same process ℓ times and deduce that there exist positive integers $M_1, M_2, \ldots, M_\ell$, and $M_{\ell+1}$ such that for all $(\nu_1, \ldots, \nu_{\ell+1})$ satisfying $\nu_{\ell+1} \geqslant \nu_\ell \geqslant \ldots \geqslant \nu_1$,

$$\int_{\{Z\in\pi^{m_0'}M_{\ell+1,\iota}(\mathcal{O}_k)\}} \pi_Q\Big(d(\nu_1,\ldots,\nu_{\ell+1})^{-1}\Big)$$

$$(\varphi)\left[\begin{array}{c|cc} Z+W & I_{\ell+1} & \\ T_1 & & \\ \hline 0 & & 0 \\ T_2 & & \end{array}\right] dZ \equiv 0$$

except possibly if $M_1 \geqslant \nu_1 \geqslant M_1'$, $M_2 \geqslant \nu_2 \geqslant M_1', \ldots$, and $M_{\ell+1} \geqslant \nu_{\ell+1} \geqslant M_1'$. For the remaining finite set of $(\nu_1, \ldots, \nu_{\ell+1})$ we use the hypothesis that $\varphi \in \mathscr{S}_{\rho_\iota}$ to see that there exists m_0' such that

$$\int_{\{Z\in\pi^{m_0'}M_{\ell+1,\iota}(\mathcal{O}_k)\}} \pi_Q\Big(d(\nu_1,\ldots,\nu_{\ell+1})^{-1}\Big)$$

$$(\varphi)\left[\begin{array}{c|cc} Z+W & I_{\ell+1} & \\ T_1 & & \\ \hline 0 & & 0 \\ T_2 & & \end{array}\right] dZ \equiv 0.$$

Then we choose m_0'' = smaller of m_0 and m_0', and we see easily that $(***)$ is valid (with m_0'' replacing m_0) Q.E.D.

Thus we have explicitly the structure of $S[M_{mn}(k)]_{U_l''}$ as an $\mathrm{Sp}_l \times \mathrm{G}\ell_{n-l} \times \mathrm{O}(Q)$ module. In particular we see from the above considerations that

$$\mathrm{Hom}_{\mathrm{Sp}_l\times \mathrm{G}\ell_{n-l}\times \mathrm{O}(Q)}(v_\ell, A\otimes B\otimes\sigma)$$

$$\cong \mathrm{Hom}_{\mathrm{Sp}_l\times \mathrm{G}\ell_{n-l}\times \mathrm{O}(Q)}(\check{A}\otimes\check{B}\otimes\check{\sigma}, \check{v}_\ell)$$

where $\check{v}_\ell$ is the $\mathrm{Sp}_l \times \mathrm{G}\ell_{n-l} \times \mathrm{O}(Q)$ module

$$\mathrm{Ind}\Bigg(\big(\mathrm{Sp}_l \times \mathrm{O}(Q|_{\Sigma_\ell})\big) \times \left\{\left(R, \left[\begin{array}{c|c} R & 0 \\ \hline X & S\end{array}\right]\right) | R\in \mathrm{G}\ell(\Sigma)\right\} \times \tilde{N}_\ell$$

$$\nearrow \mathrm{Sp}_l \times \mathrm{G}\ell_{n-l} \times \mathrm{O}(Q) | \check{\pi}_{Q|_{\Sigma(\ell)}} \otimes (R, S) \rightsquigarrow |\det R|^{(m/2)-(n+1)}$$

$$|\det S|^{-(m/2)+\ell} \langle \Delta(Q) | \det R \det S\rangle\Bigg),$$

where $\Sigma(\ell)$ denotes Σ_ℓ, and $\check{\pi}_{Q_{\Sigma(\ell)}}$ = the contragredient of the representation of $\mathrm{Sp}_l \times \mathrm{O}(Q|_{\Sigma_\ell})$ on $S[M_{m-2\ell,l}(k)]$. On the other hand,

$$\mathrm{Hom}_{\mathrm{Sp}_l\times \mathrm{G}\ell_{n-1}\times \mathrm{O}(Q)}(\check{A}\otimes\check{B}\otimes\check{\sigma}, \check{v}_\ell)$$

$$\cong \mathrm{Hom}_{\mathrm{Sp}_l\times \mathrm{O}(Q|_{\Sigma(\ell)})\times V_\Sigma}\Big(\check{A}\otimes(\check{B})_{U_{\ell,n-l-\ell}} \otimes (\check{\sigma})_{N_\ell}, \check{\pi}_{Q|_{\Sigma(\ell)}}$$

$$\otimes\big\{(R,S)\rightsquigarrow |\det R|^{(m/2)-(n+1)} |\det S|^{-(m/2)+\ell}$$

$$\times\langle\Delta(Q)|\det R \det S\rangle\big\}\Big)$$

where $V_\Sigma = \{(R, \left[\begin{array}{c|c} R & 0 \\ \hline 0 & S\end{array}\right]) | R\in \mathrm{G}\ell(\Sigma)\}$ and $U_{\ell,n-l-\ell} = \{\left[\begin{array}{c|c} I & 0 \\ \hline X & I\end{array}\right] | X \in M_{n-l-\ell,\ell}(k)\}$.

Now we return to the proof of Theorem II.4.1. That is, we assume that π gives rise to a boundary component of ρ_Q for $m = 2j \leqslant 2n$. We recall from Lemma II.4.1 that there exists an $A\otimes B$ irreducible representation of $\mathrm{Sp}_l \times \mathrm{G}\ell_{n-l}$ along with a surjective $\mathrm{Sp}_l \times \mathrm{G}\ell_{n-l}$ intertwining map

$$(\pi)_{U_l''} \to A\otimes B \to 0$$

where B has central character

$$\lambda\cdot I_{n-l} \rightsquigarrow |\lambda|^{(n-l)[-(m'/2)+n+l+1]}\langle\Delta(Q)|\lambda\rangle^{n-l}$$

with $m' = 4n + 2 - 2j$.

Then we consider the space (for $0 \leqslant \ell \leqslant \rho_i$)

$$\operatorname{Hom}_{\mathrm{Sp}_i \times \mathrm{G}\ell_{n-i} \times \mathrm{O}(Q)}(v_\ell, A \otimes B \otimes \sigma).$$

Applying the above considerations, we deduce if $\ell = 0$ then $m'/2 - (n + i + 1) = -m/2$; hence $2n + 1 = n + i + 1$ with $i \leqslant n - 1$. Thus

$$\operatorname{Hom}_{\mathrm{Sp}_i \times \mathrm{G}\ell_{n-i} \times \mathrm{O}(Q)}(v_0, A \otimes B \otimes \sigma) = \{0\}.$$

On the other hand, assuming that $\operatorname{Hom}(v_\ell, A \otimes B \otimes \sigma) \neq 0$, then there exists a nonzero functional T on the space $(\check{\sigma})_{\tilde{N}_\ell}$ such that

$$T\big((\check{\sigma})_{\tilde{N}_\ell}(\lambda \cdot I_\ell)w\big) = |\lambda|^{\ell((m/2)-(n+1))}|\lambda|^{(n-i-\ell)(\ell-(m/2))}$$

$$|\lambda|^{(n-i)(-(m'/2)+(n+i+1))}T(w) = |\lambda|^{f(\ell,i)}T(w)$$

for *all* $w \in (\check{\sigma})_{N_\ell}$ (with $\lambda \cdot I_\ell \in \operatorname{Center}(\mathrm{G}\ell(\Sigma))$) and

$$f(\ell, i) = -\ell^2 + \ell(m - i - 1) - (n - i)^2.$$

Then we observe that $f(\ell, i) \leqslant 0$ if and only if $(m - i - 1)^2 \leqslant 4(n - i)^2$. However we observe that if $m - i - 1 \leqslant 0$, then clearly $f(\ell, i) < 0$. Thus $(m - i - 1)^2 \leqslant 4(n - i)^2$ is implied by $m - i - 1 \leqslant 2(n - i)$ or $m + i - 1 \leqslant 2n$. Now $i \leqslant n - 1$ which implies that $m + n - 2 \leqslant 2n$ or $m \leqslant n + 2$. Hence $f(\ell, i) < 0$ for all ℓ and i provided $m < n + 2$. If $m = n + 2$, then $f(\ell, i) < 0$ provided $i \leqslant n - 2$ and $f(\ell, i) = 0$ for $i = n - 1$ (with $\ell = n - i$).

We recall that the simple root system Π of $\mathrm{O}(Q)$ is given by $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_i - \varepsilon_{i+1}, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_{r-1} + \varepsilon_r\}$ where $r = (m/2) -$ (dim of maximal anisotropic piece)/2. Moreover the subroot system Π' defining $\mathrm{G}\ell(\Sigma) \times \mathrm{O}(Q|_{\Sigma_\ell})$ is given by $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{\ell-1} - \varepsilon_\ell, \varepsilon_{\ell+1} - \varepsilon_{\ell+2}, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_{r-1} + \varepsilon_r\}$. Thus using §0, we deduce that for the sequence $b_m = \pi^m \cdot I_\ell \in \operatorname{Center}(\mathrm{G}\ell(\Sigma))$ (with m a large positive integer), there exist vectors $v_1 \in \check{\sigma}$ and $v_2 \in \sigma$ such that

$$\langle v_2, \check{\sigma}(b_{m'})v_1\rangle = q^{-m'f(\ell,i)}\langle v_2, v_1\rangle.$$

Thus $\lim_{m' \to +\infty}|\langle v_2, \check{\sigma}(b_{m'})v_1\rangle| = +\infty$ if $m < n + 2$ (and for $m = n + 2$ with $i \leqslant n - 2$), and $\langle v_2, \check{\sigma}(b_{m'})v_1\rangle$ does not converge to zero (as $m' \to +\infty$) when $m = n + 2$, $i = n - 1$, and $\ell = n - i$. Hence by applying the Howe unitary criterion, $\check{\sigma}$ (and hence σ) is nonunitary if $m < n + 2$ ($\check{\sigma}$ is either nonunitary or finite dimensional if $m = n + 2$). But if $m = n + 2$ (with $i = n - 1$ and $\ell = n - i$), then by the Appendix we have that $\check{\sigma}$ is either nonunitary or a character on $\mathrm{O}(Q)$. But $(\check{\sigma})_{\tilde{N}_\ell}$ must have an eigenspace under Center $\mathrm{G}\ell(\Sigma)$ (here Σ is one-dimensional) which is *trivial*! But we recall from §95 of [0-M] that there exists a surjective mapping

$$G\ell(\Sigma) \to O_+(Q)/\text{Commutator of } O(Q)$$

where $O_+(Q)$ = the subgroup of $O(Q)$ of elements having determinant = 1. Hence if $\check{\sigma}$ is a character on $O(Q)$, it must be either trivial or the sgn representation (i.e. the nontrivial character on $O_+(Q)\backslash O(Q)$). But we know from the Appendix that if $m = n + 2$, then σ must in fact be *trivial*!

Thus we have verified (3) and (4) of Theorem II.4.1.

REMARK II.4.3: We would like to point out a gap in the proof of Proposition 2.2 of [R-1]. This was kindly pointed out to the author by S. Kudla. It is easy to correct this gap. The error occurs in the induction step from $k-1$ to k. We note that the step $n = 1$ is correct. However assuming that the step $n = k-1$ is true, we must show that the case $n = k$ is valid. That is, we start with a function $f \in S[M_{mn}(\mathbb{Q}_v)]$ which is *semi-invariant* and such that $\pi_Q(G)f$ vanishes on the characteristic variety of $(P_2)_n$ (for *all* $G \in \tilde{\mathrm{S}}\mathrm{p}_n(\mathbb{Q}_v)$). We want to show that $\pi_Q(G)f$ belongs to $S[M_{mn}(\mathbb{Q}_v)][U_i^n]$. But by using the resolution of $S[M_{mn}(\mathbb{Q}_v)]_{U_i^n}$ given in II.4, it suffices to show that $T_\ell(\pi_Q(G)f) \equiv 0$ for each ℓ. But by using the definition of T_ℓ and the fact that f is semi-invariant, it suffices to show that

$$\int_{M_{\ell,i}(k)} \pi_Q(G', g)(\varphi)\begin{bmatrix} X & \begin{array}{|c|}\hline I_\ell \\ \hline\end{array} & \\ T_1 & & \\ \hline 0 & & 0 \\ T_2 & & \end{bmatrix} \mathrm{d}X \equiv 0$$

for all $(G', g) \in \mathrm{Sp}_i \times O(Q)$ and all $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in M_{m-2\ell,i}(\mathbb{Q}_v)$. But we know that the map $\varphi \rightsquigarrow T_\ell(\varphi)$ is an Sp_i intertwining map from $S[M_{mn}(\mathbb{Q}_v)]$ to $S[M_{m-2\ell,i}(\mathbb{Q}_v)]$. Now the vanishing of $\pi_Q(G)(\varphi)$ on the characteristic variety in $M_{mn}(\mathbb{Q}_v)$ (for all G) implies the vanishing of $T_\ell(\pi_Q(G', g)(\varphi))$ on the characteristic variety in $M_{m-2\ell,i}(\mathbb{Q}_v)$. By the induction hyptohesis this implies that $T_\ell(\pi_Q(G', g)(\varphi))$ vanishes on $M_{m-2\ell,i}(\mathbb{Q}_v)$. Hence we have the statement we want to prove.

References

[Ar] ARTHUR, J.: Eisenstein series and the trace formula. *Proc. of Symposium in Pure Math. 33* (1979) 253–274.

[As] ASMUTH, C.: Weil representations of symplectic p-adic groups. *Amer. Jour. of Math. 101(4)* (1979) 885–909.

[B-Z] Bernshtein, I.N. and Zelevinski, A.V.: Representations of the group $GL_2(F)$ where F is a non-Archimedean Local field. *Russian Math. Surveys 31(3)* (May-June, 1976) 1–68.

[B-J] Borel, A. and Jacquet, H.: Automorphic forms and automorphic representations. *Proc. of Symposium in Pure Math. 33* (1979) 185–202.

[C] Cartier, P.: Representations of p-adic groups: a survey. *Proc. of Symposium in Pure Math. 33* (1979) 111–157.

[F] Flath, D.: Decomposition of representations into tensor products. *Proc. of Symposium in Pure Math. 33* (1979) 179–184.

[G-J] Godement, R. and Jacquet, H.: Zeta functions of simple algebras. *Springer Lecture Notes 260* (1972).

[Gu] Gustafson, R.: *The Degenerate Principal Series for Sp(2n).* Ph.D. Thesis, Yale (May, 1979).

[HC] Harish Chandra: Harmonic analysis on reductive p-adic groups (notes by G. van Dijk). *Springer Lecture Notes 162* (1970).

[H] Howe, R.: θ-series and invariant theory. *Proc. of Symposium in Pure Math. 33* (1979) 275–287.

[H-Ps-1] Howe, R. and Piatetski-Shapiro, I.I.: A counterexample to the "generalized Ramanujan conjecture" for quasi-split groups. *Proc. of Symposium in Pure Math. 33* (1979) 315–322.

[H-Ps-2] Howe, R. and Piatetski-Shapiro, I.I.: Some examples of automorphic forms on Sp_4, *Duke Math. Jour. 50(1)* (1983) 55–106.

[J-Ps-Sh] Jacquet, H., Piatetski-Shapiro, I.I. and Shalika, J.A.: Automorphic forms on $G\ell(3)$. *Annals of Math. 109* (1979) 213–258.

[K-V] Kashiwara, M. and Vergne, M.: On the Segal-Shale-Weil representation and harmonic polynomials. *Inventiones Math. 44* (1978) 1–47.

[N] Novodvorsky, M.: New unique models of representations of unitary groups. *Comp. Math.* 33 (1976) 289–295.

[O-M] O'Meara, T.: Introduction to quadratic forms. *Die Grundlehren der Math. Wiss.*, bd 117, Springer Verlag (1963).

[Ps-1] Piatetski-Shapiro, I.I.: L-functions for GSp_2, preprint.

[Ps-2] Piatetski-Shapiro, I.I.: On the Saito Kurokawa Lifting, preprint.

[R-1] Rallis, S.: Langlands Functoriality and the Weil Representation. *Amer. Jour. of Math. 104(3)* (1982) 469–515.

[R-2] Rallis, S.: On a relation between SL_2 cusp forms and automorphic forms on orthogonal groups. *Proc. of Symposium in Pure Math.* 33 (1979) 297–314.

[R-S-1] Rallis, S. and Schiffmann, G.: Distributions invariantes par le group orthogonal. *Springer Lecture Notes 497* (1975) 494–642.

[R-S-2] Rallis, S. and Schiffmann, G.: Représentations supercuspidales der groupe métaplectique. *Kyoto Math. Journal 17* (1977) 567–603.

[S] Shalika, J.: The Multiplicity One Theorem for GL_n. *Annals of Math. 100(1)* (1974) 171–193.

[W] Warner, G.: Harmonic analysis on semisimple Lie groups I. *Grund. Math. Wiss. 188*, Springer Verlag (1972).

[We] Weil, A.: Sur certaines groupes d'operateurs unitares. *Acta Math. 111* (1964) 143–211.

Appendix

1. We now prove certain technical statements alluded to in the text.

Every finite dimensional, irreducible, admissible representation of $O(Q)$ *is given by a character on* $O(Q)$ *if* Q *has positive Witt index and* $\dim Q \geqslant 3$.

Proof: Let σ be such an admissible representation. Let C = commutator subgroup. Restrict σ to C. Then σ decomposes as a direct sum of irreducible admissible representations of C. Choose one such σ_t. Let $\mathrm{Ker}(\sigma_t)$ = Kernel of σ_t. Then $\mathrm{Ker}(\sigma_t)$ is a compact open and normal subgroup of C. Then we know that $C/\mathrm{Center}(C)$ is *simple* (except if $m = 4$ and Witt index of $Q = 2$). Hence $\mathrm{Ker}(\sigma_t)\cdot\mathrm{Center}(C) = C$; so $C/\mathrm{Ker}(\sigma_t)$ is a finite Abelian group. Thus σ_t is a character on C. But C equals its commutator group and hence σ_t is trivial. But $\mathrm{O}(Q)/C$ is an Abelian 2 group; this implies that σ is a character on $\mathrm{O}(Q)$.

If $m = 4$ and Witt index $= 2$, then there exists a surjective homomorphism from $\mathrm{Sp}_1(k)\times\mathrm{Sp}_1(k)$ onto C. In particular, each module σ_t is an irreducible representation of $\mathrm{Sp}_1(k)\times\mathrm{Sp}_1(k)$. But thus $\sigma_t = \tilde{\omega}_1 \otimes \tilde{\omega}_1^*$, where $\tilde{\omega}_1$, $\tilde{\omega}_1^*$ are irreducible, admissible, finite dimensional modules of $\mathrm{Sp}_t(k)$! But we know that $\mathrm{Sp}_1/\pm I$ is simple and Sp_1 equals its own commutator subgroup; thus $\tilde{\omega}_1$, $\tilde{\omega}_1^*$ are the trivial representations of Sp_1. Hence σ_t = trivial. Then applying the same reasoning as above, we have that σ is a character on $\mathrm{O}(Q)$. Q.E.D.

Remark: The same arguments as above show that the only finite dimensional, admissible, irreducible representation of $\mathrm{Sp}_n(k)$ is the trivial representation!

2. We fix a hyperbolic plane $H \subset K^m$ and consider the subgroup of $\mathrm{O}(Q)$ determined by $w \in \mathrm{O}(H)$ having the property that $w(v_+) = v_-$, $w(v_-) = v_+$ with $Q(v_+, v_+) = Q(v_-, v_-) = 0$ and $Q(v_+, v_-) = 1$. Then we decompose the space $S[M_{mn}(k)]$ relative to $\{w, 1\}$ into 2 eigenspaces

$$S = S_+ \oplus S_-$$

where $S_+ = \{\varphi | \pi_Q(w)\varphi = \varphi\}$, $S_- = \{\varphi | \pi_Q(w)\varphi = -\varphi\}$. Then via the restriction map to any locally closed set Z in $M_{mn}(k)$, where Z is $\mathrm{O}(Q)$ stable, we induce a similar splitting

$$\mathscr{H}(Z) = \mathscr{H}_+(Z) \oplus \mathscr{H}_-(Z)$$

where $\mathscr{H}(Z)$ is the subspace of $C^\infty(Z) = \{$locally constant functions on $Z\}$ obtained by restriction from $S[M_{mn}(k)]$. In particular, if Z is an open and dense subset, we see that $\mathscr{H}_\pm(Z) \neq \{0\}$ if and only if $S_\pm \neq \{0\}$. An example of such a Z is the set $\{X \in M_{mn}(k) |$ span of columns of X determines a subspace (of dim $= \min(m, n)$) where Q is nondegenerate$\}$. Hence we see that if $\mathscr{H}_-(Z) \neq \{0\}$, then for some $\mathrm{O}(Q)$ orbit $\mathscr{O}$ (which is automatically closed in $M_{mn}(k)$ by the nondegeneracy condition) $C_-^\infty(\mathscr{O}) \neq \{0\}$. However we know that a typical point X in $\mathscr{O}$ has an isotropy group of the form $\mathrm{O}(Q_1)$, where Q_1 is the form Q restricted to the Q perpendicular complement to the span of the columns of X. Then by applying Frobenius reciprocity

$$\mathrm{Hom}_{\mathrm{O}(Q_1)}(\text{trivial}, \mathrm{sgn}) \neq \{0\}.$$

But this latter condition is possible only if $\mathrm{O}(Q_1) = \{e\}$. But in turn this means that $m \leqslant n$.

Thus we have demonstrated that if $\mathrm{Hom}_{\mathrm{O}(Q)}(\pi_Q, \mathrm{sgn}) \neq \{0\}$, we must have that $m \leqslant n$.

(Oblatum 27-IV-1981)

The Ohio State University
Department of Mathematics
231 West 18th Ave.
Columbus, OH 43210
USA