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A REMARK ON THE PAPER OF A. TANNENBAUM

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Section 1

In his paper "On the classical characteristic linear series of plane curves with nodes and cuspidal points: two examples of Beniamino Segre" (cf. [3]) A. Tannenbaum discusses locally trivial embedded deformations of plane curves from a modern point of view. He shows that the analysis of Segre can be rigorously justified for curves with at most ordinary cusps and nodes as singularities, although Segre associates his characteristic linear series to $H^0(\pi'_* N'_\pi)$ instead of $H^0(N'_D)$, which would be correct. [1] Tannenbaum shows that for any reduced curve D in a smooth surface S there exists an exact sequence

$$0 \to N'_D \to \pi'_* N'_\pi \to T \to 0 \qquad (*)$$

where T is a torsion sheaf. Moreover he proves that $T = 0$ if D has at most ordinary cusps and nodes as singularities and he shows in an example that for higher singularities T needs not to be 0 (which lead Segre to a wrong conclusion about this example).

In this note we prove that T is never 0 if D has other singularities than only ordinary cusps and nodes. Moreover we give a general formula for $\dim_k T$. This might be of some interest since the calculations of Tannenbaum show that $H^0(N'_D)$ can be sometimes computed from the exact sequence $(*)$ and a computation of $\dim_k T$ and $H_0(\pi'_* N'_\pi)$, the latter with an embedded resolution of D.

Section 2

Let i: $D \to S$ be an embedding of a reduced curve D in a non-singular irreducible projective surface. Let π': $\tilde{D} \to D$ be the normalization of D and $\pi = i \circ \pi'$: $\tilde{D} \to S$. Let $N'_\pi = \operatorname{coker}(T_{\tilde{D}}(R) \to \pi^* T_S)$ where $T_{\tilde{D}}$ and T_S are the tangent sheafs and R is the ramification divisor, i.e. the divisor on $\tilde{D}$ defined by $F^0(\Omega^1_{\tilde{D}/D})$ where F^i denotes the i-th Fitting ideal. Let $N'_D = \ker(N_D \to T^1(D/k, \mathcal{O}_D))$, where N_D is the normal bundle of D in S.

In Lemma (1.5)(a) of [3] it is shown that

$$\pi'_* N'_\pi \cong N_D \otimes J \pi'_* \mathcal{O}_{\tilde{D}} \qquad (1)$$

[1] Since we consider this note as an appendix to the paper of Tannenbaum, we use without further reference the definitions and notations of [3].

where $J = F^1(\Omega^1_{D/k})$ is the Jacobian ideal in $\mathcal{O}_D$ and that there is an exact sequence

$$0 \to N'_D \to \pi'_* N'_\pi \to T \to 0 \tag{2}$$

where T is a torsion sheaf, concentrated in the singular points of D.

Now, for any $x \in D$ we define:

$$\delta_x = \dim_k \left(\pi'_* \mathcal{O}_{\tilde{D}} / \mathcal{O}_D \right)_x,$$

m_x = multiplicity of the local ring $\mathcal{O}_{D,x}$,
r_x = number of analytically irreducible components of (D, x),
$\tau_x = \dim_k \mathcal{O}_{D,x}/J_x$.
Note, that if $f(x_1, x_2) = 0$ is a local equation of the germ (D, x), then $\tau_x = \dim_k k[\![x_1, x_2]\!]/(\partial f/\partial x_1, \partial f/\partial x_2, f)$ which is the Tjurina number of (D, x).

PROPOSITION: *For any $x \in D$,*

$$\dim_k T_x = \dim_k \left(J\pi'_* \mathcal{O}_{\tilde{D}} \right)_x / J_x$$
$$= \tau_x + r_x - m_x - \delta_x.$$

COROLLARY: *$N'_D \to \pi'_* N'_\pi$ is an isomorphism iff D has at most ordinary cusps and nodes as singularities.*

PROOF OF THE PROPOSITION: From the definition of N'_D and the description of $T^1(D/k, \mathcal{O}_D)$ for hypersurface singularities it is clear that $N'_D = JN_D$. Since $N_{D,x} \cong \mathcal{O}_{D,x}$ we obtain from (1) and (2)

$$\dim_k T_x = \dim_k \left(J\pi'_* \mathcal{O}_{\tilde{D}} \right)_x / J_x.$$

Let ω_D be the regular differential forms in the sense of Rosenlicht, i.e. the dualizing sheaf of D. If $f = 0$ is a local equation of (D, x) then

$$\wedge df \colon \omega_{D,X} \xrightarrow{\cong} \mathcal{O}_{D,x}$$

is an isomorphism, where $\wedge df$ denotes exterior multiplication with df (cf. [2], Ch. II). There is a canonical map $\Omega^1_{D/k,x} \to \omega_{D,x}$ and we denote the image by $\Omega_{D,x}$. Then

$$\wedge df \colon \Omega_{D,x} \xrightarrow{\cong} J_x$$

is an isomorphism and hence

$$\wedge df \colon \Omega_{D,x} \cdot \pi'_* \mathcal{O}_{\tilde{D},x} / \Omega_{D,x} \xrightarrow{\cong} J_x \cdot \pi'_* \mathcal{O}_{\tilde{D},x} / J_x.$$

Consider the inclusions

$$\Omega_{D,x} \subset \Omega_{D,x} \cdot \pi'_* \mathcal{O}_{\tilde{D},x} \subset \pi'_* \Omega^{\mathrm{L}}_{\tilde{D}/k,x} \subset \omega_{D,x}.$$

Since $\dim_k \omega_{D,x}/\Omega_{D,x} = \dim_k \mathcal{O}_{D,x}/J_x = \tau_x$, $\dim_k \omega_{D,x}/\pi'_* \Omega^{\mathrm{L}}_{\tilde{D}/k,x} = \delta_x$ (by duality) and $\dim_k (\pi'_* \Omega^{\mathrm{L}}_{\tilde{D}/k,x}/\Omega_{D,x} \cdot \pi'_* \mathcal{O}_{\tilde{D},x}) = m_x - r_x$ the result follows. □

PROOF OF THE COROLLARY: For x a cusp or a node, $\tau_x + r_x = 3$ and $m_x = 2$, $\delta_x = 1$ hence $\dim_k T_x = 0$.

Now assume $\dim_k T_x = 0$, then $J_x = J_x \cdot \pi'_* \mathcal{O}_{\tilde{D},x}$. Let $\tilde{m}$ denote the Jacobson radical of $\pi'_* \mathcal{O}_{\tilde{D},x}$, and m the maximal ideal of $\mathcal{O}_{D,x}$. Then

$$2 = \dim_k J_x/mJ_x \geqq \dim_k J_x/\tilde{m}J_x = r_x$$

and $2 - r_x = \dim_k \tilde{m}J_x/mJ_x = \dim_k \tilde{m}/m\pi'_* \mathcal{O}_{\tilde{D},x} = m_x - r_x$, hence $m_x = 2$. By the holomorphic Morse lemma this implies that $f(x_1, x_2) = x_1^2 + g(x_2)$ in suitable locale coordinates. Hence $f \in (\partial f/\partial x_1, \partial f/\partial x_2)$ and therefore

$$m_x + \delta_x - r_x = \tau_x = \mu_x = 2\delta_x - r_x + 1$$

where μ_x is the Milnor number of (D, x) (cf. [1], Th. 10.5). Since $m_x = 2$, δ_x must be 1. But this is only possible if (D, x) is analytically isomorphic to an ordinary cusp ($f = x_1^2 + x_2^3$) or a node ($f = x_1^2 + x_2^2$). □

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References

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[2] J.P. SERRE: Groupes algébriques et corps de classes. Paris: Hermann 1959.

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