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P. C. KUTZKO

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THE EXCEPTIONAL REPRESENTATIONS OF Gl_2

P.C. Kutzko *

The purpose of this paper is to provide a characterization of the set of exceptional supercuspidal representations of $\mathrm{Gl}_2(F)$ where F is a local field of residual characteristic p and, in particular, to provide a proof for Lemma 4.2.2 of [5].

In §1, we describe the construction of a set of supercuspidal representations of $\mathrm{Gl}_2(F)$ by the method of Weil; supercuspidal representations which cannot be constructed in this way are said to be exceptional. In §2, we show that a "Weil representation" which belongs to a ramified quadratic extension of F may be constructed by induction from a one-dimensional representation of an open subgroup of $\mathrm{Gl}_2(F)$ and we show that the inducing representation must satisfy a certain condition ((3.01)). In §3, we show that, conversely, any supercuspidal representation which is induced from a representation satisfying (3.01) is a Weil representation. In §4, we show that condition (3.01) is equivalent to that given in Lemma 4.2.2 of [5]. In what follows we denote the ring of integers in F by $\mathcal{O}_F$, the maximal ideal of $\mathcal{O}_F$ by P_F and we set $q = [\mathcal{O}_F : P_F]$. Other notation used here is explained in [5].

Section 1

Let E/F be quadratic and separable, let τ be the nontrivial F-automorphism of E, denote by $N_{E/F}$ and $\mathrm{Tr}_{E/F}$ the norm and trace maps of E/F and let $\omega_{E/F}$ be the nontrivial character of the multiplicative group, $F^\times$, of F which is trivial on $N_{E/F}E^\times$.

Let $C_c^\infty(E)$ be the space of compactly supported, locally constant, complex-valued functions on E, let ψ be a nontrivial character of the additive group, F^+, of F and set $\psi_{E/F} = \psi \circ \mathrm{Tr}_{E/F}$. Then there is a unique choice of Haar measure, μ_ψ, on E^+ for which *Fourier inversion* holds with respect to $\psi_{E/F}$; that is, if we define the map $f \mapsto \hat{f}$ on $C_c^\infty(E)$ by $\hat{f}(\beta) = \int_E f(\alpha)\psi_{E/F}(\alpha\beta)\,d\mu_\psi(\alpha)$ then we have $\hat{\hat{f}}(x) = f(-x)$.

Now it is a consequence of the work of Weil [7] on symplectic groups

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(see [2], p. 7) that there is a representation r of $\mathrm{Sl}_2(F)$ on $C_c^\infty(E)$ such that

$$r\left(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}\right) f(\beta) = \omega_{E/F}(x)|x|_E^{1/2} f(x\beta) \tag{1.01}$$

$$r\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right) f(\beta) = \psi(yN_{E/F}\beta) f(\beta) \tag{1.02}$$

$$r\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) f(\beta) = \gamma_{E/F} \hat{f}(\beta^\tau) \tag{1.03}$$

where $\gamma_{E/F}$ is a complex number whose value may be found in Lemma 1.2 of [2].

In [2] it is shown that this representation commutes with left translations by elements α of E for which $N_{E/F}\alpha = 1$ so that $C_c^\infty(E)$ may be decomposed into a sum of $\mathrm{Sl}_2(F)$ invariant subspaces which are parametrized by characters of the subgroup ker $N_{E/F}$ of E^x. It is then shown that the representations of $\mathrm{Sl}_2(F)$ thus obtained are irreducible and that those representations which are parametrized by nontrivial characters of ker $N_{E/F}$ induce to supercuspidal representations of $\mathrm{Gl}_2(F)$ whose irreducible constituents will be referred to here as *Weil representations* of $\mathrm{Gl}_2(F)$ belonging to E/F.

Cartier has observed that the Weil representations belonging to E/F may also be obtained by first inducing the representation r to $\mathrm{Gl}_2(F)$ and then decomposing the resulting representation under a certain natural action of E^x and it is this approach, summarized in the following two lemmas, which we will use. Since this approach has been described in detail elsewhere $[N]$ we will omit proofs.

LEMMA 1.1: *There is a unique representation $\tilde{r}$ on the space $C_c^\infty(F^\times \times E)$ for which*

$$\tilde{r}\left(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}\right) f(z, \beta) = \omega_{E/F}(x)|x|_E^{1/2} f(z, x\beta) \tag{1.04}$$

$$\tilde{r}\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right) f(z, \beta) = \psi(yzN_{E/F}\beta) f(z, \beta) \tag{1.05}$$

$$\tilde{r}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) f(z, \beta) = \gamma_{E/F}\omega_{E/F}(z)|z|_E^{1/2} \hat{f}(z, z\beta^\tau) \tag{1.06}$$

$$\tilde{r}\left(\begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}\right) f(z, \beta) = f(zw, \beta) \tag{1.07}$$

where $f \mapsto \hat{f}$ is the Fourier transform in the second variable.

LEMMA 1.2: *Let θ be a character of $E^\times$ and let C_θ be the subspace of functions f in $C_c^\infty(F^\times \times E)$ for which $f(xN_{E/F}\alpha, \beta\alpha^{-1}) = \theta(\alpha)|\alpha|_E^{1/2} f(x, \beta)$, α in $E^\times$. Then C_θ is stable under $\tilde{r}$ and if θ is not of the form $\chi \circ N_{E/F}$ then C_θ is an irreducible supercuspidal* $\mathrm{Gl}_2(F)$ *subspace of $C_c^\infty(F^\times \times E)$.*

LEMMA 1.3: *Denote by $W_\psi(\theta)$ the representation of* $\mathrm{Gl}_2(F)$ *on C_θ obtained as above. Then $W_\psi(\theta)$ is equivalent to the representation $\pi(\theta)$ defined on page* 144 *of* [2]. *In particular, $W_\psi(\theta) = \pi(Ind_{W_E \uparrow W_F}\theta)$; that is, $W_\psi(\theta)$ corresponds in the sense of Langlands to the representation $Ind_{W_E \uparrow W_F}\theta$ of the Weil group, W_F, of F.*

PROOF: We recall that the representation $\pi(\theta)$ is induced from a representation $\pi(\theta, \psi)$ of the subgroup $G_{E/F}$ of $\mathrm{Gl}_2(F)$ consisting of elements g in $\mathrm{Gl}_2(F)$ for which det g lies in $N_{E/F}F^\times$. $\pi(\theta, \psi)$ acts on the subspace $\bar{C}_\theta$ of functions f in $C_c^\infty(E)$ which satisfy $f(\alpha\beta) = \theta^{-1}(\alpha)f(\beta)$ for α in ker $N_{E/F}$ and may be characterized by the following formulae ([2], p. 11):

$$\pi(\theta, \psi)\left(\begin{bmatrix} N_{E/F}\alpha & 0 \\ 0 & 1 \end{bmatrix}\right) f(\beta) = |\alpha|_E^{1/2}\theta(\alpha)f(\alpha\beta) \tag{1.08}$$

$$\pi(\theta, \psi)(g) = r(g) \quad \text{for } g \text{ in } \mathrm{Sl}_2(F). \tag{1.09}$$

(One should note that $\bar{C}_\theta$ is *invariant* under r.)

By Frobenius reciprocity, it will be enough to show that $\bar{C}_\theta$ is $G_{E/F}$-isomorphic to a subspace of C_θ. In fact, one checks easily that if C_θ^+ is the subspace of C_θ consisting of functions $f(x, \beta)$ for which $f(x, \beta) = 0$ when x is not a norm from E then C_θ^+ is the required subspace and that $f \mapsto \bar{f}$ where $\bar{f}(\beta) = f(1, \beta)$ is the required $G_{E/F}$-isomorphism from C_θ^+ to $\bar{C}_\theta$.

COROLLARY 1.4: *The equivalence class of $W_\psi(\theta)$ is independent of ψ. If θ_1, θ_2 are characters of $E^\times$ then $W_\psi(\theta_1)$ is equivalent to $W_\psi(\theta_2)$ if and only if either $\theta_2 = \theta_1$ or $\theta_2 = \theta_1^\tau$.*

We note that a Weil representation W may belong to more than one quadratic extension of F. If W belongs to the unramified quadratic extension of F, we say that W is an *unramified* Weil representation; otherwise we call W ramified. An irreducible supercuspidal representation of $\mathrm{Gl}_2(F)$ which is not a Weil representation will be called *exceptional.*

Section 2

The goal of this section is to describe a given Weil representation as an induced representation. To this end we need some preliminaries concerning the construction of supercuspidal representations by induction from open subgroups. Further details and proofs are given in [5]. Let V be the standard plane over F; i.e., $V = F \oplus F$. Then by a *lattice flag* in V we mean a sequence $L = \dots L_{-1}, L_0, L_1, \dots$ of free, rank two $\mathcal{O}_F$-sub-modules of V such that $L_k \supset L_{k+1}$, $P_F L_k = L_{k+2}$ and $\dim_{\mathcal{O}/P} L_k/L_{k+1} = 1$. There is a natural action of the ring, $M_2(F)$, of 2×2 matrices over F on the set of lattice flags which is, in fact, transitive; if we call two lattice flags L^1 and L^2 equivalent when there exists an integer m such that $L^2_k = L^1_{k+m}$ for all k then $M_2(F)$ acts transitively on the set of classes of flags as well.

Given a lattice flag L, we denote by $\mathfrak{b}_m(L)$ the subset of elements g in $M_2(F)$ for which $gL_k \subset L_{k+m}$ for all k; we set $\mathfrak{b}(L) = \mathfrak{b}_0(L)$ and note that for $k \geqslant 0$, $\mathfrak{b}_k(L)$ is a principal two-sided ideal in $\mathfrak{b}(L)$.

We set $B(L) = \mathfrak{b}^\times(L)$ and for $k \geqslant 1$ set $B_k(L) = 1 + \mathfrak{b}_k(L)$. We note that for $k \geqslant m/2 \geqslant 1$, the map $x \mapsto x - 1$ induces an isomorphism of abelian groups of $B_k(L)/B_m(L)$ and $\mathfrak{b}_k(L)/\mathfrak{b}_m(L)$. We note also that the pairing of $\mathfrak{b}_k(L)/\mathfrak{b}_m(L) \times \mathfrak{b}_{1-m}(L)/\mathfrak{b}_{1-k}(L)$ into F^+/P_F given by $(x, y) \mapsto \operatorname{tr} xy$ is nondegenerate. It follows that if ψ is a character of F^+ of conductor P_F and if for b in $\mathfrak{b}_{1-m}(L)$ we define the character ψ_b on $B_k(L)$ by $\psi_b(x) = \psi(\operatorname{tr} b(x-1))$ then $b \mapsto \psi_b$ induces an isomorphism of $\mathfrak{b}_{1-m}/\mathfrak{b}_{1-k}$ with the complex dual, $\widehat{B_k/B_m}$, of B_k/B_m whenever $k \geqslant m/2$.

Let, now, π be an irreducible supercuspidal representation of $\mathrm{Gl}_2(F)$. Call π *unramified* if it may be c-induced (see [3] for the precise definition) from the subgroup $F^\times \cdot \mathrm{Gl}_2(\mathcal{O}_F)$ and call π *ramified* otherwise. Then it is well known (see, e.g., [1]) that a Weil representation is unramified as a Weil representation if and only if it is unramified in the above sense.

On the other hand, [3], ramified supercuspidal representations may be characterized as representations which may be induced from the normalizer, $K(L)$, of some subgroup $B(L)$ of $\mathrm{Gl}_2(F)$ (all such subgroups are, of course, conjugate).

To be precise, call an element b in $M_2(F)$ $\mathfrak{b}(L)$-*generic* of level $2k+1$ if

1. $F[x]/F$ is quadratic ramified;
2. $F[x] \cap \mathfrak{b}(L) = \mathcal{O}_{F[x]}$;
3. $\nu_{F[x]}(x) = 2k+1$.

It is easy to see that x lies in $\mathfrak{b}_{2k+1}(L)$ and that, in fact, the set of $\mathfrak{b}(L)$-generic elements of level $2k+1$ is precisely $\Pi_L^{2k+1} B(L)$ where Π_L is any generator of the ideal $\mathfrak{b}_1(L)$ of $\mathfrak{b}(L)$.

Proposition 2.1: 1. *With notation as above, let n be a positive integer and let b be a $\mathfrak{b}(L)$-generic element of level $1-2n$. Let θ be a character of the subgroup $T_b = (F[b])^\times$ of $\mathrm{Gl}_2(F)$ such that $\theta(\beta) = \psi(\mathrm{Tr}_{F[b]/F} b(\beta - 1))$ for β in $U^n_{F[b]}$. Then the complex-valued function $\theta\psi_b$ on $T_b B_n(L)$ defined* *by* $\theta\psi_b(\beta k)=\theta(\beta)\psi_b(k)$, β *in* T_b, k *in* $B_n(L)$ *is in fact a well-defined character of* $T_bB_n(L)$ *which induces an irreducible supercuspidal representation* $\pi(L;\ \psi_b,\ \theta)$ *of* $Gl_2(F)$. *We have* $\pi(L;\ \psi_b,\ \theta_1)\cong\pi(L;\ \psi_b,\ \theta_2)$ *if and only if* $\theta_1=\theta_2$.

2. *Given an irreducible ramified supercuspidal representation* π *of* $\mathrm{Gl}_2(F)$ *and a lattice flag L there exist* n, b, θ *as above and a character* χ *of* $F^\times$ *so that* $\pi\cong\pi(L;\ \psi_b,\ \theta)\otimes\chi\circ\det$. *If* $f(\chi)\leqslant n$ *then* χ *may be taken to be trivial.*

PROOF: This is Proposition 3.1.1 of [5].

In order to describe a given Weil representation W as an induced representation it will be helpful to write W as $W(\theta)$ where θ enjoys certain properties. Specifically, if we denote by $f(\theta)$ the exponent of the conductor of θ and by $d(E/F)$ the exponent of the different of the extension E/F then the existence of an appropriate character θ is given by the following lemma.

LEMMA 2.2: *Let* W *be a ramified Weil representation of* $\mathrm{Gl}_2(F)$. *Then there exists an extension* E/F, *a character* θ *of* $E^\times$ *such that* $f(\theta)\geqslant 2d(E/F)-1$ *and* $f(\theta)-d(E/F)$ *is odd, and a character* χ *of* $F^\times$ *so that* W *is equivalent to the representation* $W(\theta)\otimes\chi\circ\det$. *If there exist* E', θ', χ' *with the above properties and if* $E'\neq E$ *then* $p=2$, $f(\theta)=2d(E/F)-1$ $=2d(E'/F)-1=f(\theta')$ *and* $f(\omega_{E/F}\cdot\omega_{E'/F}^{-1})=\mathrm{d}(E/F)$.

PROOF: This follows from Corollary 1.18 of [4] and the fact that $W(\theta)=\pi(Ind_{W_E\uparrow W_F}\theta)$.

In what follows, we fix a ramified quadratic extension E/F and a character θ of $E^\times$ for which $f(\theta)-d(E/F)$ is odd and $f(\theta)\geqslant 2d(E/F)-1$; we set $n(\theta)=1/2(f(\theta)+d(E/F)-1)$. In addition we fix a character ψ of F^+ of conductor P_F which if $p=2$ has the additional property that $\psi(x^2+x)=1$ for x in $\mathcal{O}_F$. We denote by $b=b_\psi(\theta)$ an element of E for which $\theta(\beta)=\psi(\mathrm{Tr}_{E/F}b(\beta-1))$ for β in $U_E^{[(f(\theta)+1)/2]}$ and by $c_\psi=c_\psi(E/F)$ an element of F for which $\omega_{E/F}(x)=\psi(c_\psi(x-1))$ for x in $U_F^{[k(E/F)+1/2]}$.

Finally, we fix a lattice flag L^n, $n=n(\theta)$, by setting $L_0^n=P_F^{1-n}\oplus\mathcal{O}_F$; $L_1^n=P_F^{1-n}\oplus P_F$. We note that then

$$\mathfrak{b}_{2k}(L^n)=P_F^k\begin{bmatrix}\mathcal{O}_F & P_F^{1-n}\\ P_F^n & \mathcal{O}_F\end{bmatrix};\qquad \mathfrak{b}_{2k+1}(L^n)=P_F^k\begin{bmatrix}P_F & P_F^{1-n}\\ P_F^n & P_F\end{bmatrix}.$$

PROPOSITION 2.3: *With notation as above, define the function* f_0 *in the space* C_θ *by* $f_0(x,\ \beta)=\theta^{-1}(\beta)|\beta|_E^{-1/2}$ *if* $xN_{E/F}\beta$ *lies in* $U_F^{[(n+1)/2]}$, $f_0(x,$

$\beta) = 0$ *otherwise. Then for k in $B_n(L^n)$ we have that*

$$W(\theta)(k)f_0 = \psi_{\bar{b}}(k)f_0$$

where

$$\bar{b} = \begin{bmatrix} 0 & -N_{E/F}b \\ 1 & \mathrm{Tr}_{E/F}b + c_\psi \end{bmatrix}.$$

PROOF: It is a straightforward computation, using formulae (1.04), (1.05) and (1.07), that

$$W(\theta)(k)f_0 = \psi_{\bar{b}}(k)f_0$$

when k lies in $B_n(L^n)$ and is upper triangular. Our result will thus follow if we show that

$$W(\theta)\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} f_0 = \psi(-yN_{E/F}b)f_0$$

when b lies in $P_F^{n+[n/2]}$. Since

$$\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

it will suffice, by (1.05), (1.06), to show that if $\hat{f}_0(z, z\beta^\tau) \neq 0$ then $\psi(-yzN_{E/F}\beta) = \psi(-yN_{E/F}b)$; that is, to show that the support of the function $\hat{f}_0(z, z\beta^\tau)$ is contained in the set of (z, β) for which $zN_{E/F}(\beta b^{-1})$ lies in $U_F^{[(n+1)/2]}$.

Now we have that

$$\hat{f}_0(z, z\beta^\tau) = \int_Y \theta^{-1}(\alpha)|\alpha|_E^{-1/2}\psi_{E/F}(\alpha z\beta^\tau)\,d\mu_\psi(\alpha)$$

where Y is the set of α for which $N_{E/F}\alpha$ lies in $z^{-1}U^{[(n+1)/2]}$. Since $f(\theta) = 2n - d(E/F) + 1 \geqslant d(E/F)$ we have that $N_{E/F}(U_E^{[(f(\theta)+1)/2]}) \subset U_F^{[(n+1)/2]}$ and thus that $\hat{f}_0(z, z\beta^\tau)$ is a nonzero multiple of

$$\begin{aligned} &\int_{P_E^{[(f(\theta)+1)/2]}}\int_Y \theta^{-1}(\alpha(1+\gamma))|\alpha|^{-1/2} \\ &\quad\times\psi_{E/F}(\alpha(1+\gamma)z\beta^\tau)\,d\mu_\psi(\alpha)\,d\mu_\psi(\gamma) \\ &= \int_Y \theta^{-1}(\alpha)|\alpha|_E^{-1/2}\psi_{E/F}(\alpha z\beta^\tau) \\ &\quad\times\int_{P_E^{[(f(\theta)+1)/2]}}\psi_{E/F}((\alpha z\beta^\tau - b)\gamma)\,d\mu_\psi(\gamma)\,d\mu_\psi(\alpha) \\ &= 0 \end{aligned}$$

unless $\alpha z\beta^{\tau} - b$ lies in $P_E^{2-d(E/F)-[(f(\theta)+1)/2]}$, that is, unless $\alpha z\beta^{\tau}b^{-1}$ lies in $U_E^{[(f(\theta)/2)]}$. (Here, one uses the fact that $\nu_E(b) = 1 - 2n$ so that $2 - d(E/F) - [(f(\theta) + 1)/2] - \nu_E(b) = f(\theta) - [(f(\theta) + 1)/2] = [f(\theta)/2]$.) Finally, since $zN_{E/F}\alpha$ lies in $U_F^{[(n+1)/2]}$ and since, in general, $N_{E/F}U_E^r \subset U_F^s$ where $s = \min([(r + d(E/F))/2],\ r)$ one checks that $\hat{f}_0(z, z\beta^{\tau}) = 0$ unless $zN_{E/F}(\beta b^{-1})$ lies in $U_F^{[(n+1)/2]}$.

COROLLARY 2.4: *With notation as above, there exists a character $\bar{\theta}$ of $T_{\bar{b}}$ such that $W(\theta)$ is equivalent with $\pi(L^n; \psi_{\bar{b}}, \bar{\theta})$.*

PROOF: We note first that $\bar{b}$ is $\mathfrak{b}(L^n)$-generic of level $1 - 2n$ since $\nu_F(\mathrm{Tr}_{E/F}b + c_{\psi}) \geqslant \min(1 - n,\ 1 - d(E/F)) = 1 - n$. Next, since $\psi_{\bar{b}}$ is stable under $T_{\bar{b}}B_n(L^n)$, the span under $T_{\bar{b}}B_n(L^n)$ of f_0 decomposes into a sum of the form $\oplus\langle f_{\bar{\theta}_j}\rangle$ where $\bar{\theta}_j$ is a character of $T_{\bar{b}}$ of the form described in Proposition 2.1 and where $W(\theta)(h)f_{\bar{\theta}_j} = \bar{\theta}_j\psi_{\bar{b}}(h)f_{\bar{\theta}_j}$ for h in $T_{\bar{b}}B_n(L^n)$. Finally, since distinct characters $\bar{\theta}_j\psi_{\bar{b}}$ induce distinct irreducible supercuspidal representations of $\mathrm{Gl}_2(F)$, we see that the span under $T_{\bar{b}}B_n(L^n)$ of f_0 is one-dimensional, that we may set $\bar{\theta} = \bar{\theta}_1$ whence $f_{\bar{\theta}_1} = f_0$, and $W(\theta)$ is equivalent to $\pi(L^n; \psi_{\bar{b}}, \bar{\theta})$.

Section 3

In this section we fix, once and for all, an integer $n \geqslant 1$ and a $\mathfrak{b}(L^n)$-generic element, $\bar{b}$, of level $1 - 2n$. Our goal is to determine whether some or all of the representations $\pi(L^n; \psi_{\bar{b}}, \theta)$ are Weil representations. From Proposition 2.3, it is clear that in order that some representation $\pi(L^n; \psi_{\bar{b}}, \theta)$ be Weil it is necessary that there exist a ramified quadratic extension E/F with $3d(E/F) \leqslant 2(n + 1)$ and an element b in E with $\nu_E(b) = 1 - 2n$ such that

$$i.\ \mathrm{tr}\,\bar{b} \equiv \mathrm{Tr}_{E/F}b + c_{\psi}(E/F)\ \left(\mathrm{mod}\ P_F^{-[(n-1)/2]}\right)$$

$$ii.\ (\det \bar{b})/N_{E/F} \equiv 1\ \left(\mathrm{mod}\ P_F^{[(n+1)/2]}\right). \tag{3.01}$$

We will say that such an element $\bar{b}$ is *Weil-generic.* Our main result in this section is

PROPOSITION 3.1: *The representation $\pi(L^n; \psi_{\bar{b}}, \theta)$ is Weil if and only if $\bar{b}$ is Weil-generic.*

We will need several lemmas.

LEMMA 3.2: *Suppose that the pair (E, b) satisfies condition* (3.01). *Let E_1/F be ramified quadratic and suppose for some b_1 in E_1 we have*

$\mathrm{Tr}_{E_1/F}b_1 \equiv \mathrm{Tr}_{E/F}b$ (mod $P_F^{-[(n-1)/2]}$), $N_{E_1/F}b_1/N_{E/F}b \equiv 1$ (mod $P_F^{[(n+1)/2]}$). *Then the pair* (E_1, b_1) *satisfies condition* (3.01).

PROOF: We must show that $c_\psi(E/F) \equiv c_\psi(E_1/F)$ (mod $P_F^{-[(n-1)/2]}$). To begin with, we note that since $2(n+1) \geqslant 3d(E/F)$ it follows that $-[(n-1)/2] > \frac{1}{2}d(E/F) - n$. In addition, we have that

$$d(E/F) = \min\Big(2\big(\nu_F(\mathrm{Tr}_{E/F}b) + n\big), 2\nu_F(2) + 1\Big),$$

$$d(E_1/F) = \min\Big(2\Big(\nu_F\big(\mathrm{Tr}_{E_1/F}b_1\big) + n\Big), 2\nu_F(2) + 1\Big).$$

One may then deduce from the congruence $\mathrm{Tr}_{E_1/F}b_1 \equiv \mathrm{Tr}_{E/F}b$ (mod $P_F^{-[(n-1)/2]}$) that $d(E_1/F) = d(E/F)$.

Now since $-[(n-1)/2] \leqslant 1 - [(d(E/F)+1)/2]$, the congruence $c_\psi(E/F) \equiv c_\psi(E_1/F)$ (mod $P_F^{-[(n-1)/2]}$) is equivalent to the statement that the restrictions of $\omega_{E/F}$ and $\omega_{E_1/F}$ to $U_F^{[(n+1)/2]}$ coincide. However $\omega_{E/F}|_{U_F^{[(n+1)/2]}}$ is determined by the data $f(\omega_{E/F}) = d(E/F)$, $\omega_{E/F}^2 = 1$, $\omega_{E/F}(1 + x\ \mathrm{Tr}_{E/F}b + x^2 N_{E/F}b) = 1$ for x with $2\nu_F(x) \geqslant 2n - 1 + [(n+1)/2]$. Since

$$\tfrac{1}{2}(2n-1) + [(n+1)/2]) + \nu_F\Big(\mathrm{Tr}_{E_1/F}b_1 - \nu_F\big(\mathrm{Tr}_{E/F}b\big)\Big)$$

$$\geqslant d(E/F);$$

$$2n - 1 + [(n+1)/2] + \nu_F\Big(N_{E_1/F}b_1 - N_{E/F}b\Big)\Big) \geqslant d(E/F),$$

we see that $\omega_{E_1/F}$ satisfies the above data, whence our result.

Let E/F be quadratic ramified with $3d(E/F) \leqslant 2(n+1)$ and let b be an element of E with $\nu_E(b) = 1 - 2n$. Denote by $W(E;\, b)$ the set of representations $W(\theta)$ where θ is a character of $E^\times$ such that $\theta(\beta) = \psi(\mathrm{Tr}_{E/F}b(\beta - 1))$ for β in $U_E^{[(2n-d(E/F)+2)/2]}$ and $\theta(\tilde{\omega}_F)\omega_{E/F}(\pi_F) = 1$ for some fixed prime element $\tilde{\omega}_F$ of F.

LEMMA 3.3: *Let* $m = [\frac{1}{2}(2n - d(E/F) + 2)]$. *Then* $W(E;\, b)$ *consists of* $(q-1)q^{m-1}$ *distinct representations if* $3d(E/F) < 2(n+1)$ *and* $\frac{1}{2}(q-1)q^{m-1}$ *distinct representations if* $3d(E/F) = 2(n+1)$.

PROOF: This follows from the fact that $[U_E : U_E^m] = (q-1)q^{m-1}$ together with Corollary 1.4 and the fact that $b \equiv b^\tau$ (mod $P_E^{2-d(E/F)-m}$) if and only if $-2n + d(E/F) \geqslant 2 - d(E/F) - m$, that is, if and only if $3d(E/F) \geqslant 2(n+1)$.

LEMMA 3.4: *Let* S *be the subgroup of* $F \times F^\times$ *consisting of pairs* (x, y) *with* x *in* $\mathrm{Tr}_{E/F}P_E^{1-n-[d(E/F)/2]}$ *and* y *in* $N_{E/F}U_E^{n-[d(E/F)/2]}$. *Suppose* E_1, E_2 *are* *ramified quadratic extensions of* F, b_i *lies in* E_i *and* $\mathrm{Tr}_{E_i/F} b_i = \mathrm{Tr}_{E/F} b \pmod{P_F^{-[(n-1)/2]}}$; $N_{E_i/F} b_i / N_{E/F} b \equiv 1 \pmod{P_F^{[(n+1)/2]}}$. *Suppose further that* $(\mathrm{Tr}_{E_1/F} b_1, N_{E_1/F} b_1) \not\equiv (\mathrm{Tr}_{E_2/F} b_2, N_{E_2/F} b_2) \pmod{S}$. *Then* $W(E_1, b_1)$ *and* $W(E_2, b_2)$ *are disjoint sets.*

PROOF: It was shown in Lemma 3.2 that $d(E_1/F) = d(E_2/F) \geqslant \frac{2}{3}(n+1)$ and that if $d(E_i/F) = \frac{2}{3}(n+1)$ then $f(\omega_{E_1/F}\omega_{E_2/F}^{-1}) < d(E_1/F)$. It follows by Lemma 2.2 that $W(E_1, b_1)$ and $W(E_2, b_2)$ are disjoint unless $E_1 = E_2$.

Suppose now that $E_1 = E_2$, that $W(\theta_i)$ lies in $W(E_i, b_i)$ and that $W(\theta_1)$ is equivalent with $W(\theta_2)$. By Corollary 1.4, there exists an element ν in the galois group of E_1/F such that

$$b_1 \equiv b_2^{\nu} \left(\mathrm{mod}\ P_E^{1-n-[d(E/F)/2]}\right)$$

which contradicts our hypothesis.

LEMMA 3.5: $[P_E^{-[(n-1)/2]} \times U_F^{[(n-1)/2]} : S] = q^{[1/2(d(E/F)-1)]}$ *if* $2(n+1) > 3d(E/F)$; $[P_E^{-[(n-1)/2]} \times U_F^{[(n+1)/2]} : S] = 2q^{[1/2(d(E/F)-1)]}$ *if* $2(n+1) = 3d(E/F)$.

PROOF: Straightforward.

PROOF OF PROPOSITION 3.1: Suppose that $\bar{b}$ is Weil-generic. Then $\bar{b}$ is $K(L^n)$ conjugate to

$$\begin{bmatrix} 0 & -\det \bar{b} \\ 1 & \mathrm{tr}\, \bar{b} \end{bmatrix}$$

and we have thus produced, by Lemmas 3.3, 3.4, 3.5, $(q-1)q^{n-1}$ distinct irreducible Weil summands of $Ind_{B_n(L^n) \uparrow \mathrm{Gl}_2(F)} \psi_{\bar{b}}$ each having central character which is trivial at $\tilde{\omega}_F$. On the other hand, the total number of such summands is

$$\left[U_{F[\bar{b}]} B_n(L^n) : B_n(L^n)\right] = \left[U_{f[\bar{b}]} : U_{f[\bar{b}]}^n\right] = (q-1)q^{n-1}.$$

Since given any representation $\pi(L^n; \psi_{\bar{b}}, \theta)$ we may find a character χ of $F^\times$ such that $f(\chi) = 0$ and $\pi(L^n; \psi_b, \theta) \otimes \chi \circ \det$ has a central character trivial on $\tilde{\omega}_F$ we have shown that all representations $\pi(L^n; \psi_b, \theta)$ are Weil representations.

Section 4

The purpose of this section is to prove the following proposition which gives a simple characterization of the property of being Weil-generic.

PROPOSITION 4.1: *Fix $n \geqslant 1$ and let L^n be the lattice flag described in* §3. *Let $\bar{b}$ be $\mathfrak{b}(L^n)$-generic of level $1-2n$ and set $\bar{E}=F(\bar{b})$. Then the following are equivalent.*

1. *$\bar{b}$ is Weil-generic.*
2. *Either $2(n+1) > 3d(\bar{E}/F)$ or the polynomial $X^3 - (\operatorname{tr} \bar{b})X^2 + \det \bar{b}$ has root in F.*
3. *There exists a ramified quadratic extension E/F with $3d(E/F) \leqslant 2(n+1)$ and an element b in E with $N_{E/F}b = \det \bar{b}$ and $\operatorname{Tr}_{E/F}b + c_\psi(E/F) \equiv \operatorname{tr} \bar{b} \pmod{P_F^{[d(E/F)/2]+1-n}}$.*

PROOF: $1 \Rightarrow 2$. Suppose that $\bar{b}$ is Weil-generic and that $2(n+1) \leqslant 3d(\bar{E}/F)$. Pick b in E satisfying (3.01). We show first that $3d(E/F) = 2(n+1)$. Suppose that $d(E/F)$ is odd. Then since, by assumption, $3d(E/F) \leqslant 2(n+1)$ we must have $3d(E/F) < 2(n+1)$. Now (see Lemma 3.2), $d(E/F) = \min(2(\nu_F(\operatorname{Tr}_{E/F}b)+n),\ 2\nu_F(2)+1)$ so that $2\nu_F(2)+1 = d(E/F) < 2(n+1)/3$ and also $\nu_F(\operatorname{Tr}_{E/F}b) \geqslant \nu_F(2)+1-n$. By (3.01) $-i$,

$$\nu_F(\operatorname{tr} \bar{b}) \geqslant \min\left(\nu_F(2)+1-n,\ -2\nu_F(2),\ -[(n-1)/2]\right).$$

However, from $2\nu_F(2)+1 < 2(n+1)/3$ we obtain that $\nu_F(2)+1-n \leqslant -2\nu_F(2)$ while $\nu_F(2)+1-n \leqslant -[(n-1)/2]$ since $n \geqslant d(E/F) = 2\nu_F(2)+1$. Thus $\nu_F(\operatorname{tr} \bar{b}) \geqslant \nu_F(2)+1-n$ whence $d(\bar{E}/F) = 2\nu_F(2)+1 = d(E/F)$. Therefore $3d(\bar{E}/F) < 2(n+1)$ which is false.

Now suppose that $d(E/F)$ is even so that $d(E/F) = 2(\nu_F(\operatorname{Tr}_{E/F}b)+n) \leqslant 2\nu_F(2)$. Then if $3d(E/F) < 2(n+1)$, we would have $\nu_F(\operatorname{Tr}_{E/F}b) = \frac{1}{2}d(E/F) - n < 1 - d(E/F) = \nu_F(c_\psi(E/F))$. Since $\frac{1}{2}d(E/F) - n \leqslant -[(n-1)/2]$ it would follow that $\nu_F(\operatorname{Tr}_{\bar{E}/F}\bar{b}) = \frac{1}{2}d(E/F) - n$ whence $d(\bar{E}/F) = d(E/F) < \frac{2}{3}(n+1)$. Thus we have shown that $3d(E/F) = 2(n+1)$ and we note that $d(E/F)$ is even.

Now by definition,

$$1 = \psi\left(c_\psi(E/F)\left(N_{E/F}(1+xb) - 1\right)\right)$$

$$= \psi\left(c_\psi(E/F)\left(x \operatorname{Tr}_{E/F}b + x^2 N_{E/F}b\right)\right)$$

for x in $P_F^{d/2+n-1}$. Setting $x = y\ \operatorname{Tr}_{E/F}b/N_{E/F}b$ and noting that $\nu_F(\operatorname{Tr}_{E/F}b) = \frac{1}{2}d(E/F) - n$ while $\nu_F(N_{E/F}b) = 1-2n$ we see that

$$\psi\left(\left(c_\psi(E/F)\left(\operatorname{Tr}_{E/F}b\right)^2/N_{E/F}b\right)\left(y+y^2\right)\right) = 1$$

for y in $\mathcal{O}_F$. Since ψ has been picked so that $\psi(y+y^2) = 1$ for y in $\mathcal{O}_F$ ($p=2$ here since $d(E/F)$ is even) we see that $c_\psi(E/F)(\operatorname{Tr}_{E/F}b)^2/$

$N_{E/F}b \equiv 1 (\mathrm{mod}\ P_F)$. By (3.01) it follows that $X = \mathrm{Tr}_{E/F}b$ satisfies the congruence $X^3 - \mathrm{tr}\ \bar{b}\ X^2 + \det \bar{b} \equiv 0\ (\mathrm{mod}\ P_F^{2-2n})$. A Hensel's lemma argument now shows that the polynomial $X^3 - \mathrm{tr}\ \bar{b}X^2 + \det \bar{b}$ has a root in F.

$2 \Rightarrow 3$. If $2(n+1) > 3d(\bar{E}/F)$ then $\nu_F(c_\psi(\bar{E}/F)) = 1 - d(\bar{E}/F) \geqslant [d(\bar{E}/F)/2] + 1 - n$ and so we may take $E = \bar{E}$, $b = \bar{b}$.

Now suppose that $2(n+1) \leqslant 3d(\bar{E}/F)$ and let s be a root in F of the polynomial $X^3 - (\mathrm{tr}\ \bar{b})X^2 + \det \bar{b}$. Then since $\nu_F(\mathrm{tr}\ \bar{b}) \geqslant [(d(\bar{E}/F) + 1)/2] - n$ while $\nu_F(\det \bar{b}) = 1 - 2n$, a standard argument shows that $\nu_F(s) = \frac{1}{3}(1 - 2n) \leqslant \nu_F(2) - n$. It follows that the polynomial $X^2 - sX + \det \bar{b}$ is irreducible over F and that if E/F is a splitting field then $3d(E/F) = 2(n+1)$. Let b be a root in E of the polynomial $X^2 - sX + \det \bar{b}$. Then since $d(E/F)$ is even we obtain, as above, that $c_\psi(E/F)(\mathrm{Tr}_{E/F}b)^2/N_{E/F}b \equiv 1\ (\mathrm{mod}\ P_F)$ whence $c_\psi(E/F) \equiv N_{E/F}b/(\mathrm{Tr}_{E/F}b)^2\ (\mathrm{mod}\ P_F^{1+(1/3)(1-2n)})$. Finally, $N_{E/F}b = \det \bar{b}$ while $\mathrm{Tr}_{E/F}b$ satisfies $X^3 - \mathrm{Tr}\ \bar{b}X^2 + \det \bar{b} = 0$ so that $N_{E/F}b/(\mathrm{Tr}_{E/F}b)^2 = \mathrm{tr}\ \bar{b} - \mathrm{Tr}_{E/F}b$. Combining this last equation with the congruence preceding it and noting that $1 + \frac{1}{3}(1 - 2n) = [d(E/F)/2] + 1 - n$ yields our result.

$3 \Rightarrow 1$. Set $d = d(E/F)$ and suppose by induction that for $1 \leqslant j \leqslant k$ we have picked quadratic extensions E_j/F and elements b_j in E_j such that $d(E_j/F) = d$, $N_{E_j/F}b_j = \det \bar{b}$ and $\mathrm{Tr}_{E_j/F}b_j + c_\psi(E_j/F) \equiv \mathrm{tr}\ \bar{b}\ (\mathrm{mod}\ P_F^{[d/2]-n+j})$. Set $\bar{s} = \mathrm{tr}\ \bar{b}$, $s_k = \mathrm{Tr}_{E_k/F}b_k$, $\Delta = \det b$, let a be an element of $P_F^{[d/2]-n+k}$ and set $s_a = s_k + a$. Let E_a be a splitting field of $X^2 - s_a X + \Delta$ over F and pick a root, b_a, of this polynomial in E_a.

Now since $\nu_F(s_a - s_k) \geqslant [d/2] - n + k$ and since $d(E_k/F) = \min(2(\nu_F(s_k) + n), 2\nu_F(2) + 1)$ while $d(E_a/F) = \min(2(\nu_F(s_a) + n), 2\nu_F(2) + 1)$ it follows that $d(E_a/F) = d(E_k/F) = d$. Since $\nu_F(s_a - s_k) \geqslant [d/2] - n + k$ while $N_{E_k/F}b_k = N_{E_a/F}b_a$ it follows that $U_F^l \cap N_{E_k/F}E_k^x = U_F^l \cap N_{E_a/F}E_a^x$ where $l = \max 2[(d+1)/2] - 2k, [(d+1)/2]$ and thus that $c_\psi(E_a/F) \equiv c_\psi(E_k/F)\ (\mathrm{mod}\ P_F^{1-l})$.

Since $1 + 2k - 2[(d+1)/2] \geqslant [d/2] - n + k + 1$ when $k \geqslant 1$ while $1 - [(d+1)/2] \geqslant [d/2] - n + k + 1$ when $k \leqslant n - d$ we see that if we set $a = \bar{s} - c_\psi(E_k/F) - s_k$, $b_{k+1} = b_a$, $E_{k+1} = E_a$ then the pair (b_{k+1}, E_{k+1}) satisfies our inductive hypothesis whenever $k \leqslant n - d$. Finally since $-[(n - 1)/2] + n - [(d)/2] \leqslant n - d + 1$ we see that we may find (b_k, E_k) as above for $k = [(n-1)/2] + n - [(d)/2]$. The pair (b_k, E_k) then satisfies (3.01) whence $\bar{b}$ is Weil-generic.

We may now state our main result.

THEOREM 4.2: *Let π be an irreducible ramified supercuspidal representation of* $\mathrm{Gl}_2(F)$ *and let L be a lattice flag. Pick n, b, θ as in Proposition* 2.1 *so that $\pi \cong \pi(L; \psi_b, \theta) \otimes \chi \circ \det$ and set $E = F(b)$. Then π is an exceptional representation of* $\mathrm{Gl}_2(F)$ *if and only if $2(n+1) \leqslant 3d(E/F)$ and the polynomial $X^3 - (\mathrm{tr}\ b)X^2 + \det b$ is irreducible over F.*

PROOF: Propositions 3.1, 4.1.

We note, in conclusion, that we obtain as a consequence

COROLLARY 4.3: $Gl_2(F)$ *has no exceptional representations unless* $p = 2$.

PROOF: If $p \neq 2$, then we have $d(E/F) = 1$ for all quadratic ramified extensions E/F. Since $n \geqslant 1$ in all cases we have that $2(n+1) > 3d(E/F)$.

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Department of Mathematics
University of Iowa
Iowa City, IA 52242
USA