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INFINITESIMAL VARIATIONS OF HODGE STRUCTURE (III): DETERMINANTAL VARIETIES AND THE INFINITESIMAL INVARIANT OF NORMAL FUNCTIONS

Phillip A. Griffiths

This is the third in our series of papers concerning infinitesimal variations of Hodge structure. In Section 1 of the first paper we associated five invariants to an infinitesimal variation of Hodge structure $V = \{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ (actually, the fourth invariant $\delta\nu$ was associated to an infinitesimal normal function, which is a slight refinement of V). In the preceeding papers we studied the first two invariants, and in this paper we shall study the third and fourth of these.

Recall that V is given by a polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ of weight n together with a linear map

$$\delta : T \to \bigoplus_{p} \operatorname{Hom}(H^{p,q}, H^{p-1,q+1})$$

satisfying certain conditions. [(1)]* The $(n-2k)^{\text{th}}$ iterate of δ induces

$$\delta^{n-2k} : \operatorname{Sym}^{n-2k} T \to \operatorname{Hom}^{(s)}(H^{n-k,k}, H^{k,n-k}).$$

Associated to this data are the determinantal varieties

$$\Xi_{k,l} = \{\xi \in \mathbb{P}T : \operatorname{rank} \delta^{n-2k}(\xi) \leqslant l\},$$

and in Section 5 we shall discuss these in the extreme cases

$$k = 0, l = 1 \quad \text{and} \quad k = 0, l = h^{n,0} - 1,$$

for infinitesimal variations of Hodge structure that arise from geometry. Therefore, for $L \to X$ a line bundle over a smooth n-dimensional variety X (assume $|L|$ has no base points), we are interested in the map

$$\kappa : H^n(X, KL^{-2}) \to \operatorname{Hom}^{(s)}(H^0(X, L), H^n(X, KL^{-1}))$$

given by cup product. The points $p \in \mathbb{P}H^n(X, KL^{-2}) = \mathbb{P}H^0(X, L^2)^*$ such that $\kappa(p)$ has rank $\leqslant l$ turn out to have a very nice projective interpretation when $l = 1$ and when $l = h^0(X, L) - 1$ (cf. Theorems (5.2)

* These numbers refer to notes at the end of the paper.

and (5.17)). (2) Applying these results to variation of Hodge structures requires understanding the image

$$\sigma^n H^1(X,\Theta)\cap\{p:\operatorname{rank}\kappa(p)\leqslant l\} \tag{$*$}$$

where

$$\sigma^n:\operatorname{Sym}^n H^1(X,\Theta)\to H^n(X,K^*)$$

is the polarized determinant map used in Section 2, and we are only able to determine $(*)$ in special cases (cf. the fundamental problem (5.21)). (3)

In the case of curves the rank one transformations in the image of

$$H^1(C,\Theta)\to\operatorname{Hom}^{(s)}\big(H^0(C,K),H^1(C,\mathcal{O})\big)$$

turn out to be closely related to the classical Shiffer variations, which by definition are nonzero classes $\theta_p\in H^1(C,\Theta)$ lying over points $\varphi_{2k}(p)\in\mathbb{P}H^1(C,\Theta)$ on the bicanonical curve (cf. (5.13)–(5.15)). A corollary is yet another construction of a general curve of genus $g\geqslant 5$ from its infinitesimal variation of Hodge structure. (4)

In Section 6 we discuss the infinitesimal invariant $\delta\nu$ associated to a normal function ν. (5) Although motivated by elementary considerations, this invariant seems to be rather subtle and its construction occupies all of Section 6(a). To illustrate the problem, we let $D_s=\sum_i p_i(s)-q_i(s)\in\operatorname{Div}^0(C_s)$ be a family of degree zero divisors on a family $\{C_s\}$ of curves, and we consider the classical abelian sum

$$u_s(D_s)=\left(\ldots,\sum_i\int_{q_i(s)}^{p_i(s)}\omega_\alpha(s),\ldots\right)$$

where the $\omega_\alpha(s)\in H^0(C_s,K)$ give a basis. What we want to do is simply *intrinsically* interpret the differential $du_s(D_s)$. The problem in doing this is that there is no obvious canonical identification of all the $\mathbb{P}^{g-1}$'s of the canonical curves $\varphi_K(C_s)$. More or less equivalently, the universal family $(\mathcal{H}_g\times\mathbb{C}^g)/\mathbb{Z}^{2g}\to\mathcal{H}_g$ of principally polarized abelian varieties is *not* a homogeneous manifold (it is only acted on by $\mathrm{Sp}(2g,\mathbb{Z})$). (6)

However, we are able to intrinsically interpret at least part of $du_s(D_s)$, and to give in general a definition of $\delta\nu$ for a normal function ν. The problem then arises of geometrically interpreting $\delta\nu$ in case ν arises from geometry; i.e., arises as the Abel-Jacobi images of a family $Z_s\in\mathfrak{Z}_h(X_s)$ of algebraic cycles on a family X_s of smooth varieties. For curves this is accomplished by Theorem 6.16, a result that involves somewhat unusual cohomological considerations.

As a first application of this formula, in Section 6(c) we consider the map $\nu\to(\delta\nu)(s_0)$ for the Kuranishi family of curves $\{C_s\}_{s\in S}$ centered at

$C = C_{s_0}$, and where ν ranges over all normal functions associated to families of divisors $D_s \in \mathrm{Div}^0(C_s)$ with D_{s_0} fixed. The result (cf. Theorem (6.28)) may be viewed as an extension of the Brill-Noether matrix to a variable family of curves. The proof of (6.28) uses Shiffer variations in an essential way.

Another application of the formula (6.17) is to genus four curves. On two occasions (in Sections 2 and 5) we have seen now to reconstruct a general curve of genus $g \geqslant 5$ from its infinitesimal variation of Hodge structure. On the other hand, for obvious reasons this is not possible for genus $g \leqslant 3$. Now when $g = 4$ the two g_3^1's define (at least up to ± 1) a normal function ν over moduli, and in Section 6(d) we show how to reconstruct a general such curve from $\delta\nu$. (7)

In Section 6(e) we extend the basic formula (6.17) to higher dimensions (cf. (6.45)). As a small application we give some conditions under which $\delta\nu = 0$ (cf. (6.49)). We have, however, not yet applied $\delta\nu$ to any substantial geometric problems in higher dimensions; in (6.51) we mention one interesting possibility along these lines.

Finally, Section 6(f) is purely speculative. A primitive algebraic cycle Z on a smooth variety $Y \subset \mathbb{P}^r$ defines a normal function ν over the space $S \subset \mathbb{P}^{r^*}$ of smooth hyperplane sections of Y, and ν essentially depends only on its fundamental class $z \in H^{m,m}_{\mathrm{prim}}(Y) \cap H^{2m}(Y, \mathbb{Z})$. Conversely, every such Hodge class z arises from a normal function ν. Therefore, for obvious reasons one would like to have some method of constructing Z (or, more precisely, all the cycles whose support has bounded degree and which are homologous to Z) from ν. What we do in Section 6(f) is show how at least some of the "equations" of Z (or of cycles homologous to Z) may be constructed from $\delta\nu$. This result is in the same spirit as Theorem 4.e.1, which however dealt with very special circumstances.

To show that this construction is not vacuous, we conclude the paper by computing an example (cf. (6.59)) where, in a very simple case, we construct cycles with given fundamental class by using $\delta\nu$. (8) The construction makes sense in general, but of course we have no idea whether or not it always leads to non-trivial algebraic cyles.

It is a pleasure to thank the referee for numerous corrections and suggestions concerning the original version of this paper.

5. Determinantal varities associated to an infinitesimal variation of Hodge structure

(a) Let $V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ be an infinitesimal variation of Hodge structure of weight n (cf. Section 1 for definitions and notations). Given $\xi \in \mathbb{P}T$ we shall denote by $\delta(\xi): H^{p,q} \to H^{p-1,q+1}$ a linear transformation given by $\delta(\tilde{\xi})$ for any non-zero vector $\tilde{\xi} \in T$ lying over ξ. Recall the iterated differential

$$\delta^{n-2k}: \mathrm{Sym}^{n-2k} T \to \mathrm{Hom}^{(s)}(H^{n-k,k}, H^{k,n-k}),$$

where $\mathrm{Hom}^{(s)}(V, V^*)$ denotes the symmetric maps from a vector space V to its dual, and where the isomorphism $H^{k,n-k} \cong (H^{n-k,k})^*$ is given by the bilinear form Q restricted to $H^{n-k,k} \times H^{k,n-k}$.

DEFINITIONS: (i) We define $\Xi_{k,l} \subset \mathbb{P}T$ by

$$\Xi_{k,l} = \{\xi \in \mathbb{P}T : \operatorname{rank}\{\delta^{n-2k}(\xi) : H^{n-k,k} \to H^{k,n-k}\} \leqslant l\};$$

(ii) with $h^{n,0} = \dim H^{n,0}$ we set

$$\Psi = \Xi_{0,h^{n,0}-1} = \{\xi \in \mathbb{P}T : \det \delta^n(\xi) = 0\};$$

(iii) and finally we set

$$\Xi = \Xi_{0,1} = \{\xi \in \mathbb{P}T : \operatorname{rank} \delta^n(\xi) \leqslant 1\}.$$

REMARKS: We may view the linear transformations $\delta(\xi_1)\ldots\delta(\xi_{n-2k}) \in \mathrm{Hom}^{(s)}(H^{n-k,k}, H^{k,n-k})$, where $\xi_i \in T$, as generating a linear system of quadrics parametrized by $\mathbb{P}(\mathrm{Sym}^{n-2k}T)$ on the projective space $\mathbb{P}(H^{k,n-k})^*$. Among these the quadrics of rank $\leqslant l$ form in a natural way a determinantal subvariety $\Sigma_{k,l}$ (the equations of this subvariety are given by all $(l+1)\times(l+1)$ principal minors of a symmetric matrix whose entries are linear functions on $\mathrm{Sym}^{n-2k}T$). The variety $\Xi_{k,l}$ is thus the intersection of $\Sigma_{k,l}$ with the image of the Veronese embedding $\mathbb{P}T \to \mathbb{P}\,\mathrm{Sym}^{n-2k}T$.

There is a somewhat more natural subvariety

$$\tilde{\Xi}_{k,l} \subset \mathbb{P}T \times G(\operatorname{codim} l, H^{n-k,k})$$

defined by

$$\tilde{\Xi}_{k,l} = \{(\xi, \Lambda) : \Lambda \subseteq \ker \delta^{n-2k}(\xi)\}$$

(here, $G(m, H^{n-k,k})$ is the Grassmannian of m-planes in $H^{n-k,k}$ and $\Lambda \in G(m, H^{n-k,k})$ is a typical m-plane). This subvariety $\tilde{\Xi}_{k,l}$ will be used in Section 6 below (under the notation Σ_l). We remark that, via the obvious projection

$$\tilde{\Xi}_{k,l} \to \Xi_{k,l},$$

$\tilde{\Xi}_{k,l}$ is a natural candidate for a desingularization of $\Xi_{k,l}$ (cf. Chapter 2 of [1]).

Our goal in this section is to interpret geometrically the extreme cases Ψ and Ξ of the varieties $\Xi_{k,l}$ in case the infinitesimal variation of Hodge structure arises from geometry.

(b) Suppose that $L \to X$ is a holomorphic line bundle over an n-dimensional smooth variety X. We denote by

$$\varphi_{kL}: X \to \mathbb{P}\big(H^n(X, KL^{-k})\big)$$

the natural mapping given by the sections in $H^0(X, L^k)$, where we recall that via Serre duality there is a natural identification

$$H^n(X, KL^{-k}) \cong H^0(X, L^k)^*. \tag{5.1}$$

In particular we shall consider the two maps

$$\varphi_L: X \to \mathbb{P}H^n(X, KL^{-1})$$

$$\varphi_{2L}: X \to \mathbb{P}H^n(X, KL^{-2}),$$

together with the natural map

$$\kappa: H^n(X, KL^{-2}) \to \mathrm{Hom}^{(s)}\big(H^0(X, L), H^n(X, KL^{-1})\big),$$

where the right-hand side are the symmetric maps given by the identification (5.1) when $k = 1$, and where κ is given by the cup-product

$$H^n(X, KL^{-2}) \otimes H^0(X, L) \to H^n(X, KL^{-1}).$$

For $\xi \in \mathbb{P}H^n(X, KL^{-2})$, we write $\kappa(\xi)$ for any transformation $\kappa(\tilde{\xi})$ where $\tilde{\xi} \in H^n(X, KL^{-2})$ is a non-zero vector lying over ξ.

DEFINITION: We define $\Xi_L \subset \mathbb{P}H^n(X, KL^{-2})$ by

$$\Xi_L = \{\xi \in \mathbb{P}H^n(X, KL^{-2}) : \mathrm{rank}\, \kappa(\xi) \leqslant 1\}.$$

THEOREM: *Assume that (i) $|L|$ has no base points, and (ii) the natural map*

$$\mathrm{Sym}^2\big(H^0(X, L)\big) \to H^0(X, L^2)$$

is surjective. Consider the image $\varphi_{2L}(X) \subset \mathbb{P}H^n(X, KL^{-2})$. Then

$$\varphi_{2L}(X) \subseteq \Xi_L$$

with equality holding if, and only if, $\varphi_L(X)$ is cut out by quadrics. (5.2)

The following argument was kindly communicated by the referee:

PROOF: We consider the *commutative* diagram

$$\begin{tikzcd}
 & \mathbb{P}_1 \arrow[d, equal] & \\
 & \mathbb{P}H^n(X, KL^{-1}) \arrow[dr, "\nu"] & \\
X \arrow[ur, "\varphi_L"] \arrow[dr, "\varphi_{2L}"'] & & \mathbb{P}\,\mathrm{Sym}^2 H^n(X, KL^{-1}) = \mathbb{P}_2 \\
 & \mathbb{P}H^n(X, KL^{-2}) \arrow[ur, "\kappa"'] \arrow[d, equal] & \\
 & \mathbb{P}_3 &
\end{tikzcd} \tag{5.3}$$

where: $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ denote the indicated projective spaces, ν is the Veronese map (whose image consists of the quadrics of rank one), and κ is induced by dualizing the natural map

$$\mathrm{Sym}^2\big(H^0(X, L)\big) \to H^0(X, L^2). \tag{5.4}$$

The commutativity of (5.3) is immediate by writing the maps out in terms of sections. Write

$$W = \kappa(\mathbb{P}_3)$$

and consider

$$\Xi_L' = W \cap \nu(\mathbb{P}_2).$$

Then

$$\Xi_L = \kappa^{-1}(\Xi_L'). \tag{5.5}$$

From the commutativity of (5.3) it follows that

$$\nu\big(\varphi_L(X)\big) = \kappa\big(\varphi_{2L}(X)\big) \subseteq \Xi_L', \tag{5.6}$$

and hence by (5.5)

$$\varphi_{2L}(X) \subseteq \Xi_L.$$

Moreover, keeping in mind that $\varphi_{2L}(X)$ spans $\mathbb{P}_3$ we have:

$$\kappa\big(\varphi_{2L}(X)\big) = \Xi_L' \Leftrightarrow \left\{\begin{array}{l} \kappa\big(\varphi_{2L}(X)\big) \text{ is cut out on} \\ \nu(\mathbb{P}_1) \text{ by hyperplanes} \end{array}\right\}$$

$$\Leftrightarrow \varphi_L(X) \text{ is an intersection of quadrics.}$$

Under our assumption that (5.4) is surjective, κ is injective so that by (5.5)

$$\kappa(\varphi_{2L}(X)) = \Xi'_L \Leftrightarrow \varphi_{2L}(X) = \Xi_L.$$

Comparing these last two equivalences completes the proof of the theorem. Q.E.D.

We note that the proof gives the following

COROLLARY: *Under the assumptions of* (5.2), *the variety of rank one transformations in*

$$\mathbb{P}(\kappa H^n(X, KL^{-2})) \subset \mathbb{P}\,\mathrm{Hom}^{(s)}(H^0(X, L), H^n(X, KL^{-1}))$$

is isomorphic to the intersection of the quadrics through $\varphi_L(X)$. (5.7)

Finally, suppose we denote by $\overline{\varphi_L(p_1)\dots\varphi_L(p_k)}$ the secant plane to $\varphi_L(X)$ determined by $\varphi_L(p_1),\dots,\varphi_L(p_k)$. Then since any linear combination of k transformations of rank one has rank at most k, we have the

COROLLARY: *Every point* $q \in \overline{\varphi_L(p_1)\dots\varphi_L(p_k)}$ *determines a transformation* $\nu(q)$ *of rank* $\leqslant k$ *in*

$$\mathbb{P}(\kappa H^n(X, KL^{-2})) \subset \mathbb{P}\,\mathrm{Hom}^{(s)}(H^0(X, L), H^n(X, KL^{-2})). \quad (5.8)$$

In particular, when $k = h^0(X, L) - 1$ the transformation $\nu(q)$ is singular; we will return to these in Section 5(d) below.

(c) For applications to variation of Hodge structures we are interested in rank one transformations in the image of the composite mapping [(9)]

$$\mathrm{Sym}^n H^1(X, \Theta) \xrightarrow{\sigma} H^n(X, K^*) \xrightarrow{\kappa} \mathrm{Hom}^{(s)}(H^0(X, K), H^n(X, \mathcal{O})).$$

As we have seen in Section 2, except when $n = 1$ the mapping σ will generally fail to be surjective (cf. Theorems 2.b.10 and 2.c.8). In fact, it is not at all clear that in general the intersection

$$\text{image}\ \sigma \cap \varphi_{2K}(X)$$

will be non-empty, so there may well be no rank one transformations in the image $\delta^n T$ of the n^{th} iterate of the differential in an infinitesimal variation of Hodge structure arising from geometry. In this case, according to Corollary (5.8) we should look for the intersection of image σ with various secant varieties to $\varphi_{2K}(X)$.

However, in case $n = 1$ the mapping σ is an isomorphism, and the points on the bicanonical curve

$$\varphi_{2K}(C) \subset \mathbb{P}H^1(C, \Theta)$$

have a very beautiful interpretation (we write C for X in the curve case).

DEFINITION: Any non-zero vector $\theta_p \in H^1(C, \Theta)$ lying over $\varphi_{2K}(p) \in \mathbb{P}H^1(C, \Theta)$ is called a *Shiffer variation* associated to $p \in C$.

Denoting by $\Theta(p)$ the sheaf of meromorphic vector fields with a simple pole at p, the quotient

$$\Theta(p)/\Theta \cong (T_p)^2$$

is a sky-scraper sheaf supported at p with stalk naturally isomorphic to the 2nd symmetric power of the tangent space to C at p. The exact cohomology sequence of

$$0 \to \Theta \to \Theta(p) \to (T_p)^2 \to 0$$

gives (assuming that the genus $g \geqslant 1$)

$$0 \to (T_p)^2 \xrightarrow{\delta} H^1(C, \Theta) \to H^1(C, \Theta(p)) \to 0. \tag{5.9}$$

The image $\delta((T_p)^2)$ is the line corresponding to $\varphi_{2K}(p) \in \mathbb{P}H^1(C, \Theta)$, and from the exactness of (5.9) we conclude that:

A Shiffer variation is given by a class $\theta \in H^1(C, \Theta)$ such that $\theta = 0$ in $H^1(C, \Theta(p))$ for some $p \in C$. (5.10)

Equivalently, in Dolbeault cohomology we may write

$$\theta = \bar{\partial}\eta \tag{5.11}$$

where η is a section of Θ that is C^∞ on $C \backslash \{p\}$ and looks like

$$\eta = \frac{1}{z}\frac{\partial}{\partial z} + (C^\infty \text{ function}) \tag{5.12}$$

near p (here, z is a local holomorphic coordinate centered at p). Intuitively, we may think of θ_p as an infinitesimal deformation of complex structure that leaves $C \backslash \{p\}$ fixed but changes the structure by a "δ-function" at p (this can be made precise – cf. Spencer and Shiffer [14]).

Since $\varphi_{2K}(C)$ spans $\mathbb{P}H^1(C, \Theta)$, every infinitesimal deformation is a linear combination of at most $3g - 3$ Shiffer variations when $g \geqslant 2$.

In a sense, Shiffer variations are the analogue for deformations of complex structure on C of the canonical curve $\varphi_K(C) \subset \mathbb{P}H^1(C, \mathcal{O})$ for deformations of line bundles on C. From this point of view it is noteworthy that they have not played a more significant role.

From the theorem of Enriques-Babbage-Petri (cf. [1], [6], or [13]) and Theorem (5.2) we have the following conclusions for C a curve of genus $g \geqslant 2$:

Every Shiffer variation $\theta_p \in H^1(C, \Theta)$ *gives in* $\mathrm{Hom}^{(s)}(H^0(C, K), H^1(C, \mathcal{O}))$ *a transformation of rank one with kernel* $\varphi_K(p) = \{\omega \in H^0(C, K) : \omega(p) = 0\}$; (5.13)

In case C does not have a g_2^1, g_3^1, *or* g_5^2 *every rank one transformation in* $H^1(C, \Theta)$ *is a Shiffer variation*; (5.14)

Under the conditions of (5.14) *the curve C is uniquely determined by its infinitesimal variation of Hodge structure. In particular, the generic global Torelli theorem holds when* $g \geqslant 5$. (5.15)

Rank one transformations for curves of genus 4 will be discussed in detail in Section 6(d) below.

PROOF OF (5.13): Referring to the proof of (5.2), for each $p \in X$ there are the hyperplanes

$$\varphi_L(p) = \{s \in H^0(X, L) : s(p) = 0\}$$

$$\alpha(p) = \{s \in H^0(X, L) : \kappa(\varphi_{2L}(p))(s) = 0\}$$

in $H^0(X, L)$. To prove (5.13) it will suffice to show in general

$$\varphi_L(p) = \alpha(p), \tag{$*$}$$

and then apply this result when $X = C$ and $L = K$. To prove $(*)$ we consider the long exact sequence of global Ext's associated to

$$0 \to I_p \to \mathcal{O} \to \mathbb{C}_p \to 0$$

where $I_p \subset \mathcal{O}$ is the ideal sheaf of p. This sequence together with its dual gives (we omit reference to X)

$$\mathrm{Ext}^{n-1}\big(I_p, KL^{-2}\big) \to \mathrm{Ext}^n\big(\mathbb{C}_p, KL^{-2}\big) \xrightarrow{r_p^*} \mathrm{Ext}^n(\mathcal{O}, KL^{-2})$$

$$H^1(X, L^2) \longleftarrow (L_p)^2 \xleftarrow{r_p} H^0(X, L^2)$$

Using the identifications

$$\operatorname{Ext}^n(\mathcal{O}, KL^{-2}) = H^n(X, KL^{-2})$$

$$\operatorname{Ext}^n\left(\mathbb{C}_p, KL^{-2}\right) \cong \mathbb{C}_p$$

we have

$$\varphi_{2L}(p) = \text{line } r_p^*\left(\mathbb{C}_p\right) \text{ in } H^n(X, KL^{-2}).$$

By naturality of cup product and duality, the mapping $\alpha(p) \in \operatorname{Hom}^{(s)}(H^0(X, L), H^n(X, KL^{-1}))$ is given by composing the usual Ext pairing

$$\operatorname{Ext}^n\left(\mathbb{C}_p, KL^{-2}\right) \otimes H^0(X, L) \to \operatorname{Ext}^n\left(\mathbb{C}_p, KL^{-1}\right)$$

with the inclusion

$$\operatorname{Ext}^n\left(\mathbb{C}_p, KL^{-1}\right) \to H^n(X, KL^{-1}).$$

Under this pairing, for s and r in $H^0(X, L)$ (and abusing notation slightly by choosing an identification $\mathbb{C}_p \cong \mathbb{C}$ and letting $p \in \mathbb{C}_p$ correspond to $1 \in \mathbb{C}$)

$$\langle p \otimes s, r \rangle = s(p) r(p) \in \left(L_p\right)^2.$$

It follows that $\langle p \otimes s, r \rangle = 0$ for all r if, and only if, $s(p) = 0$. This proves $(*)$. Q.E.D.

PROOF OF (5.14): By the aforementioned theorem of Enriques-Babbage-Petri, the hypothesis is equivalent to the surjectivity of (5.4) (in the case $X = C$, $L = K$) and the image $\varphi_K(C)$ being cut out by quadrics. Now apply Theorem 5.2. Q.E.D.

PROOF OF (5.15): The infinitesimal variation of Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ gives the rank one transformation in $\delta(T) \subset \operatorname{Hom}^{(s)}(H^{1,0}, H^{0,1})$. By (5.14)

$$\{\xi \in \mathbb{P}T : \operatorname{rank} \delta(\xi) = 1\}$$

is just the bicanonical curve. In this way we may reconstruct C satisfying (5.14) from its infinitesimal variation of Hodge structure. Q.E.D.

(d) We shall now give an analogue of Theorem (5.2) for the determinantal variety Ψ associated to an infinitesimal variation of Hodge sıructure arising from geometry.

We retain the notations from Section 5(c) above, and as before consider the cup-product mapping

$$\kappa: H^n(X, KL^{-2}) \to \mathrm{Hom}^{(s)}(H^0(X, L), H^n(X, KL^{-1})).$$

We want to determine the $\psi \in H^n(X, KL^{-2})$ such that $\det \kappa(\psi) = 0$. To state the answer, under the assumption that $|L|$ has no base points it will be a consequence of the proof of (5.17) below that

$$\det \kappa(\psi) \not\equiv 0, \tag{5.16}$$

and so we may define the proper subvariety

$$\Psi_L \subset \mathbb{P}H^n(X, KL^{-2})$$

by

$$\Psi_L = \{\psi : \det \kappa(\psi) = 0\}.$$

If $\dim |L| = r$, then $\deg \Psi_L = r+1$. For each $D \in |L|$ we denote by

$$\overline{\psi_{2L}(D)} \subset \mathbb{P}H^n(X, KL^{-2})$$

the linear span of the image of D under φ_{2L}. From the exact cohomology sequence of

$$0 \to \mathcal{O}_X(L) \xrightarrow{s} \mathcal{O}_X(L^2) \to \mathcal{O}_D(L^2) \to 0$$

(where $s \in H^0(X, L)$ has divisor $(s) = D$), it follows that

$$\operatorname{codim} \overline{\varphi_{2L}(D)} = h^0(X, L).$$

Thus we may form an abstract projective bundle together with a map

$$\begin{array}{l} I \xrightarrow{\tilde{\omega}} \mathbb{P}H^n(X, KL^{-2}) \\ \pi \downarrow \\ |L| \end{array}$$

where $\pi^{-1}(D)$ is the projective space $\overline{\varphi_{2L}(D)}$ and where $\tilde{\omega} \mid \pi^{-1}(D)$ is the inclusion $\overline{\varphi_{2L}(D)} \subset \mathbb{P}H^n(X, KL^{-2})$.

THEOREM: *We give the image* (5.17)

$$\tilde{\omega}(I) = \bigcup_{D \in |L|} \overline{\varphi_{2K}(D)}.$$

the image scheme structure, and then we have the equality of schemes

$$\Psi_L = \bigcup_{D \in |L|} \overline{\varphi_{2K}(D)}.$$

PROOF (10): To establish the set-theoretic equality we argue as follows: Given $\psi \in H^n(X, KL^{-2})$ we have $\det \kappa(\psi) = 0$ if, and only if, there is $s \in H^0(X, L)$ such that for all $t \in H^0(X, L)$

$$\langle \psi s, t \rangle = 0.$$

This is equivalent to

$$\langle \psi, s \cdot H^0(X, L) \rangle = 0.$$

Now $s \cdot H^0(X, L) \subset H^0(X, L^2)$ determines and is determined by the linear subspace

$$\left(s \cdot H^0(X, L)\right)^{\perp} = \overline{\psi_{2L}(D)} \subset \mathbb{P}H^n(X, KL^{-2})$$

where $D = (s)$, and this establishes the set-theoretic equality in (5.17). In a moment we shall see that

$$\operatorname{codim}\Big(\bigcup_{D \in |L|} \varphi_{2L}(D)\Big) = 1,$$

and this establishes (5.16).

Now we argue as in the proof of the Riemann-Kempf singularity theorem as proved in Section 4 of [1]. The dual of κ is a mapping

$$\mu : H^0(X, L) \otimes H^0(X, L) \to H^0(X, L^2) \tag{5.18}$$

that is injective in each factor separately. We consider the incidence variety

$$I \subset \mathbb{P}H^0(X, L) \times \mathbb{P}H^n(X, KL^{-2})$$

defined by

$$I = \{(s, \psi) : \kappa(\psi)s = 0\}.$$

Then I is the locus of the equations

$$\langle \kappa(\psi)s, t_j \rangle = 0 \tag{5.19}$$

where $t_0, \dots, t_r \in H^0(X, L)$ is a basis. By the injectivity in each factor property of μ in (5.18), the equations (5.19) have linearly independent differentials on each fibre $\{s\} \times \mathbb{P}H^n(X, KL^{-2})$ of the projection of $\mathbb{P}H^0(X, L) \times \mathbb{P}H^n(X, KL^{-2})$ on the first factor. It follows that the scheme I is reduced and has fundamental class equal to $(\omega_1 + \omega_2)^{r+1}$ where ω_1, ω_2 are the respective standard positive generators of $H^2(\mathbb{P}H^0(X, L), \mathbb{Z})$ and $H^2(\mathbb{P}H^n(X, KL^{-2}), \mathbb{Z})$. Projection onto the second factor (which is just $\tilde{\omega}$ above) induces a surjective mapping

$$\pi_2 : I \to \bigcup_{D \in |L|} \overline{\varphi_{2L}(D)},$$

and since (i) the fibres of π_2 are linear spaces; (ii) the image $\pi_2(I)$, being the support of Ψ_L, has dimension $\geqslant N-1$ where $h^0(X, L^2) = N+1$; and (iii) I is irreducible of dimension $N-1$, we conclude that if we give $\bigcup_{D \in |L|} \overline{\varphi_{2L}(D)} = \Psi_L'$ the image scheme structure then Ψ_L' is a variety of dimension $N-1$ and $\pi_2 : 1 \to \Psi_L'$ is birational. Moreover the degree of Ψ_L' is given by

$$(\omega_1 + \omega_2)^{r+1} \omega_2^{N-1} = r + 1.$$

Since Ψ_L' is irreducible and

$$\begin{cases} \operatorname{supp} \Psi_L = \operatorname{supp} \Psi_L' \\ \deg \Psi_L = \deg \Psi_L' \end{cases}$$

we conclude that the two divisors $\Psi_L = \Psi_L'$ as subschemes of $\mathbb{P}H^n(X, KL^{-2})$. Q.E.D.

Corollary: *If X is an n-dimensional compact, complex manifold whose canonical series has no base points, then the singular transformations in the image of*

$$\kappa : \mathbb{P}H^n(X, K^*) \to \mathbb{P}\operatorname{Hom}^{(s)}(H^0(X, K), H^n(X, \mathcal{O}))$$

form a proper hypersurface given by

$$\bigcup_{D \in |K|} \overline{\varphi_{2K}(D)}.^{(11)} \tag{5.20}$$

Because of the results in Section 2 above, if we wish to apply (5.20) to variations of Hodge structure for varieties of dimension $\geqslant 2$, we are faced with the following

Fundamental Problem: *Given a "geometrically defined" (cf. the example below) linear subspace $W \subset \mathbb{P}H^n(X, K^*)$, describe the intersection*

$$W \cap \Big(\bigcup_{D \in |K|} \overline{\varphi_{2K}(D)} \Big). \tag{5.21}$$

For example, suppose that $|L|$ gives a projective embedding

$$\varphi_L : X \to \mathbb{P}H^0(X, L)^*$$

whose image has normal bundle N, and take

$$W = \text{image of } \rho^n : \mathrm{Sym}^n H^0(X, N) \to H^n(X, K^*)$$

where $\rho : H^0(X, N) \to H^1(X, \Theta)$ is the Kodaira-Spencer map. Then, by Theorem 2.b.10,

$$W \subseteq (\Gamma)^{\perp} \tag{5.22}$$

where $\Gamma \subset H^0(X, K^2)$ is the Gauss linear system associated to $\varphi_L(X) \subset \mathbb{P}H^0(X, L)^*$. Moreover, in at least a number of special cases we will have equality in (5.22). Since Γ is the linear subsystem of $|K^2|$ generated by ramification loci of all linear projections $\varphi_L(X) \to \mathbb{P}^n \subset \mathbb{P}H^0(X, L)^*$, it is at least a reasonable geometric problem to try and determine

$$(\Gamma)^{\perp} \cap \Big(\bigcup_{D \in |K|} \overline{\varphi_{2K}(D)} \Big).$$

6. The infinitesimal invariant associated to a normal function

(a) We begin by recalling from Section 1 the definition of the infinitesimal invariant $\delta\nu$ associated to a normal function ν. The construction proceeds in several steps.

Step 1: We consider the Grassmann manifold $G = G(k, H)$ of k-planes F in a complex vector space H. The standard identification

$$T_F(G) \cong \mathrm{Hom}(F, H/F) \tag{6.1}$$

will be made without further comment. Points in the projectivized tangent bundle will be denoted by $(F, \boldsymbol{\xi})$ where $\boldsymbol{\xi} \in \mathbb{P}T_F(G)$. We denote by $\xi \in \mathrm{Hom}(F, H/F)$ a vector lying over $\boldsymbol{\xi}$ (using the identification (6.1)).

Now suppose that

$$\varphi : S \to G$$

is a holomorphic mapping from a complex manifold S into the Grassmannian. We write

$$\varphi(s) = F_s \subset H$$

where the subspace F_s varies holomorphically with $s \in S$. We denote by $\mathcal{F}$, $\mathcal{H}$ the pullback under φ of the universal sub-bundle, resp. trivial bundle; thus the fibres are given by

$$\mathcal{F}_s = F_s, \quad \mathcal{H}_s = H.$$

We set $T = T(S)$ and denote points in $\mathbb{P}T$ by $(s, \boldsymbol{\xi})$ where $\boldsymbol{\xi} \in \mathbb{P}T_s(S)$. By slightly abusing notation, we shall let $\xi \in \mathrm{Hom}(F_s, H/F_s)$ be any vector lying over $\varphi_*(\boldsymbol{\xi}) \in \mathbb{P}T_{F_s}(G)$.

The constructions below are motivated by the following local differential-geometric problem:

Find the differential conditions that a holomorphic section $\nu \in H^0(S, \mathcal{H}/\mathcal{F})$ be given by the projections to H/F_s of a con̲stant *vector $v \in H$.*

Since there is no $GL(H)$-invariant connection on $\mathcal{H}/\mathcal{F}$ there does not appear to be an obvious solution to this problem. By constructing a $GL(H)$-invariant "partial connection" we shall give necessary conditions that ν be the projection of a constant $v \in H$ (cf. the Appendix to this section).

DEFINITION: We denote by $G(l, \mathcal{H}/\mathcal{F}) \to S$ the Grassmann bundle with fibres $G(l, H/F_s)$, and define the subvariety

$$\sum_l \subset \mathbb{P}T \times G(l, \mathcal{H}/\mathcal{F})$$

by

$$\sum_l = \{(s, \boldsymbol{\xi}; \Lambda) : \xi(F_s) \subseteq \Lambda\}.$$

Here $\Lambda \in G(l, H/F_s)$ is an l-plane in H/F_s and, as explained above, $\xi \in \mathrm{Hom}(F_s, H/F_s)$ is a non-zero vector lying over $\varphi_*(\boldsymbol{\xi})$. If we define

$$\Xi_l \subset \mathbb{P}T$$

by

$$\Xi_l = \{(s, \boldsymbol{\xi}) : \operatorname{rank} \xi \leqslant l\},$$

there is an obvious projection

$$\sum_l \to \Xi_l$$

which, according to the standard theory of determinantal varieties, is a natural candidate for a desingularization of Ξ_l.

With the notation

$$\Lambda + F = \pi^{-1}(\Lambda) \subset H$$

where $F \in G(k, H)$, $\Lambda \in G(l, H/F)$, and where $\pi : H \to H/F$ is the obvious projection, we define bundles $\mathcal{O}(1)$, $\mathcal{Q}$ over $\mathbb{P}T \times G(l, \mathcal{H}/\mathcal{F})$ with respective fibres

$$\begin{cases} \mathcal{O}(1)_{(s,\boldsymbol{\xi};\Lambda)} = \text{dual of line } \boldsymbol{\xi} \subset T_s(S) \\ \mathcal{Q}_{(s,\boldsymbol{\xi};\Lambda)} = H/(F + \Lambda); \end{cases}$$

these are both standard tautological bundles. Given $\nu \in H^0(S, \mathcal{H}/\mathcal{F})$ we will define

$$\delta\nu \in H^0\Big(\sum_l, \mathcal{O}(1) \otimes \mathcal{Q}\Big)$$

whose vanishing, for all l, is a necessary condition that ν be induced from a constant $v \in H$.

For this we let $v(s) \in H$ be any local lifting of $\nu(s) \in H/F_s$ and consider, for any tangent vector ξ lying in the line $\boldsymbol{\xi} \subset T_s(S)$, the vector

$$\frac{\mathrm{d}v(s)}{\mathrm{d}\xi} = \langle \mathrm{d}v(s), \xi \rangle.$$

If $\tilde{v}(s) = v(s) + f(s)$ another lifting of $\nu(s)$, then $f(s) \in F_s$ and

$$\frac{\mathrm{d}\tilde{v}(s)}{\mathrm{d}\xi} = \frac{\mathrm{d}v(s)}{\mathrm{d}\xi} + \frac{\mathrm{d}f(s)}{\mathrm{d}\xi};$$

thus

$$\frac{\mathrm{d}\tilde{v}(s)}{\mathrm{d}\xi} \equiv \frac{\mathrm{d}v(s)}{\mathrm{d}\xi} \operatorname{mod}(F_s + \xi(F_s))$$

DEFINITION: For $(s, \xi; \Lambda) \in \Sigma_l$ we define

$$(\delta\nu)(s, \xi; \Lambda) \in (\mathcal{O}(1) \otimes \mathcal{Q})_{(s,\xi;\Lambda)}$$

by projecting $\mathrm{d}v(s)/\mathrm{d}\xi$ to $H/(F_s + \Lambda)$.

REMARK: This construction is local in $s \in S$ and involves choosing a local lifting of ν to a holomorphic section v of $\mathcal{H}$. We emphasize that, by passing to the quotient as above, the end result $(\delta\nu)(s, \xi; \Lambda)$ gives a *global* holomorphic section

$$\delta\nu \in H^0\Big(\sum_l, \mathcal{O}(1) \otimes \mathcal{Q}\Big).$$

Intuitively, $\delta\nu$ is defined by taking, for v a local lifting of ν, the part of $\mathrm{d}v$ that has intrinsic meaning. It is clear that $\delta\nu = 0$ is a necessary condition that we be able to choose $v(s) \in H$ to be constant (cf. the appendix to this section).

Step 2: Suppose now that

$$\dim H = 2k$$

and

$$Q: H \times H \to \mathbb{C}$$

is a non-degenerate alternating bilinear form. Set

$$G_Q = \{F \in G(k, H): Q(F, F) = 0\}.$$

It is well-known that G_Q is a homogeneous complex manifold of dimension $k(k+1)/2$. Using Q there is, for each $F \in G_Q$, a natural isomorphism

$$F^* \cong H/F. \tag{6.2}$$

With the identifications (6.1) and (6.2),

$$T_F(G_Q) \subset T_F(G)$$

is given by the symmetric homomorphisms

$$\mathrm{Hom}^{(s)}(F, F^*) \subset \mathrm{Hom}(F, H/F). \tag{6.3}$$

Suppose now that

$$\varphi : S \to G_Q$$

is a holomorphic mapping and use (6.2) to make the identifications

$$\begin{cases} \mathcal{F}^* \cong \mathcal{H}/\mathcal{F} \\ G(l, \mathcal{F}^*) \cong G(l, \mathcal{H}/\mathcal{F}). \end{cases}$$

Also, use the bijective maps (here the " = " is a definition)

$$\begin{array}{ccl} \tilde{\Lambda} & \to & \Lambda = \operatorname{Ann} \tilde{\Lambda} \\ \cap & & \cap \\ F_s^* & & F_s \end{array}$$

to make the identification

$$G(l, \mathcal{F}^*) \cong G(k-l, \mathcal{F}).$$

When all this is done the variety

$$\sum_l \subset \mathbb{P}T \times G(k-l, \mathcal{F})$$

defined in Step 1 above has the description

$$\sum_l = \{(s, \boldsymbol{\xi}; \Lambda) : \Lambda \subseteq \ker \xi\} \tag{6.4}$$

where $\Lambda \subset F_s$ is a $(k-l)$-plane and $\xi \in \operatorname{Hom}^{(s)}(F_s, F_s^*)$ is a vector lying over $\varphi_*(\boldsymbol{\xi})$. If we define

$$\Lambda^\perp \subset H$$

to be the $(k+l)$-plane given by

$$\Lambda^\perp = \{v \in H : Q(v, \Lambda) = 0\},$$

then Q induces a natural isomorphism

$$H/\Lambda^\perp \cong \Lambda^*$$

extending (6.2) in the case $l = 0$. With these tautological identifications the fibres of $\mathcal{Q} \to \mathbb{P}T \times G(k-l, \mathcal{F})$ are given by

$$\mathcal{Q}_{(s, \boldsymbol{\xi}; \Lambda)} = \Lambda^*, \qquad \Lambda \in G(k-l, \mathcal{F}). \tag{6.5}$$

Given $\nu \in H^0(S, \mathcal{H}/\mathcal{F})$ we have defined the infinitesimal invariant $\delta\nu \in H^0(\Sigma_l, \mathcal{O}(1) \otimes \mathcal{Q})$. To evaluate $\delta\nu$, by (6.5) it will suffice to know the scalars

$$Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = Q(\delta\nu, \omega)(\xi)$$

where $\xi \in T_s(S)$ and $\omega \in \ker \varphi_*(\xi)$ (the equality here is a definition). For these there is the following useful *Leibnitz formula for* $\delta\nu$:

$$\boxed{Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = \frac{\mathrm{d}}{\mathrm{d}\xi}(Q(\nu, \omega)) - Q(\nu, \omega'_\xi)} \tag{6.6}$$

that we now explain. Given $(s, \boldsymbol{\xi}; \Lambda) \in \Sigma_l$ we choose a tangent vector $\xi \in \boldsymbol{\xi} \subset T_s(S)$ and arc$\{s(t)\}$ in S with $s'(0) = \xi$. For $\omega \in \Lambda \subseteq \ker \varphi_*(\xi)$ we choose $\omega(t) \in F_{s(t)}$ with $\omega(0) = \omega$. Then $\omega \in \Lambda$ implies that

$$\omega'_\xi = \left.\frac{\mathrm{d}\omega(t)}{\mathrm{d}t}\right|_{t=0} \in F_s,$$

where again the equality is notation. By the definition of $\delta\nu$ we have

$$\begin{aligned} Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) &= Q\left(\frac{\mathrm{d}v(t)}{\mathrm{d}t}, \omega(t)\right)_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(Q(v(t), \omega(t)))_{t=0} - Q\left(v(t), \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}\right)_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\xi}(Q(\nu, \omega)) - Q(\nu, \omega'_\xi) \end{aligned}$$

where the last equation is a final definition, one which also defines the right-hand side of (6.6).

Step 3: Retaining the notations and assumptions of Step 2 we assume additionally given a discrete subgroup $H_{\mathbb{Z}} \subset H$ with $H_{\mathbb{Z}} \otimes \mathbb{C} = H$ (i.e., in suitable bases we have $H = \mathbb{C}^{2k}$ and $H_{\mathbb{Z}} = \mathbb{Z}^{2k}$). We also assume that Q is real on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$, and define $D \subset G_Q$ by

$$D = \{F \in G(k, H) : Q(F, F) = 0 \text{ and } F \cap \bar{F} = (0)\}. \tag{6.7}$$

For $F \in D$ we set

$$J_F = H_{\mathbb{Z}} \backslash H/F.$$

By the conditions in (6.7), $H_{\mathbf{Z}}$ projects in $H/F \cong F^*$ to a lattice so that J_F is a complex torus.

Given a holomorphic mapping

$$\varphi : S \to D$$

we define $\mathcal{J} \to S$ to be the fibre space of complex tori with fibres

$$\begin{aligned} \mathcal{J}_s &= J_{F_s} \\ &= H_{\mathbf{Z}} \backslash H / F_s . \end{aligned}$$

If $\nu \in H^0(S, \mathcal{J})$ is a holomorphic section of $\mathcal{J}$, then any local lifting of $\nu(s)$ to an H-valued function $v(s)$ is determined up to a transformation

$$\tilde{v}(s) = v(s) + f(s) + \lambda(s)$$

where $f(s) \in F_s$ and $\lambda(s) \in H_{\mathbf{Z}}$. Since $d\lambda(s) = 0$ we may repeat the definition of $\delta\nu$ above to obtain the infinitesimal invariant

$$\delta\nu \in H^0\Big(\sum_l , \mathcal{O}(1) \otimes \mathcal{Q}\Big)$$

of $\nu \in H^0(S, \mathcal{J})$. Again, the vanishing of $\delta\nu$ is a *necessary* condition that ν be induced from a constant vector $v \in H$.

Step 4: At last we are ready to define the infinitesimal invariant associated to a normal function. Let $\mathcal{V} = \{\mathcal{H}_{\mathbf{Z}}, \mathcal{F}^p, \nabla, Q, S\}$ be a variation of Hodge structure of odd weight $n = 2m + 1$ (cf. Section 1 for notations), and recall the associated fibre space $\mathcal{J} \to S$ of intermediate Jacobians where, by definition,

$$\mathcal{J}_s = \mathcal{H}_{\mathbf{Z}} \backslash \mathcal{H}_s / \mathcal{F}_s^{m+1}.$$

Denoting also by $\mathcal{J}$ the sections of this fibre space, the Gauss-Manin connection ∇ induces

$$D : \mathcal{J} \to \mathcal{H}/\mathcal{F}^m \otimes \Omega^1_S .$$

Setting

$$\mathcal{J}_h = \{\nu \in \mathcal{J} : D\nu = 0\}$$

a *normal function* is, by definition, a section $\nu \in H^0(S, \mathcal{J}_h)$ (that satisfies a suitable growth condition at infinity in case S is an algebraic variety – cf. [3]).

We now recall (cf. Section 5(a)) the determinantal variety

$$\Xi_{m,l} \subset \mathbb{P}T(S)$$

defined by

$$\Xi_{m,l} = \left\{(s, \xi) \in \mathbb{P}T(S) : \operatorname{rank}\left\{\xi : \mathfrak{F}_s^{m+1}/\mathfrak{F}_s^{m+2} \to \mathfrak{F}_s^{m}/\mathfrak{F}_s^{m+1}\right\} \leqslant l\right\}.$$

With the identifications

$$\begin{aligned} G\left(l, \mathfrak{F}^{m+1}/\mathfrak{F}^{m+2}\right) &\cong G\left(l, \left(\mathfrak{F}^{m}/\mathfrak{F}^{m+1}\right)^*\right) && (\text{using } Q) \\ &\cong G\left(k-l, \mathfrak{F}^{m+1}/\mathfrak{F}^{m+2}\right) && (\text{via } \tilde{\Lambda} \to \operatorname{Ann} \tilde{\Lambda}) \end{aligned}$$

we have

$$\sum_l \subset \mathbb{P}T(S) \times G\left(k-l, \mathfrak{F}^{m+1}/\mathfrak{F}^{m+2}\right)$$

defined by

$$\sum_l = \left\{(s, \xi; \Lambda) : \Lambda \subseteq \ker\left\{\xi : \mathfrak{F}_s^{m+1}/\mathfrak{F}_s^{m+2} \to \mathfrak{F}_s^{m}/\mathfrak{F}_s^{m+1}\right\}\right\}.$$

There is a natural mapping (denoted by $\tilde{\Xi}_{m,l} \to \Xi_{m,l}$ in §5a))

$$\sum_l \to \Xi_{m,l}.$$

DEFINITION: Given a normal function $\nu \in H^0(S, \mathcal{J}_h)$, the *infinitesimal invariant* is the holomorphic section

$$\delta\nu \in H^0\Big(\sum_l, \mathcal{O}(1) \otimes \mathfrak{Q}\Big)$$

is defined by the procedure in Steps 1–3 above.

In more concrete terms, given a tangent vector $\xi \in T_s(S)$ and $\omega \in \mathfrak{F}_s^{m+1}$ with

$$\xi \cdot \omega = 0 \text{ in } \mathfrak{F}_s^{m}/\mathfrak{F}_s^{m+1},$$

we set

$$\omega'_\xi = \nabla_\xi \omega \in \mathfrak{F}_s^{m+1}.$$

Then $Q(\delta\nu, \omega)$ is defined by Steps 1–3 above, and most importantly is given by the Leibnitz formula (6.6)

$$\boxed{Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = \frac{\mathrm{d}}{\mathrm{d}\xi}(Q(\nu, \omega)) - Q(\nu, \omega'_{\xi}).}$$

In case ν is induced from a constant vector $\nu \in H$ (i.e., $v \in H^0(S, \mathcal{H})$ satisfies $\nabla v = 0$) it follows that $\delta\nu = 0$. If S is algebraic then $\mathcal{V}$ is a direct sum of irreducible variations of Hodge structure plus a trivial variation of Hodge structure, and for the irreducible factors there will be no non-zero constant sections of $\mathcal{H}$. Hence we may expect that $\delta\nu$ should contain "a lot" of the information from ν (cf. the Appendix to this section).

A more serious problem, one that will be addressed in the following section, is to geometrically interpret $\delta\nu$ in case $\mathcal{V}$ arises from geometry.

Finally, we remark that by considering base spaces $S = \operatorname{Spec} \mathbb{C}[s^1, \ldots, s^r]/m^2$ we may define $\delta\nu$ for an infinitesimal normal function. Since it is by now clear what should be done here, we shall omit further discussion of this point.

Appendix to Section 6(a)

Robert Bryant has written a very interesting letter concerning the differential geometric problem posed at the beginning of this section. Namely, let $\mathcal{U} \subset G(k, H)$ be an open set and suppose given a holomorphic section $\nu \in H^0(\mathcal{U}, Q)$ where $Q \to G(k, H)$ is the universal quotient bundle. Assuming that rank $Q \geqslant k \geqslant 2$ we consider $\delta\nu$ as a section of a suitable sheaf over $\Xi_{k-1} = \Xi \subset \mathbb{P}T$ (all of this is of course restricted to $\mathcal{U}$). Then Robert Bryant proves the following result:

THEOREM: *The necessary and sufficient condition that ν be the projection of a constant vector $v \in H$ is that $\delta\nu = 0$.*

Because of his result we may suspect that there is no other universally defined differential invariant of normal functions other than $\delta\nu$ (and, of course, functions of $\delta\nu$).

(b) In this section we shall give a geometric formula for $\delta\nu$ in case ν is the normal function associated to a family of divisors on curves. Thus, we consider the situation

$$\pi : \mathcal{C} \to S$$

where $\mathcal{C}$, S are complex manifolds and the fibres $C_s = \pi^{-1}(s)$ are smooth curves of genus g. Let $J_s = J(C_s)$ be the Jacobian variety of C_s and suppose that

$$D_s \in \operatorname{Div}^0(C_s)$$

is a holomorphic family of divisors of degree zero. Under the usual abelian sum map

$$u_s : \mathrm{Div}^0(C_s) \to J_s$$

we then have a normal function ν where

$$\nu(s) = u_s(D_s) \in J_s.$$

We want to geometrically interpret $\delta\nu$.

For this we choose a reference point $s_0 \in S$, replace S by a neighborhood (still denoted by S) of s_0, set $C = C_{s_0}$, and choose a $\underline{C^\infty}$ trivialization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & C \times S \\ \pi \downarrow & & \downarrow \\ S & = & S \end{array} \tag{6.8}$$

such that $F : C_{s_0} \to C \times \{s_0\}$ is the identity. The family $\mathcal{C} \to S$ then gives a family of complex structures on the *fixed* C^∞ manifold C. Moreover, in a sense to be explained below, we may assume that these structures vary holomorphically with s. A convenient way to describe them is as follows: first, denoting by $A^q(C)$ the C^∞ q-forms on C, the complex structure on C_s is given by the Cauchy-Riemann (= CR) operator

$$d_s'' : A^0(C) \to A^1(C).$$

Secondly, denote by

$$T(C) = T_s' \oplus T_s''$$

the decomposition of the tangent bundle of C into (1, 0) and (0, 1) vectors for the structure d_s'' (thus, for $f \in A^0(C)$ we have that $d_s'' f \in C^\infty(T_s''^*)$). For s close to s_0 the linear projection

$$T_{s_0}'' \to T_s''$$

will be an isomorphism with dual

$$\psi_s : T_s''^* \xrightarrow{\sim} T_{s_0}''^*.$$

Denoting by $A^{0,1}(C)$ the $C^\infty(0, 1)$ forms for the structure corresponding to s_0 we define

$$\bar{\partial}_s : A^0(C) \to A^{0,1}(C)$$

by

$$\bar{\partial}_s f = \psi_s d_s'' f.$$

For s close to s_0 the complex structure on C_s determines, and is determined by, $\bar{\partial}_s$. For example, the structure sheaf $\mathcal{O}(C_s)$ is given by the local C^∞ functions f that satisfy $\bar{\partial}_s f = 0$.

It is clear that we may write

$$\bar{\partial}_s = \bar{\partial} + \boldsymbol{\theta}_s$$

where $\bar{\partial}$ is the CR operator on C and $\boldsymbol{\theta}_s \in C^\infty(T' \otimes T''^*)$ is a family of vector-valued (0, 1) forms on C. Holomorphic dependence of the complex structure on $s \in S$ means that the $\boldsymbol{\theta}_s$ depend holomorphically on s (cf. Kuranishi [10]). In any local holomorphic coordinate system on C we will have

$$\bar{\partial}_s f = \left(\frac{\partial f}{\partial \bar{z}} + \theta_s(z) \frac{\partial f}{\partial z} \right) \mathrm{d}\bar{z}$$

so that, in this coordinate system,

$$\boldsymbol{\theta}_s = \theta_s(z) \frac{\partial}{\partial z} \otimes \mathrm{d}\bar{z}.$$

If we denote the Kodaira-Spencer map by

$$\rho_s : T_s(S) \to H^1(C_s, \Theta),$$

then for $\xi = \sum_i \xi_i \partial/\partial s^i \in T_{s_0}(S)$ we have

$$\begin{aligned} \rho_{s_0}(\xi) &= \frac{\partial \boldsymbol{\theta}_s}{\partial \xi} \\ &= \sum_i \xi_i \left. \frac{\partial \theta_s}{\partial s^i}(z) \right|_{s=s_0} \frac{\partial}{\partial z} \otimes \mathrm{d}\bar{z} \end{aligned} \tag{6.9}$$

where $s^1, \ldots, s^m$ are local holomorphic coordinates on S.

Remark: The point of view of doing local deformation theory by fixing the C^∞ manifold X and deforming the almost complex structure by giving a holomorphically varying decomposition

$$T(X) \otimes \mathbb{C} = T_s' \oplus T_s'' \qquad (T_s'' = \overline{T_s'})$$

(holomorphically varying means that T_s' varies holomorphically with s) is due to Fröhlicher, Nijenhuis, Kodaira, Spencer, Nirenberg, and Kuranishi. In addition to [10] a good reference is "Deformation of complex structure, III" by Kodaira, Nirenberg, and Spencer in the 1962 Annals of Math. There it may be found that the Kodaira-Spencer class is represented in Dolbeault cohomology by equations (6.9).

Via the differentiable trivialization (6.8) we identify all the cohomology groups $H^1(C_s, \mathbb{C})$ with $H^1_{DR}(C, \mathbb{C}) = H$, and on H we denote by $Q: H \otimes H \to \mathbb{C}$ the pairing given by cup-product. It is well-known that these identifications are independent of the particular trivialization. The holomorphically varying subspace

$$H^{1,0}(C_s) \subset H$$

is given by the sections ω in $C^\infty(T_s'^*)$ that satisfy $\mathrm{d}\omega = 0$. Since

$$\begin{cases} Q\big(H^{1,0}(C_s), H^{1,0}(C_s)\big) = 0 \\ H^{1,0}(C_s) \cap \overline{H^{1,0}(C_s)} = (0) \end{cases}$$

we obtain a holomorphic map

$$\varphi: S \to D$$

as given in Step 3 in Section 6(a) above (cf. (6.7)). In fact, $\varphi(s) = H^{1,0}(C_s) \in G(g, H)$ is just a fancy way of giving the classical period matrix of C_s.

For $\omega, \psi \in H^{1,0}(C)$, the differential

$$\delta: T_{s_0}(S) \to \mathrm{Hom}^{(s)}\big(H^{1,0}(C), H^{0,1}(C)\big)$$

of the variation of Hodge structure satisfies

$$\begin{aligned} Q(\delta(\xi)\omega, \psi) &= \int_C \left(\frac{\partial \theta_s}{\partial \xi}\omega\right) \wedge \psi \\ &= \int_C \sum_i \xi_i \left.\frac{\partial \theta_s(z)}{\partial s^i}\right|_{s=s_0} \omega(z)\psi(z)\mathrm{d}z \wedge \mathrm{d}\bar{z} \end{aligned} \tag{6.10}$$

where $\omega = \omega(z)\mathrm{d}z$ and $\psi = \psi(z)\mathrm{d}z$ in local coordinates on C. In fact, (6.10) follows from (6.9) and the fact

$$\delta(\xi)\omega = \rho_{s_0}(\xi) \cdot \omega$$

that the differential is given by cup-product with the Kodaira-Spencer class.

Now suppose that $\Lambda \in G(g-l, H^{1,0}(C))$ is a linear subspace satisfying $\Lambda \subseteq \ker \xi$, where as above we consider $\xi \in T_{s_0}(S)$ as the map

$$\xi : H^{1,0}(C) \to H^{0,1}(C)$$

given by $\delta(\xi) = \rho_{s_0}(\xi)$. By (6.9) this is equivalent to

$$\frac{\partial \theta_s}{\partial \xi} \cdot \omega = -\bar{\partial} f \tag{6.11}$$

for f a C^∞ function on C. Of course f is not intrinsic. First, adding any constant to f leaves $\bar{\partial} f$ unchanged. More seriously, changing the trivialization (6.8) alters $\partial \theta_s / \partial \xi$ by $-\bar{\partial} \eta$ where $\eta \in C^\infty(T')$ is a $C^\infty(1,0)$ vector field on C (cf. [10] and the aforementioned Kodaira-Nirenberg-Spencer paper). Since

$$\left(\frac{\partial \theta_s}{\partial \xi} - \bar{\partial} \eta \right) \omega = -\left(\bar{\partial}(f + \eta \cdot \omega) \right) \tag{6.12}$$

it follows that f is uniquely determined up to a transformation

$$f \to f + \eta \cdot \omega + c \tag{6.13}$$

where c is constant and $\eta \in C^\infty(T')$. Consequently, we have that:

If $D = \sum_i p_i - q_i \in \mathrm{Div}^0(C)$ is any divisor with $\operatorname{supp} D = \sum_i p_i + q_i$ *contained in $\{\omega = 0\}$, then for f satisfying* (6.11)

$$\sum_i f(p_i) - f(q_i) \tag{6.14}$$

is well-defined.

This simple observation will be of importance below.

Now let

$$D_s = \sum_i p_i(s) - q_i(s) \in \mathrm{Div}^0(C_s)$$

be a holomorphically varying divisor, and choose a holomorphically varying basis $\omega_1(s), \dots, \omega_g(s)$ for $H^{1,0}(C_s)$. Then we may express the normal function ν associated to $\{D_s\}$ by abelian sums:

$$\nu(s) = \left(\dots, \sum_i \int_{q_i(s)}^{p_i(s)} \omega_\alpha(s), \dots \right) \in J(C_s). \tag{6.15}$$

The motivation for our admittedly complicated construction of $\delta\nu$ is to be able to intrinsically "differentiate" the abelian sum (6.15). Recalling from Section 6(a) (cf. just below (6.6)) that, for $\omega \in \ker \xi$ the scalar $Q(\delta\nu/\delta\xi, \omega)$ is well-defined, our main result is given by the

THEOREM: *With f defined by* (6.11), *there are tangent vectors* $p_i' \in T_{p_i}(C)$, $q_i' \in T_{q_i}(C)$ *such that*

$$Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = \sum_i \omega\Big|_{q_i'}^{p_i'} + \sum_i f\Big|_{q_i}^{p_i} \tag{6.17}$$

where $p_i = p_i(0)$, $\omega|_{q_i'}^{p_i'} = \omega(p_i') - \omega(q_i')$, etc. [13] (6.16)

COROLLARY: *Let* $\tilde{D} = \sum_i \tilde{p}_i - \tilde{q}_i$ *be any divisor linearly equivalent to* $D = \Sigma_i p_i - q_i$ *on C, and suppose that* supp $\tilde{D} \subseteq \{\omega = 0\}$. *Then*

$$Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = \sum_i f\Big|_{\tilde{q}_i}^{\tilde{p}_i}. \tag{6.18}$$

PROOF: Suppose that $\tilde{D} - D = (g)$ where g is a meromorphic function, and choose meromorphic functions g_s on C_s varying holomorphically with s and with $g_{s_0} = g$. (It is not necessary that the polar divisors of g_s all have the same degree as that of f.) Then

$$\tilde{D}_s = D_s + (g_s)$$

is linearly equivalent to D_s, and if $\tilde{D}_s = \Sigma_i \tilde{p}_i(s) - \tilde{q}_i(s)$ then by Abel's theorem

$$\sum_i \int_{\tilde{q}_i(s)}^{\tilde{p}_i(s)} \omega_\alpha(s) = \sum_i \int_{q_i(s)}^{p_i(s)} \omega_\alpha(s).$$

Applying (6.17) to the $\tilde{D}_s$ gives the corollary. Q.E.D.

PROOF OF (6.16): We may assume that S is the s-disc with s_0 corresponding to $s = 0$ and $\xi \in T_{s_0}(S)$ to $d/ds|_{s=0}$. Choose a holomorphic family of d-closed 1-forms $\omega(s) \in H^{1,0}(C_s)$ with $\omega(0) = \omega$ and set $\theta = \partial\theta_s/\partial\xi$. The assumption $\omega \in \ker \xi$ means that, if we consider the forms $\omega(s)$ as equivalence classes

$$\omega(s) \in H^1_{DR}(C) = \{\text{closed mod exact 1-forms}\}$$

in a fixed vector space, then

$$\omega(0) \in H^{1,0}(C) \text{ and } \left.\frac{\nabla\omega(s)}{ds}\right|_{s=0} \in H^{1,0}(C).$$

Here, ∇ is the Gauss-Manin connection and the equation $\nabla\omega(s)/ds|_{s=0} \in H^{1,0}(C)$ means that the *actual* derivative $d\omega(s)/ds|_{s=0}$ satisfies

$$\left.\frac{d\omega(s)}{ds}\right|_{s=0} = dg + \eta$$

where g is a C^∞ function and η is a (1, 0) form on C. (Note that $d\eta = 0$ implies that $\eta \in H^{1,0}(C)$.) In the notation of Step 3 in Section 6(a) we have

$$\omega'_\xi = \eta \in H^{1,0}(C)$$

(here ω'_ξ is $\langle \nabla\omega, \xi\rangle$). Thus we have

$$\omega(s) = \omega + s(\omega'_\xi + dg) + O(s^2). \tag{6.19}$$

On the other hand the *CR* equations that define the complex structure on C_s are (as described above; the following equations define the sheaf $\mathcal{O}(C_s)$)

$$\frac{\partial h}{\partial \bar{z}} + \theta_s(z)\frac{\partial h}{\partial z} = 0.$$

These are equivalent to

$$dh \wedge \varphi(s) = 0$$

where

$$\varphi(s) = dz - \theta_s(z)d\bar{z}.$$

It follows that $\varphi(s)$ is a (1, 0) form on C_s. Writing

$$\theta_s = s \cdot \theta + O(s^2)$$

the linear term in the condition

$$\omega(s) \wedge \varphi(s) \equiv 0$$

gives

$$(\omega'_\xi + dg) \wedge dz = \omega \wedge \theta(z)d\bar{z}.$$

Since ω'_ξ is of type (1, 0) on C this means that

$$\frac{\partial g}{\partial \bar{z}} = -\theta(z)\omega(z)$$

where locally $\omega = \omega(z)\mathrm{d}z$. In summary, *after altering g by a constant if necessary, the expansion* (6.19) *is*

$$\omega(s) = \omega + s(\omega'_\xi + \mathrm{d}f) + O(s^2) \tag{6.20}$$

where

$$\theta \cdot \omega = -\bar{\partial} f. \tag{6.21}$$

To prove (6.17) we will use the Leibnitz formula (6.6). This gives

$$Q\left(\frac{\delta \nu}{\delta \xi}, \omega\right) = \frac{\mathrm{d}}{\mathrm{d}s}(Q(v(s), \omega(s)))_{s=0} - Q(\nu, \omega'_\xi) \tag{6.22}$$

where $v(s) \in H$ is any lifting of $\nu(s) \in H_{\mathbf{Z}} \backslash H / H^{1,0}(C_s)$. A somewhat subtle point is that such a lifting is obtained by choosing 1-chains γ_s on C with

$$\partial \gamma_s = D_s.$$

To convince ourselves of this we write the Jacobian as

$$J(C_s) = H^{1,0}(C_s)^* / H_1(C, \mathbf{Z})$$

where the map $H_1(C, \mathbf{Z}) \to H^{1,0}(C_s)^*$ is given by integrating over 1-cycles. The linear function corresponding to D_s is

$$\omega_\alpha(s) \to \int_{\gamma_s} \omega_\alpha(s),$$

and for $\{\eta_\alpha\}$ a completion of $\{\omega_\alpha(s)\}$ to a basis for $H^1_{DR}(C)$

$$\eta_\alpha \to \int_{\gamma_s} \eta_\alpha$$

clearly extends this linear function to all of H. With this lifting of $\nu(s)$ we have

$$Q(v(s), \omega(s)) = \int_{\gamma_s} \omega(s).$$

Then, *and this is the main step*, setting $\gamma = \gamma_0$

$$\frac{\mathrm{d}}{\mathrm{d}s}(Q(v(s), \omega(s)))_{s=0} - Q(\nu, \omega'_\xi)$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\gamma_s} \omega(s)\right) - \int_\gamma \omega'_\xi$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\gamma_s} \omega(s)\right)_{s=0} - \int_\gamma\left(\left.\frac{\mathrm{d}\omega(s)}{\mathrm{d}s}\right|_{s=0}\right) + \int_\gamma \mathrm{d}f$$

by (6.20). Now the expression

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\sum_i \int_{q_i(s)}^{p_i(s)} \omega(s)\right)_{s=0} - \sum \int_{q_i(0)}^{p_i(0)} \left.\frac{\mathrm{d}\omega(s)}{\mathrm{d}s}\right|_{s=0}$$

is evaluated by the fundamental theorem of calculus, and is $\Sigma_i \omega|_{q'_i}^{p'_i}$ where the p'_i, q'_i are tangent vectors as described in the statement of the theorem. Putting everything together and using $\int_\gamma \mathrm{d}f = \Sigma_i f|_{q_i}^{p_i}$ gives the desired result (6.17). Q.E.D.

Remark: Neither term on the right-hand side of (6.17) is intrinsic; changing the C^∞ trivialization (6.8) alters each term in a manner discussed above. The point is that the two alterations *cancel* so that the sum is intrinsic (as it must be, since the left-hand side is intrinsic).

(c) In this section we shall use (6.17) to establish a non-degeneracy property of $\delta\nu$ when $\pi: \mathcal{C} \to S$ is the local Kuranishi family of curves of genus g (cf. [10]). [14] For this we will use Corollary (5.8) above, which we now recall in a convenient form.

Let $E = \Sigma_\alpha r_\alpha$ be an effective divisor on the reference curve $C = \pi^{-1}(s_0)$ (in general, we retain the notations of Section 6(b)). Then (5.8) implies that:

If $\theta \in \overline{\varphi_{2K}(E)}$ *and* $\omega \in H^0(C, K)$ *satisfies* $(\omega) \geqslant E$, *then* $\theta \cdot \omega = 0$ *in* $H^1(C, \mathcal{O})$. [15] (6.23)

It is convenient to prove (6.23) directly here. Thus, let z_α be a local holomorphic on C centered at r_α, and for any constants λ_α let ρ_α be a bump function with $\rho_\alpha(z_\alpha) = \lambda_\alpha$ for $|z_\alpha| \leqslant \varepsilon/2$ and $\rho_\alpha(z_\alpha) = 0$ for $|z_\alpha| \geqslant \varepsilon$. Set

$$\eta = \sum_\alpha \frac{\rho_\alpha}{z_\alpha} \frac{\partial}{\partial z_\alpha}. \tag{6.24}$$

Then the C^∞ vector-valued (0, 1) form

$$\theta = \bar{\partial}\eta = \sum_\alpha \frac{(\bar{\partial}\rho_\alpha)}{z_\alpha} \in H^1(C, \Theta) \tag{6.25}$$

is a linear combination of the Shiffer variations θ_{r_α} (cf. Section 5(b) above). On the other hand, if $\omega(r_\alpha) = 0$ then

$$f = \eta\omega \tag{6.26}$$

is a C^∞ function with

$$\bar{\partial} f = \bar{\partial}\eta \cdot \omega = \theta \cdot \omega. \tag{6.27}$$

It follows that

$$\omega \in \ker\{\theta : H^0(C, K) \to H^1(C, \mathcal{O})\},$$

which is (6.23). Q.E.D.

We now fix a divisor

$$D = \sum_i p_i - q_i \in \mathrm{Div}^0(C).$$

We consider *all* normal functions ν that arise from families of divisors D_s with $D_0 = D$, and we simultaneously consider all classes $\theta \in \overline{\varphi_{2K}(E)}$. Denoting the support of D by $\sigma(D) = \sum p_i + q_i$, we have the

THEOREM: *The condition*

$$Q\left(\frac{\delta\nu}{\delta\theta}, \omega\right) = 0, \qquad \omega \in H^0(C, K(-E))$$

for all ν and θ as above is equivalent to

$$(\omega) \geqslant E + \sigma(D). \tag{6.28}$$

REMARK: Referring to footnote [(13)], if we take E to be empty (i.e., we so to speak consider a constant family of curves), then (6.28) reduces to the condition $(*)$ of that footnote.

PROOF: We are considering *all* families of divisors D_s satisfying the initial condition $D_0 = D$. Thus the 1st order behavior of the points $p_i(s)$, $q_i(s)$ may be arbitrarily assigned. This means that the tangent vectors p_i', q_i' in Theorem (6.16) may be arbitrarily prescribed. Consequently, using (6.17) the condition

$$Q\left(\frac{\delta\nu}{\delta\theta}, \omega\right) = \sum_i \omega\Big|_{p_i'}^{q_i'} + \sum_i f\Big|_{q_i}^{p_i} = 0$$

for all normal functions ν arising from arbitrary families D_s implies that

$$\omega(p_i) = \omega(q_i) = 0. \tag{6.29}$$

In this case, by (6.24)–(6.26)

$$Q\left(\frac{\delta\nu}{\delta\theta}, \omega\right) = \sum_i \eta\omega\Big|_{q_i}^{p_i} = 0 \tag{6.29$'$}$$

for *all* choices of constants λ_α. If $E \cap \sigma(D) = \phi$, then (6.29)′ already follows from (6.29), and clearly $(\omega) \geqslant \sigma(D) + E$. If, say $r_\alpha = p_i$ then (6.29)′ implies that ω vanishes to 2nd order at p_i. In general, then, we conclude that

$$(\omega) \geqslant \sigma(D) + E.$$

Conversely, if this condition is satisfied, then by (6.17) and (6.24)–(6.26) we have that $Q(\delta\nu/\delta\theta, \omega) = 0$. Q.E.D.

(d) In (5.15) we have seen that a general smooth curve C of genus $g \geqslant 5$ may be reconstructed from its universal infinitesimal variation of Hodge structure. On the other hand, when $g \leqslant 3$ the universal infinitesimal variation of Hodge structure contains no information beyond that of the Hodge structure, since in this case the differential of the period map is in general an isomorphism from $H^1(C, \Theta)$ to the tangent space to Siegel space. This leaves the case $g = 4$, which we now discuss.

We shall assume that C is non-hyperelliptic and shall identify C with its canonical model in $\mathbb{P}H^1(C, \mathcal{O}) \cong \mathbb{P}^3$. It is well-known that (cf. Section 2 of [1])

$$C = Q \cap V$$

is the intersection of a quadric Q and cubic V, and that for general C the quadric Q is smooth. In this case the two rulings of Q cut out two g_3^1's, say $|D^+|$ and $|D^-|$, on C. These are the only g_3^1's (loc. cit.) and so their difference $D^+ - D^- \in \mathrm{Div}^0(C)$ gives a point

$$\nu(C) = u(D^+ - D^-) \in J(C) \tag{6.30}$$

where $u: \mathrm{Div}^0(C) \to J(C)$ is the Abel-Jacobi mapping. Clearly, $\nu(C)$ is well-defined up to $\pm$ and vanishes exactly when $|D^+| = |D^-|$; i.e., when Q is singular. In summary:

Over any family of non-hyperelliptic genus four curves there is a normal function, defined up to ± 1, *constructed from by the difference of the two* g_3^1*'s.* (6.31)

It is clear that $\delta(-\nu) = -\delta\nu$, and we shall show that

For C a general genus four curve with local Kuranishi space $\mathcal{C} \to S$ and normal function (6.31), *we may reconstruct C from the infinitesimal invariant $\delta\nu$.* (6.32)

We identify $T_{s_0}(S)$ with $H^1(C, \Theta)$ and recall from (5.7) (applied to the case $X = C$ and $L = K$) that the rank one transformations in the image of

$$\kappa : \mathbb{P}H^1(C, \Theta) \to \mathbb{P}\ \mathrm{Hom}^{(s)}\big(H^0(C, K), H^1(C, \mathcal{O})\big)$$

may be identified with the quadric Q. More precisely, choosing a basis ω_α for $H^0(C, K)$ with corresponding dual basis for $H^1(C, \mathcal{O})$, the rank one transformations in

$$\begin{aligned}\mathrm{Hom}^{(s)}\big(H^0(C, K), H^1(C, \mathcal{O})\big) &\cong \mathrm{Sym}^2 H^1(C, \mathcal{O})\\ &\cong \mathbb{C}^{10}\end{aligned}$$

are the image of the Veronese mapping

$$v : H^1(C, \mathcal{O}) \to \mathrm{Sym}^2 H^1(C, \mathcal{O}) \tag{6.33}$$

given by

$$g(\psi) = \psi^2.$$

Projectively, (6.33) induces

$$\begin{array}{ccc}\mathbb{P}H^1(C, \mathcal{O}) & \xrightarrow{v} & \mathbb{P}\ \mathrm{Sym}^2 H^1(C, \mathcal{O})\\ \wr\| & & \wr\| \\ \mathbb{P}^3 & & \mathbb{P}^9\end{array}$$

given in homogeneous coordinates by

$$v[\ldots, x^\alpha, \ldots] = [\ldots, x^\alpha x^\beta, \ldots] \qquad (\alpha \leqslant \beta).$$

In particular, $Q \subset \mathbb{P}^3$ is $v^{-1}(H)$ for some hyperplane $H \subset \mathbb{P}^9$, and (5.7) gives that

$$\kappa\big(\mathbb{P}H^1(C, \Theta)\big) = H.$$

For each point $p \in Q$ we denote by θ_p a non-zero element in $H^1(C, \Theta)$ such that $\kappa\theta_p = v(p)$; i.e.,

$$\theta_p : H^0(C, K) \to H^1(C, \mathcal{O})$$

is a rank one transformation corresponding to $p \in Q$. Then

$$\mathbb{P}(\ker \theta_p) = \{\text{hyperplanes in } \mathbb{P}^3 \text{ passing through } p\}.$$

In particular, when $p \in C = Q \cap V$ then θ_p is the Shiffer variation corresponding to p and $\ker \theta_p = H^0(C, K(-p))$. In general, we denote by $p^* \cong \mathbb{C}^3$ the linear functions $\Sigma_\alpha \lambda_\alpha x^\alpha$ that vanish at $p \in \mathbb{P}^3$, so that

$$\begin{cases} \ker \theta_p = p^* & p \in Q \\ p^* = H^0(C, K(-p)) & p \in C. \end{cases}$$

With the varieties $\pi : \Sigma_1 \to S$ and $\tilde{\omega} : \Xi_1 \to S$ being defined as in Section 6(a) (cf. (6.4)), we have for the fibres over s_0 that $\tilde{\omega}^{-1}(s_0) = Q$ and, setting $\Sigma = \pi^{-1}(s_0)$,

$$\sum \subset Q \times \mathbb{P}^{3*}$$

is given by

$$\sum = \{(p, H) : p \in Q \text{ and } H \in \mathbb{P}(p^*)\}.$$

Thus,

$$\pi_1 : \sum \to Q$$

is a $\mathbb{P}^2$-bundle. The bundle $\mathcal{O}(1) \to \Sigma$ (cf. Section 6(a)) is $\pi_1^*(\mathcal{O}_Q(2))$, because $\mathcal{O}(1)$ is induced from $\mathcal{O}_{\mathbb{P}H^1(C,\Theta)}(1)$ and the Veronese map satisfies

$$v^*(\mathcal{O}_{\mathbb{P}^9}(1)) = \mathcal{O}_{\mathbb{P}^3}(2).$$

The bundle $\mathcal{Q} \to \Sigma$ is $\pi_2^*(\mathcal{O}_{\mathbb{P}^{3*}}(1))$.

In particular, we consider the subvariety

$$X \subset \sum$$

defined by

$$X = \{(p, T_p(Q)) \text{ where } T_p(Q) \text{ is the tangent plane to } Q\}.$$

Then $X \cong Q$, and over X we have

$$\mathcal{O}(1) \otimes \mathcal{O}(1) \cong \mathcal{O}_X(3).$$

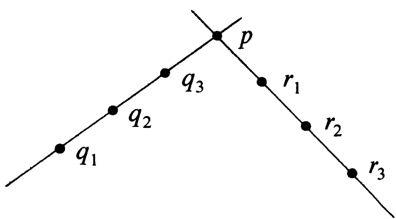
By restricting the infinitesimal invariant $Q(\delta\nu, \omega)$ to X we obtain a section

$$Q(\delta\nu, \omega) \in H^0(X, \mathcal{O}_X(3)) \cong H^0(Q, \mathcal{O}_Q(3)). \tag{6.34}$$

To establish (6.32) we shall show that:

For C a general curve of genus four, the section (6.34) *vanishes exactly on* $C = Q \cap V$. (6.35)

PROOF OF (6.35): We first show that $Q(\delta\nu, \omega)$ vanishes on C. For $p \in Q$ the intersection $T_p(Q) \cap Q$ consists of lines of the two rulings and we have a picture like



where

$$\begin{cases} q_1 + q_2 + q_3 \in |D^+| \\ r_1 + r_2 + r_3 \in |D^-|. \end{cases}$$

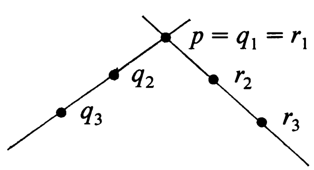
By Corollary (6.18) we have

$$Q(\delta\nu, \omega) = -\sum_i f\Big|_{r_i}^{q_i} \tag{6.36}$$

where

$$\theta_p \cdot \omega = \bar{\partial} f.$$

When $p \in C$ the above picture becomes



and θ_p is a Shiffer variation associated to p. Using the notations (6.24)–(6.26) when $E = p$, we let z be a local holomorphic coordinate centered at p and ρ a bump function with $\rho(z) = 1$ for $|z| \leqslant \varepsilon$ and $\rho(z) = 0$ for $|z| \geqslant 2\varepsilon$. Then

$$f = \eta\omega$$

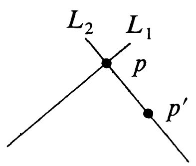
where

$$\eta = \frac{\rho(z)}{z}\frac{\partial}{\partial z}.$$

It follows that $f(q_2)=f(q_3)=f(r_2)=f(r_3)=0$ and $f(q_1)=f(r_1)$, so that by (6.36)

$$Q(\delta\nu, \omega)(p) = 0 \text{ if } p \in C. \tag{6.37}$$

It remains to show that, for C a general curve, (6.34) does not vanish identically. For this we consider curves $C = Q \in V$ having a triple branch point for one of the g_3^1's. These form a codimension-one family $\mathcal{F}$, and within $\mathcal{F}$ a codimension-one subfamily is given by those curves for which the other g_3^1 has a simple branch point on the line passing through the triple branch point:



$$p = q_1 = q_2 = q_3 = r_1$$
$$p' = r_2 = r_3$$

If $\omega \in H^0(C, K(-3p))$ defines the tangent plane to Q at p, then for $\theta = \theta_p$ the function f in (6.26) vanishes to second order at p and to first order at p'. Thus, when θ moves infinitesimally along the flex-tangent line L (i.e., we consider θ_{2p}), then $f(q_i) = 0 = f(r_1)$ but $f(p') \neq 0$. Consequently, for this curve the section (6.34) does not vanish identically. Q.E.D.

(e) In this section we will extend the basic formula (6.17) to families of intermediate Jacobians.

Let $\{X_s\}_{s\in S}$ be a family of smooth projective varieties of dimension $n = 2m + 1$. Since we are working locally around a reference point $s_0 \in S$, we set $X = X_{s_0}$ and differentiabily trivialize our family. Thus, we may think of $\{X_s\}_{s\in S}$ as given by a family of complex structures

$$d_s'' : A^0(X) \to A^1(X)$$

on the fixed C^∞ manifold X. As in Section 6(b) we use the isomorphisms

$$\psi_s : T''(X_s)^* \xrightarrow{\sim} T''(X)^*$$

to describe d_s'' by

$$\bar{\partial}_s = \psi_s \circ d_s''.$$

Then as in the curve case

$$\bar{\partial}_s = \bar{\partial} + \theta(s)$$

where $\bar{\partial}$ is the complex structure on X and $\theta(s)$ is a holomorphic family of vector-valued (0, 1) forms on X, written locally as

$$\theta(s) = \sum_{i,j} \theta_{\bar{j}}^{i}(s, z) \frac{\partial}{\partial z^i} \otimes \mathrm{d}\bar{z}^j.$$

It is well-known (cf. [10]) that the integrability condition

$$\bar{\partial}\theta(s) - \tfrac{1}{2}[\theta(s), \theta(s)] = 0$$

is satisfied, so that for $\xi = \Sigma \xi_\alpha \partial/\partial s^\alpha \in \mathrm{T}_{s_0}(\mathrm{S})$ the vector-valued (0, 1) form

$$\frac{\partial\theta(s)}{\partial\xi} = \sum_{\alpha,i,j} \xi_\alpha \partial\theta_{\bar{j}}^{i} \frac{(s, z)}{\partial s^\alpha}\bigg|_{s=0} \frac{\partial}{\partial z^i} \otimes \mathrm{d}\bar{z}^j$$

satisfies

$$\bar{\partial}\left(\frac{\partial\theta(s)}{\partial\xi}\right) = 0.$$

Thus $\partial\theta(s)/\partial\xi \in H_{\bar{\partial}}^{0,1}(X, \Theta) \cong H^1(X, \Theta)$, and in fact the map

$$\xi \to \frac{\partial\theta(s)}{\partial\xi}$$

is just the Kodaira-Spencer map

$$\rho_{s_0} : T_{s_0}(S) \to H^1(X, \Theta).$$

We now let

$$A^{p,q}(X_s) \subset A^{p+q}(X)$$

be the global $C^\infty(p, q)$ forms on X_s and set

$$F^p A^{p+q}(X_s) = A^{p+q,0}(X_s) \oplus \ldots \oplus A^{p,q}(X_s).$$

Suppose that $\omega(s) \in A^{p,q}(X_s)$ is a holomorphic family of (p, q) forms on X_s and set

$$\frac{\partial\omega(s)}{\partial\xi} = \sum_\alpha \xi_\alpha \frac{\partial\omega(s)}{\partial s^\alpha}\bigg|_{s=0}$$

For use below we have the relation

$$\frac{\partial\omega(s)}{\partial\xi} \equiv \frac{\partial\theta(s)}{\partial\xi}\cdot\omega \bmod F^pA^{p+q}(X), \tag{6.38}$$

where $\omega = \omega(0)$.

PROOF OF (6.38): The (1, 0) forms on X_s are locally spanned by

$$\varphi^i(s) = \mathrm{d}z^i - \theta^i_{\bar{j}}(s, z)\mathrm{d}\bar{z}^j.$$

Using standard multi-index notation we write

$$\omega(s) = \sum_{\substack{\#I=p\\ \#J=q}} f_{I\bar{J}}(s,z)\varphi^I(s)\wedge\overline{\varphi^J(s)}.$$

Then, since $\theta(s)$ depends holomorphically on s,

$$\frac{\partial}{\partial\xi}\overline{(\varphi^J(s))} = 0$$

and we obtain that

$$\frac{\partial\omega(s)}{\partial\xi} \equiv \sum_{I,J} f_{I\bar{J}}(z)\frac{\partial\varphi^I(s)}{\partial\xi}\wedge\mathrm{d}\bar{z}^J \bmod F^pA^{p+q}(X)$$

where $f_{I\bar{J}}(z) = f_{I\bar{J}}(0, z)$ and $\mathrm{d}\bar{z}^J = \overline{\varphi^J(0)}$. This gives

$$\frac{\partial\omega(s)}{\partial\xi} \equiv \sum_{\{\substack{i\in I\\ J,k}} (-1)^{i-1} f_{I\bar{J}}(z)\theta^i_{\bar{k}}(\xi,z)\mathrm{d}z^{I-\{i\}}\wedge\mathrm{d}\bar{z}^k$$

$$\wedge\,\mathrm{d}\bar{z}^J \bmod F^pA^{p+q}(X)$$

where $\theta^i_{\bar{k}} = \sum_\alpha \xi_\alpha \left.\frac{\partial\theta^i_{\bar{k}}(s,z)}{\partial s^\alpha}\right|_{s=0}$, from which we immediately obtain (6.38).

Now suppose that $Z_s \subset X_s$ is a codimension $(m+1)$ algebraic cycle that is homologous to zero, and define a normal function by

$$\nu(s) = u_s(Z_s) \in J(X_s)$$

where $J(X_s) = H^{2m+1}(X, \mathbb{Z})\backslash H^{2m+1}(X_s, \mathbb{C})/F^{m+1}H^{2m+1}(X_s, \mathbb{C})$ is the mth intermediate Jacobian of X_s and

$$u_s : \mathcal{Z}_h(X_s) \to J(X_s)$$

is the Abel-Jacobi map.

Writing $Z_s = Z_s^+ - Z_s^-$ where $Z_s^{\pm}$ are effective cycles, we set

$$\sigma(Z_s) = Z_s^+ + Z_s^- .$$

We shall give a formula for $\delta\nu$ over $\Sigma_1 = \Sigma$, recalling that $\Sigma = \cup_{s \in S} \Sigma_s$ where

$$\Sigma_s \subset \mathbb{P}T_s(S) \times \mathbb{P}H^{m+1,m}(X_s)$$

is defined by

$$\Sigma_s = \{(\xi, \omega) : \rho_s(\xi) \cdot \omega = 0 \text{ in } H^{m,m+1}(X_s)\}$$

(recall that $\rho_s(\xi) \in \mathbb{P}H^1(X_s, \Theta)$ and $\rho_s(\xi) \cdot \omega$ is the cup-product $H^1(X_s, \Theta) \otimes H^{m+1,m}(X_s) \to H^{m,m+1}(X_s)$). If $v(s) \in H^{2m+1}(X_s, \mathbb{C}) \cong H_{DR}^{2m+1}(X, \mathbb{C})$ is any lifting of $\nu(s)$, then by definition

$$(\delta\nu)(\omega, \xi) = \nabla_\xi v(s) \in H^{m,m+1}(X_s)/(\omega^\perp) \tag{6.39}$$

where $\omega^\perp \subset H^{m,m+1}(X_s)$ is the annihilator of ω under the pairing $Q: H^{m+1,m}(X_s) \otimes H^{m,m+1}(X_s) \to \mathbb{C}$ given by a polarization on X_s. Giving $\delta\nu$ is equivalent to knowing the quantities $Q(\delta\nu, \omega)(\xi) = Q(\delta\nu/\delta\xi, \omega)$ (the latter expression is notation), and from the Leibnitz formula (6.6) we have

$$Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = \frac{\mathrm{d}}{\mathrm{d}\xi}(Q(v, \omega)) - Q(\nu, \omega_\xi') \tag{6.40}$$

where $\omega_\xi' = \nabla_\xi \omega \in H^{m+1,m}(X)$ (here we are evaluating everything at $s = 0$).

We now imagine Z_s as a family of cycles on the fixed C^∞ manifold X (they are not complex-analytic cycles, but only currents of a suitable sort). Write

$$Z_s = \partial \Gamma_s$$

where Γ_s is a chain, and let $\omega(s) \in A^{m+1,m}(X_s)$ be closed forms giving classes in $H^{m+1,m}(X_s)$ that reduce to ω at $s = 0$. Then, as in the curve case we may consider Γ_s as a lifting of $\nu(s) \in (F^{m+1}H^{2m+1}(X_s, \mathbb{C}))^*/H_{2m+1}(X_s, \mathbb{Z})$, and

$$Q(v(s), \omega(s)) = \int_{\Gamma_s} \omega(s)$$

(the extent to which things are independent of choices will be discussed below). If η is the normal vector to $Z = Z_0$ that gives the tangent to the family of cycles Z_s, then by differentiation at $s = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(Q(v,\omega)) = \int_\Gamma \frac{\mathrm{d}\omega}{\mathrm{d}\xi} - \int_{\text{``}\mathrm{d}Z/\mathrm{d}\xi\text{''}} \omega$$

$$= \int_\Gamma \frac{\mathrm{d}\omega}{\mathrm{d}\xi} - \int_Z i(\eta)\,\omega \tag{6.41}$$

where $\Gamma = \Gamma_0$ and $i(\eta)\omega$ is the contraction of ω by η (cf. the remark at the end of this section). Now by (6.38)

$$\frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \frac{\partial\theta}{\partial\xi}\cdot\omega + \psi \tag{6.42}$$

where $\psi \in F^{m+1}A^{2m+1}(X)$. On the other hand by definition of the Gauss-Manin connection

$$\frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \nabla_\xi\omega + \mathrm{d}\tilde{\zeta}. \tag{6.43}$$

Since $\nabla_\xi\omega \in F^{m+1}H^{2m+1}(X)$ and $\mathrm{d}\omega/\mathrm{d}\xi \in F^mH^{2m+1}(X)$ we may assume (since X is Kähler and therefore the Hodge-deRham spectral sequence degenerates at E_1) that

$$\tilde{\zeta} \in F^mA^{2m}(X).$$

Letting $\tilde{\zeta} = \zeta_1 + \zeta$ where $\zeta_1 \in F^{m+1}A^{2m}(X)$ and $\zeta \in A^{m,m}(X)$, it follows from (6.42) and (6.43) that

$$\bar{\partial}\zeta = \frac{\partial\theta}{\partial\xi}\cdot\omega. \tag{6.44}$$

From (6.40), (6.41), and (6.43) we obtain our desired formula:

$$Q\left(\frac{\delta\nu}{\delta\xi},\omega\right) = \int_Z \zeta - \int_Z i(\eta)\,\omega \tag{6.45}$$

where $\zeta \in A^{m,m}(X)$ satisfies (6.44).

Remark: In contrast to the curve case it is not trivial to verify that (6.45) is independent of choices, so we shall do this in some detail.

First, if we replace ζ by $\zeta + \sigma$ where (6.44) is still satisfied, then $\sigma \in A^{m,m}(X)$ and $\bar{\partial}\sigma = 0$. Since

$$\bar{\partial}(\partial\sigma) = -\partial\bar{\partial}\sigma = 0$$

and X is Kähler, it again follows from the degeneration of the Hodge-de-

Rham spectral sequence that

$$\mathrm{d}\sigma = \partial\sigma = \mathrm{d}\gamma$$

where $\gamma \in F^{m+1}A^{2m}(X)$. Thus

$$\begin{aligned}\int_Z \zeta + \sigma - \int_Z \zeta &= \int_{\partial\Gamma} \sigma \\ &= \int_\Gamma \mathrm{d}\sigma \\ &= \int_\Gamma \mathrm{d}\gamma \\ &= \int_Z \gamma \\ &= 0\end{aligned}$$

by type considerations.

Next, $\omega(s)$ is only an equivalence class in $F^{m+1}Z^{2m+1}(X_s)/dF^{m+1}A^{2m}(X_s) \cap F^{m+1}A^{2m+1}(X_s)$ (here, "Z" denotes d-closed forms). Replacing $\omega(s)$ by $\omega(s) + \mathrm{d}\varphi(s)$ where $\varphi(s) \in F^{m+1}A^{2m}(X_s)$ and using the H. Cartan formula for Lie derivatives

$$i(\eta)\mathrm{d}\varphi + \mathrm{d}i(\eta)\varphi = \mathfrak{L}_\eta\varphi$$

(here $\varphi = \varphi(0)$) we obtain

$$\begin{aligned}\int_Z i(\eta)(\omega + \mathrm{d}\varphi) - \int_Z i(\eta)\omega &= \int_Z i(\eta)\mathrm{d}\varphi \\ &= \int_Z \mathfrak{L}_\eta\varphi - \int_Z \mathrm{d}(i(\eta)\varphi) \\ &= \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\int_{Z_s} \varphi(s)\right)\end{aligned}$$

since $\int_Z \mathrm{d}(i(\eta)\varphi) = 0$ and by the definition of the variation vector η

$$= 0$$

by type considerations.

Similarly, we may check that (6.45) is independent of all other choices.

Using (6.45) we will give sufficient conditions that $Q(\delta\nu/\delta\xi, \omega) = 0$.

For this we suppose first that

$$\omega \in H^m\big(X, \Omega_X^{m+1} \otimes I_{\sigma(Z)}\big) \tag{6.46}$$

where $I_{\sigma(Z)}$ is the ideal sheaf of the support $\sigma(Z)$ of Z. Then clearly

$$\int_Z i(\eta)\,\omega = 0.$$

Next, in the cohomology sequence

$$\to H^m(\sigma(Z), \Omega_X^m) \to H^{m+1}\big(X, \Omega_X^m \otimes I_{\sigma(Z)}\big) \xrightarrow{j} H^{m+1}(X, \Omega_X^m) \to \tag{6.47}$$

of

$$0 \to \Omega_X^m \otimes I_{\sigma(Z)} \to \Omega_X^m \to \Omega_X^m \otimes \mathcal{O}_{\sigma(Z)} \to 0,$$

for $\theta \in H^1(X, \Theta)$ and ω satisfying (6.46) the cup-product

$$\theta \cdot \omega \in H^{m+1}\big(X, \Omega_X^m \otimes I_{\sigma(Z)}\big).$$

If $\theta = \rho(\xi)$, then the assumption $\rho(\xi)\cdot\omega = 0$ in $H^{m,m+1}(X)$ means that $j(\theta\cdot\omega) = 0$ in (6.47). Consequently, there is a well-defined class

$$[\theta\cdot\omega] \in H^m(\sigma(Z), \Omega_X^m)/H^m(X, \Omega_X^m) \tag{6.48}$$

which maps to $\theta\cdot\omega$ in (6.47). We then have the

PROPOSITION: *If the conditions* (6.46) *and* $[\theta\cdot\omega] = 0$ *in* (6.48) *are satisfied, then*

$$Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = 0. \tag{6.49}$$

PROOF: By (6.45) and (6.46)

$$\begin{aligned} Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) &= \int_Z i(\eta)\,\omega + \int_Z \zeta \\ &= \int_Z \zeta \end{aligned}$$

where $\bar\partial\zeta = \theta\cdot\omega$. The condition $[\theta\cdot\omega] = 0$ implies that we may choose ζ satisfying $\bar\partial\zeta = \theta\cdot\omega$ and $\zeta|_{\sigma(Z)} = 0$. Q.E.D.

EXAMPLE/PROBLEM: We consider the family $X_s \subset \mathbb{P}^4$ of smooth quintic threefolds (the intermediate Jacobians of smooth hypersurfaces $Y \subset \mathbb{P}^4$ of degrees 3 and 4 are, in some sense, understood). On a general member, say X, there are a finite number of lines $\Lambda_i \subset X$ (according to Loring Tu, this number is 2875). Each difference $\Lambda_{ij} = \Lambda_i - \Lambda_j$ is a primitive algebraic 1-cycle, and we set

$$\nu_{ij} = u(\Lambda_{ij}) \in J(X).$$

It is a result of A. Collino that

For X general, we have for all $i < j$

$$\nu_{ij} \neq 0. \tag{6.50}$$

This suggests that we refine the Hodge structure of X to consist of the data $(J(X), \nu_{ij})$. Moreover, we may also speak about the 1st order infinitesimal variation of this data. With (6.32) in mind we pose the

PROBLEM: *Can we reconstruct a general quintic $X \subset \mathbb{P}^4$ from the infinitesimal invariants $\delta\nu_{ij}$?* (6.51)

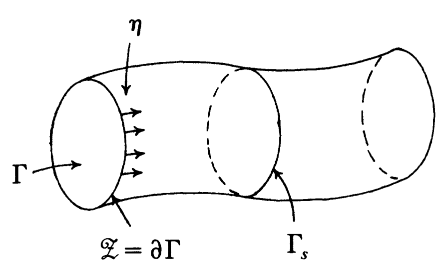
REMARK: The notation in (6.41) needs amplification. On $X \times S$ we let

$$\begin{cases} \tilde{\Delta} = \bigcup_{s \in S} \Gamma_s \\ \tilde{\mathfrak{Z}} = \bigcup_{s \in S} Z_s \\ \Omega = \text{differential form with } \Omega \,|\, X \times \{s\} = \omega(s). \end{cases}$$

Thinking of S as a disk in $\mathbb{C}$, let $\gamma \subset S$ be an arc from $s_0 = 0$ to Δs and set

$$\begin{cases} \Delta = \pi^{-1}(\gamma) \cap \tilde{\Delta} \\ \mathfrak{Z} = \pi^{-1}(\gamma) \cap \tilde{\mathfrak{Z}} \end{cases}$$

where $\pi: X \times S \to S$ is the projection



Δ = solid cylinder

Then

$$\partial\Delta = \mathfrak{Z} + \Gamma_{\Delta s} = \Gamma,$$

and the right-hand side of (6.41) is

$$\begin{aligned}
&\lim_{\Delta s \to 0} \frac{1}{\Delta s}\left(\int_{\Gamma_{\Delta s}} \Omega - \int_{\Gamma} \Omega\right) \\
&= \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left(\int_{\partial\Delta} \Omega - \int_{\mathfrak{Z}} \Omega\right) \\
&= \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left\{\int_{\Delta} d\Omega - \int_{\mathfrak{Z}} \Omega\right\} \\
&= \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left\{\int_0^{\Delta s}\left(\int_{\Gamma_t} \frac{d\Omega}{dt}\right)dt - \int_0^{\Delta s}\left(\int_{\mathfrak{Z}_t} i\left(\frac{\partial}{\partial t}\right)\omega\right)dt\right\} \\
&= \int_{\Gamma} \frac{d\omega}{ds} - \int_{\mathfrak{Z}} i(\eta)\,\omega.
\end{aligned}$$

This is the right-hand side of (6.41).

(f) In the following *highly speculative* discussion we will take up the possibility of using $\delta\nu$ to give some of the "equations" of algebraic cycles on a smooth projective variety in terms of their fundamental classes.

The idea is this: first, by the Lefschetz theorems it will suffice (over $\mathbb{Q}$) to consider the case of a primitive codimension-m algebraic cycle Z on a smooth projective variety $Y \subset \mathbb{P}^r$ where $\dim Y = 2m$ (here, primitive refers to the hyperplane class on Y). Let $z \in H^{m,m}_{\text{prim}}(Y) \cap H^{2m}(Y, \mathbb{Z})$ be the fundamental class of Z. For d a fixed large integer we consider the linear system $|\mathcal{O}_Y(d)|$ of degree d hypersurface sections of Y, and we let $S \subset |\mathcal{O}_Y(d)|$ be the dense open set of smooth $X \in |\mathcal{O}_Y(d)|$. Over S we construct the fibre space

$$\pi : \mathcal{J} \to S$$

with $\pi^{-1}(s)$ being the m^{th} intermediate Jacobian $J(X_s)$ where $X_s \in |\mathcal{O}_Y(d)|$ corresponds to $s \in S$. Then

$$Z_s = z \cdot X_s \in \mathfrak{Z}_h(X_s)$$

is a codimension-m cycle that is homologous to zero (since Z is primitive), and so we may define a normal function by

$$\nu(s) = u_s(Z_s) \in J(X_s)$$

where

$$u_s : \mathcal{Z}_h(X_s) \to J(X_s)$$

is the Abel-Jacobi map. As proved in [15] (cf. also [7] and (1.b.13) in Section 1), the variable (or essential) part of ν depends only on z; conversely, given a Hodge class $z \in H^{m,m}_{\mathrm{prim}}(Y) \cap H^{2m}(Y, \mathbb{Z})$ we may construct the normal function ν whose fundamental class is z. In other words, associated to the Hodge class z there is a global complex-analytic object $\nu \in H^0(S, \mathcal{J}_h)$, and we would like to use $\delta\nu$ to single out algebraic cycles on Y. We shall first explain how this might go for curves on a surface (the case $m = 1$); this explanation is based on the formula (6.17) in Section 6(b).

We shall write F, C, D in place of Y, X, Z; thus $C \in |\mathcal{O}_F(d)|$ is a general hypersurface section and $D \subset F$ is an algebraic 1-cycle that is homologous to zero. We denote by $\sigma(D)$ the support of D. The Poincaré residue sequence is

$$\begin{aligned} 0 \to H^0(F, \Omega_F^2) \to H^0(F, \Omega_F^2(d)) \xrightarrow{R} H^0(C, \Omega_C^1) \\ \to H^1(F, \Omega_F^2) \to 0, \end{aligned} \tag{6.52}$$

and the image of R is the variable part of $H^{1,0}(C) \subset H^1_{DR}(C)$. If $D \cdot C = \Sigma p_i - q_i$, then the normal function is expressed by abelian sums

$$\sum_i \int_{q_i}^{p_i} R(\omega), \qquad \omega \in H^0(F, \Omega_F^2(d)). \tag{6.53}$$

Intuitively, we would like to say that the condition

$$(\omega) \geqslant \sigma(D)$$

may be detected by differentiating (6.53), much in the spirit of how the Brill-Noether matrix is used to study special divisors on a fixed curve (cf. [1]).

Choosing d sufficiently large and writing $K_F = \Omega_F^2$, $K_C = \Omega_C^1$, we will have an exact sequence (obtained from the cohomology sequence of $(6.52) \otimes K_F(d)$)

$$0 \to H^0(F, K_F^2(d)) \to H^0\big(F, (K_F(d))^2\big) \to H^0(C, K_C^2) \to 0.$$

Therefore the mapping

$$\varphi_{(K_F(d))^2} : F \to \mathbb{P}H^0\big(F, (K_F(d))^2\big)^*$$

restricts to C to induce the bicanonical mapping

$$\varphi_{K_C^2}: C \to \mathbb{P}H^0\big(C, K_C^2\big)^* = \mathbb{P}H^1(C, \Theta).$$

Since $N = \mathcal{O}_C(d)$ is the normal bundle to C in F there is a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & H^0(F, \mathcal{O}_F) \arrow[d] & \\
 & H^0(F, \mathcal{O}_F(d)) \arrow[d] \arrow[dr, dashed, "\sigma"] & \\
\ldots \arrow[r] H^0(C, \Theta_S|C) \arrow[r] & H^0(C, \mathcal{O}_C(d)) \arrow[r] \arrow[d] & H^1(C, \Theta) \arrow[r] & \ldots. \\
 & \vdots &
\end{tikzcd} \tag{6.54}$$

If $C \in |\mathcal{O}_F(d)|$ corresponds to $s \in S$ ($=$ locus of smooth curves in $|\mathcal{O}_F(d)|$), then we may identify $T_s(S)$ with $H^0(F, \mathcal{O}_F(d))/H^0(F, \mathcal{O}_F)$ and σ in (6.54) becomes the Kodaira-Spencer mapping

$$\rho: H^0\big(F, \mathcal{O}_F(d)\big) \to H^1(C, \Theta).$$

We define

$$\Psi \subset \mathbb{P}H^0\big(F, K_F(d)\big) \times \mathbb{P}H^0\Big(F, \big(K_F(d)\big)^2\Big)^* \times S \tag{6.55}$$

by

$$\Psi = \Big\{(\omega, \theta, C) : \theta \in \overline{\varphi_{K_C^2}\big((\omega) \cap C\big)} \cap (\text{image}\ \rho)\Big\}.$$

According to Theorem 5.2 in Section 5 above, $(\omega, \theta, C) \in \Psi$ consists of a smooth curve $C \in |\mathcal{O}_F(d)|$ together with $\omega \in H^0(K_F(d))$ and an infinitesimal variation of C on F whose Kodaira-Spencer class θ satisfies

$$\theta \cdot \omega = 0 \text{ in } H^1(C, \mathcal{O}).$$

We may consider the infinitesimal invariant

$$Q(\delta\nu, \omega)(\theta)$$

on Ψ. Denoting by π_α ($\alpha = 1, 2, 3$) the respective projections in (6.55),

$$Q(\delta\nu, \omega)(\theta) \in H^0(\Psi, \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1) \otimes \pi_3^*\mathcal{O}(1)) \tag{6.56}$$

(the $\pi_3^*\mathcal{O}(1)$-factor comes because multiplying the section $s \in H^0(F, \mathcal{O}_F(d))$ which defines C by a scalar λ multiplies the Poincaré residue by λ^{-1}). We denote by

$$\Psi_\nu \subset \Psi$$

the subvariety where the section (6.56) vanishes. *We note that Ψ and Ψ_ν are defined purely in terms of F, $|\mathcal{O}_F(d)|$, and the fundamental class $\gamma \in H^{1,1}_{\text{prim}}(F) \cap H^2(F, \mathbb{Z})$ of D.*

We now define

$$\Psi_D \subset \Psi$$

by

$$\Psi_D = \left\{(\omega, \theta, C) : \theta \in \varphi_{K_C^2}\overline{((\omega) - \sigma(D)) \cap C}\right\}.$$

Then by Theorem 6.16 (cf. the discussion in Section 6(c)) we have the inclusion

$$\Psi_D \subset \Psi_\nu. \tag{6.56}$$

In general we do not expect equality here. However, suppose we define

$$I_\nu \subset \mathbb{P}H^0(F, K_F(d)) \times \mathbb{P}H^0\left(F, (K_F(d))^2\right)^*$$

and

$$I_D \subset \mathbb{P}H^0(F, K_F(d)) \times \mathbb{P}H^0\left(F, (K_F(d))^2\right)^*$$

by

$$I_\nu = \{(\omega, \theta) : (\omega, \theta, C) \in \Psi_\nu \text{ whenever } (\omega, \theta, C) \in \Psi\}$$

and

$$I_D = \left\{(\omega, \theta) : \theta \in \varphi_{(K_F(d))^2}\overline{((\omega) - \sigma(D))}\right\},$$

then (6.56) gives the inclusion

$$I_D \subset I_\nu. \tag{6.57}$$

We still do not expect equality. If

$$D = D' - D''$$

where D', D'' are effective and vary in respective linear systems $|D'_\lambda|$, $|D''_\mu|$, then we set

$$D_{\lambda,\mu} = D'_\lambda - D''_\mu.$$

From (6.57) we have

$$\bigcup_{\lambda,\mu} I_{D_{\lambda,\mu}} \subseteq I_\nu. \tag{6.58}$$

At this juncture, based on the examples in Section 4 we may at least in some cases expect equality to hold. If so, then under the projection

$$\pi : I_\nu \to \mathbb{P}H^0\big(F, (K_F(d))^2\big)^*$$

on the second factor, we will have in a purely Hodge-theoretic manner determined the image

$$\bigcup_{\lambda,\mu} H^0\big(F, (K_F(d))^2(-D'_\lambda - D''_\mu)\big),$$

and finally from this image we may determine primitive algebraic cyles with the given fundamental class.

More precisely, let

$$\Lambda \subset \mathbb{P}H^0\big(F, K_F(d)^2\big)^*$$

be the image of $\cup_{\lambda,\mu} H^0(F, (K_F(d))^2(-D'_\lambda - D''_\mu))$. Since d is assumed sufficiently large (estimating *a priori* how large d should be is, of course, one of the main difficulties), we may assume that each D'_λ is the base curve of the linear system

$$|K_F(d)^2 \otimes \mathcal{O}(-D''_\mu) \otimes I_{D'_\lambda}| \subset |K_F(d)^2 \otimes \mathcal{O}(-D''_\mu)|$$

($I_{D'_\lambda}$ is the ideal sheaf of D'_λ). In other words, *if* the equality

$$\pi(I_\nu) = \Lambda$$

holds (this is a big if), then on the one hand we will have determined Λ by a purely Hodge-theoretic construction, while on the other hand Λ is a *bi-ruled* subvariety of $\mathbb{P}H^0(F, (K_F(d))^2)^*$ from which we may construct cycles as follows: To say that Λ is bi-ruled means that it is the image of a rational map

$$\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}H^0\big(F, (K_F(d))^2\big)^*$$

that is *linear* in each variable separately. Each image $\mathbb{P}^r \times \{pt\}$, $\{pt\} \times \mathbb{P}^s$ then gives a linear subsystem of $\mathbb{P}H^0(F, (K_F(d))^2)^*$, and the base locus ($=$ fixed component in this case) of these linear subsystems are the curves D'_λ, D''_μ. Assuming not only that $\pi(I_\nu) = \Lambda$ but that Λ is uniquely bi-ruled, we will in this way have constructed cycles from $\delta\nu$.

To show that these considerations are not completely farfetched we shall conclude this discussion with the

EXAMPLE: Let $F \subset \mathbb{P}^3$ be a smooth quadric surface with lines L_1, L_2 chosen from the two rulings. If $\lambda_i \in H^{1,1}(F) \cap H^2(F, \mathbb{Z})$ is the fundamental class of L_i, then

$$\lambda = \lambda_1 - \lambda_2 \in H^{1,1}_{\text{prim}}(F) \cap H^2(F, \mathbb{Z})$$

is a primitive Hodge class (in some sense this is the simplest example of such). We let $S \subset |\mathcal{O}_F(3)|$ be the open set of smooth intersections

$$C = F \cap G \tag{6.60}$$

where G is a cubic surface in $\mathbb{P}^3$. Then C is a general canonical curve of genus four, and the normal function

$$\nu \in \operatorname{Hom}(S, \mathcal{J}) \tag{6.61}$$

($\mathcal{J} = \cup_{s \in S} J(C_s)$) corresponding to λ has been discussed, from a completely different viewpoint, in Section 6(d) above. Here we are adopting the viewpoint that we known the surface F, the curves (6.60), and the normal function (6.61), but we do *not* know the lines L_1, L_2 on F (the previous point of view was essentially that we knew F, ν, and the lines on F and wished to generically determine C). We shall show how to determine them from the infinitesimal invariant $\delta\nu$. Thus, at least in this very simple case we are able to give the "equations" of the 1-cycles $L_1 - L_2$ by a purely Hodge-theoretic construction, one that at least formally generalizes to higher dimensions.

Given $C \in |\mathcal{O}_F(3)|$ we shall use the Shiffer variations $\theta_p \in H^1(C, \Theta)$. Since C is general, these Shiffer variations are Kodaira-Spencer images of infinitesimal variations of C in $|\mathcal{O}_F(3)|$. Recall from Section 5(d) that the kernel of

$$\theta_p : H^0(C, K) \to H^1(C, \mathcal{O})$$

is given by

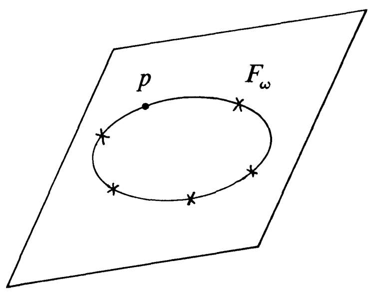
$$\ker \theta_p = H^0(C, K(-p)).$$

All the spaces $H^0(C, K)$ are isomorphic (via Poincaré residues) to $H^0(F, \mathcal{O}_F(1)) \cong H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. For $\omega \in H^0(C, K)$ we denote by $\mathbb{P}^2_\omega$ the corresponding hyperplane. We now consider triples (p, C, ω) where C is a curve (6.60) and the conditions

$$p \in C, \quad \omega \in H^0(C, K(-p))$$

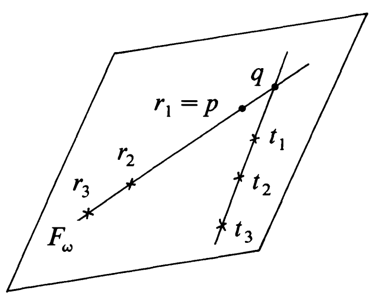
are satisfied. The possible configurations are

(i)



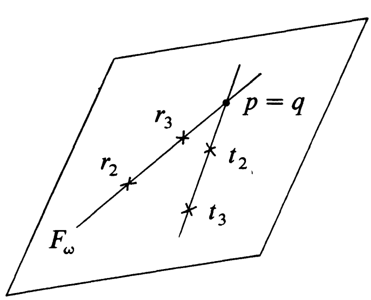
$\mathbb{P}^2_\omega$

(ii)



$\mathbb{P}^2_\omega$

(iii)



$\mathbb{P}^2_\omega$

Here $F_\omega = F \cap \mathbb{P}^2_\omega$ and in case (i) this is a smooth plane conic, while in cases (ii) and (iii) it is singular; i.e., $\mathbb{P}^2_\omega = T_q(F)$ is the tangent plane to F at some point q. The remaining points of $\mathbb{P}^2_\omega \cap C = (\omega)$ are denoted by $\times$'s; in case (ii) it is possible that q is one of the $\times$'s.

We now consider p, ω as fixed and G as variable (subject to $p \in G$). We then set

$$Q(\delta\nu, \omega)(G) = Q\left(\frac{\delta\nu}{\delta\theta_p}, \omega\right)$$

where $\theta_p \in H^1(F \cap G, \Theta)$ is a Shiffer variation.

PROPOSITION: *In cases (i), (ii) we have $Q(\delta\nu, \omega)(G) \neq 0$ for a general G, while in case (iii) $Q(\delta\nu, \omega)(G) \equiv 0$.* (6.62)

COROLLARY: *The condition $Q(\delta\nu, \omega)(G) = 0$ for a general G is equivalent to $\mathbb{P}^2_\omega = T_p(F)$.* (6.63)

Using the corollary we may detect, by purely Hodge-theoretic methods, the equations of algebraic 1-cycles with fundamental class λ. Namely, the bi-ruled variety Λ alluded to above is in this case the subvariety of $\mathbb{P}^{3*}$ consisting of $\mathbb{P}^2$'s passing through one of the lines on Q (of course, $\Lambda = Q^*$ turns out to be the dual surface to Q).

PROOF OF (6.62): In case (ii), by (6.18) we have (cf. the arguments in Section 6(d))

$$\begin{aligned} Q(\delta\nu, \omega)(\theta_p) &= \sum_i f \Big|_{t_i}^{r_i} \\ &= f(p) \\ &= \operatorname{Res}\left\{\frac{\omega(z)\mathrm{d}z}{z}\right\} \\ & 0, \end{aligned}$$

where $f = \eta\omega$ as in (6.26) with $\eta = (\rho(z)/z)\partial/\partial z$ as just above (6.37) and $\omega = \omega(z)\mathrm{d}z$ near p.

In case (iii),

$$\begin{aligned} Q(\delta\nu, \omega)(\theta_p) &= \sum_{i=2,3} f \Big|_{t_i}^{r_i} \\ &= \sum_{i=2,3} \eta\omega \Big|_{t_i}^{r_i} \\ &= 0. \end{aligned}$$

Finally, in case (i) we can use an argument similar to that at the end of the proof of (6.35) to show that $Q(\delta\nu, \omega)(\theta_p) \neq 0$ when G is general.

To conclude this paper we shall give a variant of this construction, based on (6.49), for higher codimensional cycles.

We first review the construction in Section 3(a) (cf. Theorem (3.a.7)). Let $\mathcal{O}_Y(1) \to Y^{2m}$ be a sufficiently ample line bundle and $X^{2m-1} \in |\mathcal{O}_Y(1)|$ a smooth divisor. Then there is an associated infinitesimal variation of Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ where $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ is the variable part of $H^{2m-1}(X)$ and where

$$T = H^0(Y, \mathcal{O}_Y(1))$$

with δ being determined by the map ρ in

$$\overbrace{H^0(\mathcal{O}_Y(1)) \xrightarrow{r} H^0(\mathcal{O}_X(1)) \xrightarrow{\Delta}}^{\rho} H^1(\Theta_X)$$

(here r is restriction and Δ is the coboundary in the exact cohomology sequence of $0 \to \Theta_X \to \Theta_Y | X \to \mathcal{O}_X(1) \to 0$). By residues (loc. cit.) we have

$$\mathrm{Res}_X\colon H^{2m-1-p,p} \cong H^0(K_Y(p+1))/\mathcal{U}(p, X) \tag{6.64}$$

where $\mathcal{U}(p, X) \subset H^0(K_Y(p+1))$ is a subspace depending on X (it is some sort of generalized "Jacobian ideal"). By replacing $\mathcal{O}_Y(1)$ by a power if necessary, we may assume that the induced embedding $Y \subset \mathbb{P}^N$ is projectively normal and that the *Arbarello-Sernesi module* (cf. E. Arbarello and E. Sernesi, Petri's approach to the study of the ideal associated to a special divisor, Invent. Math. 49 (1978), 99–119)

$$M = \bigoplus_{p \geqslant 0} H^0(K_Y(p+1))$$

is generated as an $R = \oplus_{l \geqslant 0} H^0(\mathcal{O}_Y(l))$-module in its lowest degree. By (6.64)

$$\bigoplus_{p \geqslant 0} H^{2m-1-p,p} = M_X$$

is a quotient vector space of M. We observe that:

The axioms of an infinitesimal variation of Hodge structure imply that M_X is a quotient R-module of M, where the pairing

$$R \otimes M_X \to M_X$$

is generated by

$$\delta : H^0(\mathcal{O}_Y(1)) \to \mathrm{Hom}(M_X, M_X). \tag{6.65}$$

Again we see the commutative algebra flavor of an infinitesimal variation of Hodge structure (cf. Donagi's proof of generic Torelli for most hypersurfaces to appear in this journal).

For $\omega \in H^0(K_Y(1))$ with divisor $D = (\omega)$ and $P \in H^0(\mathcal{O}_Y(m))$ we may consider the residue

$$\mathrm{Res}_X(P\omega) \in H^{m-1,m}(X).$$

Assuming that $P\omega|_X = 0$, the exact sequence (6.47) with D replacing $\sigma(Z)$ and $m-1$ replacing m gives

$$\{P \cdot \omega\}_X \in H^{m-1}(\Omega_X^{m-1} \otimes \mathcal{O}_D)/H^{m-1}(\Omega_X^{m-1}), \tag{6.66}$$

defined exactly as was $[\theta \cdot \omega] \in H^{m-1}(\Omega_X^{m-1} \otimes \mathcal{O}_{\sigma(Z)})/H^{m-1}(\Omega_X^{m-1})$ in (6.48). We define

$$\sum \subset \mathbb{P}H^0(K_Y(1)) \times \mathbb{P}H^0(\mathcal{O}_Y(m)) \times \mathbb{P}H^0(\mathcal{O}_Y(1)) \tag{6.67}$$

by

$$\sum = \{(P, \omega, X) : \{P \cdot \omega\}_X = 0\}.$$

Note that this construction is purely Hodge-theoretic.

We now consider a primitive codimension-m cycle $Z \subset Y$ and set $Z_X = Z \cdot X$ (by the moving lemma applied to Z, we may assume that this intersection is defined). There is the usual normal function ν given by

$$\nu(X) = u_X(Z_X) \in \mathcal{J}(X)$$

where u_X is the Abel-Jacobi mapping for codimension-m cycles on X. As noted above, ν is given by purely Hodge-theoretic data.

The main observation is that the infinitesimal invariant $\delta\nu$ may be defined on Σ. For example, suppose that $P = \xi_1 \ldots \xi_m$ where $\xi_i \in H^0(\mathcal{O}_Y(1))$. Then

$$\psi = \mathrm{Res}_X(\xi_1 \ldots \xi_{m-1} \cdot \omega) \in H^{m,m-1}$$

and

$$\delta(\xi_m)\psi = 0 \text{ in } H^{m-1,m}.$$

Hence $\delta\nu(\psi, \xi_m)$ is defined and is equal to $\delta\nu(\psi, \xi_i)$ for any other ξ_i. We denote it by $\delta\nu(\omega, P, X)$; then

$$\delta\nu \in H^0\left(\sum, \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)\right)$$

where π_i $(i = 1, 2, 3)$ are the projections of Σ in (6.67). We set

$$I(z, \mathcal{O}_Y(1)) = \left\{\omega \in |K_Y(1)| : \delta\nu(\omega, P) = 0 \text{ whenever } (\omega, P, X) \in \sum\right\}.$$

Clearly, this subvariety of $|K_Y(1)|$ is defined purely Hodge-theoretically once we have been given the Hodge class z.

On the other hand, we denote by $I_{\sigma(Z)}$ the ideal sheaf of the support of Z and set

$$I(Z, \mathcal{O}_Y(1)) = |K_Y(1) \otimes I_{\sigma(Z)}|.$$

From (6.49) we have the

Proposition: *For any primitive algebraic cycle Z with fundamental class z, we have*

$$I(Z, \mathcal{O}_Y(1)) \subseteq I(z, \mathcal{O}_Y(1)). \tag{6.69}$$

Proof: If $\sigma(Z) \subseteq D$ then from the commutative diagram

$$\begin{array}{ccc} \{P \cdot \omega\}_X & \in & H^{m-1}(\Omega_X^{m-1} \otimes \mathcal{O}_D)/H^{m-1}(\Omega_X^{m-1}) \\ \downarrow & & \downarrow \text{restriction} \\ [\mathrm{Res}_X(P \cdot \omega)] & \in & H^{m-1}(\Omega_X^{m-1} \otimes \mathcal{O}_{\sigma(Z)})/H^{m-1}(\Omega_X^{m-1}) \end{array}$$

we see that $\{P \cdot \omega\}_X = 0 \Rightarrow [\mathrm{Res}_X(P \cdot \omega)] = 0$. Now apply (6.49). Q.E.D.

In this construction the choices (i) of a particular cycle Z whose fundamental class $cl(Z)$ is equal to the Hodge class z and (ii) of sufficiently ample line bundle $\mathcal{O}_Y(1)$, are both rather arbitrary. Replacing $\mathcal{O}_Y(1)$ by $\mathcal{O}_Y(k)$ we infer from (6.69) the inclusion

$$\left\{ \bigcup_{\substack{Z = \text{codimension-}m \\ \text{algebraic cyle} \\ \text{with } cl(Z) = z}} \right\} I(Z, \mathcal{O}_Y(k)) \subseteq I(z, \mathcal{O}_Y(k)). \tag{6.70$_k$}$$

Note that the left-hand side of $(6.70)_k$ is ruled by linear subspaces of

$|K_Y(k)|$ whose base loci are the supports $\sigma(Z)$ of the cycles Z with $cl(Z)=z$. An obvious question is whether or not, for sufficiently large k, we have equality at least of the highest dimensional components on the two sides of $(6.70)_k$. If so this would give a method for digging the equations of a cycle out of its fundamental class.

References

[1] E. ARBARELLO, M. CORNALBA, P. GRIFFITHS and J. HARRIS: *Topics in the Theory of Algebraic Curves,* To appear.

[2] J. CARLSON and P. GRIFFITHS: Infinitesimal variations of Hodge structure and the global Torelli problem. *Journées de géométrie algébrique d'Angers,* Sijthoff and Nôrdhoff (1980) 51–76.

[3] F. ELZEIN and S. ZUCKER: Extendability of the Abel-Jacobi map. To appear.

[4] R. FRIEDMAN: Hodge theory, degenerations, and the global Torelli problem. Thesis, Harvard University (1981).

[5] R. FRIEDMAN and R. SMITH: The generic Torelli theorem for the Prym map. *Invent. Math. 67* (1982) 473–490.

[6] P. GRIFFITHS and J. HARRIS: *Principles of Algebraic Geometry,* John Wiley, 1978.

[7] P. GRIFFITHS: A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles. *Amer. J. Math. 101* (1979) 94–131.

[8] P. GRIFFITHS: Periods of certain rational integrals. *Ann. Math. 90* (1969) 460–541.

[9] K. KODAIRA and D.C. SPENCER: On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski. *Proc. Nat. Acad. Sci, U.S.A. 39* (1953) 1273–78.

[10] M. KURANISHI: New proof for the existence of locally complete families of complex structures. In: *Proceedings of the Conference on Complex Analysis,* Minneapolis 1964, NY, Springer-Verlag, 1965.

[11] I. KYNEV: The degree of the Prym map is equal to one. Preprint.

[12] S. LEFSCHETZ: L'Analysis Situs et la Geometrie Algebrique, Paris, Gauthier-Villars, 1924.

[13] B. SAINT-DONAT: On Petri's analysis of the linear system of quadrics through a canonical curve. *Math. Ann. 206* (1973) 157–175.

[14] D.C. SPENCER and M. SHIFFER: *Functionals on finite Riemann surfaces,* Princeton Univ. Press.

[15] S. ZUCKER: Generalized Intermediate Jacobians and the theorem on normal functions. *Invent. Math. 33* (1976) 185–222.

Notes

(1) Intuitively, δ satisfies the conditions of the differential of a variation of Hodge structure. Speaking precisely, on the classifying space for polarized Hodge structures there is a differential system I given by the infinitesimal period reation, and (in case δ is injective) V is given by an integral element of I.

(2) The dual of κ is ordinary multiplication of sections

$$\mu:\operatorname{Sym}^2 H^0(X,L)\to H^0(X,L^2),$$

so it is reasonable to hope for a good interpretation of the $\Xi_{k,l}$'s.

(3) The trouble is that, for reasons having to do with the variational theory of special divisors on curves, the mapping σ^n is almost never surjective when $n\geqslant 2$. These matters are discussed in our first paper.

(4) At latest count there are six different proofs of the weak global Torelli for curves of genus $g \geqslant 4$; of these, three use the Θ-divisor, one uses degeneration and induction on g (these proofs all work for any g), and two use infinitesimal variations of Hodge structure.

At present the best methods for proving Torelli-type results in higher dimensions seem to either be using infinitesimal variations of Hodge structure (cf. [2], and the paper by Ron Donagi which appears in this issue, Section 4 above), or to prove the result for special varieties and then establish a properness statement (cf. [4], [5], and [11]). Of course, the underlying problem is to effectively relate the Hodge theory and geometry of a higher dimensional variety in the absence of a Θ-divisor.

(5) A normal function is a cross-section $\nu(s) \in J_s$ associated to a family $\{J_s\}_{s \in S}$ of intermediate Jacobians, where ν is required to satisfy a quasi-horizontally condition $D\nu = 0$ (cf. Section 1(c)) and, in case S is a quasi-projective variety, a growth condition (cf. [3]). Normal functions typically arise by taking the Abel-Jacobi images of a family of algebraic cycles $Z_s \subset X_s$ where X_s is smooth projective and $Z_s \in \mathfrak{Z}_h(X_s)$ is homologous to zero. General references are [7], [15], and a *forthcoming paper by J. King.*

(6) Of course, we may interpret $\mathrm{d}u_s$ as the differential of a map between two manifolds, but this seems of little value. What seems necessary is some sort of homogeneity of the image manifold. Although this is not present in the situation at hand, the theory of exterior differential systems suggests a surrogate, and this is what motivated our construction of $\delta\nu$.

(7) The curve will be realized as the locus where a suitable expression $Q(\delta\nu/\delta\xi, \omega) = 0$ (this expression, which has a purely Hodge-theoretic construction, is a section of a line bundle), and since $\delta(-\nu) = -\delta(\nu)$ the sign ambiguity disappears.

(8) The point is that the proofs of the Lefschetz (1, 1) theorem either use the Θ-divisor associated to polarized Hodge structures of weight one (cf. [12] and, for a more recent exposition, [7]) or the exponential sheaf sequence ([9]); both of these methods encounter serious difficulties in codimension $\geqslant 2$, so it may be of interest to give a construction of algebraic cycles in a case covered by the Lefschetz (1, 1) theorem but by a method that at least makes sense in general.

(9) The first mapping is given by polarization of the homogeneous polynomial mapping given in Dolbeault cohomology by

$$\sigma(\theta) = \left(\det \theta\frac{i}{j}\right) \bigwedge_i \frac{\partial}{\partial z^i} \otimes \bigwedge_j \mathrm{d}\bar{z}^j$$

where $\theta = \sum_{i,j} \theta\frac{i}{j}\frac{\partial}{\partial z^i} \otimes \mathrm{d}\bar{z}^j \in H^{0,1}_{\bar\partial}(X, \Theta)$ (cf. Section 2 for further discussion).

(10) We would like to thank Joe Harris for substantial help in the proof of this result.

(11) As a dimension check, suppose that C is a smooth curve of genus $g \geqslant 2$. If $D = p_1 + \ldots + p_{2g-2} \in |K|$ is the divisor of $\omega \in H^0(C, K)$, and if $\psi \in H^0(C, K^2)$ vanishes on $p_1, \ldots, \hat{p}_i, \ldots, p_{2g-2}$, then from

$$\sum_i \mathrm{Res}_{p_i}(\psi/\omega) = 0$$

it follows that $\psi(p_i) = 0$. Alternatively, the exact cohomology sequence of

$$0 \to K \xrightarrow{\omega} K^2 \to K^2 \otimes \mathcal{O}_D \to 0$$

also gives $\psi(p_i) = 0$. In either case it follows that

$$\dim \overline{\varphi_{2K}(D)} = 2g - 4,$$

which implies that

$$\dim \bigcup_{D \in |K|} \overline{\varphi_{2K}(D)} = 3g - 5,$$

which is the dimension of a hypersurface in $\mathbb{P}H^1(C, \Theta) \cong \mathbb{P}^{3g-4}$.

(12) In this connection suppose that $\mathcal{V}$ is a trivial variation of Hodge structure; i.e., we have a fixed Hodge structure

$$\begin{cases} H = \bigoplus_{p+q=2m+1} H^{p,q} \\ H^{p,q} = \overline{H^{q,p}} \end{cases}$$

with polarizing form Q. Letting

$$J = H_{\mathbf{Z}} \backslash H / F^{m+1}$$

be the corresponding intermediate Jacobian, a normal function is given by a holomorphic mapping

$$\nu : S \to J.$$

For all l the varieties $\Xi_{m,l} = \mathbb{P}T$, and the infinitesimal invariant $\delta\nu$ may be identified with the usual differential

$$\nu_* : T_s(S) \to F^{m+1*}.$$

(13) Referring to footnote [12], suppose that $C \to S$ is *holomorphically* a product $C \times S$, so that $D_s \in \mathrm{Div}^0(C)$ is a holomorphic family of divisors on a fixed curve. Then we may take $f = 0$ in (6.11) and

$$Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = \sum_i \omega \Big|_{q_i'}^{p_i'}.$$

Put differently, in this case the infinitesimal invariant $\delta\nu$ essentially reduces to the Brill-Noether matrix that plays the fundamental role in the theory of special divisors on a fixed curve (cf. [1]). In particular, the condition

$$Q\left(\frac{\delta\nu}{\delta\xi}, \omega\right) = 0$$

for *all* families $\{D_s\} \subset \mathrm{Div}^0(C)$ with $D_0 = D = \Sigma p_i - q_i$ is equivalent to

$$(\omega) \geqslant \sigma(D) \qquad (*)$$

where $\sigma(D) = \Sigma p_i + q_i$ is the support of D. This result will be generalized in Theorem (6.28) below.

(14) Actually, all we will really use is that

$$\rho_{s_0}: T_{s_0}(S) \to H^1(C, \Theta)$$

is *surjective* (for the Kuranishi family it is an isomorphism). In particular every point $\theta \in \mathbb{P}H^1(C, \Theta)$ should be $\rho_{s_0}(\xi)$ for some $\xi \in \mathbb{P}T_{s_0}(S)$.

In our case where $\mathcal{C} \to S$ is the Kuranishi family we shall identify $T_{s_0}(S)$ with $H^1(C, \Theta)$, so that for $\theta \in H^1(C, \Theta)$ and $\omega \in \ker\{\theta: H^{1,0} \to H^{0,1}\}$ the infinitesimal invariant

$$Q\left(\frac{\delta\nu}{\delta\theta}, \omega\right)$$

is a well-defined complex number (for a given normal function ν).

(15) Here, we recall our notational convention that $\overline{\varphi_{2K}(E)}$ is the linear span of the points $\varphi_{2K}(r_\alpha) \in \mathbb{P}H^1(C, \Theta)$. Also, θ will denote both a point in $\mathbb{P}H^1(C, \Theta)$ and a non-zero vector in $H^1(C, \Theta)$; since we deal only with ranks of homogeneous equations this will cause no ambiguity.

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