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FUNCTIONS, FLOWS AND OSCILLATORY INTEGRALS ON FLAG MANIFOLDS AND CONJUGACY CLASSES IN REAL SEMISIMPLE LIE GROUPS

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0. Introduction

Let G be a real connected semisimple Lie group with finite center and $G = KAN$ its Iwasawa decomposition. Let $\mathfrak{a}$ be the Lie algebra of A. Then the "Iwasawa projection" $H: G \to \mathfrak{a}$ is defined by

$$x \in K \exp H(x) N. \tag{0.1}$$

It is well-known that this projection plays an important role in the harmonic analysis on G. For instance the matrix coefficients of the principal series of representations of G (the elementary spherical functions being a special case) may be expressed as integrals of the form

$$\int_K e^{\lambda(H(ak))} g(k)\, dk. \tag{0.2}$$

Here $a \in A$, g is an analytic function on K expressed in terms of matrix coefficients of representations of K, and λ, the eigenvalue parameter, belongs to $\mathscr{F}$, the complex dual of $\mathfrak{a}$. The asymptotics of the matrix coefficients as the group variable a tends to infinity have been studied for a long time, starting with the pioneering work of Harish–Chandra [28], [29]. He made a careful study of the system of differential equations satisfied by them. However, matrix coefficients also have the

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integral representation (0.2) which appears more suitable for studying their asymptotics when λ goes to infinity. A major aim of this paper is to do this.

Keeping $\operatorname{Re}\lambda = \eta$ fixed and absorbing the factor $e^{\eta(H(ak))}$ in the amplitude g, we see that (0.2) becomes

$$\int_K e^{i\langle H(ak), H\rangle} g(k)\, dk. \tag{0.3}$$

Here $\operatorname{Im}\lambda = \xi \in \mathfrak{a}^*$ and $H = H_\xi$ is the corresponding element of $\mathfrak{a}$ defined by ($\langle .,.\rangle$ is the Killing form of $\mathfrak{g}$):

$$\xi(Y) = \langle Y, H_\xi\rangle, \text{ for all } Y \in \mathfrak{a}. \tag{0.4}$$

If we replace H by τH in (0.3) and let $\tau \to +\infty$, then the principle of stationary phase tells us that the main contributions to the asymptotic expansion of (0.3) (in τ) come from the critical points of the "phase function" $F_{a,H}$ on K defined by

$$F_{a,H}(k) = \langle H(ak), H\rangle \qquad (k \in K). \tag{0.5}$$

Geometrically this function can be interpreted as testing the Iwasawa projection restricted to the right K-orbit of a in the symmetric space $K\backslash G$, by a linear form ξ on $\mathfrak{a}$.

The study of the critical points of the functions $F_{a,H}$, which we carry out in sections 5 and 6, reveals that the critical set of $F_{a,H}$ is equal to the union of the smooth manifolds $K_a w K_H$, where w runs through the Weyl group $\mathfrak{w}$ of (G, A). Here K_a, resp. K_H is the centralizer of a, resp. H in K. (Because $\mathfrak{w}$ is defined as the normalizer of $\mathfrak{a}$ in K modulo the centralizer M of $\mathfrak{a}$ in K, the notation wK_H makes sense, as always $K_H \supset M$ $(H \in \mathfrak{a})$.)

It is a very remarkable feature of these critical manifolds that they only depend on the sets of roots vanishing on $w^{-1}\log a$, resp. on H. So there are only finitely many possibilities, and in particular the critical sets do not change at all if the parameters a, H are varied in an equisingular way. Specifically, for regular ($=$ generic) a, H, the function $F_{a,H}$ is a Morse function on K/M whose critical points are always the Weyl group points.

Secondly, even for singular a, H, the Hessians of $F_{a,H}$ at the critical points turn out to be nondegenerate transversally to the critical manifolds, that is the $F_{a,H}$ have critical point sets which are clean in the sense of Bott.

The third important property of the $F_{a,H}$ is that, apart from their trivial left K_a-invariance, they are also right K_H-invariant if $H \in \mathrm{Cl}(\mathfrak{a}^+)$, the closure of the positive Weyl chamber. This right K_H-invariance property is somewhat surprising, and in general not true for arbitrary $H \in \mathfrak{a}$. These properties come up in our systematic study of the $F_{a,H}$ which is decisive in the investigation of (0.3) when $\|H\| \to \infty$.

It turns out that much light can be shed on the behaviour of the functions $F_{a,H}$, in particular their right K_H-invariance, if one compares them with their "infinitesimal" counterparts $f_{X,H}$ defined by

$$f_{X,H}(k) = \langle X, \mathrm{Ad}\, k(H) \rangle \qquad (k \in K). \tag{0.6}$$

(This is also the phase function in the integral formula for spherical functions on Cartan motion groups, see Gindikin [25].) Here $X, H \in \mathfrak{a}$, and we always compare $f_{X,H}$ with $F_{a,H}$ when $a = \exp X$. It is already clear from (0.6) that the $f_{X,H}$ have much simpler structure than the $F_{a,H}$. Geometrically, the functions $f_{X,H}$ can be viewed as testing the orthogonal projection of the $\mathrm{Ad}\, K$-orbit of H (in $\mathfrak{g}$) onto $\mathfrak{a}$ by linear forms on $\mathfrak{a}$. Note that actually $\mathrm{Ad}\, K(H)$ is contained in the orthogonal complement $\mathfrak{s}$ of $\mathfrak{k}$ in $\mathfrak{g}$.

The functions $f_{X,H}$ possess the same qualitative features as the $F_{a,H}$, that is they have the same critical set $K_X \mathfrak{w} K_H$, which moreover is clean. But in addition (and in contrast with the $F_{a,H}$) they are obviously left K_a- and right K_H-invariant. Furthermore, when $f_{X,H}$ is regarded as a function on the flag manifold K/K_H, there is a natural K-invariant Riemannian metric β_H on K/K_H such that the gradient flow of $f_{X,H}$ is equal to the action of the 1-parameter group $t \mapsto \exp tX$ on K/K_H. Here we identify K/K_H with the homogeneous G-space $G/G(H)$, $G(H)$ being a suitable parabolic subgroup of G. Moreover, this flow turns out to be linear (!) in the natural affine coordinates on the Bruhat cells, the cells themselves appearing as the stable manifolds at the Weyl group points.

Many of these properties of the $f_{X,H}$ have been obtained before by Bott [9] in the case of complex G (in fact for K modulo the centralizer of a torus for any compact K, but this is the same as the case of complex G, see Remark 2.2), and furthermore Hermann [35] and Takeuchi and Kobayashi [59] for general real G. Bott and Takeuchi–Kobayashi studied these functions in order to obtain information about the topology of the flag manifolds using Morse theory. More recently, the rigidity of the critical points has been used by Heckman [32] to give a simple geometric proof of a theorem of Kostant [42]. This asserts that both the image of $k \mapsto H(\exp X . k)$ and of $k \mapsto E_{\mathfrak{a}}(\mathrm{Ad}\, k^{-1}(X))$, with $E_{\mathfrak{a}}$ the orthogonal projection $\mathfrak{s} \to \mathfrak{a}$, are equal to the convex hull of the Weyl group orbit of X in $\mathfrak{a}$.

As a preparation for our study of the functions $F_{a,H}$, and also because we think these properties deserve wider attention, we give a self-contained review of the basic properties of the functions $f_{X,H}$ in sections 1 and 3. A short survey of their applications to the topology of the (complex and real) flag manifolds, both in terms of Morse theory and of the Schubert calculus, is presented in section 4. In section 2 we collect some basic facts about centralizers and parabolic subgroups which are used throughout the paper.

Now let us return to the oscillatory integrals (0.3) with H replaced by τH, $\tau \to +\infty$. The cleanness of the critical sets of the $F_{a,H}$ allows us to obtain a full asymptotic expansion (with a, H kept fixed) by a direct application of the method of stationary phase. The result is stated in section 9. Treating a, H as parameters in the phase function, one observes a "caustic" behaviour of the asymptotic expansion if a, H become singular, that is $\log a$, resp. H enter root hyperplanes. As a consequence there is a nontrivial problem of obtaining sharp estimates when $\tau \to +\infty$, which are uniform in the parameter $H \in \mathfrak{a}$ (not to speak of a). Using the rigidity of the critical sets and the right invariance properties of the $F_{a,H}$, we are able to obtain such uniform estimates when a is kept in a compact subset of A.

The upper bounds are in terms of simple functions of product type (cf. Theorem 11.1), suggested by the radial asymptotic expansions obtained in section 9. This situation is very different from what happens if the phase function belongs to a generic family of functions depending on parameters. In that case one gets uniform asymptotic expansions in terms of generalized Airy functions; these themselves have quite complicated behaviour, cf. Duistermaat [19, section 4].

For elementary spherical functions on complex groups this behaviour can be read off directly from the explicit formulae for them given in Harish–Chandra [28] and for the special case of $SL(n, \mathbb{C})$ by Gelfand and Naimark [24, §9]. For general real G no such explicit formulae are known.

The proof, involving partitions of the closures of the Weyl chambers in the H-space $\mathfrak{a}$ into suitable sectors, occupies section 11. The main feature that makes it work is that at a common critical point the phase functions can be brought into a "trigonal form"; this allows us to use the classical Morse lemma repeatedly. It would be interesting to make the phase functions $F_{a,H}$ equivalent, by a smooth coordinate transformation depending smoothly on the parameters a, H, to phase functions for which the product structure of the asymptotics of the oscillatory integrals could be read off more directly. However, we could not even make $F_{a,H}$ and $f_{X,H}$ equivalent to each other in this sense, despite the fact that their qualitative behaviour is very much the same.

In a subsequent article we hope to extend the uniform estimates in H to the case that also a is allowed to run to infinity, and to apply them to obtain sharper bounds for the remainder in the asymptotics for the spectra of compact locally symmetric spaces in DKV [20].

It may be remarked that for elementary spherical functions and for regular a, the asymptotic expansions as the eigenvalue parameter runs to infinity, can also be obtained by a careful analysis of the asymptotics of Harish–Chandra as a runs to infinity. For instance, one can use the improvements of Gangolli [23]. However, the method of stationary phase is more direct and is applicable to more general integrals, when the amplitude g is arbitrary and a is allowed to be singular. It may also be observed that our estimates are uniform in the representation, that is, the constant factor depends only on a norm on the space of amplitudes g.

We now come to the last topic of this article. In the study of spectra of compact locally symmetric spaces via the Selberg trace formulae (cf. DKV [20]) one actually needs estimates for oscillatory integrals of the type

$$\int_{C_\gamma} e^{i\langle H(x), H\rangle} g(x)\, dC_\gamma(x). \tag{0.7}$$

Here C_γ is the conjugacy class of a semisimple element γ of G, $dC_\gamma(x)$ is an invariant measure on C_γ and g is a smooth function with compact support in G. In section 8 we study the corresponding phase function $F_{H,\gamma}$ on C_γ, defined by

$$F_{H,\gamma}(x) = \langle H(x), H\rangle, \qquad (x \in C_\gamma). \tag{0.8}$$

Again it turns out remarkably that $F_{H,\gamma}$ has a clean critical point set; and it is rigid in its dependence on H because it is equal to $C_\gamma \cap G_H$, G_H = the centralizer of H in G. The corresponding asymptotics of (0.7) if H is replaced by τH and $\tau \to +\infty$, is described in section 10. In the same section we also apply this asymptotic expansion in order to obtain a detailed analysis of the singularities of the distributions T_γ which appear in DKV [20]. Finally the rigidity of the critical sets allows for sharp upper bounds which are uniform in H (only for fixed γ however); these are proved in section 12. The method of proof here in fact leads to a full asymptotic expansion of (0.7) which is uniform in $H \in \mathfrak{a}$, and applies equally to the integrals (0.3) if a is restricted to an equisingular set.

It is somewhat surprising that phase functions and oscillatory integrals associated to the Iwasawa projection have not been studied systematically before. To our knowledge the only exception to this is the work of Cohn [14], who used this point of view in his study of the asymptotics of Harish–Chandra's C-functions. These C-functions appear in the leading terms of the asymptotic expansions of the matrix coefficients as $a \to \infty$. They may be expressed as integrals similar to (0.2) but taken over $\bar{N} = \theta N$, θ being the Cartan involution of G. We compare our results with Cohn's in section 7.

NOTATION: Generally our notation is standard and is the one used in our paper DKV [20].

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1. The functions $f_{X,H}$

Consider the adjoint action of K on $\mathfrak{s}$, the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. For $H \in \mathfrak{s}$ the mapping $k \mapsto H^k = \mathrm{Ad}\, k(H)$ defines a diffeomorphism from the "flag manifold" K/K_H onto the $\mathrm{Ad}\, K$-orbit through H. (For example, if $G = SL(3, \mathbb{R})$, $H \neq 0$, H singular, then K/K_H is the projective plane and $k \mapsto H^k$ is the Veronese mapping, embedding the projective plane into a 5-dimensional Euclidean space.) Testing this map by linear forms on $\mathfrak{s}$ amounts to looking at the smooth functions

$$f_{X,H}: k \mapsto \langle X, H^k \rangle \tag{1.1}$$

on K, here $X \in \mathfrak{s}$.

LEMMA 1.1: *k is a critical point for $f_{X,H}$ if and only if* $[X, H^k] = 0$.

For any $Y \in \mathfrak{k}$,

$$\frac{d}{dt} f_{X,H}(k \exp tY)_{t=0} = \langle X, [Y,H]^k \rangle = -\langle [X, H^k], Y^k \rangle. \tag{1.2}$$

This being equal to zero for all $Y \in \mathfrak{k}$ means that

$$[X, H^k] \in \mathfrak{s} \cap [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s} \cap \mathfrak{k} = (0).$$

If X is a regular element of $\mathfrak{s}$ then its centralizer in $\mathfrak{s}$ is a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{s}$. Because $f_{X,H}$ obviously has critical points on the compact manifold K, it follows that each $\operatorname{Ad} K$-orbit in $\mathfrak{s}$ intersects $\mathfrak{a}$. In turn this implies the conjugacy under K of all maximal abelian subalgebras of $\mathfrak{s}$. This proof of the conjugacy is essentially the one by Hunt [37] for complex groups, generalized to real G by Helgason [33, Ch.V. Lemma 6.3].

In view of the conjugacy result it is sufficient to consider functions $f_{X,H}$ only for $X, H \in \mathfrak{a}$, as we shall do from now on.

Proposition 1.2: *If $X, H \in \mathfrak{a}$ then the critical set of $f_{X,H}$ is equal to*

$$K_{X,H} = \bigcup_{w \in \mathfrak{w}} K_X w K_H. \tag{1.3}$$

Here $\mathfrak{w}$ is the Weyl group, the normalizer of $\mathfrak{a}$ in K modulo the centralizer M of $\mathfrak{a}$ in K. So the notation wK_H makes sense because $K_H \supset M$. Moreover,

$$K_H = MK_H^0 \text{ and } K_X = K_X^0 M. \tag{1.4}$$

Let $[X, H^k] = 0$, that is $X^{k^{-1}} \in \mathfrak{g}_H$. G_H^0 is a connected reductive Lie group with maximal compact subgroup K_H^0 and A as the vector subgroup in its Iwasawa decomposition. Applying the conjugacy theorem to $X^{k^{-1}}$ we get $l \in K_H^0$ such that $(X^{k^{-1}})^l = X^{lk^{-1}} \in \mathfrak{a}$. According to a well-known lemma of Harish–Chandra [31] there is then an element $w \in \mathfrak{w}$ such that $(X^{lk^{-1}})^w = X$, or $x_w l k^{-1} \in K_X$ if $x_w \in K$ represents w. This shows that

$$K_{X,H} \subset \bigcup_{w \in \mathfrak{w}} K_X w K_H^0. \tag{1.5}$$

Because obviously $f_{X,H}$ is left K_X and right K_H-invariant and the Weyl group elements are stationary points, we have also

$$\bigcup_{w \in \mathfrak{w}} K_X w K_H \subset K_{X,H}, \tag{1.6}$$

combining we get (1.3).

For the second statement, taking X regular we get

$$K_H \subset \bigcup_{w \in \mathfrak{w}} MwK_H^0. \tag{1.7}$$

Now, if $MwK_H^0 \cap K_H \neq \varnothing$ then $w \in \mathfrak{w}_H$. This implies that w is generated by reflections corresponding to roots vanishing on H. But such reflections have representatives (modulo M) in K_H^0. So $K_H \subset MK_H^0$. Finally $K_X = K_X^0 M$ follows from $K_X = MK_X^0$ by applying the map $x \mapsto x^{-1}$.

$K_{X,H}$, being the union of finitely many orbits of the left-right action of the compact Lie group $K_X \times K_H$, is a union of finitely many smooth compact connected components. In general they may have different dimensions. The situation is clarified further in the following

PROPOSITION 1.3: *We have the disjoint union*

$$K_{X,H} = \coprod_{w \in \mathfrak{w}_X \backslash \mathfrak{w} / \mathfrak{w}_H} K_X w K_H, \tag{1.8}$$

where the union is over a complete set of double coset representatives. If $\pi_{K/M}$ denotes the projection: $K \to K/M$ one also has the disjoint union

$$\pi_{K/M}(K_{X,H}) = \coprod_{w \in \mathfrak{w}_X \backslash \mathfrak{w} / \mathfrak{w}_H} \pi_{K/M}(K_X^0 x_w K_H^0) \tag{1.9}$$

and all components of the $K_X \times K_H$-orbit $K_X w K_H$ have the same dimension equal to $\dim M$ *plus*

$$\dim \pi_{K/M}(K_X^0 x_w K_H^0) = \sum_{\alpha \in \Delta^+,\, \alpha(H).\, w\alpha(X) = 0} \dim \mathfrak{g}_\alpha. \tag{1.10}$$

PROOF: $K_X w K_H = w_1 K_{w_1}(w_1^{-1} w w_2) K_{w_2 H} w_2^{-1}$, which allows a reduction to the case that X and H are contained in the closure of the positive Weyl chamber. If $K_X w K_H \cap K_X w' K_H \neq \varnothing$ then $G(X) w G(H) \cap G(X) w' G(H) \neq \varnothing$ where $G(X)$, resp. $G(H)$ are the corresponding parabolic subgroups of G as defined in section 2. But then it follows from the theory of Bruhat decompositions (cf. Borel–Tits [7, 5.20]) that $w' \in \mathfrak{w}_X w \mathfrak{w}_H$.

The next statement follows from the observation that $K_X w K_H$ and $K_X^0 x_w K_H^0$ have the same image under $\pi_{K/M}$ in view of (1.4). Since the $K_X w K_H$ are right M-invariant their disjointness in K implies the disjointness of their images in K/M.

Because $K_X w K_H = w K_{X'} K_H$ with $X' = w^{-1}X$, the left hand side in (1.10) is equal to the dimension of $(\mathfrak{k}_{X'} + \mathfrak{k}_H) \ominus \mathfrak{m}$, the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}_{X'} + \mathfrak{k}_H$. Now the map $Y \mapsto Y + \theta Y$ is a linear isomorphism of $\mathfrak{n}$ with $\mathfrak{k} \ominus \mathfrak{m}$; and if E is any subset of $\mathfrak{a}$, $Y \in \mathfrak{n}$ centralizes E if and only if $Y + \theta Y$ centralizes E. From this one finds that $I + \theta$ is an isomorphism between $\mathfrak{n}_{X'} + \mathfrak{n}_H$ and $(\mathfrak{k}_{X'} + \mathfrak{k}_H) \ominus \mathfrak{m}$, proving (1.10).

Next we compute the Hessian of $f_{X,H}$ at its stationary points.

PROPOSITION 1.4: *Let $k = ux_w v$ with $u \in K_X$, x_w a representative of $w \in \mathfrak{w}$ and $v \in K_H$. Then, for each $Y \in \mathfrak{k}$:*

$$\frac{d^2}{dt^2} f_{X,H}(k \exp tY)_{t=0} = - \sum_{\alpha \in \Delta^+} \alpha(H) w\alpha(X) \,.\, \|F_\alpha(Y^v)\|^2 \tag{1.11}$$

Here F_α denotes the orthogonal projection: $\mathfrak{k} \to \mathfrak{k} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$ and the inner product in $\mathfrak{k}$ is equal to minus the Killing form.

PROOF: $f_{X,H}(ux_w v \exp tY) = f_{X,H}(x_w \exp tY^v) = f_{w^{-1}X,H}(\exp tY^v)$ reduces the computation to the case that $k = 1$. Now $\mathrm{Ad}(\exp tY) = \exp(t \,\mathrm{ad}\, Y)$, so the Taylor expansion of the exponential function yields

$$\frac{d^2}{dt^2} f_{X,H}(\exp tY)_{t=0} = \langle X, [Y,[Y,H]] \rangle = -\langle [Y,X],[Y,H] \rangle. \tag{1.12}$$

Writing $Y = \sum_{\alpha \in \Delta^+} Y_\alpha + \theta Y_\alpha$ modulo $\mathfrak{m}$, with $Y_\alpha \in \mathfrak{g}_\alpha$ (hence $\theta Y_\alpha \in \mathfrak{g}_{-\alpha}$), the right hand side is equal to

$$\begin{aligned} &- \sum_{\alpha, \beta \in \Delta^+} \alpha(X)\beta(H) \langle Y_\alpha - \theta Y_\alpha, Y_\beta - \theta Y_\beta \rangle \\ &= \sum_{\alpha \in \Delta^+} \alpha(X)\alpha(H) \{ \langle Y_\alpha, \theta Y_\alpha \rangle + \langle \theta Y_\alpha, Y_\alpha \rangle \} \\ &= \sum_{\alpha \in \Delta^+} \alpha(X)\alpha(H) \langle Y_\alpha + \theta Y_\alpha, Y_\alpha + \theta Y_\alpha \rangle. \end{aligned}$$

Here it has been used twice that $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ unless $\beta = -\alpha$, and that $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$.

COROLLARY 1.5: *At each critical point, the Hessian of $f_{X,H}$ is nondegenerate transversally to the critical manifold. In other words, $f_{X,H}$ has clean critical point set in the sense of Bott* [10].

REMARK 1.6: The dependence of the critical set of $f_{X,H}$ on the parameters X, H has a highly nongeneric rigidity. Because of (1.4), K_X only depends on K_X^0, that is on $\mathfrak{k}_X$, which in turn is determined by the set of roots vanishing on X. Because K_H behaves similarly, there is only a finite number of possibilities for the critical set $K_X \mathfrak{w} K_H$. Now for a generic family $\varepsilon \mapsto f(\varepsilon, x)$ of functions of x, one has for each ε isolated critical points x, depending smoothly on ε if ε varies in the complement of the so-called "catastrophe set" of Thom. When ε approaches the catastrophe set, some of the critical points coalesce in a root type manner, that is they come together with a velocity which runs to infinity as ε reaches the catastrophe set. In contrast, regarding the $f_{X,H}$ as functions on K/M, we have for regular X, H nondegenerate critical points which are equal to the elements of the Weyl group $\mathfrak{w}$. In particular they don't move at all as the parameters vary. Only, if the X, H enter root hyperplanes (their union being the catastrophe set in this case), the critical set becomes a smooth manifold connecting some of the previous critical points. Because $f_{X,H}$ is transversally nondegenerate, all points of these critical manifolds are indistinguishable from the qualitative point of view; so there is no special role left for the original critical points. If X, H move within these hyperplanes, the critical manifold again remains fixed, until the intersection of more hyperplanes is reached; at that moment the critical manifold changes abruptly into a higher dimensional manifold connecting more of the original critical points.

REMARK 1.7: The rigidity of the critical sets in their dependence on X, H, together with the fact (read off from (1.11)) that $f_{X,H}$ has only one local maximal (resp. minimal) value, equal to $\langle X, wH\rangle$ for some $w \in \mathfrak{w}$, have been used by Heckman [32] in his proof of the convexity theorem of Kostant [42], see also Berezin and Gel'fand [2]. This theorem states that the image of the very "roundish" object $\mathrm{Ad}\,K(H)$ under the orthogonal projection $E_{\mathfrak{a}}: \mathfrak{s} \to \mathfrak{a}$ is equal to the convex hull of the finitely many points

$$\{wH \mid w \in \mathfrak{w}\} = \mathrm{Ad}\,K(H) \cap \mathfrak{a}. \tag{1.13}$$

For instance, the equilateral triangle in the plane appears this way in the film of Banchoff [1] about the linear projections of the Veronese surface.

REMARK 1.8: As another application of (1.2), the image of the tangent map at k of the mapping $\Psi: k \mapsto E_{\mathfrak{a}}(H^k)$ is equal to the set $\{E_{\mathfrak{a}}([Y, H^k]) \mid Y \in \mathfrak{k}\}$. Using root space decompositions, one verifies easily that this space is equal to the span of the H_α, α running over the positive roots such that the $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{s}$ component of H^k is not equal to zero. That is, also the image spaces of the tangent maps of Ψ have only finitely many positions. As a consequence the set of singular values of Ψ is piecewise linear, each piece being parallel to a span of some H_α's.

REMARK 1.9: The results of this section could be extended to the case that $H \in \mathfrak{a}_c$, the complexification of $\mathfrak{a}$. Note that if H, $H' \in \mathfrak{a}$ then k is a critical point for $f_{X,H+iH'}$ if and only if it is critical for both $f_{X,H}$ and $f_{X,H'}$ simultaneously. For further information about simultaneous critical points, see Lemma 11.3.

REMARK 1.10: The flag manifold K/K_H is actually a homogeneous space for the big group G rather than K. That is $K/K_H \simeq G/G(H)$ for a so-called parabolic subgroup $G(H)$ of G, which will be introduced in the next section.

2. Parabolic subgroups and flag manifolds

In this section we give a brief review of some facts concerning parabolic subgroups of semisimple groups and the associated flag manifolds. One purpose is to explain the connection with the "flag manifolds" K/K_H of section 1, another is to establish notation. For more detailed treatments we refer to Borel [6, Ch. IV], Borel–Tits [7, §§4, 5] and Varadarajan [65, II. 6].

Let $\mathbb{G}$ be a connected linear algebraic group over $\mathbb{C}$. A parabolic subgroup of $\mathbb{G}$ can be defined as a closed subgroup P such that $\mathbb{G}/P$ is a projective variety; this space is called the corresponding flag manifold. An equivalent characterization is that P is a closed subgroup which contains a Borel (= maximal connected solvable) subgroup B of $\mathbb{G}$. Basic facts are that all Borel subgroups of $\mathbb{G}$ are conjugate to each other and that all parabolic subgroups of $\mathbb{G}$ are connected (see, for instance, (11.1) resp. (11.15) in [6]).

Because the connected solvable radical is contained in any Borel subgroup, it is always divided away when passing to $\mathbb{G}/P$. Hence it is sufficient to assume that $\mathbb{G}$ is semisimple when studying flag manifolds.

Let $\mathfrak{c}$ be a Cartan algebra in $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{c} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the corresponding root space decomposition. As usual, if Δ^+ is a choice of positive roots, we write $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Then $\mathfrak{b} = \mathfrak{c} \oplus \mathfrak{n}$ is a solvable Lie algebra.

More generally, let S be the set of simple roots in Δ^+. For a subset Φ of S, possibly empty, we write $\Delta(\Phi)$ for the set of roots in Δ that are integral linear combinations of elements of Φ only; and we put $\Delta^+(\Phi) = \Delta^+ \cap \Delta(\Phi)$. We then define

$$\mathfrak{p}_\Phi = (\mathfrak{c} \oplus \sum_{\alpha \in \Delta(\Phi)} \mathfrak{g}_\alpha) \oplus \sum_{\alpha \in \Delta^+ \setminus \Delta(\Phi)} \mathfrak{g}_\alpha = \mathfrak{b} \oplus \sum_{\alpha \in \Delta^+(\Phi)} \mathfrak{g}_{-\alpha}. \tag{2.1}$$

Obviously, $\mathfrak{p}_\Phi$ is a Lie algebra, containing $\mathfrak{b}$. Conversely, any Lie algebra in $\mathfrak{g}$ containing $\mathfrak{b}$ is of the form $\mathfrak{p}_\Phi$ for some $\Phi \subset S$. A glance at the reductive Lie algebra

$$\mathfrak{m}_\Phi = \mathfrak{c} \oplus \sum_{\alpha \in \Delta(\Phi)} \mathfrak{g}_\alpha \tag{2.2}$$

in $\mathfrak{p}_\Phi$ shows that $\mathfrak{p}_\Phi$ is solvable only if $\Phi = \varnothing$; that is, $\mathfrak{b}$ is a maximal solvable Lie algebra (Borel algebra) in $\mathfrak{g}$. Using this one obtains that B, the normalizer of $\mathfrak{b}$ in $\mathbb{G}$, is a Borel subgroup. More generally, the Lie algebra $\mathfrak{p}_\Phi$ is equal to its own normalizer in $\mathfrak{g}$; so the normalizer P_Φ of $\mathfrak{p}_\Phi$ in $\mathbb{G}$ is a closed subgroup of $\mathbb{G}$ with $\mathfrak{p}_\Phi$ as its Lie algebra. If now P is a parabolic subgroup of $\mathbb{G}$ containing B, then $\mathfrak{p} = \mathrm{Lie}\,(P)$ is of the form $\mathfrak{p}_\Phi$, for some $\Phi \subset S$; and it follows that $P = P_\Phi$.

In other words, we get a bijective correspondence $\Phi \mapsto P_\Phi$ between the subsets of S and the parabolic subgroups of $\mathbb{G}$ containing B. The P_Φ are called the "standard" parabolic subgroups with respect to the choice of $\mathfrak{c}$ and Δ^+.

Now let us turn to the case in which we actually are interested, namely that G is a connected real semisimple Lie group with finite center. A subalgebra $\mathfrak{p}$ of $\mathfrak{g} = \mathrm{Lie}\,(G)$ will be called parabolic if its complexification $\mathfrak{p}_c$ is parabolic in $\mathfrak{g}_c$, that is, if it contains a Borel subalgebra of $\mathfrak{g}_c$. A parabolic subgroup P of G is defined as the normalizer in G of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$; and G/P is called the corresponding flag manifold. In contrast to the complex case, P need not be connected.

The connection with the algebraic theory can be made by taking $\mathbb{G} = \mathrm{Aut}\,(\mathfrak{g}_c)^0$, which is a connected complex linear algebraic group, defined over $\mathbb{R}$. The adjoint representation is a homomorphism with a finite kernel from G onto $\mathrm{Ad}(G) = \mathbb{G}(\mathbb{R})^0$, the connected component of the subgroup of $\mathbb{R}$-points. The parabolic subgroups P of G then are the inverse images under Ad of the parabolic subgroups P_c of $\mathbb{G}$ that are defined over $\mathbb{R}$. Because $\mathrm{Ker\,Ad} \subset P$, the natural map $G/P \to \mathbb{G}/P_c$ is a diffeomorphism from G/P to a component of the real locus of the complex algebraic flag manifold $\mathbb{G}/P_c$.

We now proceed to a classification of the real parabolic subgroups. Let $G = KAN$ be an Iwasawa decomposition; note that A is a maximal $\mathbb{R}$-split torus in G. As usual we write $\mathfrak{a} = \mathrm{Lie}\,(A)$, Δ, the set of roots of $(\mathfrak{g}, \mathfrak{a})$ and Δ^+, the positive system defining $\mathfrak{n} = \mathrm{Lie}\,(N)$. Also M, resp. $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in K, resp. in $\mathfrak{k}$. Then $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a parabolic subalgebra with normalizer $P = MAN$.

More generally, let S be the set of simple roots in Δ^+. For any subset $\Phi \subset S$ we write $\Delta(\Phi)$ (resp. $\Delta^+(\Phi)$) for the set of roots which are integral (resp. nonnegative integral) linear combinations of the roots in Φ. We then define

$$\begin{aligned}\mathfrak{p}_\Phi &= (\mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha\in\Delta(\Phi)} \mathfrak{g}_\alpha) \oplus \sum_{\alpha\in\Delta^+\setminus\Delta(\Phi)} \mathfrak{g}_\alpha = \\ &= \mathfrak{p} \oplus \sum_{\alpha\in\Delta^+(\Phi)} \mathfrak{g}_{-\alpha}.\end{aligned} \tag{2.1 $\mathbb{R}$}$$

Obviously $\mathfrak{p}_\Phi$ is a Lie algebra containing $\mathfrak{p}$, hence parabolic; and its normalizer P_Φ in G is called the corresponding "standard" parabolic subgroup. Using Borel–Tits [7, Proposition 5.14] with $K = \mathbb{C}$ and $k = \mathbb{R}$, one proves that any parabolic subalgebra of $\mathfrak{g}$, resp. subgroup of G, is conjugate to a standard one, and that the minimal ones are all conjugate to $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, resp. $P = MAN$. Note that the conjugation can be performed by elements of K, because $G = K \cdot P$.

Now let $H \in Cl(\mathfrak{a}^+)$ be such that Φ is precisely the subset of S vanishing at H. It follows that $\Delta(\Phi)$ is the set of roots vanishing at H, while $\Delta^+\setminus\Delta(\Phi)$ is the set of roots α with $\alpha(H) > 0$. In this case we define

$$\begin{aligned}&\mathfrak{g}(H) = \mathfrak{p}_\Phi = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha(H)\geq 0} \mathfrak{g}_\alpha, \\ &G(H) = P_\Phi = \text{normalizer of } \mathfrak{g}(H) \text{ in } G.\end{aligned} \tag{2.3}$$

In order to get further information about the standard parabolic subgroups $G(H)$, we define

$$\begin{aligned}&M_\Phi = G_H = \text{centralizer of } H \text{ in } G; \mathfrak{m}_\Phi = \mathfrak{g}_H = \mathrm{Lie}(G_H); \\ &\mathfrak{n}_\Phi = \mathfrak{n}^{(H)} = \sum_{\alpha(H)>0} \mathfrak{g}_\alpha; N^{(H)} = \exp \mathfrak{n}^{(H)}.\end{aligned} \tag{2.4}$$

Note that for any $H \in \mathfrak{a}$, not necessarily in $Cl(\mathfrak{a}^+)$, we can still define $\mathfrak{g}(H)$, $G(H)$, G_H and $\mathfrak{n}^{(H)}$ by (2.3), (2.4). If we choose a chamber whose closure contains H, $G(H)$ will be a parabolic subgroup, standard with respect to the Iwasawa decomposition defined by this chamber; Φ will then be the subset of simple roots (with respect to this chamber) that vanish at H.

We begin by observing that $N^{(H)}$ is the connected unipotent radical of $G(H)$ and that we have the semi-direct product

$$G(H) = G_H N^{(H)} \qquad (H \in \mathfrak{a}). \tag{2.5}$$

This can be proved by passing, via the adjoint representation, to the corresponding decomposition for $\mathbb{G}(\mathrm{ad}\, H)$. Secondly,

$$G_H = K_H A N_H \qquad \text{for any } H \in \mathfrak{a}. \tag{2.6}$$

Indeed, let $\mathrm{Ad}(kan)(H) = H$ for $k \in K$, $a \in A$, $n \in N$. Since $\mathrm{Ad}(an)(H) \equiv H \bmod \mathfrak{n}$ and $\mathrm{Ad}\, k^{-1}(H) \in \mathfrak{s}$, while $\mathfrak{s} \cap \mathfrak{n} = (0)$, we see that necessarily $\mathrm{Ad}(an)(H) = H$, $\mathrm{Ad}\, k^{-1}(H) = H$; that is, $n \in N_H$ and $k \in K_H$.

Finally,

$$G(H) = K_H A N \qquad \text{if } H \in Cl(\mathfrak{a}^+). \tag{2.7}$$

This follows from (2.5) and (2.6) if we use

$$N = N_H N^{(H)} \qquad (H \in Cl(\mathfrak{a}^+)). \tag{2.8}$$

This in turn follows from Lemma 2.3 below, observing that $\mathfrak{n} = \mathfrak{n}_H \oplus \mathfrak{n}^{(H)}$ if $H \in Cl(\mathfrak{a}^+)$. Lemma 2.3 is well-known, we have inserted it in this section for future reference.

The decompositions (2.6), (2.7) can be regarded as obtained by restricting the Iwasawa decomposition of G to G_H, resp. $G(H)$. Notice that (2.6) is true without assuming that H has any special position with respect to the Weyl chamber $\mathfrak{a}^+$ in terms of which N is defined.

We now come to the relation with the flag manifolds K/K_H of section 1.

PROPOSITION 2.1: *The natural map $i: K/K_H \to G/G(H)$ is a diffeomorphism.*

By conjugation with a Weyl group element we can arrange that $H \in Cl(\mathfrak{a}^+)$. Combining (2.7) with the Iwasawa decomposition $G = KAN$, we get that i is a bijection. Moreover, i is of constant rank because it is K-equivariant.

Note that the adjoint representation induces a diffeomorphism of $G/G(H)$ with $\mathrm{Ad}\, G/\mathrm{Ad}\, G(H)$ (Ker Ad is divided away); it is a component of the real locus of a complex algebraic flag manifold, which moreover is defined over $\mathbb{R}$. So the complex flag manifold can be considered as a complexification of the real flag manifold K/K_H.

REMARK 2.2: If G itself is complex then $\mathfrak{a} = \mathfrak{c}_{\mathbb{R}}$ for the Cartan algebra $\mathfrak{c} = \mathfrak{a} \oplus i\mathfrak{a}$. Here $i\mathfrak{a} = \mathfrak{t}$ is the Lie algebra of a maximal torus $T\ (= M)$ in K. Therefore, in this case the real flag manifolds are identified with the complex flag manifolds of the algebraic theory. Also, since for a suitable choice of H, $K_H = K_{iH}$ = the centralizer in K of an arbitrary torus in K, and because any compact semisimple Lie group K arises as the maximal compact subgroup of a complex semisimple Lie group, the complex algebraic flag manifolds are identified with the K/Z where K is a compact connected Lie group and Z is the centralizer of a torus in K. This identification is basic in the construction of Borel–Weil [54] of irreducible representations of K.

LEMMA 2.3: *Let $\mathfrak{n}_1, \ldots, \mathfrak{n}_m$ be subalgebras of $\mathfrak{n}$ such that*
(a) *$\mathfrak{n}$ is the direct sum of the $\mathfrak{n}_j$ (as a vector space), and*
(b) *each $\mathfrak{n}_j$ is the direct sum of a collection of the $\mathfrak{g}_\alpha$.*
Let $N_j = \exp \mathfrak{n}_j$. Then the map $(n_1, \ldots, n_m) \mapsto n_1 \cdot \ldots \cdot n_m$ is an analytic diffeomorphism of $N_1 \times \ldots \times N_m$ with N.

For a proof, see for instance Séminaire Chevalley [53, Exp. 13, Théorème 1]. A similar statement is of course true for $\bar{N} = \theta N$.

3. The A-action on the flag manifolds

Although we now have two additional models for the flag manifolds K/K_H (the Ad K-orbit in $\mathfrak{s}$ and $G/G(H)$), we continue to write it as K/K_H and denote its elements by $\bar{k} = kK_H$ ($k \in K$). The diffeomorphism

$$i: K/K_H \to G/G(H) \tag{3.1}$$

of Proposition 2.1 conjugates the left action of G on $G/G(H)$ to an action of G on K/K_H. For convenience of notation *we assume throughout this section that $H \in Cl(\mathfrak{a}^+)$.*

Then $G(H) \supset AN$ and the action of $x \in G$ on $K \simeq G/AN$, resp. on $K/K_H \simeq G/G(H)$ can be written as

$$\Theta_x(k) = \kappa(xk), \text{ resp. } \Theta_x(\bar{k}) = \overline{\kappa(xk)} \qquad (x \in G, k \in K). \tag{3.2}$$

Here κ is the projection $G \to K$ defined by

$$x \in \kappa(x)AN \qquad \text{for } x \in G. \tag{3.3}$$

Note that the G-action on K/K_H is covered by the G-action on K via the natural projection $K \to K/K_H$.

For each $X \in \mathfrak{g}$, the one-parameter group $\exp tX$, $t \in \mathbb{R}$, gives rise to a flow on K, resp. K/K_H; the corresponding velocity field will be denoted by v_X. Transporting elements of $T_k K$, resp. $T_{\bar{k}}(K/K_H)$ to $\mathfrak{k}$, resp. $\mathfrak{k}/\mathfrak{k}_H$, using left multiplication by k^{-1}, we get

LEMMA 3.1: *For $X \in \mathfrak{g}$, $k \in K$, $H \in Cl(\mathfrak{a}^+)$ we get*

$$v_X(k) = E_{\mathfrak{k}}(X^{k^{-1}}), \textit{ resp. } v_X(\bar{k}) = E_{\mathfrak{k}}(X^{k^{-1}}) \bmod \mathfrak{k}_H. \tag{3.4}$$

Here $E_{\mathfrak{k}}$ is the projection $\mathfrak{g} \to \mathfrak{k}$ along $\mathfrak{a} + \mathfrak{n}$.

If $x(t)$ is a smooth curve in G with $x(0) = 1$, $\dot{x}(0) = X$, then differentiating

$$k^{-1}x(t)k = k^{-1}\kappa(x(t)k)a(x(t)k)n(x(t)k)$$

with respect to t at $t = 0$, we find

$$X^{k^{-1}} = \frac{d}{dt}k^{-1}\kappa(x(t)k)_{t=0} \text{ modulo } \mathfrak{a} + \mathfrak{n}.$$

We shall now prove that the vector field v_X on K/K_H is the gradient of the function $f_{X,H}$ with respect to a suitable K-invariant metric β_H on K/K_H, following essentially Takeuchi and Kobayashi [59]. The mapping

$$\zeta : Y + \theta Y \mapsto Y - \theta Y, \quad Y \in \mathfrak{n} \tag{3.5}$$

defines a linear isomorphism $\mathfrak{k} \ominus \mathfrak{m} \xrightarrow{\sim} \mathfrak{s} \ominus \mathfrak{a}$; it can be extended to a linear mapping $\mathfrak{k} \to \mathfrak{s}$ by defining it equal to zero on $\mathfrak{m}$. We define the form b_H on $\mathfrak{k} \times \mathfrak{k}$ by

$$b_H(Z, Z') = \langle H, [Z, \zeta(Z')]\rangle \qquad (Z, Z' \in \mathfrak{k}). \tag{3.6}$$

LEMMA 3.2: (i) *b_H is a symmetric positive semi-definite bilinear form on $\mathfrak{k} \times \mathfrak{k}$, with radical equal to $\mathfrak{k}_H$. Hence it determines an inner product $\bar{b}_H$ on $\mathfrak{k}/\mathfrak{k}_H$.*

(ii) *If F_α are the orthogonal projections $\mathfrak{k} \to \mathfrak{k}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$, we have, for all $Z, Z' \in \mathfrak{k}$*

$$b_H(Z, Z') = \sum_{\alpha \in \Delta, \alpha(H) > 0} \alpha(H) \cdot (F_\alpha(Z), F_\alpha(Z')) \tag{3.7}$$

where the inner product on $\mathfrak{k}$ is equal to minus the Killing form.

Since (ii) implies (i), we shall prove (ii). For any root α, $Y \in \mathfrak{g}_\alpha$ and $U \in \mathfrak{g}$, we obtain

$$\langle H, [Y, U]\rangle = \langle [H, Y], U\rangle = \alpha(H)\langle Y, U\rangle,$$

similarly $\langle H, [\theta Y, U]\rangle = -\alpha(H)\langle \theta Y, U\rangle$. Writing

$$Z = \sum_{\alpha \in \Delta, \alpha(H) \geq 0} Y_\alpha + \theta Y_\alpha, \; Z' = \sum_{\alpha \in \Delta, \alpha(H) \geq 0} Y'_\alpha + \theta Y'_\alpha$$

with Y_α, $Y'_\alpha \in \mathfrak{g}_\alpha$, we find

$$\zeta(Z') = \sum_{\alpha\in\Delta,\alpha(H)\geq 0} Y'_\alpha - \theta Y'_\alpha.$$

Therefore

$$\begin{aligned} b_H(Z,Z') &= \sum_{\alpha(H)>0} \alpha(H)\langle Y_\alpha - \theta Y_\alpha, \sum_{\beta(H)\geq 0} Y'_\beta - \theta Y'_\beta\rangle \\ &= -\sum_{\alpha(H)>0} \alpha(H)\langle Y_\alpha + \theta Y_\alpha, Y'_\alpha + \theta Y'_\alpha\rangle, \end{aligned}$$

using that $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta\rangle = 0$ unless $\beta = -\alpha$. Because $Y_\alpha + \theta Y_\alpha = F_\alpha(Z)$, $Y'_\alpha + \theta Y'_\alpha = F_\alpha(Z')$, the formula follows.

PROPOSITION 3.3: (i) $\bar{b}_H$ *extends to a* K*-invariant Riemannian metric* β_H *on* K/K_H.

(ii) *For every* $X \in \mathfrak{a}$, v_X *is equal to the gradient of* $\bar{f}_{X,H} = f_{X,H}$ *considered as a function on* K/K_H. *That is,*

$$d\bar{f}_{X,H} = \beta_H(\cdot, v_X). \tag{3.8}$$

For (i) we have to show that $\bar{b}_H$ is $\mathrm{Ad}\,K_H$-invariant. Now, if $k \in K_H$, $\mathrm{Ad}\,k$ normalizes $\mathfrak{n}^{(H)}$; so it commutes with ζ. Hence

$$\begin{aligned} b_H(Z^k, Z'^k) &= \langle H, [Z^k, \zeta(Z'^k)]\rangle = \langle H, [Z^k, (\zeta(Z'))^k]\rangle \\ &= \langle H, [Z, \zeta(Z')]^k\rangle = \langle H^{k^{-1}}, [Z, \zeta(Z')]\rangle = b_H(Z, Z'). \end{aligned}$$

For (ii) we start with the observation that

$$T_{\bar{k}}(K/K_H) = \{v_Z(\bar{k}) \mid Z \in \mathfrak{k}\} \qquad (k \in K),$$

because K acts transitively on K/K_H. So, using the left K-invariance of β_H and (1.2), (3.4), we have to prove that for all $k \in K$ and $Z \in \mathfrak{k}$,

$$\langle [X,Z]^{k^{-1}}, H\rangle = b_H(E^H_{\mathfrak{k}}(Z^{k^{-1}}), E^H_{\mathfrak{k}}(X^{k^{-1}})) \tag{3.9}$$

with $E^H_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k} \ominus \mathfrak{k}_H$ the projection along $\mathfrak{m} + \mathfrak{a} + \sum_{\alpha\in\Delta,\alpha(H)\geq 0} \mathfrak{g}_\alpha$. As $X^{k^{-1}} \in \mathfrak{s}$, we have $X^{k^{-1}} \equiv (Y - \theta Y) \bmod \mathfrak{s}_H$ for some $Y \in \mathfrak{n}^{(H)}$ because $(I - \theta)\mathfrak{n}^{(H)} = \mathfrak{s} \ominus \mathfrak{s}_H$. Thus

$$E^H_{\mathfrak{k}}(X^{k^{-1}}) = -(Y + \theta Y),\ \zeta E^H_{\mathfrak{k}}(X^{k^{-1}}) = -(Y - \theta Y) \equiv -X^{k^{-1}} \bmod \mathfrak{s}_H.$$

The formula (3.9) now follows immediately from the definition (3.6) of b_H.

REMARK 3.4: If G is complex, as in Remark 2.2, we can define the real-valued linear form ξ on $\mathfrak{k}$ by

$$\xi(Z) = \langle iH, Z \rangle \qquad (Z \in \mathfrak{k}).$$

Then K/K_H can be identified with the co-adjoint orbit $\{\xi^k \mid k \in K\}$ of ξ in $\mathfrak{k}^*$, on which Kirillov [38] introduced the K-invariant sympletic form Ω, given by

$$\Omega(X, Y) = -\xi([X, Y]) \qquad (X, Y \in \mathfrak{k}/\mathfrak{k}_H).$$

On the other hand, the pull-back of the complex structure on $G/G(-H)$ to K/K_H under the diffeomorphism $K/K_H \to G/G(-H)$ is the K-invariant complex structure J on K/K_H occurring in the Borel–Weil theorem. The pair (J, Ω) makes K/K_H into a Kähler manifold. It is easily verified that the corresponding Riemannian structure

$$(X, Y) \mapsto \Omega(X, JY) \qquad (X, Y \in \mathfrak{k}/\mathfrak{k}_H)$$

coincides with the metric β_H introduced in Proposition 3.3.

In case of a real group G, the restriction to $G/G(-H)$ of the above metric on $G_c/G_c(-H)$ is equal to $2\beta_H$, thus identifying β_H with the metric described in Hermann [35].

COROLLARY 3.5: *The zeroset of v_X in K is equal to the critical set $K_X \mathfrak{w} K_H = K_X^0 \mathfrak{w} K_H$ of $f_{X,H}$. It is stable under Θ_a because Θ_a commutes with the v_X-flow.*

We now use the action of the unipotent subgroups $N^{(-wH)}$, $w \in \mathfrak{w}$, to obtain a covering of the flag manifold with coordinate systems in which the vectorfields v_X, $X \in \mathfrak{a}$, are linear (!).

PROPOSITION 3.6: (i) *Given $w \in \mathfrak{w}$, the mapping*

$$\gamma_{w,H} : Y \mapsto \exp YwG(H)$$

defines an analytic diffeomorphism from $\mathfrak{n}^{(-wH)}$ onto an open subset $\Omega_{w,H}$ of $G/G(H)$ containing $wG(H)$.

(ii) *For $x \in G_{wH}$, the map $\gamma_{w,H}$ intertwines the left translation by x on*

$G/G(H)$ with the linear mapping

$$\mathrm{Ad}\, x : \mathfrak{n}^{(-wH)} \to \mathfrak{n}^{(-wH)}.$$

(iii) *If $X \in \mathfrak{g}_{wH}$, the infinitesimal action of X on $G/G(H)$ (corresponding to the vectorfield v_X via the diffeomorphism $K/K_H \to G/G(H)$) gets pulled back by $\gamma_{w,H}$ to the linear vectorfield*

$$\mathrm{ad}\, X : Y \mapsto [X, Y] \qquad (Y \in \mathfrak{n}^{(-wH)}).$$

Here, as usual, the tangent space of $\mathfrak{n}^{(-wH)}$ at any of its points is identified with $\mathfrak{n}^{(-wH)}$.

(i) It is sufficient to consider the case of $w = 1$, since $\exp YwG(H) = w \cdot \exp Y^{w^{-1}} \cdot G(H)$ and $Y^{w^{-1}} \in \mathfrak{n}^{(-H)} \Leftrightarrow Y \in \mathfrak{n}^{(-wH)}$. $\gamma_{1,H}$ is an immersion because $\exp|_{\mathfrak{n}^{(-H)}}$ is an immersion and $\mathfrak{n}^{(-H)} \cap \mathfrak{g}(H) = (0)$ because $\mathfrak{g} = \mathfrak{n}^{(-H)} \oplus \mathfrak{g}_H \oplus \mathfrak{n}^{(H)}$, $\mathfrak{g}(H) = \mathfrak{g}_H \oplus \mathfrak{n}^{(H)}$. Moreover, the image $\Omega_{1,H}$ is open in $G/G(H)$ because $\dim \mathfrak{n}^{(-H)} = \dim G/G(H)$.

What is left is to prove the injectivity of $\gamma_{1,H}$. So suppose $\gamma_{1,H}(Y_1) = \gamma_{1,H}(Y_2)$ for $Y_1, Y_2 \in \mathfrak{n}^{(-H)}$; that is $\exp(-Y_1) \cdot \exp Y_2 \in G(H)$. Because $N^{(-H)} = \exp \mathfrak{n}^{(-H)}$ is a subgroup, we can find $Y \in \mathfrak{n}^{(-H)}$ such that $\exp Y = \exp(-Y_1) \cdot \exp Y_2$. Since $G(H)$ and $N^{(-H)}$ are normalized by A we have $\exp Y^a \in G(H) \cap N^{(-H)}$ for all $a \in A$. Since the Y^a, $a \in A$, come arbitrarily close to 0, we obtain $Y^a \in \mathfrak{g}(H) \cap \mathfrak{n}^{(-H)} = (0)$ for suitable $a \in A$, showing that $Y = 0$.

(ii) follows from

$$\gamma_{w,H}(Y^x) = x \exp Y x^{-1} wG(H) = x \gamma_{w,H}(Y)$$

because $w^{-1}x^{-1}w \in G_H$ if x^{-1} (and therefore x) belongs to $wG_H w^{-1} = G_{wH}$. The assertion (iii) follows from (ii) by differentiation.

COROLLARY 3.7: *Let $X \in \mathfrak{a}$, $H \in Cl(\mathfrak{a}^+)$.* (i) *$\tilde{f}_{X,H}$ has isolated critical points on $K/K_H \cong G/G(H)$ if and only if*

$$\alpha(X) = 0 \Rightarrow w^{-1}\alpha(H) = 0 \qquad (\alpha \in \Delta, w \in \mathfrak{w}). \tag{3.10}$$

(ii) *In this case, the set of critical points of $\tilde{f}_{X,H}$, which coincides with the zeroset of v_X, is equal to $\mathfrak{w}K_H \cong \mathfrak{w}/\mathfrak{w}_H$. For each $w \in \mathfrak{w}$ the flow ϕ^t ($t \in \mathbb{R}$) of v_X on K/K_H is hyperbolic at wK_H. The stable, resp. unstable manifold $S_w^+ = S_{w,X,H}^+$, resp. $S_w^- = S_{w,X,H}^-$ defined by*

$$S_w^{\pm} = \{x \in K/K_H \mid \lim_{t \to \pm\infty} \phi^t(x) = wK_H\} \tag{3.11}$$

is given by

$$S_w^{\pm} = \gamma_{w,H}(\mathfrak{n}^{(-wH)} \cap \mathfrak{n}^{(\mp X)}) = \gamma_{w,H}\Big(\sum_{\alpha \in \Delta^+, w^{-1}\alpha(H) < 0,\ \pm\alpha(X) < 0} \mathfrak{g}_\alpha\Big). \quad (3.12)$$

(iii) *Finally, if* $-X \in \mathfrak{a}^+$ *then the stable (resp. unstable) manifolds of* v_X *on* $G/G(H)$ *coincide with the* N- *(resp.* θN-*) orbits on* $G/G(H)$.

From Proposition 1.3 we see that $\bar{f}_{X,H}$ has isolated critical point set (equal to $\mathfrak{w}K_H$) if and only if $(\mathfrak{k}_X)^{w^{-1}} \subset \mathfrak{k}_H$ for every $w \in \mathfrak{w}$. The description of the stable and unstable manifolds follows from (iii) in Proposition 3.6; indeed, the ad X-flow on $\mathfrak{n}^{(-wH)}$ is contracting, resp. expanding on $\mathfrak{n}^{(-wH)} \cap \mathfrak{n}^{(-X)}$, resp. $\mathfrak{n}^{(-wH)} \cap \mathfrak{n}^{(X)}$.

Finally, from the equality

$$\mathfrak{n}^{(X)} = (\mathfrak{n}^{(X)} \cap \mathfrak{n}^{(-wH)}) \oplus (\mathfrak{n}^{(X)} \cap \mathfrak{g}(wH))$$

we get, in view of Lemma 2.3

$$N^{(X)} = \exp(\mathfrak{n}^{(X)} \cap \mathfrak{n}^{(-wH)}) \cdot \exp(\mathfrak{n}^{(X)} \cap \mathfrak{g}(wH)).$$

Since the second factor leaves $wG(H)$ stable, (iii) follows from (ii), because $N^{(-X)} = N$ and $N^{(X)} = \theta N$ if $-X \in \mathfrak{a}^+$.

COROLLARY 3.8: *Under the assumption* (3.10) $G/G(H)$ *is equal to the disjoint union of the stable manifolds* S_w^+, $w \in \mathfrak{w}/\mathfrak{w}_H$ *as described in* (3.12). *Consequently* $G/G(H)$ *is covered by the open subsets* $\Omega_{w,H}(w \in \mathfrak{w}/\mathfrak{w}_H)$ *defined in Proposition* 3.6.(i). *Also,* $K/K_H \cong G/G(H)$ *is the disjoint union of the* N-*orbits through the points of* $\mathfrak{w}/\mathfrak{w}_H$.

The last assertion is of course the Bruhat decomposition. The proof given here is self-contained. The idea of proving the Bruhat decomposition using the A-action can be found in Séminaire Chevalley [53] in the algebraic setting and Hermann [35] in the analytic setting.

Combining (3.8), (3.7) and Proposition 3.6.(iii) it is not hard to give an alternative proof of the formula (1.11) for the Hessian of $f_{X,H}$. Also, since a linear vector field defined by a semisimple linear transformation obviously has a clean zeroset (that is the zeroset is a smooth manifold and the derivative of the vectorfield transversally to the manifold is nonsingular), Corollary 1.5 can be seen as a direct consequence of (3.8) and Proposition 3.6.(iii). Finally, it is an easy exercise to compute the zeroset of v_X directly from Lemma 3.1, leading to a proof of Proposition 1.2 via (3.8).

$\tilde{f}_{X,H}$ is a Morse function on K/K_H if and only if (3.10) holds. Thinking of $\tilde{f}_{X,H}$ as a linear form on the Ad K-orbit through H in $\mathfrak{s}$, it is interesting to compare this with the construction of Morse functions on arbitrary manifolds as in Guillemin and Pollack [26, p. 43].

In general the stable manifolds of the gradient flow of a Morse function f yield a cell decomposition of the space, as has been observed by Thom [60]. Also it is easy to see that the union of the stable manifolds of the critical points x such that $f(x) \leq c$ is closed; so, if S is a stable manifold, $Cl(S)\backslash S$ is union of (finitely many) stable manifolds of lower f-level and of lower dimension.

We conclude this section by some remarks on the action of an arbitrary element x of G on the various flag manifolds. The Jordan decomposition of x gives $x = x_s \cdot x_u$ with x_s semisimple, x_u unipotent and x_s, x_u commuting. Because the actions of x_s and x_u then commute as well, it suffices for many questions to study the actions of x_s and x_u separately. The description of the action of x_u on the flag manifolds is a nontrivial problem. For instance, if G is complex, B a Borel subgroup of G, then the set of fixpoints for the action of x_u on G/B can be identified with the variety Σ of Borel groups containing x_u. Σ is connected (cf. Springer [55, Proposition 1.5]). If, in addition, G is simple and x_u is a so-called subregular element, then Steinberg and Tits (cf. Steinberg [56, III.3.10. Proposition 2]) showed that Σ is the union of finitely many projective lines, intersecting according to the Dynkin diagram of the group G. In particular, Σ need not be smooth. Regarding x_s, one can by conjugation bring it into a standard position h such that $h = k \exp X$, $X \in \mathfrak{a}$, $k \in K_X$ (cf. DKV [20, Section 4.1]). Because $k \in K_X$, the action of k commutes with the v_X-flow on the flag manifold; this again allows one to study the action of k and the v_X-flow separately. Now the action of an elliptic element k is quite well-understood. For instance, for studying the iterates, one writes $k = k_0 \cdot l$ where k_0, $l \in K_X$, k_0 and l commute, k_0 is of finite order, and l lies in a torus T in K_X such that the powers of l are dense in T. Finally, the basic properties of the v_X-flow have been discussed in detail in this section.

4. The topology of the real flag manifolds

In general, knowledge of a Morse function and its corresponding stable manifolds leads to information about the algebraic topology of the space, especially if the Morse function and the metric are brought in a suitable position, cf. Milnor [46], [47]. The point is that for the flag manifolds K/K_H the Morse functions $\tilde{f}_{X,H}$ and the stable manifolds encountered in section 3 are in the nicest possible position for this purpose. Although it is not needed for the rest of this paper, we shall give a short survey in this section of the topological properties of the real flag manifolds which follow. Most of the results are essentially known, in particular for the complex flag manifolds. However, in the real case some of the arguments seem to need a little additional clarification. For this reason we give the proof of Lemma 4.2 and Proposition 4.4, and we also include an explicit description of the desingularization of the real Schubert varieties, preceding Proposition 4.5.

In this section $H \in \mathfrak{a}$ will be arbitrary; by conjugation with a Weyl group element we can arrange that $H \in Cl(\mathfrak{a}^+)$.

If $\bar{f}_{X,H}$ is a Morse function on K/K_H, which is the case for generic X, then its number of critical points is equal to $|\mathfrak{w}/\mathfrak{w}_H|$, see Corollary 3.7(ii). In Proposition 4.4 below it will be shown that $|\mathfrak{w}/\mathfrak{w}_H|$ is also equal to the sum of the Betti numbers of the homology modulo 2 of K/K_H. In view of the Morse inequality that the number of critical points of any Morse function is $\geq$ the sum of the Betti numbers of the homology with coefficients in any field, this shows that $\bar{f}_{X,H}$ has the minimum number of critical points. Viewing $\bar{f}_{X,H}$ as testing the mapping $k \mapsto \operatorname{Ad} k(H)$ by linear forms as in the beginning of section 1, this means that in the terminology of Kuiper [43] we have proved

PROPOSITION 4.1: *The embedding $k \mapsto \operatorname{Ad} k(H)$ of the flag manifold K/K_H into $\mathfrak{s}$ is tight.*

The embedding is even taut, as the image is contained in the sphere of radius $\|H\|$ in $\mathfrak{s}$. Taut means that the distance function to a generic point has the minimal number of critical points. Also stereographic projection leads to a tight embedding in a hyperplane in $\mathfrak{s}$. See Cecil and Ryan [12], which contains also further interesting references.

Proposition 4.1 is due to Takeuchi–Kobayashi [59]; their proof is based on Takeuchi [58]. Unfortunately in the latter paper an essential part seems to be played by the statement that the closures of the stable manifolds admit topological desingularizations by projective spaces. For the open cell, this would imply a topological desingularization of the flag manifold by a projective space, giving that the Betti numbers of the flag manifold are ≤ 1 in all dimensions. This is not true already for the space of flags in $\mathbb{R}^3$. If one replaces, however, the desingularizations by projective spaces by the desingularizations $\pi_w : \Gamma_w \to Cl(S_{\bar{w}})$ described after Proposition 4.4 below, then the proof in [58], [59] is correct.

The first ingredient in our proof is

LEMMA 4.2: *Let* $-X \in \mathfrak{a}^+$. *Each stable manifold of* v_X *intersects each unstable manifold transversally. That is, at each intersection point the sum of the tangent spaces to* $S_{\bar{w}}^+$ *and* $S_{\bar{w}'}^-$ $(w, w' \in \mathfrak{w}/\mathfrak{w}_H)$ *is equal to the tangent space of the whole space.*

Because MA normalizes N, resp. θN, its action on $G/G(H)$ permutes N-, resp. θN-orbits. Since there are only finitely many such orbits, the connected component M^0A of MA leaves all these orbits invariant. (In fact, the action of MA stabilizes the N-, resp. θN-orbits.) It follows that the image of the infinitesimal action of $\mathfrak{m} + \mathfrak{a}$ is contained in the image of the infinitesimal action of $\mathfrak{n}$, resp. of $\theta\mathfrak{n}$. So the image of the infinitesimal action of $\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} + \theta\mathfrak{n}$ is equal to the image of the infinitesimal action of $\mathfrak{n} + \theta\mathfrak{n}$. The former is equal to the tangent space of $G/G(H)$ (because G acts transitively) and the latter equals the sum of the tangent spaces of the unstable and stable manifold through the point in question.

In order to interpret the stable, resp. unstable manifolds as geometric cycles defining homology classes, we have to pass to their closures. These are called *Schubert varieties* because they coincide with the "geometrical systems" of Schubert [51, Ch. 1, §3] in many classical examples.

LEMMA 4.3: *Let* $-X \in \mathfrak{a}^+$. *If* G *is a complex (resp. real) group then the closure* $Cl(S_{\bar{w}})$ *of each Bruhat cell* $S_{\bar{w}}$ $(\bar{w} \in \mathfrak{w}/\mathfrak{w}_H)$ *is a complex (resp. real) algebraic variety in* $G/G(H)$.

Because $-X \in \mathfrak{a}^+$, we may regard $S_{\bar{w}}$ as an N-orbit. Now N is algebraically closed and $S_{\bar{w}}$ is its image under an algebraic morphism; so $S_{\bar{w}}$ is constructible and hence its closure with respect to the ordinary topology coincides with the Zariski closure in the complex case (cf. Mumford [48]). For the real case a proof can be given by passing to the complexifications of the groups and spaces involved (see Section 2) and using the combinatorial description of the inclusion relation

$$S_{\bar{w}'} \subset Cl(S_{\bar{w}}) \Leftrightarrow \bar{w}' \in Cl(S_{\bar{w}}) \Leftrightarrow Cl(S_{\bar{w}'}) \subset Cl(S_{\bar{w}}), \tag{4.1}$$

both in the real and complex case. See Borel–Tits [8, §3].

It is known that, for any finite collection $V_1, \ldots, V_N$ of compact real algebraic subsets of a compact algebraic set V_0, there exists a triangulation of V_0 inducing subtriangulations on all of the V_i. This is true even with "algebraic" replaced by "semi-analytic", see Łojasiewicz [45], cf. also Van der Waerden [66], [67] for the algebraic case. In the real, resp. complex case the triangulation is real-, resp. complex-analytic in the interiors of the simplices, consisting in fact of regular branches of algebraic functions. This may be applied to $V_0 = K/K(H)$, $V_1 = Cl(S_{\bar{w}})$, $V_2 = Cl(S_{\bar{w}})\backslash S_{\bar{w}}$.

If G is complex then the complex manifold $S_{\bar{w}}$ is orientable; so the simplices of dimension $q = \dim_{\mathbb{R}} S_{\bar{w}}$ in $Cl(S_{\bar{w}})$ can be made into a q-chain c having its boundary in $Cl(S_{\bar{w}})\backslash S_{\bar{w}}$. Since $\dim_{\mathbb{C}} V_2 \leq \dim_{\mathbb{C}} V_1 - 1$ we have $\dim_{\mathbb{R}} V_2 \leq \dim_{\mathbb{R}} V_1 - 2$, implying that c is a cycle, which will be denoted simply by $Cl(S_{\bar{w}})$. In the real case, according to Sullivan [57], the sum of the q-dimensional simplices at least forms a cycle mod 2. We are now in a position to state (still taking $-X \in \mathfrak{a}^+$)

Proposition 4.4: *If G is complex then the homology $H_*(G/G(H), \mathbb{Z})$ is a free $\mathbb{Z}$-module with the homology classes $[Cl(S_{\bar{w}})]$, $\bar{w} \in \mathfrak{w}/\mathfrak{w}_H$ coming from the stable manifolds $S_{\bar{w}}$, as a basis. Moreover the homology classes $[Cl(S'_{\bar{w}})]$, $\bar{w} \in \mathfrak{w}/\mathfrak{w}_H$, defined by the unstable manifolds $S'_{\bar{w}}$, form the Poincaré dual basis. Under the assumption that G is real, the same assertions hold with $\mathbb{Z}$ replaced by $\mathbb{Z}/2\mathbb{Z}$.*

If $S'_{\bar{w}'}$ and $S_{\bar{w}}$ have complementary dimension then Lemma 4.2 shows that $S'_{\bar{w}'} \cap S_{\bar{w}}$ consists of isolated points. This set being A-invariant, its elements are fixed points for the A-action. Consequently they are of the form $w''G(H)$ for $w'' \in \mathfrak{w}$; and this can only occur if $\bar{w}' = \bar{w}'' = \bar{w}$. Because $Cl(S_{\bar{w}})\backslash S_{\bar{w}}$, resp. $Cl(S'_{\bar{w}'})\backslash S'_{\bar{w}'}$, consists of finitely many lower-dimensional stable, resp. unstable manifolds, another application of Lemma 4.2 shows that $Cl(S'_{\bar{w}'}) \cap Cl(S_{\bar{w}}) = S'_{\bar{w}'} \cap S_{\bar{w}}$. Applying the classical intersection theory of Lefschetz [44] we have now

$$[Cl(S_{\bar{w}})] \cap [Cl(S'_{\bar{w}'})] = \delta_{\bar{w},\bar{w}'};$$

and this proves the assertions of the proposition.

It is interesting to compare this proof with the Morse theoretic proof of Poincaré duality for general compact manifolds in Milnor [47]; apparently here the Morse function and corresponding stable and unstable manifolds are already in the required nice position.

An alternative way to show that the Schubert varieties define cycles is to use a desingularization $\pi_w : \Gamma_w \to Cl(S_{\bar{w}})$ which generalizes a construction of Bott–Samelson [11], Hansen [27] and Demazure [16] to the case of $G/G(H)$ for real semisimple G. We shall only give the definition and basic properties, leaving the proofs as an exercise for the reader.

Assume that $H \in Cl(\mathfrak{a}^+)$ and $-X \in \mathfrak{a}^+$. Given $w \in \mathfrak{w}$, let w_l be the unique element in $w\mathfrak{w}_H$ of shortest length l. By downward induction, set $w_{i-1} = w_i \circ s_{\alpha_i} = s_{\beta_i} \circ w_i$ with α_i simple and $\beta_i = -w_i\alpha_i \in \Delta^+$. Writing

$$\Delta^+(w) = \{\alpha \in \Delta^+ \mid w^{-1}\alpha(H) < 0\}, \tag{4.2}$$

one has $\Delta^+(w_{i-1}) = s_{\beta_i}(\Delta^+(w_i)\backslash\{\beta_i, 2\beta_i\})$, cf. Borel–Tits [8,(3.9)]. So in view of (3.12) the $S_{\bar{w}_i}$ have decreasing dimension with decreasing i; and the process ends when $w_0 = 1$ and $S_{\bar{w}_0}$ is a point. Now let

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}, \ \mathfrak{p}_i = (\mathfrak{g}_{-\alpha_i} + \mathfrak{g}_{-2\alpha_i}) + \mathfrak{p}. \tag{4.3}$$

Then $\mathfrak{p}$, resp. $\mathfrak{p}_i$ is equal to $\mathfrak{g}(Y)$ for any $Y \in \mathfrak{a}^+$, resp. $Y \in Cl(\mathfrak{a}^+)$ such that α_i is the only simple root vanishing on Y. Therefore the normalizer $P = MAN$ (resp. P_i) of $\mathfrak{p}$ (resp. $\mathfrak{p}_i$) in G is a parabolic subgroup of G. Clearly $P_i \supset P \cup P^{s_{\alpha_i}}$, therefore $P_i^{w_i} \supset P^{w_i} \cup P^{w_{i-1}}$. This allows us to define a free action of $P^{w_1} \times \ldots \times P^{w_i}$ on $P_1^{w_1} \times \ldots \times P_i^{w_i}$ by

$$((p_1,\ldots,p_i),(x_1,\ldots,x_i)) \mapsto (x_1p_1^{-1}, p_1x_2p_2^{-1},\ldots,p_{i-1}x_ip_i^{-1}). \tag{4.4}$$

We set

$$\Gamma_{w_i} = (P_1^{w_1} \times \ldots \times P_i^{w_i})/(P^{w_1} \times \ldots \times P^{w_i}) \tag{4.5}$$

for the corresponding orbit space. The action is chosen in such a way that the mapping

$$\pi_{w_i}: (x_1,\ldots,x_i) \mapsto x_1 \cdot \ldots \cdot x_i \cdot x_{w_i} \cdot G(H) \tag{4.6}$$

from $P_1^{w_1} \times \ldots \times P_i^{w_i}$ into $G/G(H)$ induces a smooth mapping $\pi_{w_i}: \Gamma_{w_i} \to G/G(H)$. Also, forgetting the last factor we obtain a smooth fibration

$$\Gamma_{w_i} \xrightarrow{P_i^{w_i}/P^{w_i}} \Gamma_{w_{i-1}} \tag{4.7}$$

where every fiber $P_i^{w_i}/P^{w_i}$ is a *sphere* of dimension equal to $\dim(\mathfrak{g}_{\alpha_i} + \mathfrak{g}_{2\alpha_i})$, exhibiting the Γ_{w_i} as iterated sphere bundles.

PROPOSITION 4.5: $\pi_{w_i}(\Gamma_{w_i}) = Cl(S_{\bar{w}_i})$ *and* π_{w_i} *is a diffeomorphism from an open dense subset* Γ'_{w_i} *of* Γ_{w_i} *onto* $S_{\bar{w}_i}$.

In fact, one can take Γ'_{w_i} equal to the image of the mapping

$$(Y_1 + \ldots + Y_i) \mapsto (\exp Y_1,\ldots,\exp Y_i), \ Y_j \in \mathfrak{g}_{\beta_j} + \mathfrak{g}_{2\beta_j}, \tag{4.8}$$

from $\mathfrak{n}^{(-wH)} \cap \mathfrak{n}$ to $P_1^{w_1} \times \ldots \times P_i^{w_i}$, followed by the projection to Γ_{w_i}. Note that combined with π_{w_i} one gets the diffeomorphism $\gamma_{w_i,H}$ in (3.12). The proofs of the various statements are by induction on i. The desingularization $\pi_w : \Gamma_w \to Cl(S_{\bar{w}})$ is obtained by taking $i = l$.

Γ_w being a compact manifold, we can fix its orientation cycle (if Γ_w is not orientable then take its $\mathbb{Z}/2\mathbb{Z}$-orientation), whose image under π_w gives a cycle in $G/G(H)$. Replacing the intersection theory of triangulated cycles by the differential topological intersection theory of "cycles defined by mappings" (cf. Schwartz [52]) the statements of Proposition 4.4 will hold again. While not using the algebraicity of the Schubert varieties, this approach in fact leads to somewhat more refined results. For instance, if

$$\begin{array}{l}\text{for all } \alpha \in \Delta \text{ for which there exists } w \in \mathfrak{w} \text{ such} \\ \text{that } \alpha(wH) \neq 0, \text{ we have } \dim(\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}) > 1\end{array} \tag{4.9}$$

then Γ_w is simply connected and hence orientable, for every $w \in \mathfrak{w}$. So if (4.9) holds, then Proposition 4.4 is valid with coefficients in $\mathbb{Z}$.

What is more important, in the complex case Bott–Samelson [11] (and Demazure [16] in a more general algebraic setting) determine the ring structure of the cohomology of the desingularizations Γ_w, by induction on the number of fibrations and in terms of the root structure. This leads to a description of the ring structure of the cohomology of the complex flag manifolds in terms of the Schubert varieties and the root structure. In the real case obstructions to this program are: (i) the possible failure of $\mathbb{Z}$-orientability and (ii) the varying dimensions of the fibers in the iterated fibrations for the Γ_w. Historically, Bott and Samelson worked entirely in the framework of K/(centralizer of a torus) and their cycles were only later identified with the complex Schubert varieties by Hansen [27].

Van der Waerden [66] interpreted the "problem of characteristics" of Schubert [51, Ch. 6] as "find a basis for the homology (of cycles defined by complex algebraic varieties)", see also Kleiman [39]. For the complex Grassmannians it had been solved by Schubert himself. For arbitrary complex G the first formulation of Proposition 4.4 (without Poincaré duality) is due to Borel [5], who also determined the cohomology ring structure of G/B for G complex and B a Borel subgroup of G [4]. Of course, the ring structure, expressing the intersections of all the cycles, is the ultimate goal of Schubert's calculus. Further references for the relation between the ring structure and the Schubert varieties are Kostant [40], [41] and Bernstein, Gel'fand and Gel'fand [3].

The inclusion relations (4.1) between Schubert varieties imply

$$\langle w'H, X\rangle \leq \langle wH, X\rangle \qquad \text{for all } X \in \mathfrak{a}^+, \tag{4.10}$$

because the functions $f_{X,H}$ are decreasing along the flow of $-\operatorname{grad} f_{X,H}$. That is, $wH - w'H$ is a positive linear combination of simple root vectors, or $w'H \preccurlyeq wH$ in the notation of Varadarajan [64, p. 376]. One might conjecture that conversely (4.10) implies (4.1): in general, however, this is not true, see Deodhar [17].

As observed by Ehresmann [21], any q-chain is homotopic to one which is contained in the Schubert varieties of dimension $\leq q$. Applying this for $q = 1$, we get that the homotopy group $\pi_1(K/K_H)$ is generated by the one-dimensional Schubert varieties; these form an explicitly given bouquet of circles intersecting each other in general position at the unique 0-dimensional cell. This implies that K/K_H is simply connected if and only if (4.9) holds, the "only if" part following by looking at $H_1(K/K_H, \mathbb{Z}/2\mathbb{Z})$. The assumption that K/K_H is simply connected in turn implies that K_H is connected, in view of the covering $K/K_H^0 \to K/K_H$ with fiber K_H/K_H^0. It follows then that $G(H)$ and G_H are connected as well. Also, the homotopies leading to the relations between the generators of $\pi_1(K/K_H)$, being 2-chains, can be taken inside the union of the 2-dimensional Schubert varieties. Examples of computations of the π_1 along these line are given in [21]; for instance, the π_1 of the space of flags in R^3 is isomorphic to the noncommutative group of the quaternions ± 1, $\pm i$, $\pm j$, $\pm k$.

It is somewhat confusing that the flag manifolds $K/K_H \simeq G/G(H)$ are known in the literature under various names. In the complex case Tits [62] called them $\mathscr{C}$-spaces. Over more general fields the name R-spaces has been used [61], from R = racine = root, because they are classified by subsystems of roots. Also the name D-space, from D = drapeau = flag, occurs in the literature [63].

5. The functions $F_{a,H}$

The Iwasawa decomposition $G = KAN$ leads to a smooth projection $H: \mathrm{G} \to \mathfrak{a}$ defined by

$$x \in K \exp H(x) N \qquad (x \in G). \tag{5.1}$$

The corresponding projection from the symmetric space $K\backslash G$ onto $\mathfrak{a}$ along the right N-orbits is called the horospherical projection. For any

$H \in \mathfrak{a}$ we define the function F_H on G by

$$F_H(x) = \langle H(x), H \rangle \qquad (x \in G). \tag{5.2}$$

Moreover, for $x \in G$, we define the function $F_{x,H}$ on K by

$$F_{x,H}(k) = F_H(xk) = \langle H(xk), H \rangle \qquad (k \in K). \tag{5.3}$$

The functions F_H, resp. $F_{x,H}$, can be viewed as testing the Iwasawa projection, resp. the horospherical projection restricted to the right K-orbits in $K\backslash G$, by linear forms. As in Remark 1.9, the results can easily be extended to the case of complex H by passing to the real and imaginary parts.

One has the obvious invariances

$$H(kx\,man) = H(x) + \log a \;\; (k \in K,\ m \in M,\ a \in A,\ n \in N). \tag{5.4}$$

Indeed, the effect of right multiplication by a follows, using that A normalizes N. The invariance under the right M-action is because M normalizes N, centralizes A, and is contained in K. It follows that $F_{x,H}$ is left K_x- and right M-invariant, but in contrast with the functions $f_{X,H}$ it is, in general, not right K_H-invariant; see Proposition 5.6 below though.

We begin with an explicit formula for the derivatives of the Iwasawa projection. We let the elements of the real universal enveloping algebra $U(\mathfrak{g})$ act as left invariant differential operators on G as follows. If $b = X_1 X_2 \dots X_r$ ($X_i \in \mathfrak{g}$) then

$$f(x;b) = \frac{\partial^r}{\partial t_1 \dots \partial t_r} f(x \exp t_1 X_1 \cdot \ldots \cdot \exp t_r X_r)_{t_1 = \ldots = t_r = 0} \tag{5.5}$$

for any smooth function f on G, which might be vector-valued.

The Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ gives rise to a direct sum decomposition

$$U(\mathfrak{g}) = \{\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}\} \oplus U(\mathfrak{a}). \tag{5.6}$$

It makes sense therefore to speak of the projection $E_\mathfrak{a}: U(\mathfrak{g}) \to U(\mathfrak{a})$ along $\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$; note that its restriction to $\mathfrak{g}$ is equal to the projection $\mathfrak{g} \to \mathfrak{a}$ along $\mathfrak{k} \oplus \mathfrak{n}$. It is clear that

$$\deg E_\mathfrak{a}(b) \le \deg(b) \text{ and } E_\mathfrak{a}(b) \in U(\mathfrak{a})^+ \text{ if } b \in U(\mathfrak{g})^+. \tag{5.7}$$

Here deg denotes the degree and for any Lie algebra $\mathfrak{l}$, $U(\mathfrak{l})^+$ is the ideal $\mathfrak{l}U(\mathfrak{l}) = U(\mathfrak{l})\mathfrak{l}$ of "elements without constant term". Finally, since $\mathfrak{a}$ is abelian, $U(\mathfrak{a})$ is canonically isomorphic to the symmetric algebra over $\mathfrak{a}$, so we can speak of the homogeneous components v_m of order m of $v \in U(\mathfrak{a})$.

LEMMA 5.1: *Let $y \in G$, $b \in U(\mathfrak{g})^+$. Then*

$$H(y; b) = (E_{\mathfrak{a}}(b^{t(y)}))_1, \tag{5.8}$$

where $t(y) = a(y)n(y)$ is the "triangular part" of y and the suffix 1 means the homogeneous component of degree 1.

If $z \in G$, then $H(yz) = H(t(y)z) = H(z^{t(y)}t(y)) = H(z^{t(y)}) + H(y)$. Furthermore it is clear that $H(1; b) = 0$, if $b \in \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$; so $H(1; b) = H(1; E_{\mathfrak{a}}(b)) = (E_{\mathfrak{a}}(b))_1$, because $H(\exp Y_1 \cdot \ldots \cdot \exp Y_r) = Y_1 + \ldots + Y_r$ if $Y_i \in \mathfrak{a}$.

COROLLARY 5.2: *For any $y \in G$, $Y \in \mathfrak{g}$,*

$$F_H(y; Y) = \langle Y^{t(y)}, H \rangle = \langle Y, H^{n(y)^{-1}} \rangle. \tag{5.9}$$

For the proof we note that $\langle E_{\mathfrak{a}}(Y^{t(y)}), H \rangle = \langle Y^{t(y)}, H \rangle$, since $\mathfrak{a}$ is orthogonal to $\mathfrak{k} \oplus \mathfrak{n}$ with respect to $\langle .,. \rangle$, while $H^{t(y)^{-1}} = H^{n(y)^{-1}a(y)^{-1}} = H^{n(y)^{-1}}$.

We refer to Lemma 6.1 for formulae giving the second derivatives of F_H.

LEMMA 5.3: *$k \in K$ is a critical point for $F_{x,H}$ if and only if $xk \in KAN_H$.*

Write $n = n(xk)$. Then from (5.9) we see that k is a critical point for $F_{x,H}$ if and only if $H^{n^{-1}}$ is orthogonal to $\mathfrak{k}$, i.e. $H^{n^{-1}} \in \mathfrak{s}$. Because $n \in N$, $H^{n^{-1}} \in H + \mathfrak{n}$; and so, as $H \in \mathfrak{s}$, we see that $H^{n^{-1}} \in \mathfrak{s}$ if and only if $H^{n^{-1}} - H \in \mathfrak{s} \cap \mathfrak{n}$. But $\mathfrak{s} \cap \mathfrak{n} = (0)$, as follows either by inspecting the root space decomposition or by observing that elements of $\mathfrak{s}$ are semisimple, while those of $\mathfrak{n}$ are nilpotent. So $H^{n^{-1}} \in \mathfrak{s} \Leftrightarrow n \in N_H$.

Since $G = KAK$ and

$$F_{u_1 x u_2, H} = l(u_2)^{-1} F_{x,H} \qquad (u_1, u_2 \in K) \tag{5.10}$$

where $l(u)$ is left translation by $u \in K$, it is enough to study the $F_{a,H}$ with $a \in A$.

PROPOSITION 5.4: *If $X, H \in \mathfrak{a}$ then the critical set of $F_{\exp X, H}$ is equal to the critical set*

$$K_{X,H} = \bigcup_{w \in \mathfrak{w}} K_X w K_H$$

of $f_{X,H}$ as described in Proposition 1.3.

In view of (2.6) and the relations $G_H = K_H A K_H = K_H \exp \mathfrak{s}_H$ we conclude that

$$KAN_H = KG_H = KAK_H = K \exp \mathfrak{s}_H. \tag{5.11}$$

Now $\exp X \quad k \in K \quad \exp \mathfrak{s}_H \Leftrightarrow \exp X^{k^{-1}} \in K \quad \exp \mathfrak{s}_H \Leftrightarrow X^{k^{-1}} \in \mathfrak{s}_H$, since $X^{k^{-1}} \in \mathfrak{s}$ and $(u, Y) \mapsto u \exp Y$ is a diffeomorphism of $K \times \mathfrak{s}$ with G. But $[X^{k^{-1}}, H] = 0$ just means that k is a critical point of $f_{X,H}$, see Lemma 1.1.

This proposition enables Heckman [32] to prove the theorem of Kostant [42] that also the horospherical image of the K-orbit through $\exp X$ is equal to the convex hull of the wX, $w \in \mathfrak{w}$. He uses a homotopy argument starting from the convexity theorem mentioned in Remark 1.7. In [32, Ch. 2] he obtains also a generalization of the convexity theorem to projections of the A-orbits in the flag manifolds.

From Lemma 5.3 and (5.11) we see that the critical set of $F_{a,H}$ is equal to the intersection of K and $a^{-1}KAK_H$. It can be shown that this intersection is clean; that is, the intersection is a smooth manifold (as we already known from the description of $K_{X,H}$ following Proposition 1.2) and the tangent space of the intersection is equal to the intersection of the tangent spaces of K and $a^{-1}KAK_H$, at each intersection point.

Recall the action of $x \in G$ on K defined by

$$\Theta_x : k \mapsto \kappa(xk), \tag{5.12}$$

see (3.2). Using the action of A on K, we obtain a close relationship between the functions $F_{a,H}$ and the functions $f_{X,H}$.

PROPOSITION 5.5: *If $a \in A$ and $X, H \in \mathfrak{a}$ are arbitrary, then*

$$\frac{d}{dt} F_{a \exp tX, H}(k)_{t=0} = f_{X,H}(\Theta_a(k)), \tag{5.13}$$

$$F_{a \exp X, H}(k) - F_{a,H}(k) = \int_0^1 f_{X,H}(\Theta_{a \exp sX}(k))\, ds, \tag{5.14}$$

$$F_{\exp X,H}(k) = \int_0^1 f_{X,H}(\Theta_{\exp sX}(k))\,ds. \tag{5.15}$$

Using that $a\exp tX\,k = ak\cdot\exp tX^{k^{-1}}$, we get in view of Corollary 5.2

$$\frac{d}{dt}F_{a\exp tX,H}(k)_{t=0} = \langle X^{k^{-1}}, H^{t(ak)^{-1}}\rangle = \langle X, H^{kt(ak)^{-1}}\rangle.$$

Now $ak = \kappa(ak)t(ak)$ leads to $kt(ak)^{-1} = a^{-1}\kappa(ak)$; hence we continue the sequence of identities by

$$\langle X, H^{a^{-1}\kappa(ak)}\rangle = \langle X^a, H^{\kappa(ak)}\rangle = \langle X, H^{\kappa(ak)}\rangle,$$

proving (5.13). Replacing a by $a\exp sX$ in (5.13) and integrating over s from 0 to 1 yields (5.14), whereas (5.15) is a special case of (5.14).

Although $f_{X,H}$ is right K_H-invariant in a trivial way, $F_{a,H}$ is not right K_H-invariant, in general. The following proposition shows that the question is quite subtle.

PROPOSITION 5.6: *Fix $H\in\mathfrak{a}$. The following statements are then equivalent*:
(a) *$F_{a,H}$ is right K_H-invariant for each $a\in A$,*
(b) *F_H is right K_H-invariant,*
(c) *If $\alpha\in\Delta^+$, $\alpha(H)=0$ and $\alpha=\beta+\gamma$ for $\beta,\gamma\in\Delta^+$, then $\beta(H)=0=\gamma(H)$,*
(d) *There exists $H'\in Cl(\mathfrak{a}^+)$ such that $K_H = K_{H'}$,*
(e) *$\Theta_a(kK_H) = \Theta_a(k)K_H$ for each $k\in K$, $a\in A$.*
In particular, F_H is right K_H-invariant if $H\in Cl(\mathfrak{a}^+)$.

(c) $\Rightarrow$ *(d)*: Writing $\Delta_0^+ = \{\alpha\in\Delta^+ \mid \alpha(H)=0\}$, we say $\alpha\in S_0$ if $\alpha\in\Delta_0^+$ and α cannot be written as $\beta+\gamma$ with $\beta,\gamma\in\Delta_0^+$. Clearly each element of Δ_0^+ can be written as a linear combination of elements of S_0 with non-negative integral coefficients. Now (c) implies that $S_0\subset S$, the set of simple roots in Δ^+. Let $H'\in Cl(\mathfrak{a}^+)$ be such that S_0 is precisely the set of simple roots vanishing at H'. Then $\Delta_0^+ = \{\alpha\in\Delta^+\mid\alpha(H')=0\}$ as well, or $K_H^0 = K_{H'}^0$, or $K_H = K_{H'}$ using (1.4).

(d) $\Rightarrow$ *(e)*: If $H'\in Cl(\mathfrak{a}^+)$ then the A-flow on K is intertwined with the A-flow on $K/K_{H'}$ (defined in (3.2)) by the projection $K\to K/K_{H'}$. This implies (e) with H replaced by H', but $K_H = K_{H'}$ for some $H'\in Cl(\mathfrak{a}^+)$.

(e) $\Rightarrow$ *(a)*: Apply (5.15) and the right K_H-invariance of $f_{X,H}$.

(a) $\Rightarrow$ *(b)*: Obvious from $G = KAK$.

(b) $\Rightarrow$ *(c)*: In view of (5.9), condition (b) implies

$$\langle Y, H^{n^{-1}}\rangle = 0 \text{ for all } Y\in\mathfrak{k}_H, n\in N. \tag{5.16}$$

We set $n^{-1} = \exp Z$, $Z \in \mathfrak{n}$; then the second order term in the Taylor expansion of (5.16) with respect to Z gives

$$\langle Y, (\operatorname{ad} Z)^2(H)\rangle = 0 \text{ for all } Y \in \mathfrak{k}_H, Z \in \mathfrak{n}. \tag{5.17}$$

Now suppose $\alpha \in \Delta^+$, $\alpha(H) = 0$, $\alpha = \beta + \gamma$ for $\beta, \gamma \in \Delta^+$. Choose $Y_\alpha \in \mathfrak{g}_\alpha$, $Z_\beta \in \mathfrak{g}_\beta$, $Z_\gamma \in \mathfrak{g}_\gamma$ such that $\langle \theta Y_\alpha, [Z_\beta, Z_\gamma]\rangle \neq 0$. With $Y = Y_\alpha + \theta Y_\alpha$, $Z = Z_\beta + Z_\gamma$, the left hand side of (5.17) becomes

$$\begin{aligned}
&\langle Y_\alpha + \theta Y_\alpha, [Z_\beta + Z_\gamma, -\beta(H)Z_\beta - \gamma(H)Z_\gamma]\rangle = \\
&\quad = \langle \theta Y_\alpha, [Z_\beta, -\gamma(H)Z_\gamma] + [Z_\gamma, -\beta(H)Z_\beta]\rangle = \\
&\quad = (\beta(H) - \gamma(H)) \cdot \langle \theta Y_\alpha, [Z_\beta, Z_\gamma]\rangle.
\end{aligned}$$

The Y_α-term drops out because $\langle \mathfrak{g}_\alpha, \mathfrak{g}_{\beta+\gamma}\rangle = 0$ since $\alpha + \beta + \gamma \neq 0$. Using that $\beta(H) - \gamma(H) = 2\beta(H)$, we find $\beta(H) = 0$.

We call H *aligned* (with $\mathfrak{a}^+$) if one of these equivalent conditions is satisfied. Note that each $H \in \mathfrak{a}$ is aligned if and only if $\mathfrak{g}$ is a direct sum of real-rank 1 algebras.

PROPOSITION 5.7: *Let $H \in Cl(\mathfrak{a}^+)$. Then*

$$F_H(xg) = F_H(x) + F_H(g) \textit{ for any } x \in G,\ g \in G(H). \tag{5.18}$$

Moreover, if $G_{\text{inv}}(H)$ denotes the closed subgroup of those $g \in G$ such that F_H is invariant under right translation by g, then

$$G_{\text{inv}}(H) = \{g \in G(H) | F_H(g) = 0\} = K_H \exp(\mathfrak{a} \cap H^\perp)N. \tag{5.19}$$

In particular, $\bar{N}_H = \theta N_H \subset G_{\text{inv}}(H)$.

First consider $g \in G(H)$. Using (2.7), we can write $g = kan$ with $k \in K_H$, $a \in A$, $n \in N$. (5.18) is now obtained by successive applications of (5.4). This shows also that

$$G_{\text{inv}}(H) \supset \{g \in G(H) | F_H(g) = 0\} \supset K_H \exp(\mathfrak{a} \cap H^\perp)N.$$

Because F_H is right N-invariant, it is enough to show that any $g \in G_{\text{inv}}(H) \cap KA$ lies in $K_H \exp(\mathfrak{a} \cap H^\perp)$. We now differentiate the relation $F_H(x) = F_H(xg) = F_H(gx^{g^{-1}})$ with respect to x at $x = 1$. In view of (5.9) we get, for any $Y \in \mathfrak{g}$,

$$\langle Y, H\rangle = \langle Y^{g^{-1}}, H^{n(g)^{-1}}\rangle = \langle Y^{g^{-1}}, H\rangle = \langle Y, H^g\rangle;$$

here $n(g) = 1$ because $g \in KA$. As $Y \in \mathfrak{g}$ was arbitrary, this gives $H = H^g = H^{\kappa(g)}$. So $g \in K_H A$. But then (5.18) shows that $g \in K_H \cdot \exp(\mathfrak{a} \cap H^\perp)$, as we wanted. For the last statement, notice that on taking the tangent spaces at 1,

$$\mathfrak{g}_{\mathrm{inv}}(H) = \mathrm{Lie}(G_{\mathrm{inv}}(H)) = \mathfrak{k}_H \oplus (\mathfrak{a} \cap H^\perp) \oplus \mathfrak{n}.$$

Since $\theta \mathfrak{n}_H$ is contained in the right hand side, $\theta N_H \subset G_{\mathrm{inv}}(H)$.

REMARK: The relation (5.18) for $H \in Cl(\mathfrak{a}^+)$ can also be proved using the finite-dimensional representation theory of G. Here is a sketch of the argument.

We may clearly take $g \in G_H$ and even $g \in \bar{N}_H MAN_H$ by the density of the big cell in G_H, and finally $g \in \bar{N}_H$. If we prove that F_H is invariant under $g \in \bar{N}_H$, it follows that $\theta N_H \subset G_{\mathrm{inv}}(H)$; and this implies (5.18). We now use the notations of section 2 and denote by S the set of simple roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$, S_0 the subset of those roots vanishing at H, $\mathfrak{h}$ being the Cartan subalgebra $\mathfrak{h}_{\mathfrak{m}} \oplus \mathfrak{a}$. If Λ is a dominant integral linear form on $\mathfrak{h}_c$, and π_Λ is the corresponding irreducible finite-dimensional representation, then one has Harish–Chandra's well-known formula (cf. [28, Lemma 2])

$$e^{\Lambda(H(x))} = \|\pi_\Lambda(x) 1_\Lambda\| \qquad (x \in G).$$

Here 1_Λ is a unit vector of highest weight Λ and the norm is the Hilbert space norm relative to which $\pi_\Lambda(u)$ is unitary for u in the compact subgroup of G_c with Lie algebra $\mathfrak{k} + i\mathfrak{s}$. (We assume $G \subset G_c$, as we may, for this argument.) If, for some $\alpha \in S$, $\langle \alpha, \Lambda \rangle = 0$, then $\pi_\Lambda(X_{-\alpha})1_\Lambda = 0$; this implies $\pi_\Lambda(\bar{n})1_\Lambda = 1_\Lambda$ for $\bar{n} \in \bar{N}_\Lambda$. So $x \mapsto \Lambda(H(x))$ is right $\bar{N}_\Lambda$-invariant for each dominant integral Λ. The proof is completed by observing that each subset S_0 of S occurs as $\{\alpha \in S \mid \langle \Lambda, \alpha \rangle = 0\}$ for some dominant integral Λ.

If $H \in Cl(\mathfrak{a}^+)$, then, in view of Proposition 5.6, we can work in K/K_H and the 1-parameter family $\Phi_s = \Theta_{\exp sX}$, $s \in \mathbb{R}$, is equal to the v_X-flow. As another application of Proposition 5.5 we have:

COROLLARY 5.8: *Let $H \in Cl(\mathfrak{a}^+)$. Then v_X is also the gradient of $F_{\exp X, H}$ (regarded as a function on K/K_H), but now with respect to the Riemannian metric $B_{X,H}$ on K/K_H defined by*

$$B_{X,H} = \int_0^1 \Phi_s^* \beta_H \, ds. \tag{5.20}$$

In particular, if k is not a critical point for $F_{\exp X,H}$ then $F_{\exp X,H}(k; v_X(k)) > 0$.

Differentiating (5.15) and using (3.8) we get

$$dF_{\exp X,H} = d\left(\int_0^1 \Phi_s^* f_{X,H}\, ds\right) = \int_0^1 \Phi_s^*(df_{X,H})\, ds$$
$$= \int_0^1 \Phi_s^*(\beta_H(.,v_X))\, ds = \int_0^1 (\Phi_s^*\beta_H)(.,\Phi_s^* v_X)\, ds.$$

Now use that v_X is Φ_s-invariant to conclude (5.20).

In particular, $F_{\exp X,H}$ has a clean critical point set in the sense of Bott (compare Corollary 1.5), because v_X has a clean zeroset. In the general case, with $H \notin Cl(\mathfrak{a}^+)$, the function $F_{\exp X,H}$ on K still has clean critical point set, see Corollary 6.4. This can also be proved using Lemma 5.9, because "clean critical set" means clean intersection of the derivative with the zero section.

We conclude this section with another remarkable relation between the $F_{a,H}$ and the $f_{X,H}$.

LEMMA 5.9: *Let $X, H \in \mathfrak{a}$, write $a = \exp X$. Then*

$$F_{a,H}(k; Y) = f_{X,H}(\Theta_a(k); Z) \text{ if} \tag{5.21}$$
$$Y = \operatorname{Ad} k^{-1} \circ (\operatorname{ad} X/\sinh \operatorname{ad} X) \circ \operatorname{Ad} \Theta_a(k)(Z) \qquad (Y, Z \in \mathfrak{k}). \tag{5.22}$$

Indeed, using that $t(ak) = \kappa(ak)^{-1}ak = \Theta_a(k)^{-1}ak$, we obtain from (5.9)

$$F_{a,H}(k; Y) = \langle Y^{t(ak)}, H\rangle = \langle Y^{ak}, H^{\Theta_a(k)}\rangle$$
$$= \langle \sinh \operatorname{ad} X \circ \operatorname{Ad} k(Y), H^{\Theta_a(k)}\rangle,$$

since $(\operatorname{ad} X)^{2n}\mathfrak{k} \subset \mathfrak{k} \perp \mathfrak{s}$, for every integer $n \geq 0$. On the other hand, (1.2) implies that

$$f_{X,H}(\Theta_a(k); Z) = \langle [X, Z^{\Theta_a(k)}], H^{\Theta_a(k)}\rangle$$
$$= \langle \operatorname{ad} X \circ \operatorname{Ad} \Theta_a(k)(Z), H^{\Theta_a(k)}\rangle.$$

Noting that $\operatorname{ad} X/\sinh \operatorname{ad} X$ is a linear isomorphism: $\mathfrak{k} \to \mathfrak{k}$, Lemma 5.9 yields an automorphism of the tangent bundle of K, which maps $df_{X,H}$ to $dF_{a,H}$, these one-forms being regarded as functions on TK. The automorphism depends smoothly on X, does not depend on H and covers the transformation Θ_a in K, that is, it maps the fiber over k to the fiber over $\Theta_a(k)$. Note that it is not equal to the tangent map $T\Theta_a: TK \to TK$.

As an application one gets that the image of the tangent map at k_0 of $k \mapsto H(ak)$ is equal to the image of the tangent map at k_0 of the map $k \mapsto X^{k^{-1}}$ followed by orthogonal projection: $\mathfrak{s} \to \mathfrak{a}$. As observed in Remark 1.8, such an image is equal to a linear span of H_α, α running through some subset of Δ^+. So again the set of singular values of $k \mapsto H(ak)$ is piecewise linear, each piece being parallel to the linear span of some H_α's.

The similarities between $F_{\exp X,H}$ and $f_{X,H}$ strongly suggest the existence of a diffeomorphism $\psi_{X,H}$ of K onto itself such that $F_{\exp X,H} = f_{X,H} \circ \psi_{X,H}$. For fixed X, H this can be proved using the homotopy of Heckman [32] between $f_{X,H}$ and $F_{\exp X,H}$, combined with the cleanness of the critical set of $F_{\exp X,H}$, cf. Corollary 6.4. If one can find $\psi_{X,H}$ depending smoothly on H, then replacing H by tH, dividing the functions by t and then putting $t = 0$ gives $F_{\exp X,H} = f_{X,H} \circ \psi_{X,0}$. In other words, one would find $\psi_{X,H} = \psi_X$ that does not depend on H. The existence of ψ_X depending also smoothly on X would simplify several proofs. For instance, in Section 11 we then would not need a separate argument for the case that $F_{\exp X,H}$ is not right K_H-invariant, because by a substitution of variables we could replace $F_{\exp X,H}$ by $f_{X,H}$. Also if we would have a ψ_X with control on its derivatives as $\|X\| \to \infty$, then the results in Section 11 could be given with uniform control over the estimates for $\|X\| \to \infty$.

Although we tried, we could not find an explicit diffeomorphism ψ_X such that $F_{\exp X,H} = f_{X,H} \circ \psi_X$. Heckman's description of the orthogonal and Iwasawa projections of the A-orbits in K/K_H onto $\mathfrak{a}$ suggests a diffeomorphism ψ_X of the form $k \mapsto \Theta_a(k)$, where $a = a(X, k) \in A$ is allowed to depend on $X \in \mathfrak{a}$ and $k \in K$. This choice fails already in the case of $\mathrm{SL}(3, \mathbb{R})$. Comparison of the Hessians at 1 of $F_{\exp X,H}$ (Proposition 6.2 and 6.5) with the Hessian of $f_{X,H}$ (Proposition 1.4) shows that $a(X, k)$ does not depend in a C^2 fashion on k.

6. Hessians of the $F_{a,H}$

We begin by investigating second order derivatives of F_H using Lemma 5.1. Write $E_{\mathfrak{k}}, E_{\mathfrak{a}}, E_{\mathfrak{n}}$ for the projections $\mathfrak{g} \to \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ according to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

LEMMA 6.1: *Let $y \in G$, $Y, Z \in \mathfrak{g}$. Then*

$$F_H(y; YZ) = B_H(Y^{t(y)}, Z^{t(y)}) \tag{6.1}$$

where the bilinear form B_H on $\mathfrak{g} \times \mathfrak{g}$ is given by

$$\begin{aligned} B_H(Y, Z) &= \langle E_{\mathfrak{a}}(YZ)_1, H\rangle = \langle [E_{\mathfrak{n}}(Y), Z], H\rangle = \\ &= \langle [Y, E_{\mathfrak{k}}(Z)], H\rangle = \langle [E_{\mathfrak{n}}(Y), E_{\mathfrak{k}}(Z)], H\rangle = \\ &= \sum_{\alpha \in \Delta^+} \alpha(H)\langle R_\alpha(Y) - \theta R_{-\alpha}(Y), R_{-\alpha}(Z)\rangle \quad (Y, Z \in \mathfrak{g}). \end{aligned} \tag{6.2}$$

Here R_α is the projection $\mathfrak{g} \to \mathfrak{g}_\alpha$ corresponding to the direct sum decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

Modulo $\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$ and second order terms in $U(\mathfrak{a})$,

$$YZ \equiv (E_{\mathfrak{a}}(Y) + E_{\mathfrak{n}}(Y))Z \equiv E_{\mathfrak{a}}(Y)Z + [E_{\mathfrak{n}}(Y), Z],$$

while

$$E_{\mathfrak{a}}(Y)Z \equiv E_{\mathfrak{a}}(Y)(E_{\mathfrak{k}}(Z) + E_{\mathfrak{a}}(Z)) \equiv E_{\mathfrak{a}}(Y)E_{\mathfrak{k}}(Z) \equiv [E_{\mathfrak{a}}(Y), E_{\mathfrak{k}}(Z)].$$

The equality of $B_H(Y, Z)$ with the last term in the top row in (6.2) follows now as $\mathfrak{k} + \mathfrak{n} \perp \mathfrak{a}$ and $\langle [E_{\mathfrak{a}}(Y), E_{\mathfrak{k}}(Z)], H\rangle = -\langle E_{\mathfrak{k}}(Z), [E_{\mathfrak{a}}(Y), H]\rangle = 0$. The equalities in the middle row follow by similar reasoning. The last identity follows because $B_H(Y, Z) = \langle [H, E_{\mathfrak{n}}(Y)], Z\rangle$, and

$$Y = U + \sum_{\alpha \in \Delta} V_\alpha \ \ (U \in \mathfrak{m} + \mathfrak{a}, V_\alpha \in \mathfrak{g}_\alpha) \Rightarrow E_{\mathfrak{n}}(Y) = \sum_{\alpha \in \Delta^+} (V_\alpha - \theta V_{-\alpha}).$$

Although we could work on K/M with the functions $F_{a,H}$, we prefer to work on K for simplicity of notation. This enables us, for instance, to identify the tangent space $T_k K$ to K at $k \in K$ with the Lie algebra $\mathfrak{k}$ using left translation over k^{-1}. In this fashion the derivative at k of a smooth map $\Theta: K \to K$ is identified with a linear map $\dot{\Theta}(k): \mathfrak{k} \to \mathfrak{k}$.

PROPOSITION 6.2: *Let $a = \exp X$, $X \in \mathfrak{a}$. Then, for any $H \in \mathfrak{a}$, $k \in K$, $Y, Z \in \mathfrak{k}$*

$$F_{a,H}(k; YZ) = (Y, L_{a,H,k}(Z)), \tag{6.3}$$

where $L_{a,H,k}$ is the linear map $\mathfrak{k} \to \mathfrak{k}$ defined by

$$L_{a,H,k} = -\operatorname{Ad} k^{-1} \circ \sinh(\operatorname{ad} X) \circ \operatorname{Ad} \Theta_a(k) \circ \operatorname{ad} H \circ \dot{\Theta}_a(k). \tag{6.4}$$

In particular, if k is a critical point for $F_{a,H}$ then $L_{a,H,k}$ is symmetric with respect to the inner product $(= -\langle .,. \rangle)$ on $\mathfrak{k}$.

According to (6.2), $F_{a,H}(k; YZ) = F_H(ak; YZ) = \langle Y^{t(ak)}, [E_{\mathfrak{k}}(Z^{t(ak)}), H]\rangle = (Y, L'(Z))$, where

$$L'(Z) = \operatorname{Ad} k^{-1} \circ \operatorname{Ad} a^{-1} \circ \operatorname{Ad} \Theta_a(k) \circ \operatorname{ad} H(E_{\mathfrak{k}}(Z^{t(ak)})),$$

using that $t(ak)^{-1} = k^{-1}a^{-1}\kappa(ak)$. Furthermore the equality $\kappa(ak \exp sZ)\, t(ak \exp sZ) = ak \exp sZ = \kappa(ak)t(ak)\exp sZ = \kappa(ak)\exp sZ^{t(ak)}t(ak)$, implies $\kappa(ak)^{-1}\kappa(ak \exp sZ) = \kappa(\exp sZ^{t(ak)})$. Differentiating this with respect to s at $s = 0$, we find $\dot{\Theta}_a(k)(Z) = E_{\mathfrak{k}}(Z^{t(ak)})$. The proof is completed by observing that $-\frac{1}{2}(\operatorname{Ad} a - \operatorname{Ad} a^{-1}) = \operatorname{Ad} a^{-1} - \frac{1}{2}(\operatorname{Ad} a + \operatorname{Ad} a^{-1})$ maps $\mathfrak{s}$ into $\mathfrak{k}$ while $\frac{1}{2}(\operatorname{Ad} a + \operatorname{Ad} a^{-1})$ maps $\mathfrak{s}$ into $\mathfrak{s} = \mathfrak{k}^{\perp}$.

LEMMA 6.3: *For any $a \in A$, $H \in \mathfrak{a}$, $u \in K_a$, $w \in \mathfrak{w}$, the diffeomorphism Θ_a leaves uwK_H invariant. In particular, writing $a = \exp X$, $X \in \mathfrak{a}$, the critical set $K_{X,H} = K_X \mathfrak{w} K_H$ of $F_{a,H}$ is invariant under the flow $\Theta_{\exp tX}$, $t \in \mathbb{R}$; so Θ_a also leaves the components of $K_{X,H}$ invariant. (Cf. Corollary 3.5.)*

If $v \in K_H$, $av \in G_H$; and so, by (2.6), $\Theta_a(v) = \kappa(av) \in K_H$. Secondly, if $u \in K_X$, $w \in \mathfrak{w}$, then $aux_w v = uax_w v = ux_w a^{w^{-1}} v$; and since $a^{w^{-1}}$ leaves K_H invariant as well, the lemma follows.

COROLLARY 6.4: *For each $a \in A$, $H \in \mathfrak{a}$, the Hessian of $F_{a,H}$ at the critical points is transversally nondegenerate to the critical set of $F_{a,H}$.*

For $k \in K_{X,H}$ ($a = \exp X$, $X \in \mathfrak{a}$), write

$$T_{X,H,k} = \mathfrak{k}_X^{k^{-1}} + \mathfrak{k}_H \tag{6.5}$$

for the tangent space $T_k(K_{X,H})$ pulled back to $\mathfrak{k}$ by translation over k^{-1}. We have to prove that $\ker L_{a,H,k} = T_{X,H,k}$. Now we have the equality $L_{a,H,k} = L_{a',H,v}$, if $a' = a^{w^{-1}}$ and $k = ux_w v$ with $u \in K_a$, $w \in \mathfrak{w}$, and $v \in K_H$, because $F_{a,H}(ux_w v) = F_{a',H}(v)$. On the other hand:

$$T_{X,H,k} = \mathfrak{k}_X^{k^{-1}} + \mathfrak{k}_H = \mathfrak{k}_{X'}^{v^{-1}} + \mathfrak{k}_H = T_{X',H,v} \quad (X' = w^{-1}X).$$

Therefore we have reduced the proof to the case that $k \in K_H$. But then $\Theta_a(k) \in K_H$ (Lemma 6.3), so $\operatorname{Ad} \Theta_a(k)$ and $\operatorname{ad} H$ commute. Now $\operatorname{Ker}(\sinh(\operatorname{ad} X)) \cap \mathfrak{s} = \mathfrak{s}_X$; and hence:

$$\operatorname{Ker} L_{a,H,k} = \{Y \in \mathfrak{k} \mid \operatorname{ad} H \circ \operatorname{Ad} \Theta_a(k) \circ \dot{\Theta}_a(k)(Y) \in \mathfrak{s}_X\}. \tag{6.6}$$

A root space calculation shows

$$\{Z \in \mathfrak{k} \mid \operatorname{ad} H(Z) \in \mathfrak{s}_X\} = \mathfrak{k}_X + \mathfrak{k}_H.$$

We get

$$\begin{aligned}\operatorname{Ker} L_{a,H,k} &= \dot{\Theta}_a(k)^{-1} \circ \operatorname{Ad} \Theta_a(k)^{-1}(\mathfrak{k}_X + \mathfrak{k}_H) = \\ &= \dot{\Theta}_a(k)^{-1}(\operatorname{Ad} \Theta_a(k)^{-1}\mathfrak{k}_X + \mathfrak{k}_H) = \dot{\Theta}_a(k)^{-1}(T_{X,H,\Theta_a(k)}).\end{aligned}$$

But the latter space is equal to $T_{X,H,k}$ because $K_{X,H}$ is Θ_a-invariant.

PROPOSITION 6.5: *For any* $X, H \in \mathfrak{a}$, $w \in \mathfrak{w}$,

$$L_{\exp X, H, x_w} = -\tfrac{1}{2} \sum_{\alpha \in \Delta^+} \alpha(H)(1 - e^{-2w\alpha(X)}) F_\alpha \tag{6.7}$$

where F_α *is the orthogonal projection:* $\mathfrak{k} \to \mathfrak{k}_\alpha = (\mathfrak{g}_\alpha + \theta\mathfrak{g}_\alpha) \cap \mathfrak{k}$.

Because $\Theta_a(x_w \exp tY) = x_w \Theta_{a'}(\exp tY)$ for $a' = a^{w^{-1}}$, we have $\dot{\Theta}_a(x_w) = \dot{\Theta}_{a'}(1)$. From the proof of Proposition 6.2 we obtain

$$\dot{\Theta}_{a'}(1) = E_{\mathfrak{k}} \circ \exp(\operatorname{ad} X') \qquad (X' = w^{-1}X). \tag{6.8}$$

It maps $Y_\alpha + \theta Y_\alpha$ to $\exp(-\alpha(X'))(Y_\alpha + \theta Y_\alpha)$ if $Y_\alpha \in \mathfrak{g}_\alpha$. So $\dot{\Theta}_a(x_w)$ acts on $\mathfrak{k}_\alpha$ as scalar multiplication by $e^{-\alpha(X')}$, and it leaves $\mathfrak{m}$ pointwise fixed. Next, $\Theta_a(x_w) = x_w$ and $\operatorname{Ad} x_w^{-1} \circ \sinh(\operatorname{ad} X) \circ \operatorname{Ad} x_w = \sinh(\operatorname{ad} X')$. Furthermore, $\sinh \operatorname{ad} X' \circ \operatorname{ad} H$ acts on $\mathfrak{k}_\alpha$ as scalar multiplication by $\sinh \alpha(X') \cdot \alpha(H)$ and it annihilates $\mathfrak{m}$. The formula (6.7) is proved.

Proposition 6.5 could also have been obtained from (1.11) using (5.15). Using the left $K_a(=K_X)$- and right M-invariance, Proposition 6.5 also gives the Hessian of $F_{\exp X, H}$ at $K_X \mathfrak{w} M$. If $F_{\exp X, H}$ is right K_H-invariant (see Proposition 5.6) we get the Hessian at all critical points. However, even if $F_{\exp X, H}$ is not right K_H-invariant, we can still conclude:

COROLLARY 6.6: *Let* $X, H \in \mathfrak{a}$, $w \in \mathfrak{w}$. *Then at all points of* $K_X w K_H$, *the value of* $F_{\exp X, H}$ *and the signature, resp. rank of its Hessian stay constant and are, respectively, equal to*

$$\langle wH, X \rangle, \sigma_w = -\sum_{\alpha \in \Delta_w^+} n(\alpha) \cdot \operatorname{sgn}(\alpha(H) w\alpha(X)), \sum_{\alpha \in \Delta_w^+} n(\alpha), \tag{6.9}$$

where

$$\Delta_w^+ = \Delta_w^+(X, H) = \{\alpha \in \Delta^+ \mid \alpha(H) \cdot w\alpha(X) \neq 0\}; \; n(\alpha) = \dim \mathfrak{g}_\alpha. \tag{6.10}$$

At the points x_w this follows from Proposition 6.5 since $\dim \mathfrak{k}_\alpha = \dim \mathfrak{g}_\alpha$. Since the critical set is smooth and the Hessians transversally nonsingular, the value, signature and rank are locally constant on the critical set; hence we get the result on $K_X^0 w K_H^0$. As this set and $K_X w K_H$ have the same image in K/M (see (1.4)), we are through.

For $k \in K_{X,H}$, the endomorphism $L_{X,H,k}$ of $\mathfrak{k}$ is semisimple because it is symmetric with respect to the usual inner product on $\mathfrak{k}$. Since its kernel is equal to $T_{X,H,k}$, it induces an automorphism of $\mathfrak{k}/T_{X,H,k}$ (resp. $T_{X,H,k}^{\perp}$) which we denote by $L_{X,H,k}^{\perp}$. Furthermore, $\dot{\Theta}_a(k)$ induces a linear isomorphism of $\mathfrak{k}/T_{X,H,k}$ with $\mathfrak{k}/T_{X,H,\Theta_a(k)}$ which we denote by $(\dot{\Theta}_a(k))^{\perp}$. The determinants of these isomorphisms, calculated with respect to orthonormal bases for $T_{X,H,k}^{\perp}$ and $T_{X,H,\Theta_a(k)}^{\perp}$, are easily seen to be invariant, up to a change of sign, when these bases are changed.

PROPOSITION 6.7: *Let* $X, H \in \mathfrak{a}$, $a = \exp X$, $k \in K_X w K_H$. *Then, with notation as in* (6.10),

$$|\det(\dot{\Theta}_a(k))^{\perp}| = \exp\{-\sum_{\alpha\in\Delta^+,\alpha(H)\neq 0} n(\alpha)\alpha(H(ak))\}. \tag{6.11}$$

Moreover

$$|\det L_{X,H,k}^{\perp}| = |\det(\dot{\Theta}_a(k))^{\perp}| \cdot \prod_{\alpha\in\Delta_w^+} |\alpha(H)\sinh w\alpha(X)|^{n(\alpha)}. \tag{6.12}$$

Finally, if $F_{a,H}$ is right K_H-invariant (for instance if $H \in Cl(\mathfrak{a}^+)$), then

$$|\det(\dot{\Theta}_a(k))^{\perp}| = \exp\{-\sum_{\alpha\in\Delta_w^+} n(\alpha) w\alpha(X)\}. \tag{6.13}$$

We begin with (6.12). As in the proof of Corollary 6.4, write $k = ux_w v$ with $u \in K_X$, $w \in \mathfrak{w}$, $v \in K_H$, so that $\dot{\Theta}_a(k) = \dot{\Theta}_{a'}(v)$, $L_{a,H,k} = L_{a',H,v}$, $T_{X,H,k} = T_{X',H,v}$, with $a' = a^{w^{-1}}$, $X' = w^{-1}X$. This reduces the proof of (6.12) to the case that $k \in K_H$ and $w = 1$. Furthermore, $\operatorname{Ad}\Theta_a(k)$ commutes with $\operatorname{ad} H$. So writing $\gamma = \dot{\Theta}_a(1)^{-1} \circ \operatorname{Ad}\Theta_a(k) \circ \dot{\Theta}_a(k)$,

$$L_{X,H,k} = -\operatorname{Ad} k^{-1} \circ L_{X,H,1} \circ \gamma.$$

Now γ maps $T_{X,H,k}$ into $T_{X,H,1}$ and $-\operatorname{Ad} k^{-1}$ maps $T_{X,H,1}$ into $T_{X,H,k}$, cf. (6.5). Because $-\operatorname{Ad} k^{-1}$ is an isometry, it maps $T_{X,H,1}^{\perp}$ into $T_{X,H,k}^{\perp}$; and we get

$$|\det L_{X,H,k}^{\perp}| = |\det \gamma^{\perp}| \cdot |\det L_{X,H,1}^{\perp}|,$$

if we write $\gamma^{\perp}$ for the isomorphism induced by γ from $\mathfrak{k}/T_{X,H,k}$ onto

$\mathfrak{k}/T_{X,H,1}$. Similarly

$$|\det \gamma^{\perp}| = |\det \dot{\Theta}_a(1)^{\perp}|^{-1} |\det \dot{\Theta}_a(k)^{\perp}|.$$

Using the computations in the proof of Proposition 6.5, we see that

$$|\det \dot{\Theta}_a(1)^{\perp}| = \exp\{-\sum_{\alpha\in\Delta_1^+(X,H)} n(\alpha)\alpha(X)\},$$

whereas (6.7) implies

$$|\det L_{X,H,1}^{\perp}| = \prod_{\alpha\in\Delta_1^+(X,H)} |\alpha(H)\cdot\tfrac{1}{2}(1-e^{-2\alpha(X)})|^{n(\alpha)}.$$

Combining all these we obtain (6.12).

It remains to compute $|\det \dot{\Theta}_a(k)^{\perp}| = |\det \dot{\Theta}_{a'}(v)^{\perp}|$. As $H(aux_w v) = H(a'v)$ we may assume $w = e$, $k\in K_H$ in proving (6.11). By Lemma 6.3 the restriction Θ_a' of Θ_a to K_H is a diffeomorphism of K_H. Clearly

$$|\det \dot{\Theta}_a(k)^{\perp}| = |\det \dot{\Theta}_a(k)|/|\delta(k)|,$$

where $\delta(k)$ is the determinant of the isomorphism $T_{X,H,k} \xrightarrow{\sim} T_{X,H,\Theta_a(k)}$ calculated by using orthonormal bases in these spaces. Now, $\Theta_a(uv) = u\Theta_a'(v)$ if $u\in K_X$, $v\in K_H$; and so $\dot{\Theta}_a(k)$ is given by

$$\dot{\Theta}_a(k)(Y^{k^{-1}} + Z) = Y^{\Theta_a(k)^{-1}} + \dot{\Theta}_a'(k)(Z) \qquad (Y\in\mathfrak{k}_X, Z\in\mathfrak{k}_H).$$

This implies that $|\delta(k)| = |\det \dot{\Theta}_a'(k)|$, giving

$$|\det \dot{\Theta}_a(k)^{\perp}| = |\det \dot{\Theta}_a(k)|/|\det \dot{\Theta}_a'(k)|. \tag{6.14}$$

On the other hand, $|\det \dot{\Theta}_a(k)|$ has a well-known expression for all $k\in K$. In fact, the integral formula

$$\int_K g(k)\,dk = \int_K g(\Theta_a(k))e^{-2\rho(H(ak))}\,dk \qquad (g\in C^\infty(K/M))$$

(cf. Varadarajan [65, Corollary II . 6.27]) shows at once that

$$|\det \dot{\Theta}_a(k)| = \exp\{-\sum_{\alpha\in\Delta^+} n(\alpha)\alpha(H(ak))\} \qquad (k\in K). \tag{6.15}$$

If we replace G by G_H and observe that $G_H = K_H A N_H$ is an Iwasawa decomposition of G_H, we obtain also:

$$|\det \dot{\Theta}_a'(k)| = \exp\{-\sum_{\alpha\in\Delta^+,\alpha(H)=0} n(\alpha)\alpha(H(ak))\}\ (k\in K_H). \tag{6.16}$$

Substituting (6.15) and (6.16) in (6.14) gives (6.11).

Suppose finally that $H\in Cl(\mathfrak{a}^+)$. Then (6.13) can be proved from (6.12) and the formula

$$\begin{aligned}|\det L^{\perp}_{\exp X,H,k}| &= \prod_{\alpha\in\Delta_1^+(w^{-1}X,H)} |\alpha(H)\cdot\tfrac{1}{2}(1-e^{-2w\alpha(X)})|^{n(\alpha)}\\ &= \exp\{-\sum_{\alpha\in\Delta_w^+} n(\alpha)w\alpha(X)\}\cdot\prod_{\alpha\in\Delta_w^+}|\alpha(H)\sinh w\alpha(X)|^{n(\alpha)}\end{aligned}$$

which follows directly from (6.7).

7. Calculations on $\bar{N}$

For $w\in\mathfrak{w}$, let γ_w be the map $\bar{N}=\theta N\to K/M$ given by

$$\gamma_w(\bar{n}) = x_w\kappa(\bar{n})M \qquad (\bar{n}\in\bar{N}). \tag{7.1}$$

LEMMA 7.1: (i) *γ_w is an analytic diffeomorphism from $\bar{N}$ onto an open subset Ω_w of K/M containing w.*

(ii) *γ_w intertwines the conjugation with $a^{w^{-1}}$ on $\bar{N}$ with the action of a on K/M defined by Θ_a.*

(iii) *For all $a\in A$, $\bar{n}\in\bar{N}$,*

$$H(a\gamma_w(\bar{n})) = H(\bar{n}^{(a^{w^{-1}})}) - H(\bar{n}) + w^{-1}\log a. \tag{7.2}$$

This lemma is known, see Harish–Chandra [28, pp. 284–289, and Lemma 44 and its corollaries]. The statements (i), (ii) correspond to Proposition 3.6(i), (ii) if we take $H\in\mathfrak{a}^+$ there. Indeed, then γ_w is equal to the composition of conjugation by x_w, $\log:\bar{N}\to\bar{\mathfrak{n}}$, $\gamma_{w,H}:\bar{\mathfrak{n}}\to G/P$ and finally the inverse of $K/M\to G/P$, if $P=MAN$.

Restricting our attention to $\gamma=\gamma_1$, we get from (7.2):

$$F_{a,H}\circ\gamma = \psi_{a,H} + \langle\log a, H\rangle \tag{7.3}$$

if we define

$$\psi_{a,H}(\bar{n}) = \langle H(\bar{n}^a) - H(\bar{n}), H\rangle \qquad (\bar{n}\in\bar{N}). \tag{7.4}$$

From Sections 5, 6 it is clear that for arbitrary $H\in\mathfrak{a}$ the function $\psi_{a,H}$

has $\gamma^{-1}(K_a\mathfrak{w}K_H)$ as critical set; this set is clean in the sense of Bott. The map $\tau:(\bar{n},m)\mapsto\kappa(\bar{n})m$ from $\bar{N}\times M$ into K has differential given by

$$\dot{\tau}_{(\bar{n},m)}(\bar{Y},Z)=(E_t(\bar{Y}^{t(\bar{n})}))^{m^{-1}}+Z \qquad (\bar{Y}\in\bar{\mathfrak{n}}, Z\in\mathfrak{m}), \tag{7.5}$$

identifying tangent spaces with Lie algebras via left translation. Using this, the Hessians of $\psi_{a,H}$ at the critical points can be computed explicitly from the formulae in Section 6.

When $H\in Cl(\mathfrak{a}^+)$, the results are somewhat neater.

PROPOSITION 7.2: *Fix $a\in A$ and let $H\in\mathfrak{a}$ be aligned, that is $\{\alpha\in\Delta^+|\alpha(H)=0\}=\{\alpha\in\Delta^+|\alpha(H')=0\}$ for some $H'\in Cl(\mathfrak{a}^+)$. Then $\psi_{a,H}$ is right $\bar{N}_H$-invariant and its critical set is $\bar{N}_a\bar{N}_H$. Moreover $\kappa(\bar{N}_a\bar{N}_H)\subset K_a^0K_H^0$. The Hessian $\mathscr{H}_{\bar{n}}$ of $\psi_{a,H}$ at $\bar{n}\in\bar{N}_a$ is given by, for $\bar{Y},\bar{Y}'\in\bar{\mathfrak{n}}$,*

$$\mathscr{H}_{\bar{n}}(\bar{Y},\bar{Y}')=-\sum_{\alpha\in\Delta^+}\alpha(H)(1-e^{-2\alpha(\log a)})\times(R_{-\alpha}(\bar{Y}^{t(\bar{n})}),R_{-\alpha}(\bar{Y}'^{t(\bar{n})})). \tag{7.6}$$

Here the inner product is $\bar{Z},\bar{Z}'\mapsto-\langle\bar{Z},\theta\bar{Z}'\rangle$ and R_β is the orthogonal projection: $\mathfrak{g}\to\mathfrak{g}_\beta$.

Consider the commutative diagram

$$\begin{array}{ccc}\bar{N} & \xrightarrow{\gamma} & K/M\\ \downarrow & & \downarrow\\ \bar{\mathfrak{n}}\ominus\bar{\mathfrak{n}}_H & \rightarrow & K/K_H\end{array} \tag{7.7}$$

Here the left vertical arrow is the inverse of the diffeomorphism $(\exp\bar{\mathfrak{n}}\ominus\bar{\mathfrak{n}}_H)\times\bar{N}_H\to\bar{N}$ of Lemma 2.3, followed by projection to the first factor and by log. Because $\bar{\mathfrak{n}}_H=\bar{\mathfrak{n}}_{H'}$ and $K_H=K_{H'}$, the lower horizontal arrow is $\gamma_{1,H'}$ of Proposition 3.6(i) followed by the inverse of the diffeomorphism $K/K_{H'}\to G/G(H')$. The natural projection in the right vertical arrow then corresponds to the projection $G/P\to G/G(H')$.

The right $\bar{N}_H$-invariance of $F_{a,H}\circ\gamma$ then follows immediately from the right K_H-invariance of $F_{a,H}$. Furthermore, if $a=\exp X$, the critical set of $F_{a,H}$, considered as function on K/K_H, is equal to the zeroset of the vector field v_X (Corollary 3.5) which under the lower horizontal arrow is pulled back to $\operatorname{Ker}\operatorname{ad}X\cap(\bar{\mathfrak{n}}\ominus\bar{\mathfrak{n}}_H)=\bar{\mathfrak{n}}_X\ominus\bar{\mathfrak{n}}_H$ in view of Proposition 3.6(iii). So the critical set of $\psi_{a,H}$ is equal to $\exp(\bar{\mathfrak{n}}_X\ominus\bar{\mathfrak{n}}_H)\bar{N}_H$. Now $\bar{N}_H=\exp(\bar{\mathfrak{n}}_X\cap\bar{\mathfrak{n}}_H)\exp(\bar{\mathfrak{n}}_H\ominus\bar{\mathfrak{n}}_X)$ and $\exp(\bar{\mathfrak{n}}_X\ominus\bar{\mathfrak{n}}_H)\exp(\bar{\mathfrak{n}}_X\cap\bar{\mathfrak{n}}_H)=\bar{N}_X$. So the critical set is also left $\bar{N}_X$-invariant and therefore equal to $\bar{N}_X\bar{N}_H=\bar{N}_a\bar{N}_H$.

It is clear that $\kappa(\bar{N}_a \bar{N}_H)$ is contained in the connected component of $K_a \mathfrak{w} K_H$ through 1, hence $\kappa(\bar{N}_a \bar{N}_H) \subset K_a^0 K_H^0$. The straightforward calculation of the Hessian is omitted.

COROLLARY 7.3: *For $\bar{n} \in \bar{N}_a$, $\mathscr{H}_{\bar{n}}$ is diagonal with respect to the orthogonal decomposition $\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_a \oplus \bigoplus_{\lambda \neq 0} \bar{\mathfrak{n}}_\lambda$ where $\bar{\mathfrak{n}}_\lambda$ for any $\lambda \in \mathbb{R}$ is the eigenspace of* ad(log a) *in $\bar{\mathfrak{n}}$ for the eigenvalue λ. The quadratic form of $\mathscr{H}_{\bar{n}}$ is zero on $\bar{\mathfrak{n}}_a$, and for any $\lambda \neq 0$ its restriction to $\bar{\mathfrak{n}}_\lambda$ is the quadratic form*

$$\bar{Y} \mapsto (e^{2\lambda} - 1) \sum_{\alpha \in \Delta^+,\, -\alpha(\log a) = \lambda} \alpha(H) \| R_{-\alpha}(\bar{Y}^{t(\bar{n})}) \|^2 \quad (\bar{Y} \in \bar{\mathfrak{n}}_\lambda).$$

For $\lambda \neq 0$, $\mathfrak{n}_\lambda \oplus \bar{\mathfrak{n}}_\lambda = \mathfrak{g}_\lambda$ where $\mathfrak{g}_\lambda$ (resp. $\mathfrak{n}_\lambda$) is the eigenspace of ad(log a) in $\mathfrak{g}$ (resp. $\mathfrak{n}$) for the eigenvalue λ. So $\mathfrak{n}_\lambda \oplus \bar{\mathfrak{n}}_\lambda$ is stable under G_a, in particular under $t(\bar{n})$. This gives $\mathscr{H}_{\bar{n}}(\bar{Y}, \bar{Y}') = 0$ if $\bar{Y} \in \bar{\mathfrak{n}}_\lambda$, $\bar{Y}' \in \bar{\mathfrak{n}}_{\lambda'}$ if λ, λ' are distinct and nonzero.

An interesting feature of the $\psi_{a,H}$ is that, by letting $a \in A$ go to infinity in various modes, one can generate new phase functions from the $\psi_{a,H}$. For instance, taking $a = \exp tX$, $X \in \mathfrak{a}^+$, we get

$$\lim_{t \to +\infty} \psi_{\exp tX, H} = -\psi_H, \tag{7.8}$$

where ψ_H is equal to the restriction of F_H to $\bar{N}$,

$$\psi_H(\bar{n}) = \langle H(\bar{n}), H \rangle \quad (\bar{n} \in \bar{N}). \tag{7.9}$$

The function ψ_H is the phase in the integral defining Harish–Chandra's c-function. Most of the calculations concerning the critical points and Hessians of the ψ_H were carried out by Cohn [14, see §19]. The following proposition gives the complete description of facts concerning ψ_H. Again the case of complex H is dealt with by passing to the real and imaginary parts of the functions. See Remark 1.9.

PROPOSITION 7.4: *Let $H \in \mathfrak{a}$. Then the critical set of ψ_H is equal to $\bar{N}_H$ and is clean. For the Hessian of ψ_H at the critical points we have the formula*

$$\mathrm{Hess}_{\bar{n}}(\bar{Y}, \bar{Y}') = -\sum_{\alpha \in \Delta^+} \alpha(H)(\theta R_\alpha(\bar{Y}^{t(\bar{n})}) - R_{-\alpha}(\bar{Y}^{t(\bar{n})}), R_{-\alpha}(\bar{Y}'^{t(\bar{n})})), \tag{7.10}$$

valid for $\bar{n} \in \bar{N}_H$, $\bar{Y}, \bar{Y}' \in \bar{\mathfrak{n}}$. Moreover, the index of the Hessian at any point of $\bar{N}_H$ is equal to

$$\sum_{\alpha \in \Delta^+, \alpha(H) < 0} \dim \mathfrak{g}_\alpha. \tag{7.11}$$

In view of Lemma 5.1, $\bar{n}$ is a critical point for ψ_H if and only if $H^{n(\bar{n})^{-1}} \in (\bar{\mathfrak{n}})^{\perp} = \mathfrak{a} + \bar{\mathfrak{n}}$ (orthogonal complement with respect to the Killing form). As $H^{n(\bar{n})^{-1}} \in H + \mathfrak{n}$, this is equivalent to $n(\bar{n}) \in N_H$. Clearly $n(\bar{n}) \in N_H$ if $\bar{n} \in \bar{N}_H$; conversely, suppose $n(\bar{n}) \in N_H$. Then $H^{\bar{n}} = H^{\kappa(\bar{n})} \in \mathfrak{s}$, while $H^{\bar{n}} \in H + \bar{\mathfrak{n}}$. Hence $H^{\bar{n}} - H \in \mathfrak{s} \cap \bar{\mathfrak{n}} = (0)$. The formula for the Hessian follows from Lemma 6.1. To check that it is clean, write $\bar{\mathfrak{n}}_\lambda$ (resp. $\mathfrak{g}_\lambda$, $\mathfrak{n}_\lambda$) for the eigenspace of $\operatorname{ad} H$ in $\bar{\mathfrak{n}}$ (resp. $\mathfrak{g}, \mathfrak{n}$) for the eigenvalue $\lambda \in \mathbb{R}$. Arguing as in the proof of Corollary 7.3 we find that $\mathrm{Hess}_{\bar{n}}$ is diagonal with respect to the decomposition $\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_H \oplus \sum_{\lambda \neq 0} \bar{\mathfrak{n}}_\lambda$. Thus for cleanness we need only to prove that if $\lambda \neq 0$, $\bar{Y} \in \bar{\mathfrak{n}}_\lambda$, $\mathrm{Hess}_{\bar{n}}(\bar{Y}, \bar{Z}) = 0$ for all $\bar{Z} \in \bar{\mathfrak{n}}_\lambda$, then $\bar{Y} = 0$. Since $t(\bar{n}) \in G_H$, it stabilizes $\mathfrak{g}_\lambda = \mathfrak{n}_\lambda \oplus \bar{\mathfrak{n}}_\lambda$; and one checks that $\mathrm{Hess}_{\bar{n}}(\bar{Y}, \bar{Z}) = \lambda((\theta - I)(\bar{Y}^{t(\bar{n})}), L\bar{Z}^{t(\bar{n})})$, where L is the projection $\mathfrak{n}_\lambda \oplus \bar{\mathfrak{n}}_\lambda \to \bar{\mathfrak{n}}_\lambda$ along $\mathfrak{n}_\lambda$. As $\operatorname{Ad} t(\bar{n})$ leaves $\mathfrak{n}_\lambda$ invariant, it is clear that $L \circ \operatorname{Ad} t(\bar{n})$ restricted to $\bar{\mathfrak{n}}_\lambda$ is invertible. Hence $\mathrm{Hess}_{\bar{n}}(\bar{Y}, \bar{Z}) = 0$ for all $\bar{Z} \in \bar{\mathfrak{n}}_\lambda$, implies $(\theta - I)(\bar{Y}^{t(\bar{n})}) = 0$, or $\bar{Y}^{t(\bar{n})} \in \mathfrak{k}$. But then $\bar{Y}^{\bar{n}} = (\bar{Y}^{t(\bar{n})})^{\kappa(\bar{n})} \in \mathfrak{k} \cap \bar{\mathfrak{n}} = (0)$, hence $\bar{Y} = 0$. Finally, for the index it is enough to calculate it at 1 since, due to the connectedness and cleanness of $\bar{N}_H$, it stays constant on $\bar{N}_H$. As

$$\mathrm{Hess}_1(\bar{Y}, \bar{Z}) = \sum_{\alpha \in \Delta^+} \alpha(H)(R_{-\alpha}(\bar{Y}), R_{-\alpha}(\bar{Z})) \quad (\bar{Y}, \bar{Z} \in \bar{\mathfrak{n}}), \tag{7.12}$$

the index at 1 is as stated.

8. The function F_H on a conjugacy class in G

For any $\gamma \in G$ we write

$$C_\gamma = \{x\gamma x^{-1} \mid x \in G\} \tag{8.1}$$

for the G-conjugacy class of γ. We recall that C_γ is a regular analytic submanifold of G (constructible in the algebraic case), which is closed if and only if γ is a semisimple element. For any $\gamma \in G$ we put

$$F_{H,\gamma} = F_H|_{C_\gamma}, \tag{8.2}$$

where, as usual, $F_H(x) = \langle H(x), H \rangle$ for $x \in G$, $H \in \mathfrak{a}$.

The action of G on itself by inner automorphisms gives rise to a homomorphism τ of $\mathfrak{g}$ into the Lie algebra of analytic vector fields on G. Using the identification of tangent vectors to G with elements of $\mathfrak{g}$ by left translations, we have

$$\tau(X)_x = \frac{d}{dt}(x^{-1} \cdot \exp - tX \cdot x \cdot \exp tX)_{t=0} = X - X^{x^{-1}} \underset{\mathrm{def}}{=} X_x. \tag{8.3}$$

Writing

$$(Xg)(x) = \frac{d}{dt} g(x \exp tX)_{t=0}, \quad (gX)(x) = \frac{d}{dt} g(\exp tXx)_{t=0}$$

for any $X \in \mathfrak{g}$ and any smooth function g on G, we have $\tau(X)g = Xg - gX$. So if $X, X' \in \mathfrak{g}$ then $\tau(X)\tau(X')g = XX'g - XgX' - X'gX + gX'X$ from which we get, for all $x \in G$,

$$g(x; \tau(X)\tau(X')) = g(x; X_x X' - X'^{x^{-1}} X_x). \tag{8.4}$$

Finally, we remark that if $x \in G$, the vectors $X - X^{x^{-1}} = X_x (X \in \mathfrak{g})$ constitute the tangent space at x to the conjugacy class C_x; in fact, the linear transformation $X \mapsto X - X^{x^{-1}}$ is the differential at e of the map $y \mapsto y^{-1}xy$ of G onto C_x.

Let $\gamma \in G$ be fixed. We shall now determine the set of critical points of $F_{H,\gamma}$. If $x, y \in G$, we often write y^x for xyx^{-1}. For $H \in \mathfrak{a}$ let G_H denote as usual the centralizer of H in G.

LEMMA 8.1: *Let $H \in \mathfrak{a}$, $x \in G$. Then*

$$x \in G_H \Leftrightarrow x^{n(x)} \in G_H.$$

If $x \in G_H = K_H A N_H$ (see (2.6)), $n(x) \in N_H \subset G_H$ so that $x^{n(x)} \in G_H$. Conversely, suppose $x \in G$ but $x^{n(x)} \in G_H$. Then $n(x)xn(x)^{-1} = n(x)\kappa(x)a(x) \in G_H$, $\Rightarrow n(x)\kappa(x) \in G_H = N_H A K_H$, showing $n(x) \in N_H$, $\kappa(x) \in K_H$; thus $x \in G_H$.

PROPOSITION 8.2: *Let $C_{H,\gamma}$ be the set of critical points of $F_{H,\gamma}$. Then*

$$C_{H,\gamma} = G_H \cap C_\gamma.$$

Let $y \in C_\gamma$. Then, by Corollary 5.2,

$$\begin{aligned} y \in C_{H,\gamma} &\Leftrightarrow \langle X - X^{y^{-1}}, H^{n(y)^{-1}} \rangle = 0 \quad \forall X \in \mathfrak{g} \\ &\Leftrightarrow \langle X, H^{n(y)^{-1}} - H^{yn(y)^{-1}} \rangle = 0 \ \forall X \in \mathfrak{g} \\ &\Leftrightarrow H = H^{n(y)yn(y)^{-1}} \Leftrightarrow y^{n(y)} \in G_H. \end{aligned}$$

Our proposition is now immediate from the previous lemma.

PROPOSITION 8.3: *$C_{H,\gamma}$ is a regular analytic submanifold of G and the intersection of C_γ and G_H is clean everywhere on $C_{H,\gamma}$. Moreover, $C_{H,\gamma}$ is* *the union of finitely many G_H-conjugacy classes. If γ is a semisimple element of G, $C_{H,\gamma}$, as well as the G_H-conjugacy classes into which it splits, are all closed; and all the G_H-conjugacy classes in $C_{H,\gamma}$ have the same dimension.*

In view of the well-known results of Richardson [49] only the last statement needs a proof. For this, recall that if G_c is the complexification of G, and $G_{c,H}$ is the centralizer of H in G_c, any semisimple G_c-conjugacy class C meets $G_{c,H}$ in a *single* $G_{c,H}$-conjugacy class. Hence the stabilizers in $G_{c,H}$ of the points of $G_{c,H} \cap C$ all have the same complex dimension. But then the stabilizers in G_H of the points of $C_\gamma \cap G_H$ all have the same real dimension, when $\gamma \in G$ is semisimple.

We shall now proceed to determine the Hessian form of $F_{H,\gamma}$ at the points of $C_{H,\gamma}$. We fix $\gamma \in G$, $H \in \mathfrak{a}$, and let $x \in C_{H,\gamma} = G_H \cap C_\gamma$. The Hessian form $\mathrm{Hess}_x(F_{H,\gamma})$ can be pulled back to a symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$ through the map $I - \mathrm{Ad}(x^{-1})$ of $\mathfrak{g}$ onto $T_x(C_\gamma)$. Let $Q_{x,H,\gamma}$ denote this form:

$$Q_{x,H,\gamma}(X, X') = \mathrm{Hess}_x(F_{H,\gamma})(X_x, X'_x) \quad (X, X' \in \mathfrak{g}) \tag{8.5}$$

Clearly, $Q_{x,H,\gamma}$ is the Hessian form at 1 of the function

$$y \mapsto F_{H,\gamma}(y^{-1}xy) \qquad (y \in G).$$

We can then define the endomorphism $L_{x,H,\gamma}$ of $\mathfrak{g}$ by

$$Q_{x,H,\gamma}(X, X') = \langle X, L_{x,H,\gamma}(X')\rangle = \langle X', L_{x,H,\gamma}(X)\rangle. \tag{8.6}$$

PROPOSITION 8.4: *We have*

$$L_{x,H,\gamma} = \mathrm{ad}\, H \circ \Omega_x \circ (I - \mathrm{Ad}(x^{-1}))$$

where Ω_x is the endomorphism of $\mathfrak{g}$ defined by

$$\Omega_x = \mathrm{Ad}(t(x)^{-1})E_{\mathfrak{n}} \circ \mathrm{Ad}(t(x)) + \mathrm{Ad}(\kappa(x))E_{\mathfrak{k}} \circ \mathrm{Ad}(t(x)).$$

For brevity write $Q = Q_{x,H,\gamma}$, $\mathscr{H} = \mathrm{Hess}_x(F_{H,\gamma})$, and let $X, X' \in \mathfrak{g}$. The vector fields $\tau(X), \tau(X')$ are tangent to C_γ everywhere and define the tangent vectors X_x, X'_x at x. Hence

$$\begin{aligned} Q(X, X') &= \mathscr{H}(X_x, X'_x) = F_{H,\gamma}(x; \tau(X)\tau(X')) \\ &= F_H(x; \tau(X)\tau(X')) = F_H(x; X_x X' - X'^{x^{-1}} X_x) \end{aligned}$$

by (8.4). This last expression can be evaluated using Lemma 6.1. We obtain

$$\begin{aligned} Q(X,X') &= B_H(X_x^{t(x)}, X'^{t(x)}) - B_H(X'^{t(x)x^{-1}}, X_x^{t(x)}) \\ &= \langle X'^{t(x)}, [H, E_{\mathfrak{n}}(X_x^{t(x)})]\rangle + \langle X'^{\kappa(x)^{-1}}, [H, E_{\mathfrak{k}}(X_x^{t(x)})]\rangle \\ &= \langle X', L_{x,H,\gamma}(X)\rangle \end{aligned}$$

if we remember that x (and hence $\kappa(x)$ as well as $t(x)$) centralizes H (cf. formula (2.6)).

From this expression for the Hessian we can prove that the set of critical points of $F_{H,\gamma}$ is clean.

PROPOSITION 8.5: *The Hessian of $H_{H,\gamma}$ is transversally nonsingular at all points of its critical manifold.*

For any $x \in C_{H,\gamma}$ we must prove that the radical of $\mathrm{Hess}_x(F_{H,\gamma})$ is $T_x(C_{H,\gamma})$. By Proposition 8.3, $T_x(C_{H,\gamma}) = T_x(C_\gamma) \cap T_x(G_H)$. So we are reduced to proving the following:

$$X \in \mathfrak{g}, L_{x,H,\gamma}(X) = 0 \Rightarrow X_x \in \mathfrak{g}_H. \tag{8.7}$$

If $L_{x,H,\gamma}(X) = 0$, we have $\Omega_x(X_x) \in \mathfrak{g}_H$. Since $\mathrm{Ad}(t(x))$ stabilizes $\mathfrak{n}$ and $\mathrm{Ad}(\kappa(x))$ stabilizes $\mathfrak{k}$, it is clear from the expression for Ω_x that $\Omega_x(X_x) = R_1 + R_2$ where $R_1 = (E_{\mathfrak{n}}(X_x^{t(x)}))^{t(x)^{-1}} \in \mathfrak{n}$ and $R_2 = (E_{\mathfrak{k}}(X_x^{t(x)}))^{\kappa(x)} \in \mathfrak{k}$. Since $\Omega_x(X_x) \in \mathfrak{g}_H$, we conclude from the uniqueness of Iwasawa decompositions that $R_1 \in \mathfrak{n}_H$, $R_2 \in \mathfrak{k}_H$. As $t(x)$ and $\kappa(x)$ respectively lie in AN_H and K_H, we find

$$E_{\mathfrak{n}}(X_x^{t(x)}) \in \mathfrak{n}_H,\ E_{\mathfrak{k}}(X_x^{t(x)}) \in \mathfrak{k}_H,$$

which implies that $X_x^{t(x)} \in \mathfrak{g}_H$. Thus $X_x \in \mathfrak{g}_H$, proving (8.7).

Our aim now is to determine the signature and transversal determinant of the Hessian forms $Q_{x,H,\gamma}$.

LEMMA 8.6: *Let $x \in G_H$ be semisimple. Then, writing $\perp$ for orthocomplementation with respect to the Killing form, we have*

$$(\mathfrak{g}_x + \mathfrak{g}_H)^\perp \cap (\mathfrak{g}_x + \mathfrak{g}_H) = (0).$$

Since H is a semisimple element of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}_H \oplus \mathfrak{g}_H^\perp$. As $x \in G_H$, $\mathrm{Ad}(x)$ respects this splitting. So, writing $(\mathfrak{g}_H^\perp)_x = \mathfrak{g}_x \cap \mathfrak{g}_H^\perp$, we have $\mathfrak{g}_x + \mathfrak{g}_H = \mathfrak{g}_H + (\mathfrak{g}_H^\perp)_x$. Further, the semisimplicity of $\mathrm{Ad}(x)$ allows us to conclude that

$\mathfrak{g}_H^\perp = (\mathfrak{g}_H^\perp)_x \oplus \mathfrak{q}$ where $\mathfrak{q} = (I - \mathrm{Ad}(x))(\mathfrak{g}_H^\perp)$. A simple calculation shows that this decomposition is an orthogonal one. So $\mathfrak{q} = (\mathfrak{g}_x + \mathfrak{g}_H)^\perp$.

We shall from now on suppose that γ is semisimple. Let $x \in C_{H,\gamma} = G_H \cap C_\gamma$ and put

$$\mathfrak{q}(x) = (\mathfrak{g}_x + \mathfrak{g}_H)^\perp. \tag{8.8}$$

COROLLARY 8.7: *$L_{x,H,\gamma}$ leaves $\mathfrak{q}(x)$ invariant and induces an automorphism of it.*

The argument of the previous lemma actually shows that $\mathfrak{g}_x + \mathfrak{g}_H = (I - \mathrm{Ad}(x^{-1}))^{-1} T_x(C_{H,\gamma})$ and hence $\mathfrak{g}_x + \mathfrak{g}_H = \mathrm{Ker}(L_{x,H,\gamma})$. As $L_{x,H,\gamma}$ is symmetric with respect to $\langle .,. \rangle$, the corollary follows from the above lemma.

PROPOSITION 8.8: *Let $\gamma \in G$ be semisimple, $x \in C_{H,\gamma}$. Then*

$$|\det(L_{x,H,\gamma})_{\mathfrak{q}(x)}| = \\ = |\det(\mathrm{ad}\, H \circ (I - \mathrm{Ad}(x^{-1})))_{\mathfrak{q}(x)}|\, |\det(\mathrm{Ad}(t(x))_{\mathfrak{g}_H^\perp \cap \mathfrak{n}})|^{-1}$$

where the suffixes denote endomorphisms induced on the corresponding vector spaces.

Write $L = L_{x,H,\gamma}$, $t = \mathrm{Ad}(t(x))$, $k = \mathrm{Ad}(\kappa(x))$, $\mathfrak{q} = \mathfrak{q}(x)$, $R = t^{-1}E_{\mathfrak{n}}t + kE_{\mathfrak{k}}t$. Then $L = \mathrm{ad}\, H \circ R \circ (I - \mathrm{Ad}(x^{-1}))$. As L, $\mathrm{ad}\, H$ and $I - \mathrm{Ad}(x^{-1})$ leave $\mathfrak{q}$ invariant and all are automorphisms of $\mathfrak{q}$, R leaves $\mathfrak{q}$ invariant and we have

$$\det(L_{\mathfrak{q}}) = \det(\mathrm{ad}\, H \circ (I - \mathrm{Ad}(x^{-1})))_{\mathfrak{q}} \cdot \det(R_{\mathfrak{q}}). \tag{8.9}$$

So we need to establish that $\det(R_{\mathfrak{q}}) = \pm \det((t)_{\mathfrak{g}_H^\perp \cap \mathfrak{n}})^{-1}$.

We begin by observing that $\mathfrak{g}_H^\perp$ is stable under $E_{\mathfrak{k}}$ and $E_{\mathfrak{n}}$ and that $E_{\mathfrak{k}} + E_{\mathfrak{n}} = I_{\mathfrak{g}_H^\perp}$, the identity on $\mathfrak{g}_H^\perp$. So R leaves $\mathfrak{g}_H^\perp$ invariant and

$$R_{\mathfrak{g}_H^\perp} = I_{\mathfrak{g}_H^\perp} + (k - t^{-1})E_{\mathfrak{k}}t = I_{\mathfrak{g}_H^\perp} + (\mathrm{Ad}(x) - I)t^{-1}E_{\mathfrak{k}}t.$$

So, if $Z \in (\mathfrak{g}_H^\perp)_x$, $R(Z) \equiv Z \bmod(\mathfrak{q})$. So, with respect to the decomposition $\mathfrak{g}_H^\perp = (\mathfrak{g}_H^\perp)_x \oplus \mathfrak{q}$, $R_{\mathfrak{g}_H^\perp}$ has the matrix

$$R_{\mathfrak{g}_H^\perp} \sim \begin{pmatrix} I & 0 \\ * & R_{\mathfrak{q}} \end{pmatrix}$$

which shows that $\det(R_{\mathfrak{q}}) = \det(R_{\mathfrak{g}_H^\perp})$. Hence we come down to checking that

$$\det(R_{\mathfrak{g}_H^\perp}) = \pm\det((t)_{\mathfrak{g}_H^\perp \cap \mathfrak{n}})^{-1}.$$

But $\mathfrak{g}_H^\perp = (\mathfrak{n} \cap \mathfrak{g}_H^\perp) \oplus (\mathfrak{k} \cap \mathfrak{g}_H^\perp)$, and with respect to this decomposition, $(t^{-1}E_{\mathfrak{n}} + kE_{\mathfrak{k}})_{\mathfrak{g}_H^\perp}$ has the matrix

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & k \end{pmatrix}$$

whose determinant is $\det(t^{-1})_{\mathfrak{n}\cap\mathfrak{g}_H^\perp} \cdot \det(k)_{\mathfrak{k}\cap\mathfrak{g}_H^\perp} = \pm\det(t^{-1})_{\mathfrak{n}\cap\mathfrak{g}_H^\perp}$. Hence

$$\det(R_{\mathfrak{g}_H^\perp}) = \pm\det(t)_{\mathfrak{g}_H^\perp}\det(t^{-1})_{\mathfrak{n}\cap\mathfrak{g}_H^\perp}.$$

As $\mathrm{Ad}(t(x))$ is unimodular on $\mathfrak{g}$ as well as $\mathfrak{g}_H$, $\det(t)_{\mathfrak{g}_H^\perp} = 1$. So we are done.

COROLLARY 8.9: *Assume in addition that* $H \in Cl(\mathfrak{a}^+)$. *Then* $\mathfrak{q}(x) \cap \mathfrak{n}$ *is stable under* $\mathrm{Ad}(x^{-1})$ *(as well as* $\mathrm{ad}\, H$*), and*

$$|\det(L_{x,H,\gamma})_{\mathfrak{q}(x)}| = (\det(\mathrm{ad}\, H \circ (I - \mathrm{Ad}(x^{-1})))_{\mathfrak{q}(x)\cap\mathfrak{n}})^2.$$

We have a standard parabolic subalgebra $\mathfrak{g}(H)$ associated with H whose nilradical is $\mathfrak{n}_+ = \sum_{\alpha(H)>0}\mathfrak{g}_\alpha$. Let $\mathfrak{n}_- = \sum_{\alpha(H)>0}\mathfrak{g}_{-\alpha}$. Then $\mathfrak{n}_\pm$ are stable under G_H, hence under $\mathrm{Ad}(x^{-1})$; and their direct sum is $\mathfrak{g}_H^\perp$. Clearly $\mathfrak{n}_\pm = (\mathfrak{n}_\pm)_x \oplus (I - \mathrm{Ad}(x^{-1}))(\mathfrak{n}_\pm)$. Write $\mathfrak{q} = \mathfrak{q}(x)$. Then we see that $\mathfrak{q} = (\mathfrak{q}\cap\mathfrak{n}) \oplus (\mathfrak{q}\cap\bar{\mathfrak{n}})$ where $\mathfrak{q}\cap\mathfrak{n} = \mathfrak{q}\cap\mathfrak{n}_+$, $\mathfrak{q}\cap\bar{\mathfrak{n}} = \mathfrak{q}\cap\mathfrak{n}_-$, and $\mathfrak{q}\cap\mathfrak{n}_\pm = (I - \mathrm{Ad}(x^{-1}))(\mathfrak{n}_\pm)$. Now $\langle .,. \rangle$ is nondegenerate on $\mathfrak{q}\times\mathfrak{q}$ by Lemma 8.6, and it is easily seen that $\mathrm{Ad}(x)$ (resp. $-\mathrm{ad}\, H$) is the transpose of $\mathrm{Ad}(x^{-1})$ (resp. $\mathrm{ad}\, H$) with respect to $\langle .,. \rangle$; moreover, $\mathfrak{q}\cap\mathfrak{n}_+$ and $\mathfrak{q}\cap\mathfrak{n}_-$ are in duality with respect to $\langle .,. \rangle$. Hence

$$\begin{aligned}\det(\mathrm{ad}\, H \circ (I - \mathrm{Ad}(x^{-1})))_{\mathfrak{q}\cap\mathfrak{n}_-} &= \det(-\mathrm{ad}\, H \circ (I - \mathrm{Ad}(x)))_{\mathfrak{q}\cap\mathfrak{n}_+} \\ &= \pm\det(\mathrm{ad}\, H \circ (I - \mathrm{Ad}(x^{-1})))_{\mathfrak{q}\cap\mathfrak{n}_+} \cdot \det(\mathrm{Ad}(x))_{\mathfrak{q}\cap\mathfrak{n}_+}\end{aligned}$$

On the other hand, as $\kappa(x)$ and $t(x)$ are in G_H, $\mathrm{Ad}(\kappa(x)) = k$ and $\mathrm{Ad}(t(x)) = t$ leave $\mathfrak{n}_\pm$ invariant, so that

$$\det((t)_{\mathfrak{n}_+}) = \pm\det(\mathrm{Ad}(x))_{\mathfrak{n}_+} = \pm\det(\mathrm{Ad}(x))_{\mathfrak{q}\cap\mathfrak{n}_+}.$$

The corollary follows from these formulae.

PROPOSITION 8.10: *Let $\gamma \in G$ be semisimple. Then*

(i) *The signature of $Q_{x,H,\gamma}$ is constant when x varies in any G_H-conjugacy class of $C_{H,\gamma}$. It is moreover* 0 *on all of $C_{H,\gamma}$ if $H \in Cl(\mathfrak{a}^+)$.*

(ii) *The critical value of $F_{H,\gamma}$ is constant on each G_H-conjugacy class in $C_{H,\gamma}$.*

Both the critical value and the signature are constant on each G_H^0-conjugacy class in $C_{H,\gamma}$. But $G_H = MG_H^0$ and $F_{H,\gamma}$ is invariant under the inner automorphisms induced by M. It therefore remains only to verify that $\mathrm{sgn}(Q_{x,H,\gamma}) = 0$ for all $x \in C_{H,\gamma}$ when $H \in Cl(\mathfrak{a}^+)$. In this case, suppose $X \in \mathfrak{q} \cap \mathfrak{n}$. Then $X_x = X - X^{x^{-1}}$ is also in $\mathfrak{q} \cap \mathfrak{n}$ so that $X_x^{t(x)} \in \mathfrak{n}$. This gives $E_{\mathfrak{n}}(X_x^{t(x)}) = X_x^{t(x)}$ and $E_{\mathfrak{k}}(X_x^{t(x)}) = 0$ from which we get

$$Q_{x,H,\gamma}(X,X) = \langle [H, X_x], X\rangle = 0$$

since $\langle \mathfrak{n}, \mathfrak{n}\rangle = 0$. So $Q_{x,H,\gamma}$ is zero on $\mathfrak{q} \cap \mathfrak{n}$ whose dimension is $\frac{1}{2}\dim(\mathfrak{q})$. It follows at once that $Q_{x,H,\gamma}$ has signature 0.

REMARK: The form $Q_{x,H,\gamma}$ is the pullback of $\mathrm{Hess}_x(F_{H,\gamma})$ via $I - \mathrm{Ad}(x^{-1})$. Hence the absolute value of $\det(\mathrm{Hess}_x(F_{H,\gamma}))_{\mathfrak{q}(x)}$ calculated with respect to pseudo-orthogonal bases of $\mathfrak{q}(x)$ (bases (X_i) with $\langle X_i, X_j\rangle = \pm\delta_{ij}$) is given by

$$|\det(\mathrm{Hess}_x(F_{H,\gamma}))_{\mathfrak{q}(x)}| = |\det(L_{x,H,\gamma})_{\mathfrak{q}(x)}| \cdot |\det(I - \mathrm{Ad}(x^{-1}))_{\mathfrak{q}(x)}|^{-2}.$$

We note that $\mathrm{Ad}\, n(x)$ is unimodular on any subspace it stabilizes, so that we get

$$\begin{aligned} |\det(\mathrm{Hess}_x(F_{H,\gamma}))_{\mathfrak{q}(x)}| &= |\det(\mathrm{ad}\, H \circ (I - \mathrm{Ad}(x^{-1}))^{-1})_{\mathfrak{q}(x)}| \cdot \\ &\cdot \exp\Big(- \sum_{\alpha \in \Delta^+, \alpha(H) \neq 0} n(\alpha)\alpha(H(x))\Big). \end{aligned} \tag{8.10}$$

Although $\mathfrak{q}(x)$ depends on x, the argument (via G_c) used in the proof of Proposition 8.3 shows that the first term on the right side of (8.10) does not depend on $x \in C_{H,\gamma}$.

9. Asymptotic expansions for matrix coefficients

The results obtained in the sections 5 and 6 allow us to obtain the asymptotics, as $\tau \to +\infty$, of the integrals

$$I_{a,H,\tau}(g) = \int_K e^{i\tau\langle H(ak),H\rangle} g(k)\,dk, \tag{9.1}$$

where $a \in A$, $H \in \mathfrak{a}$ are fixed and $g \in C^\infty(K)$.

The asymptotic expansion can be given a distributional formulation. To explain this, let $\mu_0, \mu_1, \ldots \in \mathbb{R}$ such that $\mu_n \to -\infty$, let $c_n \in \mathbb{R}$ and let $\Psi_n = \Psi_{n,a,H}$ be distributions on K. Then we shall write

$$I_{a,H,\tau} \sim \sum_{n=0}^{\infty} e^{i\tau c_n} \tau^{\mu_n} \Psi_n \text{ as } \tau \to +\infty, \tag{9.2}$$

if for each integer $N \geq 0$ there exists a continuous seminorm ν_N on $C^\infty(K)$ such that

$$I_{a,H,\tau}(g) - \sum_{n=0}^{N} e^{i\tau c_n} \tau^{\mu_n} \Psi_n(g) = 0(\nu_N(g)\tau^{\mu_{N+1}}),\ \tau \geq 1 \tag{9.3}$$

for all $g \in C^\infty(K)$. It is clear that if such an asymptotic expansion exists, the c_n and Ψ_n are uniquely determined. If the 0 in (9.3) is uniform with respect to the parameters a, H (ranging over some domain), then we say that the expansion is uniform in these parameters.

The asymptotic expansion of $I_{a,H,\tau}$ as $\tau \to +\infty$ is obtained by a straightforward application of the method of stationary phase (see Hörmander [36, p. 144], Chazarain [13], Colin de Verdière [15]). In order to formulate it we need some notation. For fixed $a \in A$, $H \in \mathfrak{a}$, and any $w \in \mathfrak{w}$, we put (cf. (6.9) and (6.10))

$$\begin{aligned} \Delta_w^+ &= \Delta_w^+(a,H) = \{\alpha \in \Delta^+ \mid \alpha(H)\cdot w\alpha(\log a) \neq 0\}, \\ n_w &= \sum_{\alpha\in\Delta_w^+} n(\alpha),\quad \sigma_w = -\sum_{\alpha\in\Delta_w^+} n(\alpha)\cdot \operatorname{sgn}(\alpha(H)\,w\alpha(\log a)). \end{aligned} \tag{9.4}$$

Here, as usual $n(\alpha) = \dim \mathfrak{g}_\alpha$ for any root α of $(\mathfrak{g}, \mathfrak{a})$. Finally, dk is the normalized Haar measure on K, that is $\int_K dk = 1$; and d_0k is the Euclidean measure induced by the bi-invariant Riemannian metric on K which on $\mathfrak{k}$ is equal to minus the Killing form.

As a consequence of Proposition 5.4, Corollary 6.4, Corollary 6.6, Proposition 6.7 and Proposition 5.6 we then have:

THEOREM 9.1: *Fix* $a\in A$, $H\in\mathfrak{a}$. *The function* $k\mapsto\exp i\tau\langle H(ak),H\rangle$, *regarded as a distribution* $I_{a,H,\tau}$ *on* K, *has an asymptotic expansion of the form*

$$I_{a,H,\tau}\sim\sum_{\mathfrak{w}_a\backslash\mathfrak{w}/\mathfrak{w}_H}e^{i\tau\langle wH,\log a\rangle}\sum_{l=0}^{\infty}\tau^{-\frac{1}{2}n_w-l}c_{w,l}\tag{9.5}$$

as $\tau\to+\infty$. *Here* $c_{w,l}=c_{w,l,a,H}$ *is a distribution of order* $\leq 2l$ *with support contained in* K_awK_H. *To be more precise*: $c_{w,l}=Qv$ *where* Q *is a linear differential operator with smooth coefficients in* K *of order* $\leq 2l$, *and* v *is a smooth density on* K_awK_H. *The top order coefficient* $c_{w,0}$ *is the measure given by the formula, valid for all* $g\in C^\infty(K)$, *and with* $\mathrm{vol}_0(K)=\int_K d_0k$,

$$\begin{aligned}c_{w,0,a,H}(g)=e^{\frac{i\pi}{4}\sigma_w}\prod_{\alpha\in\Delta_w^+}\left|\frac{\alpha(H)}{2\pi}\sinh w\alpha(\log a)\right|^{-\frac{1}{2}n(\alpha)}\\ \cdot\mathrm{vol}_0(K)^{-1}\int_{K_awK_H}g(k)\exp\{\tfrac{1}{2}\sum_{\substack{\alpha\in\Delta^+\\ \alpha(H)\neq 0}}n(\alpha)\alpha(H(ak))\}\,d_0k.\end{aligned}\tag{9.6}$$

Let A_a (*resp.* $\mathfrak{a}_H$) *be the vector subgroup* (*resp. linear subspace*) *of* A (*resp.* $\mathfrak{a}$) *of elements centralizing* K_a (*resp.* K_H), *and* A_a' (*resp.* $\mathfrak{a}_H'$) *the open subset of* A_a (*resp.* $\mathfrak{a}_H$) *of elements with centralizer exactly* K_a (*resp.* K_H). *Then the* $c_{w,l,a',H'}$ *depend smoothly on* $(a',H')\in A_a'\times\mathfrak{a}_H'$; *and the expansion is locally uniform in* $A_a'\times\mathfrak{a}_H'$. *Moreover, with* $l(\cdot)$ *denoting left translation in* $C^\infty(K)$,

$$c_{w,l,a,H}(g)=c_{e,l,a',H}(l(x_w)^{-1}g)\quad(a'=a^{w^{-1}})\tag{9.7}$$

and the $c_{w,l,a,H}$ *are invariant under left translations by elements of* K_a. *Finally, if* $H\in Cl(\mathfrak{a}^+)$, *or more generally, if* H *is aligned with* $\mathfrak{a}^+$, *then the* $c_{w,l,a,H}$ *are also invariant under right translations by elements of* K_H; *and the top order coefficient* $c_{w,0,a,H}$ *is given by*

$$\begin{aligned}c_{w,0,a,H}(g)=e^{\frac{i\pi}{4}\sigma_w}\prod_{\alpha\in\Delta_w^+}\left|\frac{\alpha(H)}{4\pi}(1-e^{-2w\alpha(\log a)})\right|^{-\frac{1}{2}n(\alpha)}\cdot\\ \cdot\mathrm{vol}_0(K)^{-1}\int_{K_awK_H}g(k)\,d_0k\qquad(g\in C^\infty(K)).\end{aligned}\tag{9.8}$$

EXAMPLE: For the elementary spherical function $\phi_{i\tau\xi+\lambda}$, $\xi\in\mathscr{F}_R=\mathfrak{a}^*$, $\lambda\in\mathscr{F}=\mathfrak{a}_{\mathbb{C}}^*$, the theorem can be applied with $H=H_\xi$, $g(k)=\exp(\lambda-\rho)(H(ak))$.

Here, as usual,

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} n(\alpha)\alpha. \tag{9.9}$$

The formula for the top order term becomes particularly simple if a and ξ are regular; in this case

$$\begin{aligned} \phi_{i\tau\xi+\lambda}(a) \sim \sum_{w \in \mathfrak{w}} & e^{w(i\tau\xi+\lambda)(\log a)}\tau^{-\frac{1}{2}\dim N} \cdot \\ & \cdot e^{\frac{i\pi}{4}\sigma_w} \prod_{\alpha \in \Delta^+} \left| \frac{\langle \alpha, \xi \rangle}{2\pi} \sinh w\alpha(\log a) \right|^{-\frac{1}{2}n(\alpha)} \cdot \frac{\mathrm{vol}_0(M)}{\mathrm{vol}_0(K)} \end{aligned} \tag{9.10}$$

modulo lower order terms in τ. This formula has a formal resemblance with the famous asymptotics of Harish–Chandra for $\phi_{i\xi}(a)$ as $a \to \infty$ in A^+. Careful analysis of the latter asymptotics, for instance in the formulation of Gangolli [23], also leads to an asymptotic expansion for $\phi_{i\tau\xi}(a)$, as $\tau \to \infty$ and $a \in A^+$ keeps away from the walls of the Weyl chamber. However, Theorem 9.1 is more general; it applies also to singular a and can be applied directly to spherical functions related to K-representations as in (11.5). Also, it is our opinion that even in the case of $\phi_{i\tau\xi}(a)$, $a \in A^+$, the proof using the method of stationary phase is more elementary.

In the above theorem, as long as (a', H') varies around (a, H) in such a way that $K_{a'} = K_a$ and $K_{H'} = K_H$, the asymptotic expansion varies smoothly. For small but arbitrary variations of (a', H') on the other hand, it is natural to expect that at least the absolute value of $I_{a', H', \tau}(g)$ is dominated by the growth order at (a, H).

In fact, let $k_0 \in K_{a,H}$. We can then select coordinates $x_1, \ldots, x_q, y_1, \ldots, y_s$ around k_0 such that $x_i(k_0) = y_j(k_0) = 0$ and $K_{a,H}$ is locally given by $y_1 = \ldots = y_s = 0$. The phase function $F_{a',H'}$ becomes a function of $x_1, \ldots, x_q, y_1, \ldots, y_s$; hence, as a function of $y_1, \ldots, y_s$, it has $y_1 = \ldots = y_s = 0$ as a nondegenerate critical point when $x_1 = \ldots = x_q = 0$, $a' = a$, $H' = H$. So, treating $x_1, \ldots, x_q, a', H'$ as parameters, using the Morse lemma with parameters and applying the method of stationary phase to the integral over $y_1, \ldots, y_s$, we get the following

PROPOSITION 9.2: *Fix $a \in A$, $H \in \mathfrak{a}$. Then we can find a neighborhood Θ of (a, H) in $A \times \mathfrak{a}$, and a continuous seminorm ν on $C^\infty(K)$ such that for all $(a', H') \in \Theta$, $g \in C^\infty(K)$, and $\tau \geq 1$,*

$$\left| \int_K e^{i\tau\langle H(a'k), H' \rangle} g(k)\, dk \right| \leq \nu(g) \sum_{\mathfrak{w}_a \backslash \mathfrak{w} / \mathfrak{w}_H} \tau^{-\frac{1}{2}n_w}. \tag{9.11}$$

As H' varies around H, then the $\tau H'$, $\tau \geq 1$ fill up a conical neighborhood of H minus some compact subset. We then obtain the following corollary.

COROLLARY 9.3: *Fix $a \in A$, $H \in \mathfrak{a}$. Then we can find a neighborhood ω of a in A, a conical neighborhood Γ of H in $\mathfrak{a}$, and a continuous seminorm ν on $C^\infty(K)$ such that for all $a' \in \omega$, $H' \in \Gamma$, $g \in C^\infty(K)$*

$$\left| \int_K e^{i\langle H(a'k), H'\rangle} g(k)\, dk \right| \leq \nu(g) \sum_{\mathfrak{w}_a \backslash \mathfrak{w} / \mathfrak{w}_H} \prod_{\alpha \in \Delta_w^+} (1 + |\alpha(H')|)^{-\frac{1}{2}n(\alpha)}.$$

These estimates are sharp as long as a', H' are restricted to the "equisingular set" $A'_a \times \mathfrak{a}'_H$ mentioned in Theorem 9.1, but give the wrong (too high) growth order along other rays for which $K_{a'}\mathfrak{w}K_{H'}$ is smaller.

In the terminology of high-frequency optics, the set of parameters ε where the growth order of the oscillatory integral $\int e^{i\tau f(\varepsilon, x)} g(x)\, dx$ is larger (less negative) than at the generic neighboring point, is called the *caustic set*, see for instance [19, 1.6]. In our case $\varepsilon = (a, H)$ or, if a is fixed, $\varepsilon = H$, and the caustic set is a union of hyperplanes. When approaching the caustic set, the growth order in τ remains constant (equal to $-\frac{1}{2}\dim N$), but at least one of the top order coefficients goes to infinity, cf. (9.6). When reaching the caustic set the growth order in τ changes discontinuously to a less negative number, with new leading coefficients. This behaviour is repeated when entering the intersection of more root hyperplanes. In Section 11 we shall derive upper bounds which are uniform in all $H \in \mathfrak{a}$, with a in a compact subset of A. These uniform bounds have a rather simple form but at the same time are sharp in the sense that at least for equisingular a they follow the absolute value near the caustics accurately modulo constant factors.

The situation at the caustics in our case is in striking contrast with what happens for generic families $f(\varepsilon, .)$; for instance, for low dimensions of the parameter space, the caustics are the elementary catastrophes of Thom and the asymptotics is locally uniform in terms of new classes of special functions of Airy type. Also the behaviour of the critical set $K_X\mathfrak{w}K_H$ ($a = \exp X$) of the phase function $F_{a,H}$ in its dependence on the parameters a, H is highly nongeneric, as has already been observed in section 1.

10. Testing the distributions T_γ (γ semisimple)

Fix a semisimple element $\gamma \in G$. The results of section 8 enable us to obtain the asymptotics of the integral

$$\int_{C_\gamma} e^{i\tau\langle H(x), H\rangle} g(x)\, dC_\gamma(x) \qquad (g \in C_c^\infty(G))$$

as $\tau \to +\infty$, dC_γ being an invariant measure on C_γ. We shall normalize it as follows. Let $\langle .,.\rangle$ be the left invariant pseudo-Riemannian metric on G such that it coincides with the Killing form $\langle .,.\rangle$ at 1. On any semisimple conjugacy class C this induces a nondegenerate pseudo-Riemannian metric $\langle .,.\rangle_C$. The measure dC_γ is the measure defined by $\langle .,.\rangle_{C_\gamma}$. Since $\langle .,.\rangle_G$ is invariant under inner automorphisms of G, $\langle .,.\rangle_{C_\gamma}$, and hence dC_γ, are G-invariant. For any $H \in \mathfrak{a}$, we see from Lemma 8.6 that the manifold $C_{H,\gamma} = G_H \cap C_\gamma$ inherits a pseudo-Riemannian metric from $\langle .,.\rangle_G$; the corresponding G_H-invariant measure on $C_{H,\gamma}$ is denoted by $dC_{H,\gamma}$. With notation as in section 8 we then have the following theorem.

THEOREM 10.1: *Let $H \in \mathfrak{a}$, and let x_j ($1 \le j \le m = m(H,\gamma)$) be a complete set of representatives for the G_H-conjugacy classes in $C_{H,\gamma}$. Then we have the asymptotic expansion, as $\tau \to +\infty$, of distributions*

$$\int_{C_\gamma} e^{i\tau\langle H(x), H\rangle} g(x)\, dC_\gamma(x) \sim \sum_{1 \le j \le m} e^{i\tau\langle H(x_j), H\rangle} \sum_{l=0}^{\infty} c_{j,l,H}(g)\tau^{-\frac{1}{2}d-l}. \quad (10.1)$$

Here the $c_{j,l,H}$ are distributions on the manifold C_γ, of order $\le 2l$, with support contained in $C_{H,\gamma,j}$ = the G_H-conjugacy class of x_j; $d = d_{H,\gamma}$ is the common value of $\dim(\mathfrak{q}(x))$ *as x varies in $C_{H,\gamma}$; and the top term $c_{j,0,H}$ is given for all $g \in C_c^\infty(G)$ by*

$$c_{j,0,H}(g) = \\ = e^{i\sigma_j\frac{\pi}{4}} B_{H,\gamma} \int_{C_{H,\gamma,j}} \exp\Big(\tfrac{1}{2} \sum_{\alpha\in\Delta^+,\, \alpha(H)\neq 0} n(\alpha)\alpha(H(x))\Big) g(x)\, dC_{H,\gamma}(x) \quad (10.2)$$

where $\sigma_j = \operatorname{sgn}(Q_{x_j,H,\gamma})$ and $B_{H,\gamma}$ is the common value of $(2\pi)^{d/2} |\det(\operatorname{ad} H \circ (I - \operatorname{Ad}(x^{-1}))^{-1})_{\mathfrak{q}(x)}|^{-\frac{1}{2}}$ for x in $C_{H,\gamma}$ (cf. last remark in section 8). The distributions $c_{j,l,H}$ depend smoothly on H, and the error estimates are locally uniform in H, as long as H varies in such a way as to keep G_H fixed. With H' varying in some full neighborhood of H in $\mathfrak{a}$ we *have a uniform estimate of the form*

$$\left|\int_{C_\gamma} e^{i\tau\langle H(x),H'\rangle} g(x)\,dC_\gamma(x)\right| \le \nu(g)(1+\tau)^{-d/2}$$

for all $\tau \geq 0$, ν *being a continuous seminorm on* $C_c^\infty(G)$. *Finally, if* $H \in Cl(\mathfrak{a}^+)$, *the signature* $\sigma_j = 0$ *for all* $j = 1, \ldots, m$.

The $\mathfrak{w}$-invariant distribution T_γ on A is defined by the formula

$$\langle T_\gamma, f\rangle = \int_{G/G_\gamma} (\mathscr{A}^{-1}f)(x\gamma x^{-1})\,d\bar{x} \qquad (f \in C_c^\infty(A)^{\mathfrak{w}}) \tag{10.3}$$

where $\mathscr{A}^{-1}$ is the inverse of the Abel transform on G, $\mathscr{A}: C_c^\infty(G//K) \simeq C_c^\infty(A)^{\mathfrak{w}}$, and $d\bar{x}$ is the standard measure on G/G_γ (cf. DKV, Section 4). We can rewrite this as

$$\langle T_\gamma, f\rangle = c(G,\gamma)\int_{C_\gamma} (\mathscr{A}^{-1}f)(x)\,dC_\gamma(x) \quad (f \in C_c^\infty(A)^{\mathfrak{w}}) \tag{10.4}$$

where $c(G,\gamma)$ is a constant >0 independent of f. One also knows that if $b > 0$, $U = \{a \in A \mid \|\log a\| \leq b\}$, and $V = KUK$, then one has $\operatorname{supp}(\mathscr{A}^{-1}f) \subset V$ for all $f \in C_c^\infty(A)^{\mathfrak{w}}$ with $\operatorname{supp}(f) \subset U$. Fix U, V as above and let $\psi \in C_c^\infty(G//K)$ be such that $\psi = 1$ on V. Then, using the theorems of Helgason (*), Gangolli and Harish–Chandra, we find,

$$\langle T_\gamma, f\rangle = c(G,\gamma)|\mathfrak{w}|^{-1} \times$$
$$\int_{C_\gamma} \psi(x) \times \left(\int_{\mathscr{F}_R} \hat{f}(\lambda)\phi_{-i\lambda}(x)\beta(i\lambda)\,d\lambda\right) dC_\gamma(x) \tag{10.5}$$

where $\hat{f}(\lambda) = \int_A f e^{i\lambda}\,da$; this is true even for not necessarily $\mathfrak{w}$-invariant f in $C_c^\infty(U)$. To test the singularities of T_γ, we replace f by $fe^{i\tau\xi}$, with $\xi \in \mathscr{F}_R$ and let $\tau \to +\infty$. We get

$$\langle T_\gamma, fe^{i\tau\xi}\rangle = c(G,\gamma)|\mathfrak{w}|^{-1} \times$$
$$\int_{C_\gamma} \psi(x) \times \left(\int_{\mathscr{F}_R} \hat{f}(\tau\xi + \lambda)\phi_{-i\lambda}(x)\beta(i\lambda)\,d\lambda\right) dC_\gamma(x)$$
$$= c(G,\gamma)|\mathfrak{w}|^{-1} \times$$
$$\int_{C_\gamma} \psi(x) \times \left(\int_{\mathscr{F}_R} \hat{f}(-\lambda)\phi_{i(\tau\xi+\lambda)}(x)\beta(i(\tau\xi + \lambda))\,d\lambda\right) dC_\gamma(x).$$

* As is also clear from Gangolli [23], this Paley–Wiener theorem originates from Helgason's article in *Math. Ann.* 165 (1966) 297–308.

We rewrite this as

$$\begin{aligned}\langle T_\gamma, fe^{i\tau\xi}\rangle &= c(G,\gamma)|\mathfrak{w}|^{-1}\int_{\mathscr{F}_R}\hat{f}(-\lambda)\beta(i(\tau\xi+\lambda))\\ &\times\left(\int_{C_\gamma}\psi(x)\phi_{i(\tau\xi+\lambda)}(x)\,dC_\gamma(x)\right)d\lambda.\end{aligned} \tag{10.6}$$

The integrals appearing in the right side of (10.6) with respect to $dC_\gamma(x)$ can be simplified. We have, for all $\nu\in\mathscr{F}$, $g\in C_c^\infty(G)$,

$$\int_{C_\gamma} g(x)\phi_\nu(x)\,dC_\gamma(x) = \int_{C_\gamma}\bar{g}(x)e^{(\nu-\rho)(H(x))}\,dC_\gamma(x) \tag{10.7}$$

where $\bar{g}(x) = \int_K g(kxk^{-1})\,dk$. In fact,

$$\begin{aligned}\int_{C_\gamma} g(x)\phi_\nu(x)\,dC_\gamma(x) &= \int_{C_\gamma}\int_K g(x)e^{(\nu-\rho)(H(xk))}\,dk\,dC_\gamma(x)\\ &= \int_{C_\gamma}\int_K g(x)e^{(\nu-\rho)(H(k^{-1}xk))}\,dk\,dC_\gamma(x)\\ &= \int_{C_\gamma}\int_K g(kxk^{-1})e^{(\nu-\rho)(H(x))}\,dk\,dC_\gamma(x)\\ &= \int_{C_\gamma}\bar{g}(x)e^{(\nu-\rho)(H(x))}\,dC_\gamma(x).\end{aligned}$$

Hence

$$\begin{aligned}&\int_{C_\gamma} g(x)\phi_{i(\tau\xi+\lambda)}(x)\,dC_\gamma(x)\\ &\quad= \int_{C_\gamma} e^{i\tau\xi(H(x))}\bar{g}(x)e^{(i\lambda-\rho)(H(x))}\,dC_\gamma(x).\end{aligned} \tag{10.8}$$

Theorem 10.1 can now be applied to develop the asymptotic expansion of the right side of (10.8). With notations as before we have the following theorem.

THEOREM 10.2: *We have the asymptotic expansion, as $\tau\to+\infty$, of distributions*

$$\begin{aligned}&\int_{C_\gamma}\phi_{i(\tau\xi+\lambda)}(x)g(x)\,dC_\gamma(x)\sim\\ &\sim\sum_{1\le j\le m} e^{i\tau\xi(H(x_j))}\sum_{l=0}^{\infty} c'_{j,l,\xi,\lambda}(g)\tau^{-\frac{1}{2}d-l}.\end{aligned} \tag{10.9}$$

Here the $c'_{j,l,\xi,\lambda}$ are distributions on C_γ, of order $\leq 2l$, supported by $C_{H,\gamma,j}$, given for $g \in C_c^\infty(G)$ by

$$c'_{j,l,\xi,\lambda}(g) = c_{j,l,H_\xi}(\bar{g}\,e^{(i\lambda-\rho)\circ H}). \tag{10.10}$$

In particular, they depend smoothly on ξ, λ as long as ξ varies in $\mathscr{F}_R$ without changing G_{H_ξ}, the error estimates and coefficients being locally uniform in ξ and of polynomial growth at most in λ. Finally, there is a neighborhood Θ of ξ in $\mathscr{F}_R$, an integer $R \geq 0$ and a continuous seminorm ν on $C_c^\infty(G)$, such that for all $\xi' \in \Theta$, $\lambda \in \mathscr{F}_R$, $g \in C_c^\infty(G)$, $\tau \geq 0$,

$$\begin{aligned}&\left|\int_{C_\gamma} \phi_{i(\tau\xi'+\lambda)}(x) g(x)\, dC_\gamma(x)\right| \\ &\qquad \leq \nu(g)(1+\|\lambda\|)^R (1+\tau)^{-d/2}.\end{aligned} \tag{10.11}$$

COROLLARY 10.3: *The top coefficient $c'_{j,0}$ is given for all $\lambda \in \mathscr{F}_R$, $g \in C_c^\infty(G)$, writing $B_{H_\xi,\gamma} = B_{\xi,\gamma}$, $C_{H_\xi,\gamma,j} = C_{\xi,\gamma,j}$, by*

$$c'_{j,0,\xi,\lambda}(g) = e^{i\sigma_j\pi/4} B_{\xi,\gamma} \int_{C_{\xi,\gamma,j}} e^{(i\lambda-\rho_\xi)(H(x))}\bar{g}(x)\, dC_{\xi,\gamma,j}(x), \tag{10.12}$$

where $\rho_\xi = \frac{1}{2}\sum_{\alpha\in\Delta^+,\langle\xi,\alpha\rangle=0} n(\alpha)\alpha$. If $\xi \in Cl(\mathscr{F}_R^+)$, then $\sigma_j = 0$.

The above results may be used to obtain an asymptotic expansion of $\langle T_\gamma, fe^{i\tau\xi}\rangle$ when $\tau \to +\infty$. Since T_γ is $\mathfrak{w}$-invariant we may suppose that $\xi \in Cl(\mathscr{F}_R^+)$. Theorem 10.2 gives the asymptotics of the inner integrals in (10.6), and so it remains to develop the asymptotics of $\beta(i(\tau\xi+\lambda))$. Let us fix $\xi_0 \in Cl(\mathscr{F}_R^+)$, let Δ^{++} be the set of positive short roots (DKV, Section 3.8), and let

$$\Delta_0^{++} = \{\alpha \in \Delta^{++} \mid \langle\alpha,\xi_0\rangle > 0\}.$$

If L is the subspace of $\mathscr{F}_R$ of all ξ with $\langle\alpha,\xi\rangle = 0\,\forall\,\alpha\in\Delta_0^{++}$ and L' is the subset of L where $\langle\alpha,\xi\rangle \neq 0$ for $\alpha \notin \Delta_0^{++}$, L' is an open neighborhood of ξ_0 in L. By an equisingular neighborhood of ξ_0 in $Cl(\mathscr{F}_R^+)$ we mean a neighborhood of ξ_0 in $L' \cap Cl(\mathscr{F}_R^+)$. We fix such a compact neighborhood Θ. For $\xi \in \Theta$, $G_\xi = G_{\xi_0}$, if we write for convenience $G_\xi = G_{H_\xi}$.

We begin by noting that the product structure of β (cf. DKV, Section 3.8) gives

$$\beta(\nu) = I(\rho)^2 \prod_{\alpha\in\Delta^{++}} f_\alpha(\langle\alpha,\nu\rangle) \qquad (\nu \in \mathscr{F}_I) \tag{10.13}$$

where, for $z \in (-1)^{\frac{1}{2}}\mathbb{R}$ and $|z| \to +\infty$ we have the asymptotic expansion

$$f_\alpha(z) \sim 2^{-n(2\alpha)}|z|^{d(\alpha)}(1 + \sum_{k=1}^{\infty} c_{\alpha,k}|z|^{-2k}). \tag{10.14}$$

In particular, for all $\lambda \in \mathscr{F}_R$, with $c = I(\rho)^2 I(\rho_\xi)^{-2}$, $\xi \in \Theta$

$$\beta(i(\tau\xi + \lambda)) = c\beta_\xi(i\lambda) \prod_{\alpha \in \Delta_0^{++}} f_\alpha(i\langle\alpha, \tau\xi + \lambda\rangle). \tag{10.15}$$

Here β_ξ is the counterpart of β for the group G_ξ. We shall now obtain, for the product in the right side of (10.15) an asymptotic expansion when $\tau \to +\infty$ with coefficients and error terms that are of polynomial growth in λ and uniform in ξ when ξ varies in Θ. We shall do this however only for $\tau > 0$, $\xi \in \Theta$ and $\lambda \in \mathscr{F}_R$ varying in such a way that

$$\frac{|\langle\alpha, \lambda\rangle|}{\tau\langle\alpha, \xi\rangle} < 1 \text{ and } |\langle\alpha, \tau\xi + \lambda\rangle| \geq \tau\delta \text{ for all } \alpha \in \Delta_0^{++}, \tag{10.16}$$

where

$$\delta = \tfrac{1}{2} \min_{\alpha \in \Delta_0^{++}, \xi \in \Theta} \langle\alpha, \xi\rangle > 0. \tag{10.17}$$

The restricted regime (10.16) is perfectly adequate for our purposes as we shall see presently.

Now, if $z \in \mathbb{C}$ and $|z| < 1$, we have, for any integer $N \geq 1$,

$$(1 - z)^{-1} = \sum_{n=0}^{N-1} z^n + (1 - z)^{-1} z^N$$

So, if $\alpha \in \Delta_0^{++}$ and τ, ξ, λ satisfy (10.16), we have, taking $z = -\langle\alpha, \lambda\rangle/\tau\langle\alpha, \xi\rangle$,

$$\begin{aligned}\langle\alpha, \tau\xi + \lambda\rangle^{-1} &= \sum_{n=0}^{N-1} \frac{(-\langle\alpha, \lambda\rangle)^n}{\langle\alpha, \xi\rangle^{n+1}} \tau^{-(n+1)} + \\ &\quad + \langle\alpha, \tau\xi + \lambda\rangle^{-1} \left(\frac{-\langle\alpha, \lambda\rangle}{\langle\alpha, \xi\rangle}\right)^N \tau^{-N}.\end{aligned} \tag{10.18}$$

Assuming (10.16), the coefficient of $\tau^{-(n+1)}$ is majorized by $(2\delta)^{-(n+1)}\|\alpha\|^n \|\lambda\|^n$ while the error term, which is the last one appearing on the right side of (10.18), is majorized by $\delta^{-1}(2\delta)^{-N}\|\alpha\|^N \|\lambda\|^N \tau^{-(N+1)}$. Further $|\langle\alpha, \tau\xi + \lambda\rangle|^{2k} = \langle\alpha, \tau\xi + \lambda\rangle^{2k}$ since everything is real. Hence, raising (10.18) to the $2k$-th power, we get the asymptotic expansion

$$|\langle\alpha, \tau\xi + \lambda\rangle|^{-2k} \sim \sum_{n=0}^{\infty} h'_{k,n}(\alpha, \xi, \lambda)\tau^{-(n+2k)} \tag{10.19}$$

where the coefficients $h'_{k,n}$ and the error terms are of polynomial growth in λ, uniformly for $\xi \in \Theta$, everything being valid under the assumption of (10.16). Moreover, (10.16) also gives

$$|\langle\alpha, \tau\xi + \lambda\rangle| = \tau\langle\alpha, \xi\rangle \left|1 + \frac{\langle\alpha, \lambda\rangle}{\tau\langle\alpha, \xi\rangle}\right| = \tau\langle\alpha, \xi\rangle\left(1 + \frac{\langle\alpha, \lambda\rangle}{\tau\langle\alpha, \xi\rangle}\right).$$

Hence

$$f_\alpha(i\langle\alpha, \tau\xi + \lambda\rangle) \sim 2^{-n(2\alpha)}\langle\alpha, \xi\rangle^{d(\alpha)}\tau^{d(\alpha)}(1 + \sum_{n=1}^{\infty} h_n(\alpha, \xi, \lambda)\tau^{-n}) \tag{10.20}$$

where the coefficients h_n and the error terms are of polynomial growth in λ, uniformly for $\xi \in \Theta$. This leads to the asymptotic expansion

$$\begin{aligned}\beta(i(\tau\xi + \lambda)) \sim b\beta_\xi(i\lambda)\tau^{\dim(\mathfrak{n}/\mathfrak{n}_\xi)} \prod_{\alpha\in\Delta_0^{++}} \langle\alpha, \xi\rangle^{d(\alpha)} \times \\ \times \{1 + \sum_{n=1}^{\infty} h_n(\xi, \lambda)\tau^{-n}\}.\end{aligned} \tag{10.21}$$

Here

$$b = I(\rho)^2 I(\rho_{\xi_0})^{-2} \prod_{\alpha\in\Delta_0^{++}} 2^{-n(2\alpha)}; \tag{10.22}$$

moreover, the coefficients h_n and the error terms are of polynomial growth in λ, uniformly for $\xi \in \Theta$, everything being valid under (10.16).

To obtain the asymptotic expansion for $\langle T_\gamma, fe^{i\tau\xi}\rangle$, let $\Omega(\xi, \tau)$ be the subset of $\mathscr{F}_R$ described by (10.16); its complement in $\mathscr{F}_R$ can be written as a disjoint union of measurable sets $\Omega_\alpha(\xi, \tau)$, $\Omega'_\alpha(\xi, \tau)$ $(\alpha \in \Delta_0^{++})$ where

$$\lambda \in \Omega_\alpha(\xi, \tau) \Rightarrow |\langle\alpha, \tau\xi + \lambda\rangle| < \tau\delta,$$
$$\lambda \in \Omega'_\alpha(\xi, \tau) \Rightarrow |\langle\alpha, \lambda\rangle| \geq \tau\langle\alpha, \xi\rangle.$$

If $\lambda \in \Omega_\alpha(\xi, \tau)$, we get, using (10.17), $\tau\delta > \tau\langle\alpha, \xi\rangle - |\langle\alpha, \lambda\rangle| \geq 2\tau\delta - |\langle\alpha, \lambda\rangle|$ so that $|\langle\alpha, \lambda\rangle| \geq \tau\delta$. Clearly, this is also true for $\lambda \in \Omega'_\alpha(\xi, \tau)$.

Hence there is a constant $a > 0$ such that

$$\|\lambda\| \geq a\tau \quad \forall\, \lambda \in \mathscr{F}_R \backslash \Omega(\xi, \tau), \qquad \xi \in \Theta. \tag{10.23}$$

It is immediate from (10.23) that if ϕ is a continuous function of λ which is $O(\|\lambda\|^{-m})$ as $\|\lambda\| \to \infty$ for each $m \geq 1$, then, as $\tau \to +\infty$,

$$\int_{\mathscr{F}_R \backslash \Omega(\xi,\tau)} |\phi(\lambda)|\, d\lambda = O(\tau^{-m}) \tag{10.24}$$

for each $m \geq 1$, uniformly for $\xi \in \Theta$. We now substitute for the inner integral in (10.6) from Theorem 10.2, split the integral with respect to λ as a sum of integrals over $\Omega(\xi, \tau)$ and $\mathscr{F}_R \backslash \Omega(\xi, \tau)$, and use the expansion (10.21) for $\beta(i(\tau\xi + \lambda))$ on $\Omega(\xi, \tau)$. We then obtain the following theorem. Notation is as in Theorem 10.1.

THEOREM 10.4: *Let* $\xi \in Cl(\mathscr{F}_R^+)$. *Write* $\mathfrak{n}_+ = \sum_{\langle \alpha, \xi \rangle > 0} \mathfrak{g}_\alpha$, $\mathfrak{n}_{+,1} = \mathfrak{n}_+ \cap \mathfrak{g}_{x_1}$. *Then, we have the following asymptotic expansion for* $\langle T_\gamma, fe^{i\tau\xi} \rangle$ *as* $\tau \to +\infty$ $(f \in C_c^\infty(U))$.

$$\begin{aligned} \langle T_\gamma, fe^{i\tau\xi} \rangle \sim\ & \tau^{\dim(\mathfrak{n}_{+,1})} |\det(\operatorname{ad} H_\xi)_{\mathfrak{n}_{+,1}}| \times \\ & \times \sum_{1 \leq j \leq m} b_j e^{i\tau\xi(H(x_j))} \Big\{ \langle T_{x_j}^{G_\xi}, f \rangle + \sum_{n=1}^{\infty} \langle \theta_{j,n}^\xi, f \rangle \tau^{-n} \Big\}. \end{aligned} \tag{10.25}$$

Here b_j are constants > 0 depending on G, γ, x_j, and on ξ only through G_ξ; $T_{x_j}^{G_\xi}$ are distributions on A which are the counterparts in G_ξ of the T_γ; $\theta_{j,n}^\xi$ are distributions on A; and the error terms are majorized by seminorms in f locally uniformly in ξ as long as ξ varies in an equisingular manner.

To get the exponent of τ in the above formula we note that this exponent is

$$\dim(\mathfrak{n}) - \dim(\mathfrak{n}_\xi) - \tfrac{1}{2}\dim(\mathfrak{q}(x))$$

for any $x \in C_{\xi,\gamma}$, say $x = x_1$; it then simplifies to $\dim(\mathfrak{n}_{+,1})$. The leading term appearing as the coefficient of $e^{i\tau\xi(H(x_j))}$ comes out as

$$b'c(G,\gamma)|\mathfrak{w}|^{-1} \int_{\mathscr{F}_R} \hat{f}(-\lambda)\beta_\xi(i\lambda) \left(\int_{C_{\xi,\gamma,j}} e^{(i\lambda - \rho_\xi)(H(x))} \psi(x)\, dC_{\xi,\gamma}(x) \right) d\lambda$$

with

$$b' = b(2\pi)^{d/2} |\det(I - \operatorname{Ad}(x_1^{-1}))_{\mathfrak{q}(x_1)}|^{\frac{1}{2}}.$$

On the other hand, applying (10.6) and (10.8) to the group G_ξ with $\tau = 0$ we get,

$$\langle T_{x_j}^{G_\xi}, f\rangle = c(G_\xi, x_j)|\mathfrak{w}_\xi|^{-1}\int_{\mathscr{F}_R}\hat{f}(-\lambda)\beta_\xi(i\lambda)\times \times\left(\int_{C_{\xi,\gamma,j}} e^{(i\lambda-\rho_\xi)(H(x))}\psi(x)\,dC_{\xi,\gamma}(x)\right)d\lambda. \tag{10.26}$$

Hence, we get the desired expression for the leading term with

$$b_j = \frac{c(G,\gamma)}{c(G_\xi, x_j)}\frac{|\mathfrak{w}_\xi|}{|\mathfrak{w}|}b(2\pi)^{d/2}|\det(I - \mathrm{Ad}(x_1^{-1}))_{\mathfrak{q}(x_1)}|^{\frac{1}{2}}.$$

Remarks: Theorem 10.4 can be regarded as a testing of the singularities of T_γ. It is interesting to compare this with the description of the T_γ in DKV, Section 4.5. Since there is only a finite number of possibilities for the G_ξ as ξ varies in $\mathfrak{a}^*$, only finitely many conjugates $h = x_j$ of γ have to be considered. Each of them can be put in a θ-stable Cartan subgroup L of G such that $\mathfrak{l}_R \subset \mathfrak{a}$. Writing $^*\mathfrak{l}$ for the orthogonal complement of $\mathfrak{l}_R$ in $\mathfrak{a}$, the formula (4.27) in DKV expresses that T_γ is equal to a sum of transversal derivatives of some distributions in the affine subspace $^*\mathfrak{l} + \log h_R$ averaged with respect to $\mathfrak{w}$. From Theorem 10.4 we now read off that $\langle T_\gamma, fe^{i\tau\xi}\rangle$ is not rapidly decreasing as $\tau \to \infty$ (locally uniformly in ξ), only if $H_\xi \in \mathrm{Ker}(\mathrm{Ad}\,h - I) \cap \mathfrak{a}$ for some of the h. In the terminology of Hörmander [36, 2.5] the wave front set of T_γ therefore is contained in the union of the finitely many affine subspaces of $\mathfrak{a} \times \mathfrak{a}$ which are equal to the Cartesian product of $^*\mathfrak{l} + \log h_R$ with $\mathrm{Ker}(\mathrm{Ad}\,h - I) \cap \mathfrak{a}$. Here we use the identification of $\mathfrak{a}^*$ with $\mathfrak{a}$ via the Killing form.

Now always $\mathrm{Ker}(\mathrm{Ad}\,h - I) \cap \mathfrak{a} = \mathrm{Ker}(\mathrm{Ad}\,h_I - I) \cap \mathfrak{a} \supset \mathfrak{l}_R$. If equality holds, then the distributions on $^*\mathfrak{l} + \log h_R$ (of which transversal derivatives were taken) actually are smooth densities on $^*\mathfrak{l} + \log h_R$. Because $\mathrm{Ker}(\mathrm{Ad}\,h - I) \cap \mathfrak{a} = \mathfrak{l}_R$ if h is regular, we thus find back Theorem 4.12 of DKV, but the result of Theorem 10.4 here is sharper; it establishes also non-smoothness and quite detailed further information about the distributions in $^*\mathfrak{l} + \log h_R$ if $\mathrm{Ker}(\mathrm{Ad}\,h - I) \cap \mathfrak{a} \supsetneqq \mathfrak{l}_R$. Note that if $\mathrm{Ker}(\mathrm{Ad}\,h - - I) \cap \mathfrak{a} = \mathfrak{l}_R$, then the number of transversal derivatives apparently is equal to $\dim(\mathfrak{n} \cap \mathfrak{g}_{\mathfrak{l}_R})$. Also the polynomial dependence on H_ξ of the coefficients in (10.25) (viz. $\det(\mathrm{ad}\,H_\xi)_{\mathfrak{n}_{+,1}}$) reflects the differentiations appearing in the description of the T_γ in DKV (4.27).

We leave the further explicating as an exercise. This will lead to a quite detailed description of the singularities of the T_γ, in awaiting of the full explicit formula for the Fourier transform of the T_γ, which we think in principle is obtainable. Cf. Sally and Warner [50], Herb [34].

11. Uniform estimates for matrix coefficients with $\lambda \in \mathscr{F}_I$, $\|\lambda\| \to \infty$, and a bounded

As observed at the end of section 9, it is not possible to get sharp estimates for the integral

$$I_{a,H}(g) = \int_K \exp(iF_{a,H}(k))g(k)\,dk, \tag{11.1}$$

uniform for $H \in \mathfrak{a}$, a bounded, by simply rewriting it as

$$I_{a,\tilde{H},\tau}(g) = I_{a,\tau\tilde{H}}(g) = \int_K \exp(i\tau F_{a,\tilde{H}}(k))g(k)\,dk, \tag{11.2}$$

letting $\tau \to \infty$ and treating $\tilde{H}$ as a parameter on the unit sphere in $\mathfrak{a}$. The problem is caused by the singular asymptotic behaviour for $\tau \to \infty$ of (11.2) if $\tilde{H}$ is in a root hyperplane; with increasing order of asymptotics if more roots vanish at $\tilde{H}$. This originates from the sudden change in dimension of the critical set of $F_{a,\tilde{H}}$ as $\tilde{H}$ enters these (intersections of) root hyperplanes. On the other hand the geometric simplicity of the caustic set and the remarkable rigidity of the critical sets encourage one to hope for uniform upper bounds for (11.1) in terms of much simpler functions than the Airy-type of functions; these would be needed in the generic case (cf. Duistermaat [19]). Making essential use of suitable right invariance properties of the $F_{a,H}$ in order to get the estimates locally uniform in a, we can prove:

THEOREM 11.1: *Fix a compact subset ω of A. For any $w \in \mathfrak{w}$, let*

$$\Delta_w^+(\omega) = \{\alpha \in \Delta^+ \mid w\alpha(\log a) \neq 0 \text{ for all } a \in \omega\}. \tag{11.3}$$

Then we can find a C^m-norm v on $C^m(K)$, $m = \frac{1}{2}\dim N$, such that for all $a \in \omega$, $g \in C^m(K)$ and all $H \in \mathfrak{a}$,

$$\left| \int_K e^{iF_{a,H}(k)} g(k)\,dk \right| \leq v(g) \sum_{w \in \mathfrak{w}} \prod_{\alpha \in \Delta_w^+(\omega)} (1 + |\alpha(H)|)^{-\frac{1}{2}n(\alpha)}. \tag{11.4}$$

If $\dim N$ is odd, then $g \in C^m$ if and only if $g \in C^{[m]}$ and the derivatives of g of order $\leq [m]$ are Hölder continuous of order $\frac{1}{2}$. C^m-norms are then defined in the obvious manner.

In view of the asymptotic expansions along rays which were obtained in section 9, the estimate (11.4) is sharp modulo constant factors if we let a vary in an equisingular compact subset of A. However, in this case an even more refined asymptotic expansion holds which is uniform in $H \in \mathfrak{a}$, as will be shown in section 12.

Let E be a finite-dimensional vector space and let $R = (R_1, R_2)$ be a double representation of K on E. For any $\lambda \in \mathscr{F}$, the complex dual of $\mathfrak{a}$, and any endomorphism T of E, the (R, T)*-spherical function with eigenvalue parameter* λ is defined as the matrix-valued function $\Phi = \Phi_\lambda^{R,T}$ on G given by (cf. Harish–Chandra [30])

$$\Phi_\lambda^{R,T}(x) = \int_K e^{(\lambda-\rho)(H(xk))} R_1(\kappa(xk)) \circ T \circ R_2(k)^{-1}\, dk. \tag{11.5}$$

Because $\Phi(k_1 x k_2) = R_1(k_1) \circ \Phi(x) \circ R_2(k_2)$ for all $k_1, k_2 \in K$, $x \in G$, Φ is determined by its values on A in view of the Cartan decomposition $G = KAK$.

COROLLARY 11.2: *For any compact subset ω of A there is a constant $C > 0$ such that for all $a \in \omega$, $\lambda = i\xi + \eta \in \mathscr{F}$ with $\xi, \eta \in \mathscr{F}_R$, we have*

$$\begin{aligned} &|\Phi_{i\xi+\eta}^{R,T}(a)| \leq C(1 + \|\eta\|)^m \\ &\max_{w', w \in \mathfrak{w}} [\exp(\eta(\log a^{w'})) \cdot \prod_{\alpha \in \Delta_w^+(\omega)} (1 + |\langle \xi, \alpha \rangle|)^{-\frac{1}{2}n(\alpha)}]. \end{aligned} \tag{11.6}$$

Apply Theorem 11.1 with $F_{a,H} = F_{a,H_\xi}$ and

$$g(k) = \exp(\eta - \rho)(H(ak)) R_1(\kappa(ak)) \circ T \circ R_2(k)^{-1}.$$

Use the relation

$$\sup_{k \in K} \exp \eta(H(ak)) = \max_{w' \in \mathfrak{w}} \exp \eta(\log a^{w'}), \tag{11.7}$$

which follows immediately from the determination of the critical set of F_{a,H_η} in Proposition 5.4, and observe that derivatives of order $\leq m$ applied to g lead to polynomial factors in η of order $\leq m$. So we obtain (11.6) as a consequence of (11.4).

The exponential factor $\max_{w' \in \mathfrak{w}} \exp \eta(\log a^{w'})$ in (11.6) is sharp. This can be seen by applying the method of steepest descent (cf. Erdélyi [22]) to the points where $k \mapsto \eta(H(ak))$ takes its maximum value. The same method should also lead to improvements in the polynomial factor $(1 + \|\eta\|)^m$. However, an extension of Theorem 11.1 to the case of complex H would lead to additional complications in its already long proof, so we settle for (11.4), (11.6).

The proof of Theorem 11.1 simplifies considerably if we restrict H even somewhat more, namely to the closure of the positive Weyl chamber in $\mathfrak{a}$. Indeed, for those H the function $F_{a,H}$ is always right K_H-invariant (see Proposition 5.6). For this reason we shall first give the proof for $H \in Cl(\mathfrak{a}^+)$; we indicate the modifications needed in the case of general $H \in \mathfrak{a}$ at the end of this section.

It may be noted that for the elementary spherical functions ϕ_λ (when $R_1 = R_2 = 1$ and $T = I$), Corollary 11.2 can be obtained already from the version of Theorem 11.1 for $H \in Cl(\mathfrak{a}^+)$, because $\phi_\lambda = \phi_{w\lambda}$, for any $w \in \mathfrak{w}$, and each $H \in \mathfrak{a}$ is in the w-image of $Cl(\mathfrak{a}^+)$ for some $w \in \mathfrak{w}$.

Since the proof of Theorem 11.1 in the case of $H \in Cl(\mathfrak{a}^+)$ depends strongly on the right K_H-invariance of the $F_{a,H}$, we start with a lemma concerning the subgroups K_H when H is varying over certain subsets of $\mathfrak{a}$.

LEMMA 11.3: *For any subset $B \subset \mathfrak{a}$ and any closed subgroup L of G (resp. subalgebra $\mathfrak{l}$ of $\mathfrak{g}$) let*

$$L_B = \bigcap_{H \in B} L_H, \text{ resp. } \mathfrak{l}_B = \bigcap_{H \in B} \mathfrak{l}_H. \tag{11.8}$$

Then, if $X \in \mathfrak{a}$ and B is a finite subset of $\mathfrak{a}$,

$$K_X \mathfrak{w} K_B = \bigcap_{H \in B} K_X \mathfrak{w} K_H \tag{11.9}$$

and the intersection is clean. If $B \subset C$ with C the closure of a Weyl chamber, and $H' = \sum_{H \in B} H$, then $G_B = G_{H'}$, $K_B = K_{H'}$.

Combining Proposition 1.2 and Lemma 1.1 we see that $k \in K_X \mathfrak{w} K_H \Leftrightarrow X^{k^{-1}} \in \mathfrak{g}_H$. So k belongs to the right hand side of (11.9) if and only if $X^{k^{-1}} \in \mathfrak{g}_B$. Replacing now G_H by G_B in the proof of Proposition 1.2 (first part), we get the equivalence of this with $k \in K_X \mathfrak{w} K_B$.

For the cleanness, let $k = u x_w v$ with $u \in K_X$, $w \in \mathfrak{w}$, $v \in K_B$. Setting $Y = w^{-1}X$ and using that $K_Y K_B$ is the $K_Y \times K_B$-orbit through 1, we have to prove that the $K_Y K_H$ $(H \in B)$ meet cleanly at 1, that is, $\bigcap_{H \in B} (\mathfrak{k}_Y + \mathfrak{k}_H) = \mathfrak{k}_Y + \mathfrak{k}_B$. But $\bigcap_{H \in B} (\mathfrak{g}_Y + \mathfrak{g}_H) = \mathfrak{g}_Y + \mathfrak{g}_B$ using root space decompositions; and the desired relation is obtained on intersection with $\mathfrak{k}$, noting that all subspaces in sight are stable under the Cartan involution θ.

For the last assertion we may take $B \subset Cl(\mathfrak{a}^+)$ because all Weyl chambers are conjugate to each other. Note that for any B, $G_B \subset G_{H'}$. Moreover, for $\alpha \in \Delta^+$, $\alpha(H') = 0 \Leftrightarrow \alpha(H) = 0$ for all $H \in B$, so that $\mathfrak{g}_B = \mathfrak{g}_{H'}$. But then $G_{H'} = G^0_{H'}M = G^0_B M \subset G_B$.

The proof of Theorem 11.1 starts with a series of reductions. The first one is that, using a partition of unity for g, it is sufficient to prove Theorem 11.1 in a local form, that is to prove

PROPOSITION 11.4: *Fix $a_0 \in A$, $k_0 \in K$. Then there exist neighborhoods A_0 of a_0 in A, resp. V of k_0 in K and a C^m-norm ν on $C^m(K)$, such that for all $a \in A_0$, $g \in C^m_c(V)$, $H \in Cl(\mathfrak{a}^+)$ the estimate* (11.4) *holds.*

The next reduction consists in showing that instead of considering arbitrary $H \in Cl(\mathfrak{a}^+)$, we may restrict ourselves to H in certain conical subsets $C(\pi, \mu, \gamma)$ of $Cl(\mathfrak{a}^+)$, defined as follows.

Let S be the set of simple roots in Δ^+. Because S is a basis of $\mathfrak{a}^*$, its number of elements is equal to $r = \dim \mathfrak{a} =$ real rank G. Let Π be the set of all ordered partitions $\pi = (S_1, \ldots, S_s)$ of S into disjoint nonempty subsets S_j. Let μ, γ be positive real numbers, $\mu \geq 1$. Then we say $H \in C(\pi, \mu, \gamma)$ if and only if

$$\begin{aligned} &\text{(i) } H \in Cl(\mathfrak{a}^+), \text{ that is } \alpha(H) \geq 0 \text{ for all } \alpha \in S; \\ &\text{(ii) if } \alpha, \beta \in S_j \text{ for some } j, \text{ then } \alpha(H) \leq \mu\beta(H); \\ &\text{(iii) if } \alpha \in S_j,\ \beta \in S_k, \text{ and } j < k, \text{ then } \alpha(H) \geq \gamma\beta(H). \end{aligned} \tag{11.10}$$

In the applications γ will usually be large. This means that the coordinates $\alpha(H)$ of $H \in C(\pi, \mu, \gamma)$ with respect to the dual basis $S^\vee$ can be grouped in such a way that the coordinates in the same group are of comparable size, while the coordinates in a given group are much larger than the coordinates in the next group.

The next combinatorial lemma shows that, if, for every $\pi \in \Pi$ and $\mu \geq 1$, we can prove Proposition 11.4 with $H \in Cl(\mathfrak{a}^+)$ replaced by $H \in C(\pi, \mu, \gamma)$, for some γ no matter how large, then Proposition 11.4 itself is true.

LEMMA 11.5: *Suppose for each $\pi \in \Pi$ and $\mu \geq 1$ we are given a number $\gamma(\pi, \mu) > 0$. Then there is a mapping $\pi \mapsto \mu(\pi)$ with $\mu(\pi) \geq 1$ such that*

$$Cl(\mathfrak{a}^+) = \bigcup_{\pi \in \Pi} C(\pi, \mu(\pi), \gamma(\pi, \mu(\pi))). \tag{11.11}$$

For any $j \in \{0, \ldots, r-1\}$ let Π_j be the subset of Π consisting of all partitions of S into $r-j$ subsets. We set $\mu_0 = 1$, $\gamma_{-1} = 1$ and define by induction over j the numbers μ_{j+1} and γ_j by

$$\gamma_j = \max\{\max_{\pi \in \Pi_j} \gamma(\pi, \mu_j), \gamma_{j-1}, 1\}, \tag{11.12}$$

$$\mu_{j+1} = (\gamma_j)^{j+1}. \tag{11.13}$$

Note that γ_j and μ_j are ≥ 1 and that

$$C(\pi, \mu_j, \gamma_j) \subset C(\pi, \mu_j, \gamma(\pi, \mu_j)) \text{ for } \pi \in \Pi_j. \tag{11.14}$$

Hence it suffices to prove that

$$Cl(\mathfrak{a}^+) = \bigcup_{0 \leq j \leq r-1} \bigcup_{\pi \in \Pi_j} C(\pi, \mu_j, \gamma_j). \tag{11.15}$$

In fact we shall prove by induction on $j = 0, \ldots, r-1$ that

$$\bigcup_{0 \leq k \leq j} \bigcup_{\pi \in \Pi_k} C(\pi, \mu_k, \gamma_k) \supset C_j \tag{11.16}$$

where

$$C_j = \bigcup_{\pi \in \Pi_j} C(\pi, \infty, \gamma_j). \tag{11.17}$$

This would prove what we want since $C_{r-1} = Cl(\mathfrak{a}^+)$.

(11.16) is trivial for $j = 0$. Let $H \in C_j$. Then $H \in C(\pi, \infty, \gamma_j)$ for some $\pi \in \Pi_j$; write $\pi = (S_1, \ldots, S_{r-j})$. If we can subdivide S_i into disjoint non-void subsets A, B such that $\alpha \in A$, $\beta \in B \Rightarrow \alpha(H) \geq \gamma_{j-1}\beta(H)$, then $H \in C_{j-1}$. Therefore, by induction hypothesis, H is contained in the left hand side of (11.16), even with j replaced by $j-1$. So we may assume that no such subdivision exists. Then, given any i, start with an arbitrary $\alpha_1 \in S_i$. There is an $\alpha_2 \in S_i$ different from α_1 such that $\alpha_2(H) < \gamma_{j-1}\alpha_1(H)$. Continuing by induction, we find $\alpha_l \in S_i$ different from $\alpha_1, \ldots, \alpha_{l-1}$ such that $\alpha_l(H) < \gamma_{j-1}\alpha_m(H)$ for some $m < l$. Consequently, again by induction, $\alpha_l(H) \leq (\gamma_{j-1})^{l-1}\alpha_1(H)$. It follows that $\alpha(H) \leq (\gamma_{j-1})^{r_i - 1}\alpha_1(H)$, if $\alpha \in S_i$ and r_i denotes the number of elements of S_i. Since π is a partition into $r-j$ subsets S_i, $r_i \leq j+1$; so from the definition of μ_j we see that $H \in C(\pi, \mu_j, \gamma_j)$.

Let $\{\beta^\vee \mid \beta \in S\}$ *be the basis in* $\mathfrak{a}$ *dual to the basis* S *of* $\mathfrak{a}^*$, that is

$$\alpha(\beta^\vee) = \delta_{\alpha\beta} \text{ for all } \alpha, \beta \in \mathrm{S}. \tag{11.18}$$

In general the $\beta^\vee$ are distinct from the H_β. Note also that

$$Cl(\mathfrak{a}^+) = \{\sum_{\alpha\in S} t_\alpha \alpha^\vee \,|\, t_\alpha \geq 0\},\ H = \sum_{\alpha\in S} \alpha(H)\alpha^\vee. \tag{11.19}$$

In order to describe the next localization in the variable H, we define, for a given ordered partition $\pi = (S_1, \dots, S_s)$, the "S_j-component" of $H \in \mathfrak{a}$ as

$$H_j = \sum_{\alpha\in S_j} \alpha(H)\alpha^\vee \tag{11.20}$$

and for $\tau = (\tau_1, \dots, \tau_s) \in \mathbb{R}^s$ the "inflated vector" $\tau \cdot H \in \mathfrak{a}$ by

$$\tau \cdot H = \sum_{j=1}^{s} \tau_j H_j. \tag{11.21}$$

For $\mu \geq 1$ consider the parallelepiped

$$P(\mu) = \left\{ H \in \mathfrak{a} \,\middle|\, \frac{1}{\mu} \leq \alpha(H) \leq 1 \text{ for all } \alpha \in S \right\}. \tag{11.22}$$

Of course $P(\mu) \subset \mathfrak{a}^+$ and $\alpha(H) \leq \mu\beta(H)$ for all $H \in P(\mu)$ and $\alpha, \beta \in S$. For any $\gamma \geq 1$ define

$$P(\pi, \mu, \gamma) = \{\tau \cdot H \,|\, H \in P(\mu) \text{ and } \tau_1 \geq \gamma\tau_2 \geq \ldots \geq \gamma^{s-1}\tau_s \geq 0\}. \tag{11.23}$$

The $P(\pi, \mu, \gamma)$ are comparable to the $C(\pi, \mu, \gamma)$, in fact

$$C(\pi, \mu, \gamma) \subset P(\pi, \mu, \gamma) \subset C(\pi, \mu, \gamma/\mu). \tag{11.24}$$

Indeed, let $H \in C(\pi, \mu, \gamma)$. If $\alpha(H) = 0$ for some $\alpha \in S_j$ then we read off from (11.10) that $\beta(H) = 0$ for all $\beta \in S_j \cup \ldots \cup S_s$. So if p is the smallest j for which this occurs, we have $\alpha(H) > 0$ for $\alpha \in S_1 \cup \ldots \cup S_{p-1}$ and $\beta(H) = 0$ for $\beta \in S_p \cup \ldots \cup S_s$. For $1 \leq j \leq p-1$, define $\tau_j = \max_{\alpha\in S_j} \alpha(H)$; then $\frac{\tau_j}{\mu} \leq \beta(H) \leq \tau_j$ for all $\beta \in S_j$. Therefore

$$H' = \sum_{j=1}^{p-1} \tau_j^{-1} H_j + \sum_{\alpha \in S_p \cup \ldots \cup S_s} \alpha^\vee \in P(\mu).$$

If we take $\tau_i = 0$ for $p \leq i \leq s$ and observe that $\tau_j \geq \gamma\tau_k$ for $j < k$, it is immediate that $H = \tau \cdot H'$ and $H \in P(\pi, \mu, \gamma)$. The second inclusion is even easier.

The advantage of the $P(\pi, \mu, \gamma)$ is that they are defined in terms of the compact subsets $P(\mu)$ of $\mathfrak{a}^+$, rather than of $Cl(\mathfrak{a}^+)$. This reduces the proof of Proposition 11.4 to proving Proposition 11.6 below.

Fix $a_0 \in A$, $k_0 \in K$, let $\pi = (S_1, \ldots, S_s)$ be an ordered partition of S, and let $H^0 \in \mathfrak{a}^+$. *Let t be the largest j such that k_0 is a critical point for $F_{a_0, \alpha^\vee}$ for all $\alpha \in S_1 \cup \ldots \cup S_j$*; we write $t = 0$ if k_0 is noncritical for $F_{a_0, \alpha^\vee}$ for all $\alpha \in S$.

Let $B = \{\alpha^\vee | \alpha \in S_1 \cup \ldots \cup S_t\}$. Then, from (11.9) we conclude that $k_0 \in K_{a_0} \mathfrak{w} K_B$; in the sequel let $w \in \mathfrak{w}$ be such that $k_0 \in K_{a_0} w K_B$. Define

$$n_{\pi, j} = \sum_{\alpha \in [\Delta^+_{\pi, j} \backslash \Delta^+_{\pi, j+1}] \cap \Delta^+(a_0^{w^{-1}})} n(\alpha), \tag{11.25}$$

where

$$\Delta^+_{\pi, j} = \Delta^+ \cap \sum_{\beta \in S_j \cup \ldots \cup S_s} \mathbb{Z} \cdot \beta, \tag{11.26}$$

and

$$\Delta^+(a_0^{w^{-1}}) = \{\alpha \in \Delta^+ | \alpha(\log a_0^{w^{-1}}) \neq 0\}. \tag{11.27}$$

Proposition 11.6: *There exist neighborhoods A_0 of a_0 in A, resp. V of k_0 in K, resp. Ξ of H^0 in $\mathfrak{a}^+$, and furthermore a C^μ-norm ν on $C^\mu(K)$ ($\mu = \frac{1}{2} \sum_{j=1}^t n_{\pi, j}$) and a constant $\gamma \geq 1$ such that for all $a \in A_0$, $g \in C^\mu_c(V)$, $H \in \Xi$ and $\tau_1, \ldots, \tau_s$ satisfying $\tau_1 \geq \gamma \tau_2 \geq \ldots \geq \gamma^{s-1} \tau_s \geq 0$, we have*

$$\left| \int_K \exp(i \sum_{j=1}^s \tau_j F_{a, H_j}(k)) g(k) dk \right| \leq \nu(g) \prod_{j=1}^s (1 + \tau_j)^{-\frac{1}{2} n_{\pi, j}}. \tag{11.28}$$

It is clear that the right hand side of (11.28) is dominated by the right hand side of (11.4) with H replaced by $\sum_{j=1}^s \tau_j H_j$. Then also $\mu = \frac{1}{2} \sum_{j=1}^s n_{\pi, j} \leq m$.

For any $H \in \mathfrak{a}^+$, with H_j defined as in (11.20), F_{a, H_j} is right K_j-invariant if we define

$$K_j = \text{centralizer in } K \text{ of all } \alpha^\vee, \alpha \in S_1 \cup \ldots \cup S_j. \tag{11.29}$$

Moreover, $K = K_0 \supset K_1 \supset \ldots \supset K_t = K_B \supset \ldots \supset K_s = M$; so F_{a, H_j} is in fact right K_l-invariant for $j \leq l$. This is the first basic ingredient of the proof of Proposition 11.6; the second is given in the following two lemmas.

LEMMA 11.7: *For $1 \leq j \leq t$, the Hessian at k_0 of the restriction of F_{a_0, H_j^0} to $k_0 K_{j-1}$ has rank $n_{\pi,j}$.*

Since F_{a_0, H_j^0} is left K_{a_0}- and right K_B-invariant and $K_B \subset K_{j-1}$, we may assume that $k_0 = w$. In view of Proposition 6.5 Q, the Hessian at w, is diagonal with respect to the decomposition $\mathfrak{k} = \sum_{\alpha \in \Delta^+}^{\oplus} \mathfrak{k}_\alpha \oplus \mathfrak{m}$. Because

$$\mathfrak{k}_{j-1} = \sum_{\beta \in \Delta^+, \beta(\alpha^\vee) = 0 \text{ for all } \alpha \in S_1 \cup \ldots \cup S_{j-1}} \mathfrak{k}_\beta \oplus \mathfrak{m}, \tag{11.30}$$

the rank of $Q|_{\mathfrak{k}_{j-1}}$ is equal to the sum of the $n(\beta)$ where $\beta(H_j^0) \neq 0$, $\beta \in \Delta^+(a_0^{w^{-1}})$ and $\beta(\alpha^\vee) = 0$ for all $\alpha \in S_1 \cup \ldots \cup S_{j-1}$. Writing such β as $\beta = \sum_{\gamma \in S} k_\gamma \gamma$ with $k_\gamma \in \mathbb{N} \cup \{0\}$, the last condition amounts to $k_\gamma = 0$ for all $\gamma \in S_1 \cup \ldots \cup S_{j-1}$, whereas $\beta(H_j^0) \neq 0$ means that $k_\gamma \neq 0$ for some $\gamma \in S_j$. But this means that $\beta \in \Delta^+(a_0^{w^{-1}}) \cap [\Delta_{\pi,j}^+ \backslash \Delta_{\pi,j+1}^+]$; so we get the number $n_{\pi,j}$ in (11.25).

LEMMA 11.8: *Let E be the subset of S such that k_0 is a critical point for $F_{a_0, \alpha^\vee}$ if $\alpha \in E$ and not critical if $\alpha \in S \backslash E$. Then there exists $Z \in \mathfrak{k}$ such that*

(a) *$Z \in \mathfrak{k}_E$, the centralizer of the $\alpha^\vee$, $\alpha \in E$;*

(b) *$F_{a_0, \alpha^\vee}(k_0; Z) > 0$ for $\alpha \in S \backslash E$.*

We take $Z = v_{X_0}(k_0) = E_{\mathfrak{k}}(X_0^{k_0^{-1}})$, see (3.4). Because $k_0 \in K_{a_0} w K_E$ for some $w \in \mathfrak{w}$, $X_0^{k_0^{-1}} = Y_0^{v^{-1}}$ for suitable $v \in K_E$; here $Y_0 = w^{-1} X_0 \in \mathfrak{a} \subset \mathfrak{g}_E$. Hence $X_0^{k_0^{-1}} \in \mathfrak{g}_E$. Because the restriction of $E_{\mathfrak{k}}$ to $\mathfrak{g}_E$ is equal to the corresponding projection for $\mathfrak{g}_E$, the conclusion $Z \in \mathfrak{k}_E$ follows. (b) is an immediate consequence of Corollary 5.8.

Let X_j be a local analytic section through k_0 for the bundle $k_0 K_{j-1} \to k_0 K_{j-1}/K_j$ $(1 \leq j \leq s)$. By making X_j sufficiently small, the map

$$\zeta_j : (x_j, \ldots, x_s) \mapsto k_0 x_j \ldots x_s \tag{11.31}$$

is an analytic diffeomorphism of $X_j \times \ldots \times X_s$ with an open neighborhood of $k_0 M$ in $k_0 K_{j-1}/M$. Let $f_j = f_{j,\theta}$ be the pull-back of F_{a, H_j} to $X_1 \times \ldots \times X_s$ under ζ_1; here $\theta \in U$ stands for the parameters a, H_j in suitable coordinates, so that (a_0, H_j^0) corresponds to the origin 0. Using also local coordinates in the X_j such that $k_0 M$ corresponds to 0, and writing $\delta_j = n_{\pi,j}$, we have abstracted to the following situation.

> $f_{j,\theta}$ $(1 \leq j \leq s)$ is a system of smooth functions on $X = X_1 \times \ldots \times X_s$, depending smoothly on parameters $\theta \in U$, such that:

$$\begin{array}{ll}\text{(i) For each } \theta \in U,\ f_{j,\theta} \text{ depends on } x_1, \ldots, x_j \text{ only;} & \\ \text{(ii) } \phi_j : x_j \mapsto f_{j,0}(0,\ldots,0,x_j) \text{ has } x_j = 0 \text{ as critical point with} & \\ \quad \text{rank equal to } \delta_j, \text{ if } 1 \le j \le t; & (11.32) \\ \text{(iii) If } t < s \text{ then } \phi_{t+1} \text{ is noncritical at } x_{t+1} = 0. & \end{array}$$

Indeed, (i) is immediate from the definition of the ζ_j in (11.31) and the right K_l-invariance of F_{a,H_j} if $l \geq j$.

For (ii), we note that taking $x_1 = \ldots = x_{j-1} = 0$ is equivalent to restriction to $k_0 K_{j-1}$; so that the statement in question asserts that F_{a_0,H_j^0}, restricted to $k_0 K_{j-1}$, has k_0 as a critical point of rank $\delta_j = n_{\pi,j}$. This is just Lemma 11.7.

To check (iii) we have to prove that the restriction of F_{a_0,H_{t+1}^0} to $k_0 K_t$ is noncritical at k_0. Let E, Z be as in Lemma 11.8. Then $S_1 \cup \ldots \cup S_t \subset E$, hence $Z \in \mathfrak{k}_E \subset \mathfrak{k}_t$. On the other hand the definition of t implies that $\alpha \in S \backslash E$ for some $\alpha \in S_{t+1}$. Since H_{t+1}^0 is a linear combination of the $\alpha^\vee$, $\alpha \in S_{t+1}$, with strictly positive coefficients, while $F_{a_0,\alpha^\vee}(k_0; Z) = 0$ for $\alpha \in S_{t+1} \cap E$, the conclusion from Lemma 11.8(b) is that $F_{a_0,H_{t+1}^0}(k; Z) > 0$.

Proposition 11.6 now follows immediately from

PROPOSITION 11.9: *Let the $f_{j,\theta}$ be a "trigonalized" system as in* (11.32). *Then there exist neighborhoods U_0 of 0 in U, resp. V of 0 in X, and furthermore, for each $L \geq 0$, a C^μ-norm v on $C_c^\mu(V)$ (with $\mu = \frac{1}{2}\sum_{j=1}^t \delta_j + L$) and a constant $\gamma \geq 1$, such that for all $\theta \in U_0$, $g \in C_c^\mu(V)$ and $\tau_1, \ldots, \tau_s \in \mathbb{R}$ satisfying $\tau_1 \geq \gamma\tau_2 \geq \ldots \geq \gamma^{s-1}\tau_s \geq 0$, we have the estimate*

$$\left| \int_X \exp\Big(i \sum_{j=1}^s \tau_j f_{j,\theta}(x)\Big) g(x)\, dx \right| \leq v(g) \prod_{j=1}^t 1 + \tau_j)^{-\frac{1}{2}\delta_j} (1 + \tau_{t+1})^{-L}. \quad (11.33)$$

Here again, if μ is not an integer then $g \in C^\mu$ means that $g \in C^{[\mu]}$ and the derivatives of order $\leq [\mu]$ are Hölder continuous of order $\mu - [\mu]$.

We first prove Proposition 11.9 for $s = 1$; it then follows from the next two lemmas. They express the method of stationary phase in a form which is suitable for our purposes.

LEMMA 11.10: *Let f_0 be a smooth real-valued function on an open neighborhood U of 0 in $\mathbb{R}^n$. If $df_0(0) \neq 0$, then there exist a compact neighborhood U_0 of 0 in U and a neighborhood $\mathscr{F}$ of f_0 in $C^1(U_0)$, such that we have the following. For all real $N \geq 0$ the mapping $f, g \mapsto I_{f,g}$, defined by*

$$I_{f,g}(\tau) = \int_{U_0} e^{i\tau f(x)} g(x)\, dx, \quad (11.34)$$

is a C^N *mapping from* $(\mathscr{F} \cap C^{N+1}(U_0)) \times C_c^N(U_0)$ *to the space* $\mathscr{I}_N$ *of functions* $I: \mathbb{R} \to \mathbb{C}$ *such that*

$$\|I\|_N = \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^N |I(\tau)| < \infty. \tag{11.35}$$

LEMMA 11.11: *Let* f *be as in the previous lemma but assume* $df_0(0) = 0$ *and the Hessian* Q *of* f_0 *at* 0 *has rank* d. *Then there exist a compact neighborhood* U_0 *of* 0 *in* U *and a neighborhood* $\mathscr{F}$ *of* f_0 *in* $C^3(U_0)$, *such that for all* $0 \leq N \leq \frac{1}{2}d$ *the mapping* $f, g \mapsto I_{f,g}$ *is a continuous mapping from* $(\mathscr{F} \cap C^{N+3}(U_0)) \times C_c^N(U_0)$ *to* $\mathscr{I}_N$.

If $df_0(0) \neq 0$ then, if $f_0(0) = c$, the implicit function theorem ensures the existence of a local C^{N+1} change of coordinates, depending C^{N+1} on f, such that $f(x) = c + x_1$ on the new coordinates. Observe that this exists for f in a C^1-neighborhood in C^{N+1} of f_0. The new amplitude depends C^N on x, resp. f, g. Note that the Jacobian involves one loss of derivative. Because $e^{i\tau x} = (i\tau)^{-1} \dfrac{d}{dx} e^{i\tau x}$ we get, iterating

$$\int_{\mathbb{R}} e^{i\tau x} g(x)\,dx = \frac{-1}{i\tau} \int_{\mathbb{R}} e^{i\tau x} \frac{dg}{dx}(x)\,dx = \left(\frac{-1}{i\tau}\right)^N \int_{\mathbb{R}} e^{i\tau x} \frac{d^N g}{dx^N}(x)\,dx.$$

Because the factor $e^{i\tau c}$ in front of the integral does not depend on f, g, differentiations with respect to f, g do not increase the growth order in τ. This proves Lemma 11.10 in the case that N is an integer. For Hölder estimates, derivatives have to be replaced by differences. The typical estimate to be used is the following.

$$\begin{aligned} \left| \int_{\mathbb{R}} e^{i\tau x} g(x)\,dx \right| &= \left| (1 - e^{-i\tau u})^{-1} \int_{\mathbb{R}} e^{i\tau x} (g(x) - g(x+u))\,du \right| \\ &\leq |1 - e^{-i\tau u}|^{-1} \left(\int_{\operatorname{supp} g} 1\,dx + |u| \right) \|g\|_{H^\alpha} |u|^\alpha, \end{aligned} \tag{11.36}$$

where

$$\|g\|_{H^\alpha} = \sup_{x,u} |g(x) - g(x+u)| / |u|^\alpha$$

denotes the Hölder norm of g of order α, $0 \leq \alpha \leq 1$. Taking $u = \pi\tau^{-1}$ gives an estimate by $\|g\|_{H^\alpha} \tau^{-\alpha}$.

If $df_0(0) = 0$, we start writing $\mathbb{R}^n = Y \times Z$ with $Z = \operatorname{Ker} Q$. Then, performing the integration over Y and treating the Z-components as parameters, we have reduced the problem to the case when Q is nondegenerate. Indeed, the continuity of (11.34) then implies local uniformity of the estimates with respect to $z \in Z$. So integration over z afterwards will not destroy the conclusion.

Now the proof of the Morse lemma in Hörmander [36, Lemma 3.2.3] based as it can be on the implicit function theorem in Banach spaces (cf. Dieudonné [18]) leads to the following. There exists a C^{N+1} change of coordinates depending in a C^{N+1} fashion on $f \in C^{N+3}$, such that on the new coordinates

$$f(x) = f(0) + \sum_{k=1}^{d} \varepsilon_k x_k^2, \qquad \varepsilon_k = \pm 1. \tag{11.37}$$

Then the new amplitude is C^N. Now write

$$\int e^{i\tau f(x)} g(x)\, dx = e^{i\tau f(0)} \int e^{i\tau\varepsilon_1 x_1^2} \int \dots \int e^{i\tau\varepsilon_d x_d^2} g(x)\, dx_d \dots dx_1 \tag{11.38}$$

We then have reduced the proof to the case $d = 1$, observing that, if $g \in C^{\alpha+\beta}$, then $x_1 \mapsto g(x_1, .)$ is a C^α mapping from $\mathbb{R}$ to the space of C^β functions of $(x_2, \dots, x_d)$.

Let $g \in C_c^\alpha(\mathbb{R})$, $0 \le \alpha < 1$, $\tau \ge 1$. Then

$$\begin{aligned}\int_0^\infty e^{i\tau x^2} g(x)\, dx &= \int_0^\infty e^{i\tau y} g(y^{\frac12}) \tfrac12 y^{-\frac12}\, dy = \\ &= \int_0^{M\tau^{-1}} e^{i\tau y} g(y^{\frac12}) \tfrac12 y^{-\frac12}\, dy + \int_{M\tau^{-1}}^\infty e^{i\tau y} g(y^{\frac12}) \tfrac12 y^{-\frac12}\, dy\end{aligned}$$

with $M > \pi$. The first integral is obviously $0(\tau^{-\frac12})$, the second one is estimated as in (11.36) with $u = \pi\tau^{-1}$. Using that $(y+u)^{\frac12} - y^{\frac12} \sim \frac12 y^{-\frac12} u$, $g((y+u)^{\frac12}) - g(y^{\frac12}) = 0((y^{-\frac12}u)^\alpha)$, the second integral comes out to be $0(\tau^{-\alpha}) + 0(\tau^{-\frac12})$. More precisely, there is a C^α-norm ν_α such that

$$\left| \int_{\mathbb{R}} e^{i\tau x^2} g(x)\, dx \right| \le \nu_\alpha(g)(1 + |\tau|)^{-\alpha} (g \in C_c^\alpha(\mathbb{R}), 0 \le \alpha \le \tfrac12). \tag{11.39}$$

Remark: The power $-\alpha$ in (11.39) is sharp in view of the example $g(x) = e^{-i\tau x^2} h(x)$, $h \in C_c^\infty(\mathbb{R})$, because the ν_α of this g is of order τ^α as $\tau \to \infty$. Also note that differentiating (11.38) with respect to f leads to additional powers of τ; this is the reason for stating only the continuity of the map (11.34) in Lemma 11.11. Finally, the critical value of f is an invariant under coordinate changes in the domain of f.

We start the *proof of Proposition 11.9* for general s, t with a reduction to the case $t = s$. Writing

$$\bar{X} = X_1 \times \ldots \times X_t,\ Y = X_{t+1} \times \ldots \times X_s,$$

we get

$$\int_X \exp(i \sum_{j=1}^{s} \tau_j f_{j,\theta}(x)) g(x)\, dx = \int_{\bar{X}} \exp(i \sum_{j=1}^{t} \tau_j f_{j,\theta}(\bar{x})) \psi(\bar{x})\, d\bar{x}. \tag{11.40}$$

Here $\psi = \psi_{\theta, \tau_{t+1}, \ldots, \tau_s, g}$ is defined by

$$\psi(\bar{x}) = \int_Y \exp(i\tau_{t+1} \phi_{\theta, \tau_{t+2}/\tau_{t+1}, \ldots, \tau_s/\tau_{t+1}}(\bar{x}, y)) g(\bar{x}, y)\, dy, \tag{11.41}$$

with

$$\phi_{\theta, \tau_{t+2}/\tau_{t+1}, \ldots, \tau_s/\tau_{t+1}}(\bar{x}, y) = f_{t+1,\theta}(\bar{x}, y) + \sum_{k=t+2}^{s} \frac{\tau_k}{\tau_{t+1}} f_{k,\theta}(\bar{x}, y). \tag{11.42}$$

In (11.41, 42), the $\theta, \tau_{t+2}/\tau_{t+1}, \ldots, \tau_s/\tau_{t+1}$, and $\bar{x}$ are treated as parameters. Note that the $\tau_{t+2}/\tau_{t+1}, \ldots, \tau_s/\tau_{t+1}$ are kept between 0 and $1/\gamma$; so the localization near 0 corresponds to the choice of sufficiently large γ in Proposition 11.9. Lemma 11.10 now implies that any C^μ-norm of ψ ($\mu \geq 0$) can be estimated by $(1 + \tau_{t+1})^{-L}$ times a C^L-norm of g ($L \geq 0$), thus reducing the proof of Proposition 11.9 to the case $t = s$.

Assuming $t = s$, the proof now will be given by induction on s.

$$\int_X \exp(i \sum_{j=1}^{s} \tau_j f_{j,\theta}) g\, dx = \int_{\tilde{X}} \exp(i \sum_{j=1}^{s-1} \tau_j f_{j,\theta}(\tilde{x})) \psi_{\theta,\tau_s}(\tilde{x})\, d\tilde{x} \tag{11.43}$$

where $\tilde{x} = (x_1, \ldots, x_{s-1}) \in X_1 \times \ldots \times X_{s-1} = \tilde{X}$, and

$$\psi_{\theta,\tau_s}(\tilde{x}) = \int_{X_s} \exp(i\tau_s f_{s,\theta}(\tilde{x}, x_s)) g(\tilde{x}, x_s)\, dx_s. \tag{11.44}$$

Applying the Morse lemma with parameters to the function $x_s \mapsto f_{s,\theta}(\tilde{x}, x_s)$, treating $\theta, \tilde{x}$ as parameters, we can write, with q_s a standard quadratic form in δ_s variables,

$$\psi_{\theta,\tau_s}(\tilde{x}) = e^{i\tau_s \phi_\theta(\tilde{x})} \int_{X_s} e^{i\tau_s q_s(x_s)} h_\theta(\tilde{x}, x_s)\, dx_s. \tag{11.45}$$

Here $\phi_\theta(\tilde{x})$ is the critical value of $x_s \mapsto f_{s,\theta}(\tilde{x}, x_s)$, which depends smoothly on the parameters $\theta, \tilde{x}$. Also h_θ depends in a bounded linear way on g, with respect to any C^μ-norm. We now have

$$\int_X \exp\Big(i \sum_{j=1}^{s} \tau_j f_{j,\theta}\Big) g\, dx = \int_{\tilde{X}} \exp\Big(i \sum_{j=1}^{s-1} \tau_j \tilde{f}_{j,\theta,\tau_s/\tau_{s-1}}\Big) \tilde{g}_{\theta,\tau_s}\, d\tilde{x}, \tag{11.46}$$

with

$$\tilde{f}_{j,\theta,\tau_s/\tau_{s-1}}(\tilde{x}) = f_{j,\theta}(\tilde{x}) \text{ for } 1 \le j < s-1, \tag{11.47}$$

$$\tilde{f}_{s-1,\theta,\tau_s/\tau_{s-1}}(\tilde{x}) = f_{s-1,\theta}(\tilde{x}) + \frac{\tau_s}{\tau_{s-1}} \phi_\theta(\tilde{x}), \tag{11.48}$$

$$\tilde{g}_{\theta,\tau_s}(\tilde{x}) = \int_{X_s} \exp(i\tau_s q_s(x_s)) h_\theta(\tilde{x}, x_s)\, dx_s. \tag{11.49}$$

Treating τ_s/τ_{s-1} as another parameter, which remains between 0 and $1/\gamma$, the system of phase functions $\tilde{f}_j$, $1 \le j \le s-1$ again is a trigonalized system as in (11.32). Note that $\tilde{f}_{s-1,0,0}(\tilde{x}) = f_{s-1,0}(\tilde{x})$. The localization of τ_s/τ_{s-1} near 0 corresponds to the choice of sufficiently large γ in Proposition 11.9. This result now follows by applying Lemma 11.11 to (11.49) and then applying the induction hypothesis to the integral on the right hand side of (11.46).

Modification of the proof for general $H \in \mathfrak{a}$

Now let C be the closure of any Weyl chamber, and let $P(C)$ be the corresponding choice of positive roots, that is

$$\alpha \in P(C) \Leftrightarrow \alpha \in \Delta \text{ and } \alpha(H) \ge 0 \text{ for all } H \in C. \tag{11.50}$$

Replacing S by the set $S(C)$ of simple roots in $P(C)$, the reduction of Theorem 11.1 with $H \in C$ to Proposition 11.6 with $\mathfrak{a}^+$ replaced by C^{int}, is done as above in the text running from Proposition 11.4 to Proposition 11.6.

However, our proof that the F_{a,H_j} can be viewed as a trigonalized system of functions as in (11.32) with $\delta_j = n_{\pi,j}$ depends in an essential way on their right K_{H_j}-invariance, but this is no longer available now. Nevertheless we can still make the F_{a,H_j} into a trigonalized system as in (11.32) with $\delta_j = n_{\pi,j}$, by using suitable local coordinates.

The idea is to use the relation between $F_{a,H}$ and $f_{X,H}$ (with $a = \exp X$) expressed in Lemma 5.9. For each $Z \in \mathfrak{k}$ let $Y_{X,Z}$ be the vectorfield on K

such that $Y_{X,Z}(k)$, pulled back to $\mathfrak{k}$ by left multiplication by k^{-1}, is given by

$$Y_{X,Z}(k) = \mathrm{Ad}\, k^{-1} \circ (\mathrm{ad}\, X/\sinh \mathrm{ad}\, X) \circ \mathrm{Ad}\, \Theta_a(k)(Z). \tag{11.51}$$

The point is that since $f_{X,H}$ is right K_H-invariant, the function $F_{a,H}$ will be invariant under the flow $\Phi^t_{X,Z}$, $t \in \mathbb{R}$, of $Y_{X,Z}$ if $Z \in \mathfrak{k}_H$. Indeed, we read off from Lemma 5.9 that

$$F_{a,H}(k; Y_{X,Z}(k)) = 0 \text{ for } k \in K, Z \in \mathfrak{k}_H. \tag{11.52}$$

Defining K_j as in (11.29), let $\mathfrak{k}_j$ be their respective Lie algebras and let $Z^{(j)}_1, \ldots, Z^{(j)}_{n_j}$ be a basis of $\mathfrak{k}_{j-1} \ominus \mathfrak{k}_j$, $j = 1, \ldots, s+1$. Together they form a basis of $\mathfrak{k}$, and because $Z \mapsto Y_{X,Z}(k)$ is a linear isomorphism $\mathfrak{k} \to \mathfrak{k}$, the $Y_{X,Z^{(j)}_m}(k_0)$, $m = 1, \ldots, n_j$, $j = 1, \ldots, s+1$ form a basis of $\mathfrak{k}$ as well. As a consequence, the map ζ_X defined by

$$\begin{aligned} &\zeta_X(\ldots, t^{(j)}_m, \ldots) = \\ &= \Phi^{t^{(s+1)}_{n_s+1}}_{X, Z^{(s+1)}_{n_s+1}} \circ \ldots \circ \Phi^{t^{(s+1)}_1}_{X, Z^{(s+1)}_1} \circ \Phi^{t^{(s)}_{n_s}}_{X, Z^{(s)}_{n_s}} \circ \ldots \circ \Phi^{t^{(1)}_1}_{X, Z^{(1)}_1}(k_0) \end{aligned} \tag{11.53}$$

defines a diffeomorphism from an open neighborhood of 0 in $\mathbb{R}^n$, $n = \dim \mathfrak{k}$, to an open neighborhood of k_0 in K. Using (11.52), F_{a,H_j} is $\Phi^t_{X,Z^{(l)}_m}$-invariant for all $j < l$, $m = 1, \ldots, n_l$. So, writing $x_l = (t^{(l)}_1, \ldots, t^{(l)}_{n_l})$, the functions $f_{j,\theta}$ defined by $(1 \le j \le s)$

$$f_{j,\theta}(x_1, \ldots, x_{s+1}) = F_{a,H_j}(\zeta_X(\ldots, t^{(j)}_m, \ldots)) \tag{11.54}$$

form a trigonalized system as in (11.32)(i). Here again $\theta = (a, H)$ in suitable local coordinates, while x_{s+1} actually is a dummy variable because of right M-invariance.

To check (11.32)(iii) we recall Proposition 3.3 which states that the gradient of $f_{X,H}$, with respect to a suitable Riemannian metric on K/K_H, is equal to the vector field v_X. Now, in section 3 we assumed that $H \in Cl(\mathfrak{a}^+)$. But if H is in the closure of another Weyl chamber then everything in section 3 remains true if all parabolic subgroups as well as the Iwasawa decomposition used in section 3 are chosen to be the standard ones with respect to this Weyl chamber. This shows that Lemma 11.8 remains true if we replace S by the set of simple roots with respect to any Weyl chamber, and replace $F_{a_0,\alpha^\vee}$ by $f_{X_0,\alpha^\vee}$. Because of the invariance of the critical sets under Θ_{a_0} we may also replace k_0 by $\Theta_{a_0}(k_0)$, and get $Z \in \mathfrak{k}_t$ such that

$$F_{a_0, H^0_{t+1}}(k_0; Y) = f_{X_0, H^0_{t+1}}(\Theta_{a_0}(k_0); Z) > 0$$

for $Y = Y_{X_0,Z}(k_0)$. Writing $Z = \sum c_i Z_i^{(t+1)} \bmod \mathfrak{k}_{t+1}$ we get that differentiating F_{a_0,H_{t+1}^0} with respect to Y at k_0 gives the same result as applying the differential operator $\sum c_i \partial/\partial t_i^{(t+1)}$ to ϕ_{t+1} at the origin.

What is left is the computation of the ranks δ_j in (11.32). As before, $k_0 \in K_{a_0} w K_B^0$ for some $w \in \mathfrak{w}$. Using the left K_{a_0}-invariance of F_{a_0,H_j^0} and the formula $F_{a,H}(x_w k) = F_{a^{w^{-1}},H}(k)$, we may assume that actually $k_0 \in K_B^0$, replacing a_0 by $a_0^{w^{-1}}$.

Corresponding to the K_j we have $G_j = G_{(S_1 \cup \ldots \cup S_j)^\vee}$. Keeping in mind that $K_B^0 \subset G_{j-1}^0$, we transfer now everything to the connected reductive group G_{j-1}^0. Because G_{j-1} is an intersection of centralizers of elements of $\mathfrak{a}$, (2.6) shows that the Iwasawa projection of G_{j-1}^0, is equal to the Iwasawa projection of G restricted to G_{j-1}^0.

So the functions $F_{a,H}$ on K_{j-1}^0 defined starting from G_{j-1}^0 are equal to the restrictions to K_{j-1}^0 of the previously defined functions $F_{a,H}$ on K. Also the A-action on K_{j-1}^0 defined in terms of G_{j-1}^0 is equal to the restriction to K_{j-1}^0 of the previously defined A-action $a \mapsto \Theta_a$ on K; this is the same as saying that Θ_a leaves K_{j-1}^0 invariant, see Lemma 6.3. So for $Z \in \mathfrak{k}_{j-1}$ the vector field $Y_{w^{-1}X,Z}$ can be viewed as a vector field on K_{j-1}^0. At each point $k \in K_{j-1}^0$ they span the tangent space to K_{j-1}^0, and the problem is therefore to compute the rank of the Hessian of $F_{a_0^{w^{-1}},H_j^0}$ (with G now replaced by G_{j-1}^0) at the critical point $k_0 \in K_B^0$.

Now, according to Corollary 6.6 with G replaced by G_{j-1}^0, this rank is constant along $K_{H_j}^0$ and since $K_B^0 \subset K_{H_j}^0$ it is therefore sufficient to compute the rank at the identity element. There the computation of Lemma 11.7 shows that it is equal to $n_{\pi,j}$. Because restriction to K_{j-1}^0 means putting $x_1 = 0, \ldots, x_{j-1} = 0$, $\delta_j = n_{\pi,j}$ and the proof of Theorem 11.1 is complete.

12. Uniform expansions for orbital integrals and for matrix coefficients with equisingular a

Now we turn to uniform estimates for the orbital integrals

$$J_{\gamma,H}(g) = \int_{C_\gamma} e^{i\langle H(x),H\rangle} g(x)\, dC_\gamma(x) \qquad (g \in C_c^\infty(G)) \tag{12.1}$$

of section 10. Let γ be a semisimple element of G in standard position, cf. DKV, Lemma 4.1, that is

$$\gamma = \gamma_I \gamma_R,\ \gamma_I \in K,\ \gamma_R \in A,\ \gamma_I \text{ and } \gamma_R \text{ commute.} \tag{12.2}$$

There are always elements of this form in a semisimple conjugacy class; although such elements are not unique, their components in A are unique up to conjugacy by an element of $\mathfrak{w}$ (cf. DKV, Lemma 4.2). Hence the assumption (12.2) on γ means no loss of generality. As in section 11 we put, for any $w \in \mathfrak{w}$,

$$\Delta_w^+(\gamma_R) = \{\alpha \in \Delta^+ \mid w\alpha(\log \gamma_R) \neq 0\}. \tag{12.3}$$

It is our purpose to prove uniform estimates of the following form.

THEOREM 12.1: *Let $\gamma \in G$ be a semisimple element satisfying* (12.2). *Then we can find a continuous norm ν on $C_c^\infty(G)$ of order $\leq \dim N$, such that for all $g \in C_c^\infty(G)$, $H \in \mathfrak{a}$,*

$$\left| \int_{C_\gamma} e^{i\langle H(x), H\rangle} g(x)\, dC_\gamma(x) \right| \leq \nu(g) \sum_{w \in \mathfrak{w}} \prod_{\alpha \in \Delta_w^+(\gamma_R)} (1 + |\alpha(H)|)^{-n(\alpha)}. \tag{12.4}$$

It follows from the remarks just made that the sum at the right side of (12.4) does not depend on the choice of γ as long as γ satisfies (12.2). Clearly the assertion is equivalent to the one for $g \in C_c^{\dim N}(Z)$, Z a compact subset of G, the norm on $C_c^{\dim N}(Z)$ being allowed to depend on Z.

Not having the K_H-invariances available, the proof of Section 11 will not be applicable. On the other hand, lacking an additional parameter like a in the $F_{a,H}$, which we wanted to vary arbitrarily in compact subsets, we will be able to obtain a more refined result that can be interpreted as a full asymptotic expansion for $\|H\| \to \infty$, $H \in \mathfrak{a}$. Since the final statement needs more explanations, it will be given at the end of the proof, in Proposition 12.2. Also the exponents in (12.4) are not always optimal, but the "true" ones are more complicated to describe.

The natural additional parameter here is γ, but varying γ in the neighborhood of a singular semisimple element of G will lead to conjugacy classes with jumping dimensions, a rather awkward situation when studying the dependence on the parameters of integrals over such families of manifolds. Nevertheless, the method of proof in this section will lead to analogous uniform asymptotic expansions for the integrals $I_{a,H}(g)$ of section 11, when a is restricted to equisingular subsets of A.

Like in section 11, we start by restricting H to the closure C of some Weyl chamber; reducing to the study of integrals of the form

$$\int_{C_\gamma} \exp\Big(i \sum_{j=1}^{s} \tau_j F_{H_j,\gamma}(x)\Big) g(x)\, dC_\gamma(x). \tag{12.5}$$

Here we have used an ordered partition $\pi = (S_1, \ldots, S_s)$ of the set of simple roots $S(C)$ corresponding to the Weyl chamber C^{int}. Moreover, in terms of the S_j we have defined

$$H_j = \sum_{\alpha \in S_j} \alpha(H)\alpha^{\vee}. \tag{12.6}$$

H may be restricted to a sufficiently small neighborhood of a given $H^0 \in C^{\mathrm{int}}$, and the $\tau_1, \ldots, \tau_s \in \mathbb{R}$ are restricted by

$$\tau_1 \geq \eta\tau_2 \geq \ldots \geq \eta^{s-1}\tau_s \geq 0 \tag{12.7}$$

for sufficiently large $\eta \geq 1$. Finally g has support close to some fixed $x_0 \in C_\gamma$. This reduction is made in the same way as the reduction from Theorem 11.1 to Proposition 11.6, with $Cl(\mathfrak{a}^+)$ replaced by C.

Let us explain the idea of the proof with the $F_{H_j,\gamma}$ replaced by a general family of smooth functions $\psi_{j,\theta}$ ($j = 1, \ldots, s$) on an n-dimensional smooth manifold X, also depending smoothly on the parameters θ in some parameter manifold Θ. That is, we study the integral

$$\int_X \exp\Big(i \sum_{j=1}^{s} \tau_j \psi_{j,\theta}(x)\Big) g(x)\, dx, \tag{12.8}$$

with $g \in C_c^\infty(X)$ of sufficiently small support, dx being a fixed positive smooth density on X.

We split the integration variable x in local coordinates into two groups, y_1, x_1, of variables. Next we write the integral (12.8) as the repeated integral

$$\int_{X_1} \int_{Y_1} \exp(i\tau_1 \psi_{1,\tilde{\tau}_1,\theta}(y_1, x_1)) g(y_1, x_1)\, dy_1\, dx_1. \tag{12.9}$$

Here $\tilde{\tau}_1 = (\tau_2/\tau_1, \ldots, \tau_s/\tau_1)$ are treated as new parameters; and

$$\psi_{1,\tilde{\tau}_1,\theta}(y_1, x_1) = \psi_{1,\theta}(y_1, x_1) + \sum_{j=2}^{s} \frac{\tau_j}{\tau_1} \psi_{j,\theta}(y_1, x_1) \tag{12.10}$$

is treated as a function of y_1, with parameters $\tilde{\tau}_1, \theta, x_1$. Observe that in contrast with the proof in section 11 we start by treating the largest (τ_1) of the frequency variables $\tau_1, \ldots, \tau_s$.

Now assume that $y_1 \mapsto \psi_{1,\tilde{\tau}_1,\theta}(y_1, x_1)$ has a nondegenerate critical point at $y_1 = c_{\tilde{\tau}_1,\theta}(x_1)$, which then depends smoothly on $\tilde{\tau}_1, \theta, x_1$. Write d_1 for the dimension of the y_1-space. An application of the method of stationary phase allows us to rewrite the inner integral in (12.9) as, suppressing the $\tilde{\tau}_1, \theta$ dependence for a moment in the notation,

$$\exp(i \sum_{j=1}^{s} \tau_j \psi_{j,\theta}(c(x_1), x_1)) \sum_{m=0}^{M-1} l(\tau_1)^{-\frac{1}{2}d_1 - m} \chi_m(x_1) + R, \tag{12.11}$$

where the remainder term R can be estimated by

$$|R| \le \nu(g) l(\tau_1)^{-\frac{1}{2}d_1 - M} \tag{12.12}$$

for $g \in C_c^\infty(U)$, U compact $\subset X$ and ν is a suitable norm on $C_c^\infty(U)$. Here we have introduced for convenience of notation a function l such that

$$l \in C^\infty([0, \infty)),\ l(\tau) > 0 \text{ for all } \tau \ge 0, l(\tau) = \tau \text{ for } \tau \ge 1. \tag{12.13}$$

Now we can hope to continue the procedure with τ_2 as a frequency variable, if $x_1 \mapsto \psi_{1,\theta}(c(x_1), x_1)$ is constant. Then, indeed, in each term in (12.11), the τ_1-dependence can be exhibited as a factor of the form

$$\exp(i\tau_1 \psi_{1,\theta}(x^0)) l(\tau_1)^{-\frac{1}{2}d_1 - m} \qquad (0 \le m < M). \tag{12.14}$$

On substitution in (12.9) these expressions then appear in front of the integrals over x_1, so that for the remaining integrands the frequency variable τ_2 becomes the largest. Here x^0 is just any of the points $(c_{\tilde{\tau}_1,\theta}(x_1), x_1)$ along which $\psi_{1,\theta}$ was assumed to be constant.

The following set-up will assure the continuation of the procedure described above. Let $X_1, \ldots, X_s$ be C^∞ closed submanifolds of X such that

$$X = X_0 \supset X_1 \supset \ldots \supset X_s \supset X_{s+1} = \emptyset, \tag{12.15}$$

all X_j being provided with positive smooth densities dx_j. The X_j need not be connected; but for each j, all connected components of X_j have the same dimension. We assume now that for each $j = 1, \ldots, s$ and each $\theta \in \Theta$:

$$\begin{cases} \text{(i)} \ \psi_{j,\theta}|_{X_{j-1}} \text{ has } X_j \text{ for a clean set of critical points;} \\ \text{(ii) There is a smooth subbundle } B_j \text{ of } T_{X_j}(X_{j-1}) \text{ such that:} \\ \quad \text{(a) } T_{X_j}(X_{j-1}) = T(X_j) \oplus B_j \text{ (Whitney sum),} \\ \quad \text{(b) } d\psi_{k,\theta}|_{B_j} = 0 \text{ for } k > j. \end{cases} \tag{12.16}$$

Here, for any smooth manifold Z, we write $T(Z)$ for its tangent bundle. Further, if Y is any smooth submanifold of Z, $T_Y(Z)$ is the restriction of $T(Z)$ to Y, that is, the vector bundle over Y whose fibres are the tangent spaces of Z at the points of Y.

Write

$$d_j = \dim X_{j-1} - \dim X_j \qquad (j = 1,\dots,s) \tag{12.17}$$

for the fiber dimension of B_j. For $1 \le t \le s$, $x^0 \in X_t$, we introduce (by induction on t) a smooth diffeomorphism ζ from a neighborhood of 0 in $\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_t} \times \mathbb{R}^{n_t} = \mathbb{R}^n$, $n_t = \dim X_t$, to a neighborhood in X of x^0 with the following properties. $\zeta(0) = x^0$, ζ maps $y_1 = 0,\dots, y_j = 0$ into X_j, and finally the tangent map of ζ at $u = (0,\dots,0,y_{j+1},\dots,y_t,x_t)$ maps the y_j-space $y_1 = \dots = y_{j-1} = 0$, $y_{j+1} = \dots = y_t = 0$, $x_t = 0$ to the fiber of B_j at $\zeta(u)$. Pull-back by ζ leads to a system of functions $\psi_{j,\theta}$ for which (12.16) gets translated into ($1 \le j \le t$, $\theta \in \Theta$):

$$\left\{\begin{array}{ll} \text{(i)} & (y_j, y_{j+1}, \dots, y_t, x_t) \mapsto \psi_{j,\theta}(0, \dots, 0, y_j, y_{j+1}, \dots, y_t, x_t) \text{ has} \\ & \text{all first order partial derivatives equal to zero if and only if} \\ & y_j = 0 \text{, and the Hessian of } y_j \mapsto \psi_{j,\theta}(0, \dots, 0, y_j, y_{j+1}, \dots, y_t, \\ & x_t) \text{ at } y_j = 0 \text{ is nondegenerate (for each } y_{j+1},\dots,y_t,x_t). \\ \text{(ii)} & \text{The derivative of } y_j \mapsto \psi_{k,\theta}(0, \dots, 0, y_j, y_{j+1}, \dots, y_t, x_t) \text{ at} \\ & y_j = 0 \text{ is equal to zero for } k > j \text{ and each } y_{j+1},\dots,y_t,x_t. \end{array}\right. \tag{12.18}$$

Applying this for $t = 1$, we read off from (12.18)(ii) that $y_1 \mapsto \psi_{k,\theta}(y_1, x_1)$ has a critical point at $y_1 = 0$ for all $k > 1$, whereas from (12.18)(i) we then see that the function $y_1 \mapsto \psi_{1,\tilde{\tau}_1,\theta}(y_1, x_1)$ of (12.10) has a unique and nondegenerate critical point at $y_1 = 0$ if $\tilde{\tau}_1$ is sufficiently small, i.e. if η is sufficiently large in (12.7). From (12.18)(i), however, we also see that $x_1 \mapsto \psi_{1,\theta}(0, x_1)$ is constant.

So applying the method of stationary phase to (12.9) we end up with terms equal to $\exp(i\tau_1\psi_{1,\theta}(0))l(\tau_1)^{-\frac{1}{2}d_1 - m}$ times an integral of the form

$$\int_{X_1} \exp(i \sum_{j=2}^{s} \tau_j \psi_{j,\theta}(0, x_1)) \chi_m(x_1) dx_1. \tag{12.19}$$

Here χ_m is obtained by applying a linear partial differential operator with respect to the y_1-variables, of order $2m$, to g and then restricting to $y_1 = 0$. The coefficients of the differential operator depend smoothly on x_1, θ and $\tilde{\tau}_1$.

Repeating the procedure to the newly appearing integrals, and applying partial integrations as in the proof of Lemma 11.10 at noncritical points, we obtain the proposition below. In it, the parameters $\tilde{\tau}(\tau)$ are defined by

$$\begin{aligned}\tilde{\tau}_j(\tau) &= \left(\frac{\tau_{j+1}}{\tau_j}, \ldots, \frac{\tau_q}{\tau_j}\right) \quad (j = 1, \ldots, q-1),\\ \tilde{\tau}(\tau) &= (\tilde{\tau}_1(\tau), \ldots, \tilde{\tau}_q(\tau)),\end{aligned} \tag{12.20}$$

with the convention $\tilde{\tau}_q(\tau) = 0$, where $1 \leq q \leq s$.

PROPOSITION 12.2: *Let $\psi_{j,\theta}$ be a family of functions on X as above, in particular satisfying* (12.15) *and* (12.16). *Let U, resp. Θ_0 be a compact subset of X, resp. Θ, let $1 \leq q \leq s$ and let $M_1, \ldots, M_q$ be integers ≥ 1. For each sequence $m = (m_1, \ldots, m_q)$ of integers, $0 \leq m_j < M_j$, there exists a linear differential operator $Q_m = Q^{(q)}_{m,\theta,\tilde{\tau}(\tau)}$ in X of order $\leq 2\sum_{j=1}^q m_j$ and with smooth coefficients depending also smoothly on the parameters θ and $\tilde{\tau}$ such that the following holds. There is a continuous seminorm v on $C_c^\infty(U)$ such that for all $g \in C_c^\infty(U)$, $\theta \in \Theta_0$, $\tau_1 \geq \eta\tau_2 \geq \ldots \geq \eta^{s-1}\tau_s \geq 0$ with $\eta \geq 1$ sufficiently large,*

$$\begin{aligned}\int_X \exp\Big(i\sum_{j=1}^s \tau_j\psi_{j,\theta}(x)\Big)g(x)\,dx &= \exp\Big(i\sum_{j=1}^q \tau_j\psi_{j,\theta}(x^0)\Big)\prod_{j=1}^q l(\tau_j)^{-\frac{1}{2}d_j}\cdot\\ &\cdot\Bigg\{\sum_{m_1=0}^{M_1-1}\cdots\sum_{m_q=0}^{M_q-1}\prod_{j=1}^q l(\tau_j)^{-m_j}\int_{X_q}\exp\Big(i\sum_{k=q+1}^s \tau_k\psi_{k,\theta}(x_q)\Big)\cdot\\ &\cdot(Q_m g)(x_q)\,dx_q + R\Bigg\}\end{aligned} \tag{12.21}$$

where x^0 is any point on X_q and

$$|R| \leq v(g)l(\tau_q)^{-M},\ M = \min_{j=1,\ldots,q}\Big(M_j - \tfrac{1}{2}\sum_{i=j+1}^q d_i\Big). \tag{12.22}$$

In the proof it is used repeatedly that $l(\tau_j)^{-1}$ can be estimated by $l(\tau_q)^{-1}$ if $j \leq q$. For instance, if $x^0 \in X_j \backslash X_{j+1}$, $1 \leq j \leq q-1$, then the partial integrations in the integral over y_{j+1} will lead to a contribution of g with support in a neighborhood of x^0 which can be absorbed in the remainder term R. Applying a partition of unity to g this leads to contributions coming only from $x^0 \in X_q$. Note that the functions $\psi_{j,\theta}$ $(1 \leq j \leq q)$ are constant on X_q.

We present the intermediate result (12.21), (12.22) for all $1 \leq q \leq s$ because knowledge of the behaviour of the integral in its dependence on more τ_j-variables (larger q) in (12.21) is paid for by means of a worse estimate for the remainder term in (12.22). Indeed, $l(\tau_q)^{-M}$ is less small as

q increases. Of course, if $q = s$ then the oscillatory factor in the integrals over X_q in the right hand side of (12.21) is absent. This also happens if we restrict ourselves to families of $\tau_1, \ldots, \tau_s$ for which $\tau_{q+1}, \ldots, \tau_s$ converge to a finite limit, whereas $\tau_1, \ldots, \tau_q$ are allowed to run to infinity.

Applying Proposition 12.2 with $q = s$ and using the proof of Lemma 11.11 to determine the sharp order of the norm, we get

COROLLARY 12.3: *Let assumptions be as in Proposition 12.2. There exist a number $\eta \geq 1$ and a C^μ norm ν on $C_c^\mu(U)$ such that for all $g \in C_c^\infty(U)$, $\theta \in \Theta_0$, $\tau_1 \geq \eta\tau_2 \geq \ldots \geq \eta^{s-1}\tau_s \geq 0$ we have*

$$\left| \int_X \exp\left(i \sum_{j=1}^{s} \tau_j \psi_{j,\theta}(x)\right) g(x)\, dx \right| \leq \nu(g) \prod_{j=1}^{s} l(\tau_j)^{-\frac{1}{2}d_j}. \tag{12.23}$$

Here we may take $\mu = \frac{1}{2}\sum_{j=1}^{s} d_j$ and use Hölder norms if μ is not an integer.

After these heuristic arguments, we now verify that the functions $F_{H_j}|_{C_\gamma}$ satisfy the assumptions (12.16) for a suitable sequence of submanifolds $X \supset X_1 \supset \ldots \supset X_s$.

PROPOSITION 12.4: *Let $\gamma \in G$ be semisimple, $H \in \mathfrak{a}$ and let $\mathfrak{q}_{H,\gamma}$ be the orthogonal complement of $T(C_{H,\gamma})$ in $T_{C_{H,\gamma}}(C_\gamma)$. Then $\mathfrak{q}_{H,\gamma}$ is a smooth G_H-stable subbundle of $T_{C_{H,\gamma}}(C_\gamma)$ and its fiber at $x \in C_{H,\gamma}$ is equal to the orthogonal complement $\mathfrak{q}(x) = (\mathfrak{g}_x + \mathfrak{g}_H)^\perp$ of $\mathfrak{g}_x + \mathfrak{g}_H$ in $\mathfrak{g}$. Here we have used the identification of tangent spaces with linear subspaces of $\mathfrak{g}$ via left multiplication by x^{-1}. Moreover,*

$$T_{C_{H,\gamma}}(C_\gamma) = T(C_{H,\gamma}) \oplus \mathfrak{q}_{H,\gamma} \text{ (Whitney sum)}. \tag{12.24}$$

Recall that $C_{H,\gamma}$, the critical set of $F_H|_{C_\gamma}$, is equal to $C_\gamma \cap G_H$ and that this intersection is clean (Proposition 8.2 and 8.3). Let $x \in C_{H,\gamma}$; then

$$T_x(C_\gamma) = T_x(C_x) = (I - \operatorname{Ad} x^{-1})(\mathfrak{g}) \tag{12.25}$$

while

$$T_x(C_{H,\gamma}) = T_x(C_\gamma) \cap T_x(G_H). \tag{12.26}$$

Now $I - \operatorname{Ad} x^{-1}$ stabilizes both $\mathfrak{g}_H$ and $\mathfrak{g}_H^\perp$ and consequently its range is the direct sum of the ranges of its restrictions to these subspaces.

Hence $(I - \mathrm{Ad}\, x^{-1})(\mathfrak{g}) \cap \mathfrak{g}_H = (I - \mathrm{Ad}\, x^{-1})(\mathfrak{g}_H)$, giving

$$T_x(C_{H,\gamma}) = (I - \mathrm{Ad}\, x^{-1})(\mathfrak{g}_H) = (I - \mathrm{Ad}\, x^{-1})(\mathfrak{g}_x + \mathfrak{g}_H). \tag{12.27}$$

Further, by Lemma 8.6, $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_x + \mathfrak{g}_H$ and $\mathfrak{q}(x) = (\mathfrak{g}_x + \mathfrak{g}_H)^{\perp}$; moreover both are stable under $I - \mathrm{Ad}\, x^{-1}$, which is even invertible on $\mathfrak{q}(x)$. Hence $(I - \mathrm{Ad}\, x^{-1})(\mathfrak{g})$ is the direct sum of $T_x(C_{H,\gamma})$ and $\mathfrak{q}(x)$, proving (12.24). On the other hand $T_x(C_{H,\gamma}) \subset \mathfrak{g}_H$ and $\mathfrak{q}(x) \subset \mathfrak{g}_H^{\perp}$, therefore we get that $\mathfrak{q}(x)$ is the orthogonal complement of $T_x(C_{H,\gamma})$ in $T_x(C_\gamma)$. Finally, if $g \in G_H$, it is clear that $\mathfrak{q}(gxg^{-1}) = \mathrm{Ad}(g)(\mathfrak{q}(x))$, so $\mathfrak{q}_{H,\gamma}$ is G_H-stable.

Although $C_{H,\gamma}$ is in general not connected, its connected components have all the same dimension (Proposition 8.3) and therefore also $\mathfrak{q}_{H,\gamma}$ has constant fiber dimension.

LEMMA 12.5: *Let $\gamma \in G$ be semisimple and $H \in \mathfrak{a}$. Then, for all $x \in C_{H,\gamma}$, $H' \in \mathfrak{a}$,*

$$dF_{H',\gamma}|_{\mathfrak{q}_{H,\gamma}} = 0, \tag{12.28}$$

that is, $F_{H',\gamma}(x; Z) = 0$ for all $Z \in \mathfrak{q}(x)$.

Indeed, since $x \in G_H$, $n(x)^{-1} \in G_H$, see (2.6); and so $(H')^{n(x)^{-1}} \in \mathfrak{g}_H$. But then, as $\mathfrak{q}(x) \subset \mathfrak{g}_H^{\perp}$, we have according to (5.9), for $Z \in \mathfrak{q}(x)$,

$$F_{H',\gamma}(x; Z) = \langle Z, (H')^{n(x)^{-1}} \rangle = 0.$$

Now let $\pi = (S_1, \ldots, S_s)$ be an ordered partition of $S(C)$. We define, for $1 \leq j \leq s$,

$$\begin{aligned} &G_j = \bigcap_{\alpha \in S_1 \cup \ldots \cup S_j} G_{\alpha^\vee}, \quad \mathfrak{g}_j = \bigcap_{\alpha \in S_1 \cup \ldots \cup S_j} \mathfrak{g}_{\alpha^\vee}, \\ &C_{j,\gamma} = G_j \cap C_\gamma. \end{aligned} \tag{12.29}$$

It is clear that for any $H \in C^{\mathrm{int}}$ we have, with $H_i = \sum_{\alpha \in S_i} \alpha(H)\alpha^\vee$, $H'_j = \sum_{i=1}^{j} H_i$, and using Lemma 11.3,

$$\begin{aligned} &G_j = G_{H'_j} = \bigcap_{1 \leq i \leq j} G_{H_i}, \quad \mathfrak{g}_j = \mathfrak{g}_{H'_j} = \bigcap_{1 \leq i \leq j} \mathfrak{g}_{H_i}, \\ &C_\gamma = C_{0,\gamma} \supset C_{1,\gamma} \supset \ldots \supset C_{s,\gamma} \supset C_{s+1,\gamma} = \emptyset. \end{aligned} \tag{12.30}$$

With these notations, we have

LEMMA 12.6: *Let $H \in C^{\text{int}}$, $\pi = (S_1, \ldots, S_s)$ an ordered partition of $S(C)$. Then, for $1 \le j \le s$, $F_{H_j,\gamma}|_{C_{j-1,\gamma}}$ has $C_{j,\gamma}$ as a clean set of critical points.*

The stationary set of $F_{H_j,\gamma}$ is $C_\gamma \cap G_{H_j}$ which contains $C_{j,\gamma}$. Conversely, let $x \in C_{j-1,\gamma}$ be a stationary point for the restriction of $F_{H_j,\gamma}$ to $C_{j-1,\gamma}$. This means that the derivatives at x of $F_{H_j,\gamma}$ along the directions in $T_x(C_{j-1,\gamma})$ vanish. On the other hand, Lemma 12.5 shows that the derivatives of $F_{H_j,\gamma}$ at x along the directions in $T_x(C_\gamma)$ orthogonal to $T_x(C_{j-1,\gamma})$ also vanish. Hence x is a stationary point of $F_{H_j,\gamma}$; that is, $x \in G_{H_j} \cap C_{j-1,\gamma} = C_{j,\gamma}$. For the cleanness we must verify the following assertion (cf. (8.6)): if $Z \in \mathfrak{g}_x + \mathfrak{g}_{j-1}$ and $\langle L_{x,H_j,\gamma}(Z), Z' \rangle = 0$ for all $Z' \in \mathfrak{g}_x + \mathfrak{g}_{j-1}$, then $Z - Z^{x^{-1}} \in \mathfrak{g}_j$. Since $x \in G_{j-1}$, we can conclude from the formula for $L_{x,H_j,\gamma}$ given in Proposition 8.4 that $L_{x,H_j,\gamma}$ stabilizes $\mathfrak{g}_x + \mathfrak{g}_{j-1}$. Hence $L_{x,H_j,\gamma}(Z) = 0$. Proposition 8.5 now implies that

$$Z - Z^{x^{-1}} \in \mathfrak{g}_{H_j} \cap \mathfrak{g}_{j-1} = \mathfrak{g}_j.$$

The Lemmas 12.6 and 12.5 together imply that, with $X = C_\gamma$, $X_j = C_{j,\gamma}$, $\theta = H$ and $\psi_{j,\theta} = F_{H_j,\gamma}$, the conditions (12.15) and (12.16) are fulfilled. Hence

THEOREM 12.7: *The conclusions of Proposition 12.2 and Corollary 12.3 are valid, with $X = C_\gamma$, $\psi_{j,\theta} = F_{H_j,\gamma}$, $\Theta = C^{\text{int}}$.*

In order to show that this implies Theorem 12.1, we have to investigate the numbers

$$d_j = \dim(G_{j-1} \cap C_\gamma) - \dim(G_j \cap C_\gamma). \tag{12.31}$$

These are differences of the dimensions of successive elements of the flag of linear subspaces

$$T_x(C_\gamma) \supset T_x(G_1 \cap C_\gamma) \supset \ldots \supset T_x(G_t \cap C_\gamma) \tag{12.32}$$

where x is an arbitrary element of $G_t \cap C_\gamma$ and t is the largest number j such that $G_j \cap C_\gamma \neq \emptyset$.

Since the choice of x is at our disposal, we may assume that $x = u \exp Y$ where $u \in K \cap G_t$, $Y \in \mathfrak{a}$ and $Y^u = Y$; moreover, with such an x, $\mathfrak{g}_x = \mathfrak{g}_u \cap \mathfrak{g}_Y \subset \mathfrak{g}_Y$.

Hence, for $1 \le j \le t$,

$$\mathfrak{g}_x \cap \mathfrak{g}_j \subset \mathfrak{g}_Y \cap \mathfrak{g}_j \subset \mathfrak{g}_j$$

where $\mathfrak{g}_j$ is defined in (12.29). On the other hand, $T_x(G_j \cap C_\gamma) = (I - \mathrm{Ad}\, x^{-1})(\mathfrak{g}_j) = \mathfrak{g}_j \cap \mathfrak{g}_x^\perp$ in view of (12.27). Since $Y \in \mathfrak{a} \subset \mathfrak{g}_j$ and $Y^x = Y$, ad Y stabilizes $\mathfrak{g}_j \cap \mathfrak{g}_x^\perp$, so that

$$\mathfrak{g}_j \cap \mathfrak{g}_x^\perp = (\mathfrak{g}_j \cap \mathfrak{g}_x^\perp \cap \mathfrak{g}_Y) \oplus (\mathfrak{g}_j \cap \mathfrak{g}_x^\perp \cap \mathfrak{g}_Y^\perp).$$

But $\mathfrak{g}_Y^\perp \subset \mathfrak{g}_x^\perp$ and so we have

$$T_x(G_j \cap C_\gamma) = (\mathfrak{g}_j \cap \mathfrak{g}_Y^\perp) \oplus (\mathfrak{g}_j \cap \mathfrak{g}_Y \cap \mathfrak{g}_x^\perp). \tag{12.33}$$

In other words, the flag (12.32) is the *direct sum* of the two flags $(\mathfrak{g}_j \cap \mathfrak{g}_Y^\perp)_{1 \le j \le t}$ and $(\mathfrak{g}_j \cap \mathfrak{g}_Y \cap \mathfrak{g}_x^\perp)_{1 \le j \le t}$. Consequently, if we define

$$d'_j = \dim(\mathfrak{g}_{j-1} \cap \mathfrak{g}_Y^\perp) - \dim(\mathfrak{g}_j \cap \mathfrak{g}_Y^\perp) \;\; (1 \le j \le t), \tag{12.34}$$

then

$$d_j \ge d'_j. \tag{12.35}$$

The inequalities (12.35) mean that we can replace d_j by d'_j in the estimate (12.23). Let us now write $\Delta(Y)$ for the set of roots $\alpha \in \Delta$ for which $\alpha(Y) \ne 0$; let $P_{s+1} = \emptyset$, and for $1 \le j \le s$ let P_j be the set of roots in $P(C)$ (cf. (11.50)) which are linear combinations of $S_j \cup \ldots \cup S_s$. As in the proof of Lemma 11.7,

$$d'_j = 2 \sum_{\alpha \in \Delta(Y) \cap (P_j \setminus P_{j+1})} n(\alpha) \qquad (1 \le j \le s). \tag{12.36}$$

As a consequence,

$$\prod_{j=1}^{s} l(\tau_j)^{-\frac{1}{2}d'_j} \le \text{const.} \prod_{\alpha \in \Delta(Y) \cap P(C)} (1 + \alpha(H))^{-n(\alpha)}. \tag{12.37}$$

We now observe that x and γ are conjugate in G. This implies that Y and $\log \gamma_R$ are conjugate under G, hence under $\mathfrak{w}$. But then, using also that $P(C)$ is equal to the conjugate of Δ^+ under some element of $\mathfrak{w}$, $\Delta(Y) \cap P(C) = \Delta_w^+(\gamma_R)$ for some $w \in \mathfrak{w}$; so certainly

$$\prod_{j=1}^{s} l(\tau_j)^{-\frac{1}{2}d'_j} \le \text{const.} \sum_{w \in \mathfrak{w}} \prod_{\alpha \in \Delta_w^+(\gamma_R)} (1 + |\alpha(H)|)^{-n(\alpha)}. \tag{12.38}$$

Therefore, *Theorem 12.1 is proved.*

We conclude this section by remarking that Proposition 12.2 can also be applied with $\psi_{j,\theta}$ replaced by the functions F_{a,H_j} on K (cf. (11.20)).

Then we should take a, H as parameters, with H in an open Weyl chamber C^{int} and a restricted to a compact equisingular subset ω of A (that is $K_a = K_{a'}$ if $a, a' \in \omega$). The filtration (12.15) has to be replaced by the

$$K_{a,j} = K_a \mathfrak{w} K_{(S_1 \cup \ldots \cup S_j)^\vee} \quad (1 \leq j \leq s) \tag{12.39}$$

Because $K_{a,j}$ may have different components which even can have different dimensions, a localization by means of a partition of unity is essential. However, for each $w \in \mathfrak{w}$ the components of

$$K_{w,a,j} = K_a w K_{(S_1 \cup \ldots \cup S_j)^\vee} \tag{12.40}$$

all have the same dimension. Moreover F_{a,H_i} is constant on it and equal to $\langle wH_i, \log a\rangle$ for $i \leq j$. It is by now a straightforward application of the properties of the critical points of the $F_{a,H}$ to verify (12.16)(i) locally. If F_{a,H_j} would be replaced by f_{X,H_j} ($X \in \mathfrak{a}$, $a = \exp X$), then we could take for the complementary bundles in (12.16)(ii) the orthogonal complements. So, for the F_{a,H_j}, we take the images of these orthogonal complements under the automorphism of $T(K)$ described infra Lemma 5.9; and again (12.16)(ii) will hold.

Finally, $d_{w,a,j} = \dim K_{w,a,j-1} - \dim K_{w,a,j}$ is given by

$$d_{w,a,j} = \sum_{\alpha \in [\Delta^+_{\pi,j} \setminus \Delta^+_{\pi,j+1}] \cup \Delta^+(a^{w^{-1}})} n(\alpha). \tag{12.41}$$

It is equal to the number $n_{\pi,j}$ in (11.25). With these notations, we have

THEOREM 12.8: *Let C be an open Weyl chamber and let $\pi = (S_1, \ldots, S_s)$ be an ordered partition of the set $S(C)$ of simple roots corresponding to C. For $1 \leq q \leq s$, $w \in \mathfrak{w}$ and for each sequence $m = (m_1, \ldots, m_q)$ of integers ≥ 0 there exist linear differential operators $Q_{w,m} = Q^{(q)}_{w,m,a,\underline{H},\tilde{\tau}(\tau)}$ in K of order $\leq 2\sum_{j=1}^q m_j$ and with smooth coefficients depending smoothly on the parameters a, $\underline{H}$, $\tilde{\tau}$, such that*

$$\begin{aligned} I_{a,H}(g) \sim \sum_{w \in \mathfrak{w}} e^{i\langle wH', \log a\rangle} \prod_{j=1}^{q} l(\tau_j)^{-\frac{1}{2}d_{w,a,j}} \cdot \\ \cdot \sum_{m} \prod_{j=1}^{q} l(\tau_j)^{-m_j} \int_{K_{w,a,j}} \exp(i F_{a,H''}(k))(Q_{w,m} g)(k)\, dk. \end{aligned} \tag{12.42}$$

In this expansion we restrict a to a compact equisingular subset of A, $H = \sum_{j=1}^s \tau_j \underline{H}_j$ with $\underline{H}$ in a compact subset of C and $\tau_1 \geq \gamma\tau_2 \geq \ldots \geq \gamma^{s-1}\tau_s \geq 0$ with $\gamma \geq 1$ sufficiently large. Furthermore $g \in C^\infty(K)$ and *we have written*

$$H' = \sum_{j=1}^{q} \tau_j \underline{H}_j, \quad H'' = \sum_{j=q+1}^{s} \tau_j \underline{H}_j. \tag{12.43}$$

The expansion is uniform in the sense that, for any $N \geq 0$, *the remainder term, if we replace the right hand side of* (12.42) *by a sufficiently large finite sum over m, can be estimated by* $\nu(g)l(\tau_q)^{-N}$. *Here* ν *is a continuous seminorm on* $C^\infty(K)$ *independent of* a, $\underline{H}$, τ.

The radial asymptotic expansions of Theorem 9.1 are retrieved if we substitute $\tau_j = \omega \underline{\tau}_j$, for fixed $\underline{\tau}_j$ $(1 \leq j \leq s)$, and let $\omega \to +\infty$. For $q = s$ and $\underline{\tau}_j > 0$ for all $1 \leq j \leq s$ this leads to the radial asymptotic expansions for regular H, whereas the cases of singular H are obtained by taking $q < s$ and $\underline{\tau}_{q+1} = \ldots = \underline{\tau}_s = 0$, making $H'' = 0$ in (12.42).

REFERENCES

[1] T.F. Banchoff: Computer animation and the geometry of surfaces in 3- and 4-space. *Proc. Int. Congr. of Math.*, Helsinki, 1978, pp. 1005–1013. Helsinki: Academia Scientiarum Fennica 1980.

[2] F.A. Berezin and I.M. Gel'fand: Some remarks on the theory of spherical functions on symmetrical Riemannian manifolds. *Trudy Moskov. Mat. Obšč 5* (1956) 311–351. English transl.: *Amer. Math. Soc. Translations* (*2*) *21* (1962) 193–238.

[3] I.N. Bernstein, I.M. Gel'fand and S.I. Gel'fand: Schubert cells and cohomology of the spaces *G/P*. *Uspekhi Mat. Nauk. 28* (1973) 3–26. English transl.: *Russ. Math. Surveys 28* (1973) 1–26.

[4] A. Borel: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. 57* (1953) 115–207.

[5] A. Borel: Kählerian coset spaces of semi-simple Lie groups. *Proc. Nat. Acad. Sci., U.S.A. 40* (1954) 1147–1151.

[6] A. Borel: Linear Algebraic Groups. New York: Benjamin 1969.

[7] A. Borel and J. Tits: Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math. 27* (1965) 55–151.

[8] A. Borel and J. Tits: Compléments à l'article "Groupes réductifs". *Inst. Hautes Études Sci. Publ. Math. 41* (1972) 253–276.

[9] R. Bott: On torsion in Lie groups. *Proc. Nat. Acad. Sci., U.S.A. 40* (1954) 586–588.

[10] R. Bott: On the iteration of closed geodesics and the Sturm intersection theory. *Comm. Pure Appl. Math. 9* (1956) 171–206.

[11] R. Bott and H. Samelson: Applications of the theory of Morse to symmetric spaces. *Amer. J. Math. 80* (1958) 964–1029.

[12] T.E. Cecil and P.J. Ryan: Tight spherical embeddings. In: Global Differential Geometry and Global Analysis. Berlin, 1979, pp. 94–104. *Lecture Notes in Mathematics 838*. Berlin: Springer-Verlag 1981.

[13] J. Chazarain: Formule de Poisson pour les variétés riemanniennes. *Invent. Math. 24* (1974) 65–82.

[14] L. Cohn: Analytic Theory of the Harish–Chandra *C*-function. *Lecture Notes in Mathematics 429*. Berlin: Springer-Verlag 1974.

[15] Y. Colin de Verdière: Spectre du Laplacien et longueurs des géodésiques périodiques, II. *Comp. Math. 27* (1973) 159–184.

[16] M. DEMAZURE: Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup.* *(4) 7* (1974) 53–88.
[17] V.V. DEODHAR: On Bruhat ordering and weight-lattice ordering for a Weyl group. *Nederl. Akad. Wetensch. Proc. Ser. A 81 = Indag. Math. 40* (1978) 423–435.
[18] J. DIEUDONNÉ: Foundations of Modern Analysis. New York: Academic Press 1960.
[19] J.J. DUISTERMAAT: Oscillatory integrals, Lagrange immersions and unfolding of singularities. *Comm. Pure Appl. Math. 27* (1974) 207–281.
[20] J.J. DUISTERMAAT, J.A.C. KOLK and V.S. VARADARAJAN: Spectra of compact locally symmetric manifolds of negative curvature. *Invent. Math. 52* (1979) 27–93.
[21] C. EHRESMANN: Sur la topologie de certaines variétés algébriques réelles. *J. Math. Pures Appl. (9) 16* (1937) 69–100.
[22] A. ERDÉLYI: Asymptotic Expansions. New York: Dover 1956.
[23] R. GANGOLLI: On the Plancherel formula and the Paley–Wiener theorem for spherical functions on semisimple Lie groups. *Ann. of Math. 93* (1971) 150–165.
[24] I.M. GELFAND and M.A. NEUMARK: Unitäre Darstellungen der klassischen Gruppen. Berlin: Akademie-Verlag 1957.
[25] S.G. GINDIKIN: Unitary representations of groups of automorphisms of Riemann symmetric spaces of null curvature. *Funkcional Anal. i Priložen 1: 1* (1967) 32–37. English transl.: *Functional Anal. Appl. 1* (1967) 28–32.
[26] V. GUILLEMIN and A. POLLACK: Differential Topology. Englewood Cliffs: Prentice-Hall 1974.
[27] H.C. HANSEN: On cycles in flag manifolds. *Math. Scand. 33* (1973) 269–274.
[28] HARISH–CHANDRA: Spherical functions on a semisimple Lie group, I. *Amer. J. Math. 80* (1958) 241–310.
[29] HARISH–CHANDRA: Spherical functions on a semisimple Lie group, II. *Amer. J. Math. 80* (1958) 553–613.
[30] HARISH–CHANDRA: Some results on differential equations and their applications. *Proc. Nat. Acad. Sci., U.S.A. 45* (1959) 1763–1764.
[31] HARISH–CHANDRA: Harmonic analysis on real reductive groups, I. The theory of the constant term. *J. of Functional Analysis 19* (1975) 104–204.
[32] G.J. HECKMAN: Projections of Orbits and Asymptotic Behaviour of Multiplicities for Compact Lie Groups. Thesis, Rijksuniversiteit Leiden (1980).
[33] S. HELGASON: Differential Geometry and Symmetric Spaces. New York: Academic Press 1962.
[34] R. HERB: Discrete series characters and Fourier inversion on semisimple real Lie groups. Preprint, University of Maryland (1980). To appear *T.A.M.S.*
[35] R. HERMANN: Geometric aspects of potential theory in symmetric spaces. III. *Math. Ann. 153* (1964) 384–394.
[36] L. HÖRMANDER: Fourier integral operators, I. *Acta Math. 127* (1971) 79–183.
[37] G.A. HUNT: A theorem of Élie Cartan. *Proc. Amer. Math. Soc. 7* (1956) 307–308.
[38] A.A. KIRILLOV: Unitary representations of nilpotent Lie groups. *Uspekhi Mat. Nauk. 17* (1962) 57–110. English transl.: *Russ. Math. Surveys 17* (1962) 53–104.
[39] S.L. KLEIMAN: Problem 15. Rigorous foundation of Schubert's enumerative calculus. In: Mathematical Developments arising from Hilbert Problems. *Proc. Sympos. Pure Math.*, vol. 28, pp. 445–482. Providence: Amer. Math. Soc. 1976.
[40] B. KOSTANT: Lie algebra cohomology and the generalized Borel–Weil theorem. *Ann. of Math. 74* (1961) 329–387.
[41] B. KOSTANT: Lie algebra cohomology and generalized Schubert cells. *Ann. of Math. 77* (1963) 72–144.
[42] B. KOSTANT: On convexity, the Weyl group and the Iwasawa decomposition. *Ann. Sci. École Norm. Sup. (4) 6* (1973) 413–455.
[43] N.H. KUIPER: Minimal total absolute curvature for immersions. *Invent. Math. 10* (1970) 209–238.
[44] S. LEFSCHETZ: Intersections and transformations of complexes and manifolds. *Trans. Amer. Math. Soc. 28* (1926) 1–49.
[45] S. ŁOJASIEWICZ: Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 18* (1964) 449–474.

[46] J. Milnor: Morse Theory. *Ann. of Math. Studies N° 51*. Princeton: Princeton University Press 1963.
[47] J. Milnor: Lectures on the *h*-Cobordism Theorem. Notes by L. Siebenmann and J. Sondow. Princeton: Princeton University Press 1965.
[48] D. Mumford: Introduction to Algebraic Geometry, preliminary version of Chapters I–III. Cambridge: Harvard Univ. Math. Dept. 1966.
[49] R.W. Richardson, Jr.: Conjugacy classes in Lie algebras and algebraic groups. *Ann. of Math. 86* (1967) 1–15.
[50] P.J. Sally, Jr and G. Warner: The Fourier transform on semisimple Lie groups of real rank one. *Acta Math. 131* (1973) 1–26.
[51] H. Schubert: Kalkül der Abzählenden Geometrie. Leipzig: Teubner 1879.
[52] J.T. Schwartz: Differential Geometry and Topology. New York: Gordon and Breach 1968.
[53] *Séminaire C. Chevalley:* Classification des Groupes de Lie Algébriques, vols. 1 en 2. Paris: Secrétariat Mathématique 1958.
[54] J.-P. Serre: Représentations linéaires et espaces homogènes kähleriennes des groupes de Lie compacts. Séminaire Bourbaki, 1953/1954. Exposé 100. Paris: Secrétariat Mathématique.
[55] T.A. Springer: The unipotent variety of a semisimple group. In: Algebraic Geometry. Papers presented at the Bombay Colloquium, 1968, pp. 373–391. London: Oxford University Press 1969.
[56] R. Steinberg: Conjugacy Classes in Algebraic Groups. *Lecture Notes in Mathematics 366*. Berlin: Springer-Verlag 1974.
[57] D. Sullivan: Combinatorial invariants of analytic spaces. In: Proceedings Liverpool Singularities – Symposium I, 1970, pp. 165–168. *Lecture Notes in Mathematics 192*. Berlin: Springer-Verlag 1971.
[58] M. Takeuchi: Cell decompositions and Morse equalities on certain symmetric spaces. *J. Fac. of Sci. Univ. of Tokyo 12* (1965) 81–192.
[59] M. Takeuchi and S. Kobayashi: Minimal imbeddings of *R*-spaces. *J. Differential Geometry 2* (1968) 203–215.
[60] R. Thom: Sur une partition en cellules associée à une fonction sur une variété. *C.R. Acad. Sci. Paris Sér. A-B 228* (1949) 973–975.
[61] J. Tits: Sur les *R*-espaces. *C.R. Acad. Sci. Paris Sér. A-B 239* (1954) 850–852.
[62] J. Tits: Sur certaines classes d'espaces homogènes de groupes de Lie. *Acad. Roy. Belg. Cl. Sci. Mem. Coll. 29* (1955), no. 3.
[63] J. Tits: Espaces homogènes complexes compacts. *Comment. Math. Helv. 37* (1962) 111–120.
[64] V.S. Varadarajan: Lie groups, Lie Algebras, and Their Representations. Englewood Cliffs: Prentice-Hall 1974.
[65] V.S. Varadarajan: Harmonic Analysis on Real Reductive Groups. *Lecture Notes in Mathematics 576*. Berlin: Springer-Verlag 1977.
[66] B.L. van der Waerden: Topologische Begründung des Kalküls der abzählenden Geometrie. *Math. Ann. 102* (1930) 337–362.
[67] B.L. van der Waerden: Einführung in die Algebraische Geometrie. Berlin: Springer 1939.

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Added in proof

It was brought to our attention that the method of stationary phase was applied to the integral form of the Weyl character formula by Semenov–Tjan–Šanskiĭ in Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 37 (1973) 53–65. English transl.: *J. Soviet Math.* 8 (1977) 208–218.