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POLARIZED ABELIAN VARIETIES AND THE HEAT EQUATIONS

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0. Introduction

This paper deals with the deformation theory connected with the heat equations satisfied by the classical theta functions. Our goal is to replace the analytical approach by a statement of algebraic nature, valid in any characteristic.

Basing on [3] and [4] we show: Let X be a smooth algebraic variety, T_X its tangent sheaf, and let D be an effective divisor on X. Any symmetric global section of $\otimes^2 T_X$ defines canonically – up to isomorphism – a linear infinitesimal deformation of the couple (X, D) (cf (1.9)).

(0.1) For a principally polarized abelian variety (X, θ), this leads to an explicit computation of the first order infinitesimal deformations $(X_\varepsilon, \theta_\varepsilon)$ of the couple (X, θ) along the directions of the local moduli space of X (cf (2.2) and (2.3)).

We apply this in Section 3 to show that the following theorem of Andreotti and Mayer goes through in positive characteristics different from 2 (We make no statements hereabout in the characteristic 2 case):

(0.2) THEOREM (cf [1], page 213): *Let $\mathscr{A}_g = \mathscr{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$ be the moduli space of complex principally polarized abelian varieties of dimension g, and let $\mathscr{J}_g \subset \mathscr{A}_g$ be the subspace parametrizing principally polarized jacobian varieties. The closure $\bar{\mathscr{J}}_g$ of $\mathscr{J}_g$ is an irreducible component of the locus N_{g-4} of principally polarized abelian varieties (X, θ) such that $\dim(\mathrm{Sing}(\theta)) \geq g - 4$.*

A short account of the main ideas of the proof of (0.2) may be found in: D. Mumford, Curves and their Jacobians, The University of Mich-

igan Press, Ann Arbor, 1975, pages 87–89. Andreotti and Mayer's arguments are analytical and make essential use of the heat equations. A main point in their proof is the following geometrical consequence of these equations: Let P be a point of multiplicity 2 of the theta divisor θ of a (complex) principally polarized abelian variety X. The directions of deformation in the local moduli space of X which preserve this singularity are dual with the projectivized tangent cone of θ at P.

We show that (0.1) leads to the same conclusion, in any characteristic (cf (3.3)). The rest of the proof is almost the same as the classical one. The irreducibility of the moduli space of curves in char p, needed to state the theorem in its form (0.2), is taken from [2]. The reason for skipping the case $p = 2$ is a separability question in the proof of Lemma (3.8), which we have not been able to overcome in this case.

In a different direction, but closely related with the foregoing, we study the relationship between the heat equations (cf (2.2) for this terminology) and Mumford's theory of theta structures ([10]). Let B be a k-scheme, k being an algebraically closed field. Given an abelian scheme over B with a relatively ample line bundle $\mathscr{L}$ on it, of separable type, we consider the family of effective divisors on the fibres, which are defined by the sections of the line bundles induced by $\mathscr{L}$. It follows from Mumford's work [10] (cf (2.7)) that this family is endowed with a canonical flat connection (the holonomy being the monodromy on the set of theta structures). We show in (2.12) that this connection is given by the heat equations.

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1. Linear infinitesimal deformations of sections of line bundles

(1.1) Throughout we denote by X a smooth algebraic variety, defined over an algebraically closed field k. The isomorphism classes of linear infinitesimal deformations of X are parametrized by $H^1 T_X$, where T_X denotes the sheaf of germs of k-derivations of $\mathcal{O}_X$.

If L is a line bundle on X (which we identify with its sheaf of germs of sections), the isomorphism classes of linear infinitesimal deformations of the couple (X, L) are parametrized by $H^1 \Sigma_L$, where Σ_L is the sheaf of germs of differential operators of order ≤ 1 of L.

The above goes back to Kodaira–Spencer ([6]; cf also [4], p. 13, Cor. 2). Assume now that $s \in H^0 L$ is a global section of L. We want to study linear infinitesimal deformations of the triple (X, L, s). By definition,

such a deformation is a triple $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$ where X_ε is a flat $k[\varepsilon]$-scheme, $\varepsilon^2 = 0$, L_ε is a line bundle on X_ε and s_ε is a global section of L_ε, together with isomorphisms $X \cong X_\varepsilon \otimes_{k[\varepsilon]} k$ and $L \cong L_\varepsilon \otimes_{k[\varepsilon]} k$ (compatible with the first one) such that $s_\varepsilon \otimes_{k[\varepsilon]} k$ goes over into s. We shall say, for short, that $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$ restricts to (X, L, s). An isomorphism between two deformations $(X'_\varepsilon, L'_\varepsilon, s'_\varepsilon)$ and $(X''_\varepsilon, L''_\varepsilon, s''_\varepsilon)$ is defined as a couple of $k[\varepsilon]$-isomorphisms $X'_\varepsilon \xrightarrow{\cong} X''_\varepsilon$ and $L'_\varepsilon \xrightarrow{\cong} L''_\varepsilon$ (compatible with the first one) sending s'_ε into s''_ε and restricting to the identity on (X, L, s).

Consider the morphism $d^1 s : \Sigma_L \to L$, defined by $(d^1 s)(D) = Ds$. With these notations one has:

(1.2) PROPOSITION: *The isomorphism classes of linear infinitesimal deformations of the triple (X, L, s) are parametrized canonically by the first hypercohomology group $\mathbb{H}^1(d^1 s)$ of the complex*

$$0 \longrightarrow \Sigma_L \xrightarrow{d^1 s} L \longrightarrow 0.$$

PROOF: Let $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$ be such a deformation. Consider a covering $\mathcal{U} = (U_i)_{i \in I}$ of X by open affine subsets $U_i = \operatorname{Spec} A_i$, and write $U_{ij} = U_i \cap U_j = \operatorname{Spec} A_{ij}$. The scheme X_ε is gotten by glueing the affine schemes $U_i[\varepsilon] = \operatorname{Spec} A_i[\varepsilon]$ along the open subsets $U_{ij}[\varepsilon] = \operatorname{Spec} A_{ij}[\varepsilon]$, by means of suitable isomorphisms

$$U_{ij}[\varepsilon] \xrightarrow{\cong} U_{ij}[\varepsilon]. \tag{1.3}$$

Writing $(U_i)_\varepsilon$ for the scheme structure induced on U_i by X_ε, the isomorphisms (1.3) are obtained by choosing isomorphisms $h_i : U_i[\varepsilon] \xrightarrow{\cong} (U_i)_\varepsilon$ and taking the composition $h_i^{-1} h_j$ on $U_{ij}[\varepsilon]$. The maps (1.3) correspond to isomorphisms

$$A_{ij}[\varepsilon] \xleftarrow[\cong]{\begin{pmatrix} 1 & 0 \\ v_{ij} & 1 \end{pmatrix}} A_{ij}[\varepsilon],$$

where $v_{ij} : A_{ij} \to A_{ij}$ are k-derivations. The transition conditions ensure that $\{v_{ij}\}$ yields a 1-cocycle of $\mathcal{U}$ with values in T_X. The latter gives the Kodaira–Spencer class of the deformation X_ε of X in $H^1 T_X$.

The exact sequence on $U_i[\varepsilon]$:

$$0 \to \mathcal{O}_{U_i} \to \mathcal{O}^*_{U_i[\varepsilon]} \to \mathcal{O}^*_{U_i} \to 0$$

$$f \mapsto 1 + f\varepsilon$$

yields $H^1\mathcal{O}^*_{U_i[\varepsilon]} \xrightarrow{\;\cong\;} H^1\mathcal{O}^*_{U_i}$, i.e. $\text{Pic } U_i[\varepsilon] \xrightarrow{\;\cong\;} \text{Pic } U_i$. Thus $L_\varepsilon | U_i[\varepsilon] \cong (L|U_i)[\varepsilon]$, and the bundle L_ε is described as the glueing of the bundles $(L|U_i)[\varepsilon]$ along $(L|U_{ij})[\varepsilon]$ by means of suitable isomorphisms $(L|U_{ij})[\varepsilon] \cong (L|U_{ij})[\varepsilon]$, compatible with (1.3). Writing $M_{ij} = H^0(U_{ij}, L)$, the above isomorphisms are described equivalently by

$$M_{ij}[\varepsilon] \xleftarrow{\;\cong\;} M_{ij}[\varepsilon],$$
$$\begin{pmatrix} 1 & 0 \\ \eta_{ij} & 1 \end{pmatrix}$$

where $\eta_{ij} : M_{ij} \to M_{ij}$ is a differential operator of order ≤ 1, with associated k-derivation $v_{ij} : A_{ij} \to A_{ij}$. By the transition conditions, $\{\eta_{ij}\}$ yields a 1-cocycle of $\mathcal{U}$ with values in Σ_L. This gives the deformation class of L_ε in $H^1\Sigma_L$.

Finally, writing $M_i = H^0(U_i, L)$, the section s_ε of L_ε is described as a collection of sections $a_i + b_i\varepsilon \in M_i[\varepsilon]$, $i \in I$, satisfying the transition conditions on U_{ij}:

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \eta_{ij} & 1 \end{pmatrix}\begin{pmatrix} a_i \\ b_i \end{pmatrix},$$

i.e.

$$\begin{cases} a_j = a_i \\ b_j - b_i = \eta_{ij}(a_i). \end{cases}$$

According to our notations, $a_i = s|U_i$, hence to give a section $s_\varepsilon \in H^0L_\varepsilon$ extending the section $s \in H^0L$ amounts to give a 0-cochain $\{b_i\} \in C^0(\mathcal{U}, L)$ such that, on U_{ij},

$$b_j - b_i = \eta_{ij}(s). \tag{1.4}$$

Now, since $\{\eta_{ij}\} \in C^1(\mathcal{U}, \Sigma_L)$ is a cocycle, this condition is equivalent to saying that $(\{b_i\}, \{\eta_{ij}\}) \in C^0(\mathcal{U}, L) \oplus C^1(\mathcal{U}, \Sigma_L)$ is a 1-cocycle of the total complex associated with the double complex

$$\begin{array}{ccc} C^0(\mathcal{U}, \Sigma_L) & \longrightarrow & C^1(\mathcal{U}, \Sigma_L) \longrightarrow \cdots \\ \downarrow d^1s & & \downarrow -d^1s \\ C^0(\mathcal{U}, L) & \longrightarrow & C^1(\mathcal{U}, L) \longrightarrow \cdots, \end{array}$$

hence it defines an element of $\mathbb{H}^1(d^1 s)$. We call this the deformation class of the triple $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$. It is easily checked that it depends only on the isomorphism type of the triple and that, conversely, any element of $\mathbb{H}^1(d^1 s)$ is the deformation class of a triple $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$, uniquely determined upto isomorphism.   Q.E.D.

(1.5) REMARK: We look at the two spectral sequences of hypercohomology in this case. The first one yields an exact sequence

$$H^0 \Sigma_L \xrightarrow{\;d^1 s\;} H^0 L \xrightarrow{\;\alpha\;} \mathbb{H}^1(d^1 s) \xrightarrow{\;\beta\;} H^1 \Sigma_L \xrightarrow{\;d^1 s\;} H^1 L. \qquad (1.6)$$

The map α attaches to $b \in H^0 L$ the triple $(X[\varepsilon], L[\varepsilon], s + b\varepsilon)$. The map β is the forgetful one, sending $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$ into $(X_\varepsilon, L_\varepsilon)$.

To deal with the second one, we use the self-defining exact sequence

$$0 \longrightarrow K \longrightarrow \Sigma_L \xrightarrow{\;d^1 s\;} L \longrightarrow F \longrightarrow 0. \qquad (1.7)$$

The second spectral sequence then gives

$$0 \longrightarrow H^1 K \xrightarrow{\;\gamma\;} \mathbb{H}^1(d^1 s) \xrightarrow{\;\delta\;} H^0 F \xrightarrow{\;\eta\;} H^2 K, \qquad (1.8)$$

where $\gamma(\{\eta_{ij}\}) = (\{0\}, \{\eta_{ij}\})$, $\delta(\{b_i\}, \{\eta_{ij}\}) = \{b_i\} \in H^0 F$, and η is the iterated connecting homomorphism for the sequence (1.7). Note that, if $D \hookrightarrow X$ is the divisor defined by $s \in H^0 L$, $s \neq 0$, one has: $\mathrm{Supp}(F) = \mathrm{Sing}(D)$.

A short discussion about notations: If V is a vector space of finite dimension, we denote as usual by $S^d V$ its d-fold symmetric power, i.e. the quotient space of $\otimes^d V$ by the subspace generated by the elements of the type $v_1 \otimes \ldots \otimes v_d - v_{\sigma_1} \otimes \ldots \otimes v_{\sigma_d}$, $\sigma \in S_d$. We denote instead by $\bar{S}^d V$ the subspace of $\otimes^d V$ of the elements which are invariant under the standard action of S_d on $\otimes^d V$. The natural map $\otimes^d(V^v) \cong (\otimes^d V)^v$, given by the formula $\langle \omega_1 \otimes \ldots \otimes \omega_d, v_1 \otimes \ldots \otimes v_d \rangle = \Pi \langle \omega_i, v_i \rangle$, induces a canonical isomorphism $\bar{S}^d(V^v) \cong (S^d V)^v$.

The main ingredient of this paper is the following elementary

(1.9) FACT: Any element of $H^0(\bar{S}^2 T_X)$ – with X as in (1.1) – determines canonically a linear infinitesimal deformation class both for any couple (X, L) and for any triple (X, L, s) as in (1.1).

To justify this statement, we recall from [3], Section 16:

One denotes by $\mathscr{P}_X^n$, $n \geq 0$, the sheaf of principal parts of order n of $\mathscr{O}_X$. Similarly, $\mathscr{P}_X^n(L) = \mathscr{P}_X^n \otimes L$ is the sheaf of principal parts of order n of sections of L (this tensor product is taken with $\mathscr{P}_X^n$ endowed with its right module structure over $\mathscr{O}_X$, cf Loc. cit.). We write $\Sigma_L^{(n)} = \mathscr{D}\!\mathit{iff}^n(L, L) = \mathscr{H}\!\mathit{om}_{\mathscr{O}_X}(\mathscr{P}_X^n(L), L)$ for the sheaf of differential operators of order $\leq n$ on L. There is a standard exact sequence (ibid)

$$0 \to S^n \Omega_X^1 \to \mathscr{P}_X^n \to \mathscr{P}_X^{n-1} \to 0.$$

By tensorization with L and using the functor $\mathscr{H}\!\mathit{om}_{\mathscr{O}_X}(\ , L)$, one derives an exact sequence

$$0 \to \Sigma_L^{(n-1)} \to \Sigma_L^{(n)} \to \bar{S}^n T_X \to 0.$$

We are concerned with the cases $n = 1, 2$:

$$0 \to \mathscr{O}_X \to \Sigma_L \to T_X \to 0, \tag{1.10}$$

$$0 \to \Sigma_L \to \Sigma_L^{(2)} \to \bar{S}^2 T_X \to 0. \tag{1.11}$$

The latter yields, for any couple (X, L), a natural map

$$H^0(\bar{S}^2 T_X) \to H^1 \Sigma_L, \tag{1.12}$$

and, for any triple (X, L, s), a natural map

$$H^0(\bar{S}^2 T_X) \to \mathbb{H}^1(d^1 s),$$

the first map being equal to the composition of the second one with the forgetful morphism $\beta: \mathbb{H}^1(d^1 s) \to H^1 \Sigma_L$ of (1.6). The map (1.12) is, by definition, the first connecting homomorphism of (1.11) and (1.13) is taken to be, also by definition, the first connecting homomorphism of the exact sequence of hypercohomology of the short exact sequence of complexes

$$
\begin{array}{ccccccc}
0 \to & \Sigma_L & \to & \Sigma_L^{(2)} & \to & \bar{S}^2 T_X & \to 0 \\
& \downarrow{\scriptstyle d^1 s} & & \downarrow{\scriptstyle d^2 s} & & \downarrow & \\
0 \to & L & = L & \to & & 0 \to 0.
\end{array} \tag{1.14}
$$

The image of $w \in H^0(\bar{S}^2 T_X)$ by the map (1.13) will be denoted $w \cdot s \in \mathbb{H}^1(d^1 s)$.

Composing (1.12) with the forgetful morphism $H^1 \Sigma_L \to H^1 T_X$, deduced from (1.10) – and which attaches to $(X_\varepsilon, L_\varepsilon)$ the underlying de-

formation X_ε of X –, we get a natural map

$$\mu_L : H^0(\bar{S}^2 T_X) \to H^1 T_X. \tag{1.15}$$

(1.16) LEMMA: *Write* $[L] \in H^1 \Omega_X^1$ *the cohomology class of the bundle* L. *For all* $w \in H^0(\bar{S}^2 T_X)$ *one has:* $\mu_L(w) = -w \cup [L] + \mu_{\mathcal{O}_X}(w)$.

(1.17) REMARKS: The symbol $w \cup [L]$ means cup product, considering w as an element of $H^0(\otimes^2 T_X)$. We shall see in a moment that, for abelian varieties, $\mu_{\mathcal{O}_X} = 0$. (We dont know if this holds for any X as in (1.1)).

PROOF OF (1.16): To make (1.10) and (1.11) more explicit, we recall that, if D is a local section of Σ_L, its image v in T_X is determined by the requirement, for all $a \in \mathcal{O}_X$ and all $\sigma \in L$:

$$\langle v, da \rangle \sigma = D(a\sigma) - aD(\sigma), \tag{1,18}$$

and, if δ is a local section of $\Sigma_L^{(2)}$, its image w in $\bar{S}^2 T_X$ is characterized by the formula, for all $a, b \in \mathcal{O}_X$ and all $\sigma \in L$:

$$\langle w, da \otimes db \rangle \sigma = \delta(ab\sigma) - a\delta(b\sigma) - b\delta(a\sigma) + ab\delta(\sigma). \tag{1.19}$$

Let $\mathscr{U} = U_i)_{i \in I}$ be an affine open cover trivializing L. Put $\sigma_i : \mathcal{O}_{U_i} \xrightarrow{\sim} L | U_i$, and $\sigma_j = u_{ij}\sigma_i$ on U_{ij}. Then $[L] \in H^1 \Omega_X^1$ is given by the 1-cocycle $\{du_{ij}/u_{ij}\}$.

Fix $w \in H^0(\bar{S}^2 T_X)$, and let $\tilde{w}_i \in H^0(U_i, \Sigma_{\mathcal{O}_X}^{(2)})$ be liftings of w. We may assume (and we shall do so, for simplicity) that $\tilde{w}_i(1) = 0$. Then $\mu_{\mathcal{O}_X}(w) \in H^1 T_X$ is given by the 1-cocycle $\{\tilde{v}_{ij}\}$, $\tilde{v}_{ij} = \tilde{w}_j - \tilde{w}_i \in H^0(U_{ij}, T_X)$.

We compute $\mu_L(w)$. One defines a lifting $\delta_i \in H^0(U_i, \Sigma_L^{(2)})$ of w on U_i by putting $\delta_i(a\sigma_i) = \tilde{w}_i(a)\sigma_i$ for all $a \in \mathcal{O}_{U_i}$. Then $\mu_L(w) \in H^1 T_X$ is represented by the image $\{v_{ij}\}$ in T_X of the cocycle $\{\delta_j - \delta_i\} \in C^1(\mathscr{U}, \Sigma_L)$. For $a \in \mathcal{O}_{U_{ij}}$ one has, by using (1.18) and (1.19):

$$\begin{aligned}
\langle v_{ij}, da \rangle \sigma_j &= (\delta_j - \delta_i)(a\sigma_j) - a(\delta_j - \delta_i)(\sigma_j) = \\
&= \tilde{w}_j(a)\sigma_j - \tilde{w}_i(au_{ij})\sigma_i + a\tilde{w}_i(u_{ij})\sigma_i = \\
&= \tilde{w}_j(a)\sigma_j - \langle w, da \otimes du_{ij} \rangle \sigma_i - \tilde{w}_i(a)\sigma_j = \\
&= (\tilde{v}_{ij}(a) - \langle w, da \otimes (du_{ij}/u_{ij}) \rangle)\sigma_j
\end{aligned}$$

hence

$$v_{ij} = \tilde{v}_{ij} - \langle w, du_{ij}/u_{ij} \rangle. \qquad \text{Q.E.D.}$$

(1.20) If X is an abelian variety, $\mu_{\mathcal{O}_X} = 0$. This follows from the sequence (1.11) for $L = \mathcal{O}_X$:

$$0 \to \Sigma_{\mathcal{O}_X} \to \Sigma_{\mathcal{O}_X}^{(2)} \to \bar{S}^2 T_X \to 0,$$

together with the surjectivity of $H^0(\Sigma_{\mathcal{O}_X}^{(2)}) \to H^0(\bar{S}^2 T_X)$: a global section of $\bar{S}^2 T_X$ is translation invariant, hence it suffices to lift its value at $0 \in X$ to an element of $\Sigma_{\mathcal{O}_X}^{(2)}(0) = \mathrm{Diff}_k^2(\mathcal{O}_{X,0}, k)$ and then to propagate the latter, by translations, to any point of X.

2. Polarized abelian varieties

(2.1) We specialize to the case where X is an abelian variety and L is an ample line bundle on X, of degree prime to the characteristic of the ground field k.

The isomorphism classes of linear infinitesimal deformations of (X, L) are known to correspond naturally with the elements of $H^0(\bar{S}^2 T_X)$. We recall a proof of this fact, in the present context (cf e.g. [11], [12]). The separability assumption on L amounts to say that cup product with $[L] \in H^1 \Omega_X^1$ gives an isomorphism $H^0 T_X \xrightarrow{\simeq} H^1 \mathcal{O}_X$ (cf [12], p. 172). On the other side, consider sequence (1.10). The connecting homomorphisms are given by cup product with $-[L]$, hence the cohomology sequence gives in particular

$$0 \longrightarrow H^1 \Sigma_L \longrightarrow H^1 T_X \xrightarrow{\cup(-[L])} H^2 \mathcal{O}_X.$$

Imbedding this in the commutative diagram (cf (1.16), (1.17))

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\bar{S}^2 T_X) & \longrightarrow & H^0 T_X \otimes H^0 T_X & \longrightarrow & \Lambda^2(H^0 T_X) & \longrightarrow & 0 \\
& & \downarrow & & \simeq \downarrow 1 \otimes \cup(-[L]) & & \simeq \downarrow \Lambda^2(\cup[L]) & & \\
(1.12) & & \downarrow & & H^0 T_X \otimes H^1 \mathcal{O}_X & & \Lambda^2(H^1 \mathcal{O}_X) & & \\
& & \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\
0 & \longrightarrow & H^1 \Sigma_L & \longrightarrow & H^1 T_X & \xrightarrow{\cup(-[L])} & H^2 \mathcal{O}_X, & &
\end{array}
$$

where the top sequence is the obvious one, it follows that (1.12) is an isomorphism, as claimed.

(2.2) Given (X, L) as above and a section $s \in H^0 L$, the tautological diagram

$$H^0(\bar{S}^2 T_X) \xrightarrow{\ (1.13)\ } \mathbb{H}^1(d^1 s)$$

$$(1.12) \searrow \quad \simeq \quad \swarrow \beta$$

$$H^1 \Sigma_L$$

implies therefore that to any linear infinitesimal deformation $(X_\varepsilon, L_\varepsilon)$ of the polarized abelian variety (X, L) there is attached canonically – up to isomorphisms, cf (1.1) – a deformation $s_\varepsilon \in H^0 L_\varepsilon$ of s. We call s_ε the linear infinitesimal deformation of s *"defined by the heat equations"*. (This terminology seems to be the most appropiate one – compare with (2.12).)

(2.3) It is natural to ask about the geometrical meaning of the deformation s_ε of s distinguished in this way. To this end we consider the sequence (1.6); from (1.10) (cf also (2.1)) we deduce that $H^0 \mathcal{O}_X \cong H^0 \Sigma_L$, hence, taking into account that $H^1 L = 0$, sequence (1.6) yields here:

$$0 \longrightarrow H^0 L / \langle s \rangle \xrightarrow{\ \alpha\ } \mathbb{H}^1(d^1 s) \xrightarrow{\ \beta\ } H^1 \Sigma_L \longrightarrow 0. \qquad (2.4)$$

In particular, if L gives a principal polarization and $s \neq 0$, we get $\mathbb{H}^1(d^1 s) \cong H^1 \Sigma_L$, and the heat equations define the unique linear infinitesimal deformation s_ε of s (up to isomorphisms) attached to the given deformation $(X_\varepsilon, L_\varepsilon)$.

In the general case, this is connected with Mumford's theory of theta structures ([10]):

(2.5) Let $\pi : \mathcal{X} \to B$ be an abelian scheme over a k-scheme B, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$, of degree prime to the characteristic of the base field k. Call $\mathcal{H} \hookrightarrow \mathcal{X}$ the kernel of the morphism of abelian schemes

$$\mathcal{X} \to \hat{\mathcal{X}} = \underline{\mathrm{Pic}}^0_{\mathcal{X}/B}$$

induced by $\mathcal{L}$. This is a finite étale commutative group scheme over B.

The effective divisors occurring in the systems $|\mathcal{L}(t)|$, $t \in B(k)$, are the k-points of the projective bundle

$$\mathcal{P} = \mathbb{P}((R^0_\pi \mathcal{L})^v).$$

The group scheme $\mathcal{H}$ acts naturally on $\mathcal{P}$, by translations. In fact, $\mathcal{P}$

represents the functor on B-schemes attaching to T the set $\mathscr{P}(T)$ described as follows. If

$$
\begin{array}{ccc}
\mathscr{X}_T & \xrightarrow{\ F\ } & \mathscr{X} \\
\pi_T \downarrow & & \downarrow \pi \\
T & \xrightarrow{\ f\ } & B
\end{array}
\tag{2.6}
$$

is the induced pullback diagram then, writing $\mathscr{L}_T$ for $F^*\mathscr{L}$:

$$\mathscr{P}(T) = \{\text{global sections of the bundle } \mathbb{P}((R^0_{\pi_T}\mathscr{L}_T)^v)\} =$$

$$= \left\{\begin{array}{l} \text{effective relative Cartier divisors } \mathscr{D} \text{ on } \mathscr{X}_T \text{ such} \\ \text{that } [\mathcal{O}_{\mathscr{X}_T}(\mathscr{D})] = [\mathscr{L}_T] \text{ in } \mathrm{Pic}_{\mathscr{X}/B}(T) \end{array}\right\}.$$

On the other side (cf [10], II, p.p. 76–77, whose notations we shall alter a bit) the group scheme $\mathscr{H}$ represents the functor

$$\mathscr{H}(T) = \left\{\begin{array}{l} \text{global sections } \alpha \text{ of } \pi_T \text{ such that} \\ {[T^*_\alpha\mathscr{L}_T] = [\mathscr{L}_T] \text{ in } \mathrm{Pic}_{\mathscr{X}/B}(T)} \end{array}\right\}.$$

By attaching to $\alpha \in \mathscr{H}(T)$ and $\mathscr{D} \in \mathscr{P}(T)$ the element $T^*_{-\alpha}(\mathscr{D}) \in \mathscr{P}(T)$ one defines an action of the functor $\mathscr{H}$ on the functor $\mathscr{P}$, hence the claimed action as schemes follows.

The following proposition is inferred from [10]. Since it involves only a slight variation of the viewpoint of that paper, we could refer (somewhat vaguely) to [10]. However, for completeness sake, we give a full proof of it. In doing so, we shall use freely notations, definitions and results of [10]. (If our symbols differ occasionally from the original ones, this is done to preserve our own ones, previously chosen in this paper.) A reader not familiar with Mumford's paper and willing to skip the technical details below in a first reading, may consult pages 297, 298 of [10], I for the essential fact behind this result.

(2.7) PROPOSITION: *Assume that B is connected, and that $\mathscr{H}$ is a trivial group scheme over B (i.e. $\mathscr{H} \cong \mathscr{H}' \times_k B$, for some k-group $\mathscr{H}'$). Let $0 \in B(k)$; then:*

(a) *There is a unique trivialization $\mathscr{H} \xrightarrow{\cong} \mathscr{H}(0) \times_k B$ inducing the identity on $\mathscr{H}(0)$.*

(b) *There is an étale cover U of an open neighbourhood of $0 \in B$, and a trivialization $\mathscr{P}|U \xrightarrow{\cong} \mathscr{P}(0) \times_k U$ inducing the identity on $\mathscr{P}(0)$ and equivariant with (a).*

(c) *If U is connected, the trivialization of $\mathscr{P}|U$ as in* b) *is uniquely determined by these properties.*

PROOF: Part (a) is clear. As for (b), we start with the exact sequence over B ([10], II, p. 76)

$$0 \to \mathbb{G}_{m, B} \to \mathscr{G} \to \mathscr{H} \to 0,$$

where $\mathscr{G} = \mathscr{G}(\mathscr{L})$ and $\mathscr{H} = \mathscr{H}(\mathscr{L})$ in the notations of Loc. cit. We shall write also $\mathscr{G}$ the functor on B-schemes represented by $\mathscr{G}$ (ibid). The above sequence induces a bilinear map

$$e : \mathscr{H} \times_B \mathscr{H} \to \mathbb{G}_{m, B},$$

characterized by the following property (cf [8], p. 222). For any B-scheme T and sections $\tilde{\alpha}, \tilde{\beta} \in \mathscr{G}(T)$ with respective images $\alpha, \beta \in \mathscr{H}(T)$, the relation $e_T(\alpha, \beta) = \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1}\tilde{\beta}^{-1}$ holds in $H^0(T, \mathcal{O}_T^*)$.

Let δ be the type of $\mathscr{L}(0)$ (cf [10], I, p. 294), and consider, with the notations of Loc. cit., the k-group $H(\delta)$ endowed with its natural symplectic structure. We fix a symplectic isomorphism $\mathscr{H}(0) \xrightarrow{\simeq} H(\delta)$. Writing $\mathscr{H}_\delta = H(\delta) \times_k B$, the latter one extends uniquely to an isomorphism of group schemes over B, $\mathscr{H} \xrightarrow{\simeq} \mathscr{H}_\delta$, which (by étaleness and finiteness, again) is symplectic.

The level structure for $\mathscr{L}$ gotten in this way can be lifted, locally in B, to a theta structure. That is ([10], II, p. 78), defining the k-group $\mathscr{G}(\delta)$ as in [10], I, p. 294, and putting $\mathscr{G}_\delta = \mathscr{G}(\delta) \times_k B$, there exists an étale cover U of an open neighbourhood of $0 \in B$ and an isomorphism of group schemes $\mathscr{G}|U \xrightarrow{\simeq} \mathscr{G}_\delta|U$ fitting in the diagram

$$
\begin{array}{ccccccccc}
0 & \to & \mathbb{G}_{m, U} & \to & \mathscr{G}|U & \to & \mathscr{H}|U & \to & 0 \\
& & \| & & \downarrow \simeq & & \downarrow \simeq & & \\
0 & \to & \mathbb{G}_{m, U} & \to & \mathscr{G}_\delta|U & \to & \mathscr{H}_\delta|U & \to & 0.
\end{array}
$$

Here the bottom row is gotten from

$$0 \to k^* \to \mathscr{G}(\delta) \to H(\delta) \to 0$$

([10], I, p. 294) by base extension, and the right hand side vertical arrow is the previously defined one. This follows at once, by using the functors represented by these schemes and imitating the proof of the Corollary at pp. 294–295 in [10], I (cf also [10], II, pp. 79–80).

For the sake of brevity in our notations, we assume from now on that $U = B$, in this proof. We put $\mathscr{V} = R_\pi^0 \mathscr{L}$, a locally free $\mathcal{O}_B$-module of rank $d = \deg(\mathscr{L})$. Let $V(\delta)$ be the k-vectorspace defined in [10], I, p. 297, and put $\mathscr{V}_\delta = V(\delta) \otimes_k \mathcal{O}_B$. Denote furthermore by V the functor on B-schemes defined by $V(T) = H^0(T, R_{\pi_T}^0 \mathscr{L}_T)$ (cf diagram (2.6)). The group scheme $\mathscr{G}$ acts on $\mathscr{P}$ (through $\mathscr{H}$) and this action is induced by a standard action of the functor $\mathscr{G}$ on the functor V (cf [10], II, p. 81). By Proposition 2 of Loc. cit., p. 80, there exists an invertible sheaf $\mathscr{M}$ on B and an isomorphism of $\mathcal{O}_B$-modules

$$\mathscr{V} \xrightarrow{\ \simeq\ } \mathscr{V}_\delta \otimes \mathscr{M},$$

yielding an isomorphism of projective bundles

$$\mathscr{P} = \mathbb{P}(\mathscr{V}^v) \xrightarrow{\ \simeq\ } \mathbb{P}(\mathscr{V}_\delta^v) = \mathbb{P}(V(\delta)^v) \times_k B$$

which is equivariant with the isomorphism $\mathscr{G} \xrightarrow{\simeq} \mathscr{G}_\delta$, hence with $\mathscr{H} \xrightarrow{\simeq} \mathscr{H}_\delta$. This proves (b).

Part (c) follows from the fact that the only $\mathscr{H}_\delta$-automorphisms of $\mathscr{P}_\delta = \mathbb{P}(V(\delta)^v) \times_k B$ are those given by the action of $\mathscr{H}_\delta$. To see this, consider such an automorphism. This amounts to give an invertible sheaf $\mathscr{N}$ and an isomorphism

$$f : \mathscr{V}_\delta \xrightarrow{\ \simeq\ } \mathscr{V}_\delta \otimes_{\mathcal{O}_B} \mathscr{N}$$

of $\mathcal{O}_B$-modules with the following property. For any B-scheme T such that $\mathscr{N}_T \cong \mathcal{O}_T$, the induced map

$$f_T : H^0(T, (\mathscr{V}_\delta)_T) \to H^0(T, (\mathscr{V}_\delta)_T) \otimes_{H^0(T, \mathcal{O}_T)} H^0(T, \mathscr{N}_T)$$

satisfies: for all $\tilde{\alpha} \in \mathscr{G}_\delta(T)$ there exists $u \in H^0(T, \mathcal{O}_T^*)$ such that

$$(\tilde{\alpha} \otimes 1) f_T = u \cdot f_T \tilde{\alpha}$$

(cf [10], II, p. 77, for the functor represented by $\mathscr{G}_\delta$).

It follows that $u = \chi_T(\tilde{\alpha})$, with

$$\chi_T : \mathscr{G}_\delta(T) \to \mathbb{G}_m(T)$$

a well-defined morphism. The group scheme homomorphism

$$\chi : \mathscr{G}_\delta \to \mathbb{G}_{m, B}$$

obtained in this way factors through $\mathcal{H}_\delta$, yielding a global section of the dual scheme of $\mathcal{H}_\delta$, $\chi \in \mathcal{H}_\delta^D(B)$. By self-duality, this corresponds with a global section $\gamma \in \mathcal{H}_\delta(B)$. Let $U \subset B$ be an open set such that $\mathcal{N}_U \cong \mathcal{O}_U$ and such that γ lifts to $\tilde{\gamma} \in \mathcal{G}_\delta(U)$ over U. Then, for all $\tilde{\alpha} \in \mathcal{G}_\delta(W)$, $W \subset U$ an open subset, we have (writing $\alpha \in \mathcal{H}_\delta(W)$ the image of $\tilde{\alpha}$):

$$(\tilde{\alpha} \otimes 1) f_W = \chi_W(\tilde{\alpha}) \cdot f_W \tilde{\alpha} = e_W(\alpha, \gamma) \cdot f_W \tilde{\alpha} = (\tilde{\alpha} \tilde{\gamma} \tilde{\alpha}^{-1} \tilde{\gamma}^{-1}) \cdot f_W \tilde{\alpha},$$

hence

$$(\tilde{\alpha} \otimes 1)\{(\tilde{\gamma}^{-1} \otimes 1) f_W\} = \{(\tilde{\gamma}^{-1} \otimes 1) f_W\} \tilde{\alpha}.$$

By Proposition 2 of [10], II, p. 80 again, this implies that

$$(\tilde{\gamma}^{-1} \otimes 1)(f|U): \mathcal{V}_\delta|U \to (\mathcal{V}_\delta \otimes \mathcal{N})|U$$

is multiplication by a section $u \in H^0(U, \mathcal{O}_U^*)$, hence

$$f|U = u\tilde{\gamma} \otimes 1.$$

Thus the automorphism of $\mathcal{P}_\delta$ we started with is given on U by the action of $\gamma|U$. This ends the proof of Proposition (2.7).   Q.E.D.

Hence $\mathcal{P}$ carries a canonical flat connection. In particular, it makes sense to speak of horizontal sections of $\mathcal{P}$ (in the étale topology) and hence of horizontal deformations of k-points of $\mathcal{P}$.

We claim that the horizontal deformations are given, linear-infinitesimally, by the heat equations (cf 2.12) below). To make this statement precise, let $0 \in B(k)$ be fixed, write $X = \mathcal{X}(0)$, $L = \mathcal{L}(0)$, $H = \mathcal{H}(0)$, $P = \mathcal{P}(0)$, and let $D \in |L|$. If $s \in H^0 L$ is an equation for D, there is a canonical morphism, of Kodaira–Spencer type:

$$T_{\mathcal{P}}(D) \to \mathbb{H}^1(d^1 s), \tag{2.8}$$

making the following diagram commutative (the bottom arrow being the usual Kodaira–Spencer map):

$$
\begin{array}{ccc}
T_{\mathcal{P}}(D) & \to & \mathbb{H}^1(d^1 s) \\
\downarrow & & \downarrow \beta \\
T_B(0) & \to & H^1 \Sigma_L \subset H^1 T_X.
\end{array}
\tag{2.9}
$$

To see this, let $v: \operatorname{Spec} k[\varepsilon] \to B$ be a tangent vector of B at 0, and write $X_\varepsilon = \mathcal{X} \times_B \operatorname{Spec} k[\varepsilon]$, $L_\varepsilon = \mathcal{L} \otimes_{\mathcal{X}} X_\varepsilon$. An element of $T_{\mathcal{P}}(D)$ mapping to

$v \in T_B(0)$ is given by a section $s_\varepsilon \in H^0 L_\varepsilon$ restricting to $s \in H^0 L$ and being taken modulo units of $k[\varepsilon]$. Hence $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$ so constructed determines an element of $\mathbb{H}^1(d^1 s)$, according to Proposition (1.2). The rest is straightforward.

The kernels of the vertical arrows of (2.9) are both isomorphic with $H^0 L / \langle s \rangle$ (cf (2.4)), and they are identified by the above map (2.8). Therefore, if $v \in T_B(0)$ is mapped into $w \in H^0(\bar{S}^2 T_X)$ by the Kodaira–Spencer map (cf (2.2)), (2.8) yields an isomorphism

$$\left\{\begin{matrix}\text{tangent vectors to } \mathscr{P} \text{ at } D \\ \text{mapping to } v \in T_B(0)\end{matrix}\right\} \underset{(2.8)}{\leftrightarrow} \left\{\begin{matrix}\text{classes in } \mathbb{H}^1(d^1 s) \\ \text{mapping to } w \in H^0(\bar{S}^2 T_X)\end{matrix}\right\}. \qquad (2.10)$$

In particular, if the Kodaira–Spencer map yields an isomorphism $T_B(0) \xrightarrow{\cong} H^1 \Sigma_L$, the same holds for (2.8).

(2.11) The latter is e.g. the case, if B is an étale cover of a fine moduli scheme for separably polarized abelian varieties, $\mathscr{X} \to B$ is the pullback of the universal family, and $\mathscr{L}$ is a relatively ample line bundle on $\mathscr{X}$ inducing on each fiber the given polarization (cf [9], p. 129, 139, and [11], Theorem (2.4.1)).

(2.12) PROPOSITION: *With the identification* (2.10), *the horizontal lift* $\tilde{v} \in T_{\mathscr{P}}(D)$ *of* $v \in T_B(0)$ *is given by* $w \cdot s \in \mathbb{H}^1(d^1 s)$ (*cf* (1.14) *for notations*).

PROOF: If $s' \in H^0 L$ is any equation for $D' \in P$, the image of $w \cdot s' \in \mathbb{H}^1(d^1 s')$ in $T_{\mathscr{P}}(D')$ is independent of the particular choice of s'. We denote this image by the symbol $w \cdot D'$. To prove the proposition, it suffices to show that $w \cdot D$ is a horizontal tangent vector.

By base change via $v: \operatorname{Spec} k[\varepsilon] \to B$, the B-schemes $\mathscr{X}$, $\mathscr{P}$ and $\mathscr{H}$ define $k[\varepsilon]$-schemes X_ε, P_ε, and H_ε respectively. We denote by L_ε the line bundle on X_ε obtained in the same way from $\mathscr{L}$. The group scheme H_ε is constant, hence Proposition (2.7) applies. Let

$$H(k) \to H_\varepsilon(k[\varepsilon]) \qquad (2.13)$$

be the inverse of the group isomorphism defined by pulling back sections of H_ε to sections of H.

The horizontal liftings of $v \in T_B(0)$ define a map

$$P(k) \to P_\varepsilon(k[\varepsilon]) \qquad (2.14)$$

which, in view of Proposition (2.7), is determined uniquely by the following properties:

(1) (2.14) is deduced from a k-linear morphism $H^0(X, L) \to H^0(X_\varepsilon, L_\varepsilon)$ which yields the identity when composed with the restriction map $H^0(X_\varepsilon, L_\varepsilon) \to H^0(X, L)$;

(2) (2.14) is equivariant with (2.13).

We claim that the map defined by

$$D' \mapsto w \cdot D', D' \in P(k) \tag{2.15}$$

satisfies the required properties.

To prove (1), we fix an affine open covering $\mathcal{U} = (U_i)_{i \in I}$ of X, and lift $w \in H^0(\bar{S}^2 T_X)$ to $\delta_i \in H^0(U_i, \Sigma_L^{(2)})$ for each $i \in I$. Then $w \cdot s' \in \mathbb{H}^1(d^1 s')$ is given by the 1-hypercocycle $(\{\delta_i(s')\}, \{\delta_j - \delta_i\})$. The collection of sections $(s' | U_i) + \delta_i(s')\varepsilon \in H^0(U_i[\varepsilon], (L | U_i)[\varepsilon])$ yields a section $s'_\varepsilon \in H^0(X_\varepsilon, L_\varepsilon)$, and the k-linear morphism

$$H^0(X, L) \to H^0(X_\varepsilon, L_\varepsilon), s' \mapsto s'_\varepsilon$$

shows that (2.15) verifies property (1).

To prove (2), let $u \in H(k)$, and call $u_\varepsilon \in H_\varepsilon(k[\varepsilon])$ its image by (2.13). We choose an arbitrary u-isomorphism $\phi : L \xrightarrow{\simeq} L$, i.e. an isomorphism of line bundles making the following diagram commutative

$$
\begin{array}{ccc}
L & \xrightarrow[\simeq]{\phi} & L \\
\downarrow & & \downarrow \\
X & \xrightarrow[+u]{} & X.
\end{array}
$$

If $s' \in H^0 L$, $s' \neq 0$, defines the divisor $D' \in P$, then the translate $D'_u = T^*_{-u}(D')$ of D' is defined by $s'_u = \phi(s') \in H^0 L$.

We define an u-isomorphism $\phi' : \Sigma_L \xrightarrow{\simeq} \Sigma_L$ by putting, for any section λ of $\Sigma_L : \phi'(\lambda) = \phi \lambda \phi^{-1}$. The so obtained u-isomorphism of complexes

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma_L & \xrightarrow{d^1 s'} & L & \longrightarrow & 0 \\
 & & \phi' \downarrow \simeq & & \simeq \downarrow \phi & & \\
0 & \longrightarrow & \Sigma_L & \xrightarrow{d^1 s'_u} & L & \longrightarrow & 0
\end{array}
$$

yields an isomorphism

$$\phi_* : \mathbb{H}^1(d^1 s') \xrightarrow{\simeq} \mathbb{H}^1(d^1 s'_u).$$

Let us call for a moment $\rho : P_\varepsilon(k[\varepsilon] \to P(k)$ the obvious restriction map.

It follows at once from the definitions, that the following diagram is commutative

$$\begin{array}{ccc}
\rho^{-1}(D') & \xrightarrow{(2.8)} & \mathbb{H}^1(d^1s') \\
\simeq \downarrow & & \simeq \downarrow \phi_* \\
\rho^{-1}(D'_u) & \xrightarrow{(2.8)} & \mathbb{H}^1(d^1s'_u),
\end{array} \qquad (2.16)$$

the left hand side vertical arrow being given by the action of u_ε. (Explicitly, if $\phi_\varepsilon : L_\varepsilon \xrightarrow{\simeq} L_\varepsilon$ is an u_ε-isomorphism extending ϕ, the action of u_ε on $P_\varepsilon(k[\varepsilon])$ is induced by the isomorphism $H^0 L_\varepsilon \xrightarrow{\simeq} H^0 L_\varepsilon$ defined by ϕ_ε.)

The commutativity of (2.16) reduces the sought-for equality $w \cdot (D'_u) = (w \cdot D')_{u_\varepsilon}$ to $w \cdot s'_u = \phi_*(w \cdot s')$. The latter equality states the commutativity of the diagram

$$\begin{array}{ccc}
\mathbb{H}^1(d^1s') & \xleftarrow{(1.13)} & \\
\simeq \downarrow \cdot \phi_* & & H^0(\bar{S}^2 T_X) \\
\mathbb{H}^1(d^1s'_u) & \xleftarrow{(1.13)} &
\end{array}$$

Finally, this commutativity follows from the u-isomorphism of diagrams of type (1.14) (the definition of ϕ'' being similar to that of ϕ'):

together with the fact that the global sections of $\bar{S}^2 T_X$ are invariant by translations.   Q.E.D.

3. On a theorem of Andreotti and Mayer

(3.1) We apply the preceding sections to ideas developed by Andreotti and Mayer in their paper [1]. Our goal is to extend the validity of their Theorem (cf (0.2) of our Introduction) to positive characteristics different from 2.

Let (X, L) as in (2.1); fix $s \in H^0L$, $s \neq 0$, and call $D \subset X$ the divisor defined by the equation $s \in H^0L$. Assume that $x \in D$ is a singular point of D, i.e. $(d^1s)(x) = 0$. From the commutative diagram (1.14) we infer that $(d^2s)(x)$ factors through $(\bar{S}^2 T_X)(x)$, yielding a morphism

$$(\bar{S}^2 T_X)(x) \to L(x). \tag{3.2}$$

Identifying $L(x)$ with k, this gives an element $\omega_x \in S^2\Omega^1_X(x) \cong H^0(S^2\Omega^1_X)$. If x is a point of multiplicity 2 of D, ω_x is an equation for the projectivized tangent cone of D at x; otherwise $\omega_x = 0$.

(3.3) LEMMA: *We keep the above notations and assumptions. Let $w \in H^0(\bar{S}^2 T_X)$ and let $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$ be a deformation representing $w \cdot s \in \mathbb{H}^1(d^1s)$. Then: there exists a deformation $x_\varepsilon \in X_\varepsilon(k[\varepsilon])$ of $x \in X(k)$ satisfying $s_\varepsilon(x_\varepsilon) = 0$ if and only if $\langle w, \omega_x \rangle = 0$.*

PROOF: This is an easy local computation; we stress its formal aspects.

Denote by (1.14) (x) the diagram gotten from (1.14) by replacing the sheaves by their pointwise fibre at $x \in X$. The natural morphism of diagrams (1.14) $\to$ (1.14) (x) produces a commutative square

$$
\begin{array}{ccc}
\mathbb{H}^1((d^1s)(x)) & \leftarrow & \bar{S}^2 T_X(x) \\
\uparrow & & \uparrow \\
\mathbb{H}^1(d^1s) & \longleftarrow & H^0(\bar{S}^2 T_X).
\end{array}
\tag{3.4}
$$

By assumption, $(d^1s)(x) = 0$, hence e.g. by (1.6), $\mathbb{H}^1((d^1s)(x))$ is identified in a natural way with $L(x)$. The upper horizontal arrow in (3.4) then becomes identified with (3.2), and the condition $\langle w, \omega_x \rangle = 0$ is equivalent with $w \cdot s \in \mathbb{H}^1(d^1s)$ having zero restriction in $\mathbb{H}^1((d^1s)(x)) = L(x)$.

Let $(X_\varepsilon, L_\varepsilon, s_\varepsilon)$ be any deformation of (X, L, s) and let $(\{b_i\}, \{\eta_{ij}\})$ be an associated cocycle, as in the proof of Proposition (1.2). We keep the notations of that proof, and assume also that the covering $\mathcal{U} = (U_i)_{i \in I}$ yields trivial bundles $L|U_i$. Then $s_\varepsilon | U_i[\varepsilon]$ is given by $s_i + b_i\varepsilon \in A_i[\varepsilon]$, with $s_i = s|U_i$. Suppose $x \in U_j$. Using the isomorphism $(U_j)_\varepsilon \cong U_j[\varepsilon]$, a section $x_\varepsilon \in X_\varepsilon(k[\varepsilon])$ restricting to $x \in X(k)$ is given equivalently by a tangent vector $v_x \in T_{U_j}(x)$, and the condition $s_\varepsilon(x_\varepsilon) = 0$ reads $s_j(x) + (v_x(s_j) + b_j(x))\varepsilon = 0$. Since we are assuming that $s_j(x) = 0$ and $v_x(s_j) = 0$, this is equivalent to $b_j(x) = 0$, i.e., to $(\{b_i\}, \{\eta_{ij}\}) \in \mathbb{H}^1(d^1s)$ restricting to zero in $L(x)$.   Q.E.D.

Lemma (3.3) implies, together with Proposition (2.12):

(3.5) COROLLARY: *Let $\pi: \mathscr{X} \to B$, $\mathscr{L}$ be a separably polarized abelian scheme, as in (2.5), and assume that $\mathscr{D} \subset \mathscr{X}$ is a horizontal effective relative Cartier divisor such that $\mathscr{L} \cong \mathcal{O}_{\mathscr{X}}(\mathscr{D}) \otimes \mathscr{M}$ for some line bundle $\mathscr{M}$ on B (horizontality meaning that the section of $\mathscr{P} \to B$ defined by $\mathscr{D}$ is horizontal). Let $0 \in B(k)$, and call $X = \mathscr{X}(0), D = \mathscr{D}(0)$. Suppose that $x \in D$ is a singular point of D, and let ω_x in $S^2 \Omega_X^1(x) = H^0 S^2 \Omega_X^1$ be an equation for the projectivized tangent cone of D at x if the multiplicity of D at x is 2, and $\omega_x = 0$ otherwise. Then, calling $W \subset H^0(\bar{S}^2 T_X)$ the image by the Kodaira–Spencer map $T_B(0) \to H^1 T_X$ of the projection of $T_{\mathscr{D}}(x)$ in $T_B(0)$, one has: $W \subset \langle \omega_x \rangle^\perp$.*

(3.6) REMARK: We keep the notations and assumptions made in this section. Lemma (3.3) actually implies that, if B is chosen as in (2.11), the image of $T_{\mathscr{D}}(x)$ in $H^0(\bar{S}^2 T_X)$ coincides with $\langle \omega_x \rangle^\perp$. A consequence of this fact is: x (as above) is a smooth point of $\mathscr{D}$ if and only if it is a point of multiplicity 2 of D.

Corollary (3.5) enables us to extend the classical proof of (0.2) to positive characteristics different from 2. We state:

(3.7) THEOREM (Andreotti–Mayer, [1], p. 213): *Let $\bar{\mathscr{J}}_g \subset \mathscr{A}_g$ be the closure of the Jacobian locus inside the coarse moduli scheme for principally polarized abelian varieties of dimension g defined over an algebraically closed field of characteristic $p \geq 0$, $p \neq 2$. Call $N_k \subset \mathscr{A}_g$ the subvariety parametrizing principally polarized abelian varieties such that $\dim(\mathrm{Sing}\,\theta) \geq k$. The variety $\bar{\mathscr{J}}_g$ is an irreducible component of N_{g-4}.*

PROOF: It seems convenient to repeat the original argument of Andreotti–Mayer in detail, adding the necessary complements.

It is known that $\mathscr{J}_g$ is irreducible ([2]) and that $\mathscr{J}_g \subset N_{g-4}$, by the Riemann Singularity Theorem ([5], Corollary, p. 184) and Brill–Noether theory ([7], Théorème, p. 3). Furthermore, the statement is clear for $g \leq 3$, hence we assume from now on that $g \geq 4$. For the sake of clearness, we make a precision: By dimension of a variety or a scheme we understand, as usual, the maximum of the dimensions at each point.

We replace $\mathscr{A}_g$ by a fine moduli scheme $\mathscr{A} = \mathscr{A}_g^{(n)}$ and $\mathscr{J}_g$ by an irreducible component $\mathscr{J}$ of its inverse image in $\mathscr{A}$ (cf [9]; also, for easy reference, [12], pp. 159–163). Call $\pi: \mathscr{X} \to \mathscr{A}$ the universal family. We choose n even (prime to the characteristic). In this way we dispose of a relative effective Cartier divisor $\theta \subset \mathscr{X}$ inducing on each fibre of π a copy of the corresponding theta divisor: The étale multisection of $\underline{\mathrm{Pic}}_{\mathscr{X}/\mathscr{A}} \to \mathscr{A}$ given by the symmetric theta divisors is trivial, since $\mathscr{X}$ admits a level 2 structure, and this gives such choices for θ.

We consider θ as a scheme and call $\mathscr{S} \subset \theta$ the closed subset where the projection $\pi : \theta \to \mathscr{A}$ fails to be smooth. For each $k \geq 0$, let $\mathscr{N}_k \subset \mathscr{A}$ be the closed subset where the fibres of the map

$$\pi_{\mathscr{S}} : \mathscr{S} \to \mathscr{A}$$

have dimension $\geq k$. Clearly, $\mathscr{N}_k$ is the inverse image of $N_k \subset \mathscr{A}_g$ in $\mathscr{A}$.

Since any irreducible subvariety of $\mathscr{A}_g$ is dominated by any irreducible component of its inverse image in $\mathscr{A}$, to prove the theorem it suffices to show that the closure $\bar{\mathscr{J}}$ of $\mathscr{J}$ in $\mathscr{A}$ is an irreducible component of $\mathscr{N}_{g-4}$.

Let $\mathscr{J} \subset \mathscr{N}' \subset \mathscr{N}_{g-4}$ be an irreducible component of $\mathscr{N}_{g-4}$ containing $\mathscr{J}$. We shall see that $\dim \mathscr{N}' = 3g - 3$, and $\bar{\mathscr{J}} = \mathscr{N}'$ will follow.

The sets $\mathscr{N}_k \subset \mathscr{A}$ are the images of the closed subsets $\mathscr{S}_k \subset \mathscr{S}$ defined by

$$\mathscr{S}_k = \left\{ x \in \mathscr{S} \,|\, \dim_x \pi_{\mathscr{S}}^{-1}(\pi_{\mathscr{S}}(x)) \geq k \right\},$$

hence $\mathscr{N}'$ is the image by $\pi_{\mathscr{S}}$ of some irreducible component $\mathscr{S}'$ of $\mathscr{S}_{g-4}$. For all $x \in \mathscr{N}'$ we define a linear subspace $W(X) \subset H^0(S^2\Omega_X^1)$ by putting

$$W(X) = \langle \omega_x \,|\, x \in (\mathscr{S}' \cap X)_{\mathrm{red}} \rangle.$$

The statement about the dimension of $\mathscr{N}'$ (and hence the theorem) then follows from the two lemmas below.

(3.8) LEMMA: *If $X \in \mathscr{N}'$ is a smooth point of $\mathscr{N}'$ then $T_{\mathscr{N}'}(X) \subset W(X)^{\perp}$.*

(3.9) LEMMA: *There exists a non empty open subset of $\mathscr{N}'$ such that, for any X in this subset, $\dim_k W(X)^{\perp} \leq 3g - 3$.*

PROOF OF (3.8): We endow $\mathscr{S}'$ and $\mathscr{N}'$ with their reduced scheme structures. In this proof, intersections are thought in the scheme theoretical sense. Let $U \subset \mathscr{S}'$ be the open set consisting of those points such that $\mathscr{S}'$ is smooth at x and $\mathscr{N}'$ is smooth at $X = \pi(x)$. We claim that

$$d\pi_{\mathscr{S}'} : T_{\mathscr{S}'}(x) \to T_{\mathscr{N}'}(X)$$

is surjective for general $x \in U$.

This is clear in characteristic zero. To proceed in the general case,
(a) we start observing that, for general $X \in \mathscr{N}'$, $\dim(X \cap \mathscr{S}') = g - 4$.

In fact, for all $X \in \mathcal{N}'$, $\dim(X \cap \mathcal{S}') \geq g - 4$ and, if X is the polarized Jacobian of a non hyperelliptic curve C of genus g, $\dim(\text{Sing } \theta_C) = g - 4$ ([5], [7], loc. cit.), hence $\dim(X \cap \mathcal{S}') = g - 4$.

(b) If $x \in U$ is such that $\dim_x(X \cap \mathcal{S}') = g - 4$ and x is a smooth point of $(X \cap \mathcal{S}')_{\text{red}}$, then

$$\dim T_{(X \cap \mathcal{S}')_{\text{red}}}(x) = \dim_x(X \cap \mathcal{S}') = g - 4 = \dim \mathcal{S}' - \dim \mathcal{N}' =$$

$$= \dim T_{\mathcal{S}'}(x) - \dim T_{\mathcal{N}'}(X).$$

On the other side, $\ker(d\pi_{\mathcal{S}'}) = T_{\mathcal{S}'}(x) \cap T_X(x)$, hence the surjectivity of $d\pi_{\mathcal{S}'}$ at x as above is equivalent with $T_{\mathcal{S}'}(x) \cap T_X(x) = T_{(X \cap \mathcal{S}')_{\text{red}}}(x)$.

(c) Let $x \in \mathcal{S}'$ and put $X = \pi(x)$. Choose any local coordinate system $z_1, \ldots, z_g$ of X centered at x, and let $f \in \mathcal{O}_{X,x}$ be a local equation for $\theta \cdot X$ at x. Then

$$(d^2 f)(x) = \sum_{i \leq j} c_{ij}(dz_i)(x) \cdot (dz_j)(x)$$

with $c_{ij} = (\partial^2 f / \partial z_i \, \partial z_j)(x)$ if $i < j$ and $2c_{ii} = (\partial^2 f / \partial z_i^2)(x)$, $i = 1, \ldots, g$. Thus $\omega_x \in S^2 \Omega_X^1(x)$ is given by

$$\omega_x = c \cdot \frac{1}{2} \sum_{i,j} (\partial^2 f / \partial z_i \, \partial z_j)(x)(dz_i)(x) \cdot (dz_j)(x),$$

for some non zero scalar factor c.

We identify ω_x with the symmetric bilinear form which it induces on $T_X(x)$. Let $x \in \mathcal{S}'$ be taken as in (b). It follows from the above that

$$T_{(X \cap \mathcal{S}')_{\text{red}}}(x) \subset \ker(\omega_x),$$

hence $rk(\omega_x) \leq 4$. By the lower semicontinuity of the function $rk(\omega_x)$, this inequality holds for any $x \in \mathcal{S}'$. On the other side, if X is the polarized Jacobian of a general curve C of genus g, there exist points x in $(X \cap \mathcal{S}')_{\text{red}}$ such that $rk(\omega_x) = 4$. One way of seeing this is the following: By Brill–Noether theory ([7], loc. cit.), $(X \cap \mathcal{S}')_{\text{red}}$ meets the locus $W_{g-1}^1 \setminus W_{g-1}^2$. At a point x of this intersection, we may write $\omega_x = \xi_0 \xi_1 - \xi_2 \xi_3$ (by the Riemann–Kempf Singularity theorem, [5], p. 185, Theorem 2), with $\xi_0, \ldots, \xi_3 \in H^0 \Omega_X^1$ linearly independent ("Petri's conjecture", cf [7], pp. 4, 5). Thus $rk(\omega_x) = 4$, as asserted.

(It suffices, to our purposes, to prove the existence of just one case in which $rk(\omega_x) = 4$ holds. This could be done alternatively by using non-hyperelliptic trigonal curves – cf Corollary to Proposition 8, p. 212 and Lemma 4, p. 192 of [1], cf also Lemma (3.9) (a) below.)

Therefore, at a general point $x \in \mathscr{S}'$, $rk(\omega_x) = 4$ and $T_{(X \cap \mathscr{S}')_{red}}(x) = \ker(\omega_x)$.

(d) Let $x \in U$ be a point satisfying the latter property and also the properties of (b). Then $d\pi_{\mathscr{S}'}$ is surjective at x.

To see this, let $t_1, \ldots, t_N (N = \frac{1}{2}g(g + 1))$ be coordinates of $\mathscr{A}$ centered at $X = \pi(x)$. Choosing suitable functions $z_1, \ldots, z_g$ in $\mathcal{O}_{\mathscr{X},x}$ we get a system of coordinates $z_1, \ldots, z_g, t_1, \ldots, t_N$ of $\mathscr{X}$ centered at x. Let $f \in \mathcal{O}_{\mathscr{X},x}$ be a local equation for θ at x. By hypothesis, the functions $\partial f / \partial z_i$, $i = 1, \ldots, g$ vanish at $\mathscr{S}'$. Hence, if $v \in T_{\mathscr{S}'}(x)$, then $v(\partial f / \partial z_i) = 0$. In particular, for all $v = \sum_k c_k (\partial / \partial z_k)(x) \in T_{\mathscr{S}'}(x) \cap T_X(x)$, we get:

$$\sum_k c_k (\partial^2 f / \partial z_k \partial z_i)(x) = 0, \quad i = 1, \ldots, g,$$

i.e. $v \in \ker(\omega_x)$. Thus $T_{\mathscr{S}'}(x) \cap T_X(x) \subset \ker(\omega_x) = T_{(X \cap \mathscr{S}')_{red}}(x)$, and, by (b) this proves that $d\pi_{\mathscr{S}'}$ is surjective at x.

Thus, continuing with the proof of (3.7), there exists an open subset $V \subset U$, $V \neq \emptyset$, such that, for all $x \in V$, $d\pi_{\mathscr{S}'}$ is surjective at x. Writing $X = \pi(x)$, we infer from Corollary (3.5) that $T_{\mathscr{N}'}(X) \subset \langle \omega_x \rangle^\perp$ (cf also (3.6)). Since V is dense in $\pi_{\mathscr{S}'}^{-1}(\mathrm{Reg}(\mathscr{N}'))$, this holds for any $X \in \mathrm{Reg}(\mathscr{N}')$ and any $x \in \mathscr{S}' \cap X$, hence the lemma follows.   Q.E.D.

PROOF OF (3.9): One shows equivalently that $\dim_k W(X) \geq \frac{1}{2}(g - 2)(g - 3)$, for X belonging to a certain non empty open subset of $\mathscr{N}'$.

(a) There exists at least one such $X \in \mathscr{N}'$: Choose for X the polarized Jacobian of a non hyperelliptic trigonal curve C. Examples of such curves can be constructed by taking curves of bidegree $(3, m)$ with at most one ordinary double point on a quadric surface. By the Riemann–Kempf Singularity Theorem ([5], Theorem 2, p. 185), [1], Corollary to Proposition 8, p. 212 holds for arbitrary characteristics. The result then follows by observing that $(\mathscr{S}' \cap X)_{red}$ certains at least one of the irreducible components of $\mathrm{Sing}\,\theta_C$ (cf also [1], Proposition 4, p. 199).

(b) For X as in (a), choose a basis $\omega_{x_1}, \ldots, \omega_{x_r}$ of $W(X)$, $r \geq \frac{1}{2}(g - 2)(g - 3)$. Replacing $\mathscr{N}'$ by a suitable irreducible finite extension $\mathscr{N}''$, we may assume that $\{x_1, \ldots, x_r\}$ is the fibre above X of a subscheme $M \hookrightarrow \mathscr{S}'' = \mathscr{S}' x_{\mathscr{N}'} \mathscr{N}''$ mapping finitely of degree r onto $\mathscr{N}''$. For X' belonging to an open neighbourhood of X in $\mathscr{N}''$, the fibre of M above X' consists of r distinct points $x'_i, \ldots, x'_r$ and the corresponding forms $\omega_{x'_1}, \ldots, \omega_{x'_r}$ stay independent, hence $\dim_k W(X') \geq \dim_k W(X)$. The image in $\mathscr{N}'$ of this neighbourhood contains a non empty open subset of $\mathscr{N}'$ satisfying the reuiqred property.   Q.E.D.

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