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ISOLS AND BALANCED BLOCK DESIGNS WITH $\lambda = 1$

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Abstract

The word "*number*" stands for nonnegative integer, "*set*" for collection of numbers and "*class*" for collection of sets. A *BBD on a finite set* σ of cardinality ≥ 2 is a class Γ of subsets of σ (called *blocks*) for which there exist positive numbers k, r, λ such that $k \geq 2$ and (i) all blocks have cardinality k, (ii) every element of σ occurs in exactly r blocks, and (iii) every two distinct elements of σ occur together in exactly λ blocks. The numbers $v = \operatorname{card} \sigma$, $b = \operatorname{card} \Gamma, k, r$ and λ are the *parameters* of Γ. The basic relations between the parameters of a BBD are: $bk = vr$ and $r(k-1) = \lambda(v-1)$. Using partial recursive functions we generalize the notion of a BBD on a finite set to that of an ω-BBD on an isolated set. We then prove $BK = VR$ and $R(K-1) = \lambda(V-1)$, where V, B, K, R are isols instead of numbers, while λ remains finite. As examples we discuss the cases $K = 3$, $\lambda = 1$ (Steiner triple systems) and $V = B$, $K = R$, $\lambda = 1$ (projective planes). Let c denote the cardinality of the continuum. While there are only denumerably many BBDs on finite sets, there are c ω-BBDs on isolated sets. Among these there are c Steiner triple systems (whose orders need not be $\equiv 1$ or 3 modulo 6) and c projective planes.

§1. Preliminaries

Let Γ be a class of subsets of a nonempty finite set σ and $b = \operatorname{card} \Gamma$. Choose a set β of cardinality b, but disjoint from σ and a one-to-one mapping ϕ from β onto Γ. Let G_Γ be the bigraph with $\sigma \cup \beta$ as set of vertices and $\{(x, y) \in \sigma \times \beta \mid x \in \phi(y)\}$ as class of edges. Then we can represent the class Γ by the bigraph G_Γ; in fact, we can interpret an element b of β as the name of the set $\phi(b)$ in Γ.

All graphs under consideration will be undirected, simple and countable. Let $G = \langle \sigma \cup \beta, \eta \rangle$ be a finite bigraph and d_x the degree of x, for $x \in \sigma \cup \beta$. Then G is *regular*, if d_x is the same for all $x \in \sigma \cup \beta$, *semiregular*, if the value of d_x depends only on whether $x \in \sigma$ or $x \in \beta$, i.e., if there exist positive numbers r and k such that $d_x = r$, for $x \in \sigma$, while $d_x = k$, for $x \in \beta$. The numbers $v = \text{card}\,\sigma$, $b = \text{card}\,\beta$, r and k are the *parameters* of the semiregular bigraph $G = \langle \sigma \cup \beta, \eta \rangle$. If we put $e = \text{card}\,\eta$, we clearly have $bk = vr = e$; thus G is regular iff $k = r$ or equivalently, iff $b = v$.

We shall generalize the notion of a semiregular, finite bigraph to that of an ω-semiregular, countable bigraph G by imposing some computability conditions on G which are trivially satisfied if G is finite. For a summary of the basic concepts and propositions involving RETs (i.e., recursive equivalence types) and isols, see [3, sections 1 and 2]; for a detailed exposition, see [6] or [8]. Let $\langle \rho_n \rangle$ be the canonical enumeration of the class of all finite sets [3, p. 277] and $r_n = \text{card}\,\rho_n$. Then r_n is a recursive function. For every finite set γ there is exactly one number i such that $\gamma = \rho_i$; this number is called the *canonical index* of γ and denoted by $\text{can}\,\gamma$. We write ε for the set $(0, 1, \ldots)$ and $[\alpha; k]$ for $\{n \in \varepsilon \mid \rho_n \subset \alpha \;\&\; r_n = k\}$. Henceforth, the word "*graph*" will be used in the sense of an ordered pair $G = \langle v, \eta \rangle$, where v and η are sets and $\eta = \{\text{can}(x, y) \in [v; 2] \mid x \text{ adj. } y\}$. Every vertex of G is therefore a number, while every edge of G is identified with the canonical index of the set of its endpoints. Let x, y, z be vertices of G. Then we write "$x, y \text{ adj. } z$" if both x and y are adjacent to z, and "$x \text{ adj. } y, z$" if x is adjacent to both y and z. The sets α and β are *separable* (written: $\alpha \mid \beta$), if they can be separated by r.e. sets. The graph $G = \langle v, \eta \rangle$ is an *α-graph*, if there is an effective procedure which enables us to decide, given any two vertices, whether they are adjacent, in short, if $\eta \mid [v; 2] - \eta$. A connected graph is an *ω-graph*, if there exists an effective procedure which enables us, given any two distinct vertices, to find a path of minimal length between them. Trivially, every ω-graph is an α-graph; however, a connected α-graph need not be an ω-graph [5, Prop. 2]. If γ is any set, we write $(\gamma \times \gamma)^-$ for $\{\langle x, y \rangle \in \gamma \times \gamma \mid x \neq y\}$. Let p be a vertex of G; then we denote the set of all edges of G which are incident with p by η_p. The phrase "*function of n variables*" is used for a mapping f from a subcollection of ε^n into ε; its domain and range are denoted by δf and ρf respectively. If $f(p, q, x)$ is a function of three variables, we associate with each ordered pair $\langle p, q \rangle$ a function $f_{pq}(x)$ of one variable, namely the function h such that

$$\delta h = \{x \in \varepsilon \mid (\exists p)(\exists q)[\langle p, q, x \rangle \in \delta f]\},\ h(x) = f(p, q, x).$$

DEFINITION D1: The bigraph $G = \langle \sigma \cup \beta, \eta \rangle$ is *ω-semiregular*, if (a) $\sigma | \beta$, (b) G is an α-graph, (c) there is a function $f(p, q, x)$ such that $f_{pq}(\eta_p) = \eta_q$, for $\langle p, q \rangle \in \sigma \times \sigma$ and $f(p, q, x)$ has a partial recursive extension $\bar{f}(p, q, x)$ such that $\bar{f}_{pq}(x)$ is a one-to-one (partial recursive) function of x, and (d) there is a function $g(r, s, x)$ such that $g_{rs}(\eta_r) = \eta_s$, for $\langle r, s \rangle \in \beta \times \beta$ and $g(r, s, x)$ has a partial recursive extension $\bar{g}(r, s, x)$ such that $\bar{g}_{rs}(x)$ is a one-to-one (partial recursive) function of x.

The graph $G = \langle v, \eta \rangle$ is *isolic* if the set v (hence η) is isolated. Let c denote the cardinality of the continuum. While there are only $\aleph_0$ finite, semiregular bigraphs, there are c isolic, ω-semiregular bigraphs. This follows from the observation that for any two separable, isolated sets σ and β, the complete bigraph $K_{\sigma, \beta} = \langle \sigma \cup \beta, \eta \rangle$ with $\eta = \{\text{can}(x, y) | x \in \sigma \;\&\; y \in \beta\}$ is isolic and ω-semiregular. If $G = \langle \sigma \cup \beta, \eta \rangle$ is an ω-semiregular bigraph, the RETs

$$V = \text{Req}\, \sigma,\ B = \text{Req}\, \beta,\ R = \text{Req}\, \eta_x, \text{ for } x \in \sigma,\ K = \text{Req}\, \eta_x, \text{ for } x \in \beta,$$

are the *parameters* of G.

DEFINITION D2: Let $G = \langle \sigma \cup \beta, \eta \rangle$ be a bigraph and $v = \sigma \cup \beta$. Then G is *ω-regular*, if (a) $\sigma | \beta$, (b) G is an α-graph, and (c) there is a function $m(c, d, x)$ such that $m_{cd}(\eta_c) = \eta_d$, for $\langle c, d \rangle \in v \times v$ and $m(c, d, x)$ has a partial recursive extension $\bar{m}(c, d, x)$ such that $\bar{m}_{cd}(x)$ is a one-to-one (partial recursive) function of x.

REMARK R1: Let $G = \langle \sigma \cup \beta, \eta \rangle$ be an ω-semiregular bigraph with parameters V, B, K, R and let $v = \sigma \cup \beta$. We claim that *G is ω-regular iff* $R = K$. For ω-regularity trivially implies ω-semiregularity and $R = K$. Now assume that G is ω-semiregular with $R = K$ and that the functions $f(p, q, x)$ and $g(r, s, x)$ are related to G as specified in conditions (*c*) and (*d*) of D1. Put $a = \min \sigma$, $b = \min \beta$, then $R = \text{Req}\, \eta_a$ and $K = \text{Req}\, \eta_b$. Since $R = K$, there is a partial recursive one-to-one function h with $\eta_a \subset \delta h$ and $h(\eta_a) = \eta_b$. Define for $\langle c, d \rangle \in v \times v$,

$$m(c, d, x) = \begin{cases} f(c, d, x), & \text{if } c, d \in \sigma, \\ g(c, d, x), & \text{if } c, d \in \beta, \\ g_{bd} h f_{ca}(x), & \text{if } c \in \sigma, d \in \beta, \\ f_{ad} h^{-1} g_{cb}(x), & \text{if } c \in \beta, d \in \sigma. \end{cases}$$

Then $m_{cd}(x)$ is a one-to-one function from η_c onto η_d and $m(c, d, x)$ has a partial recursive extension $\bar{m}(c, d, x)$ such that $\bar{m}_{cd}(x)$ is a one-to-one function of x. Hence G is ω-regular.

PROPOSITION P1: *Let the ω-semiregular bigraph $G = \langle \sigma \cup \beta, \eta \rangle$ have parameters V, B, R, K and let $E = \mathrm{Req}\,\eta$. Then $VR = BK = E$.*

PROOF: Put $a = \min \sigma$, $b = \min \beta$, $\gamma = \{j(x,y) \in j(\sigma \times \beta) \mid x \,\mathrm{adj.}\, y\}$, then we claim: (1) $\gamma \simeq \eta$, (2) $j(\sigma \times \eta_a) \simeq \gamma$, (3) $j(\beta \times \eta_b) \simeq \gamma$. If we can prove these relations, we are done, for they imply $\mathrm{Req}\,\gamma = E$, $VR = \mathrm{Req}\,\gamma$, $BK = \mathrm{Req}\,\gamma$ respectively.

Re (1). Let $\bar{\sigma}$ and $\bar{\beta}$ be disjoint r.e. sets such that $\sigma \subset \bar{\sigma}$ and $\beta \subset \bar{\beta}$. Define the function $\bar{f}$ by: $\delta \bar{f} = j(\bar{\sigma} \times \bar{\beta})$, $\bar{f} j(x,y) = \mathrm{can}(x,y)$, then $\bar{f}$ is a partial recursive, one-to-one function with $\gamma \subset \delta \bar{f}$ and $\bar{f}(\gamma) = \eta$.

Re (2). Let the functions f, $\bar{f}$, g, $\bar{g}$ be related to G as described in conditions (c) and (d) of D1. Define m_1, m_2, $\bar{m}_1$, $\bar{m}_2$ by

$$e = \mathrm{can}(m_1(e), m_2(e)), \text{ for } m_1(e) \in \sigma, m_2(e) \in \beta,$$

$$e = \mathrm{can}(\bar{m}_1(e), \bar{m}_2(e)), \text{ for } \bar{m}_1(e) \in \bar{\sigma}, \bar{m}_2(e) \in \bar{\beta},$$

then m_1, m_2 have the partial recursive extensions $\bar{m}_1$, $\bar{m}_2$ respectively. Let $\delta h = j(\sigma \times \eta_a)$ and $hj(x,y) = j[x, m_2 f_{ax}(y)]$. If we can prove (4) $\rho h = \gamma$, (5) h is one-to-one and (6) h has a partial recursive one-to-one extension, we are done, for they imply $\delta h \simeq \gamma$, i.e. $j(\sigma \times \eta_a) \simeq \gamma$.

Re (4). Let $j(x,y) \in \delta h$, i.e., $x \in \sigma$ and $y \in \eta_a$. Then $f_{ax}(y) \in \eta_x$ and $hj(x,y) = j(x, y^*)$, where y^* is the vertex in β of $f_{ax}(y)$, hence $hj(x,y) \in \gamma$. Thus $\rho h \subset \gamma$. Now assume $j(u,v) \in \gamma$, then $u \in \sigma$, $v \in \beta$ and $u \,\mathrm{adj.}\, v$, i.e., $\mathrm{can}(u,v) \in \eta_u$. Put $w = f_{au}^{-1}\,\mathrm{can}(u,v)$, then $f_{au}(w) = \mathrm{can}(u,v)$ and

$$hj(u,w) = j[u, m_2 f_{au}(w)] = j[u, m_2\,\mathrm{can}(u,v)] = j(u,v).$$

However, $\mathrm{can}(u,v) \in \eta_u$ implies $f_{au}^{-1}\,\mathrm{can}(u,v) \in \eta_a$, i.e., $w \in \eta_a$; then $j(u,w) \in \delta h$, since $u \in \sigma$. Thus $j(u,v) \in \rho h$ and the relation $\rho h \subset \gamma$ can be strengthened to $\rho h = \gamma$.

Re (5). Let $\langle x_1, y_1 \rangle$, $\langle x_2, y_2 \rangle$ be different elements of δh, then $x_1 \neq x_2$, or $x_1 = x_2 \,\&\, y_1 \neq y_2$. In the former case we trivially have $hj(x_1, y_1) \neq hj(x_2, y_2)$; in the latter case we put $x = x_1 = x_2$. Then y_1 and y_2 are distinct edges through a, hence $f_{ax}(y_1)$ and $f_{ax}(y_2)$ distinct edges through x so that $m_2 f_{ax}(y_1) \neq m_2 f_{ax}(y_2)$ and again $h(x_1, y_1) \neq h(x_2, y_2)$.

Re (6). The functions $m_2(e)$ and $f_{ax}(y)$ have partial recursive extensions, hence so has h. Let the number $j(u,v) \in \rho h$ be given. By our proof of (4) we have $h^{-1} j(u,v) = j(u,w)$, where $w = f_{au}^{-1}\,\mathrm{can}(u,v)$. Since $\mathrm{can}(u,v)$ and $f_{au}^{-1}(x)$ have partial recursive extensions so has h^{-1}. Both the one-to-one functions h and h^{-1} have partial recursive extensions, hence h has a partial recursive one-to-one extension.

Re (3). Since the bigraph $G = \langle \sigma \cup \beta, g \rangle$ is ω-semiregular, so is the bigraph $G^* = \langle \beta \cup \sigma, \eta \rangle$. Hence (3) follows from (2).

§2. Balanced block designs

Henceforth the letter σ will only be used to denote a set of cardinality ≥ 2. As observed in the beginning of section 1, every finite class Γ of subsets of a finite set σ can after a suitable choice of a set β be represented by (or identified with) a bigraph $G = \langle \sigma \cup \beta, \eta \rangle$. This is also the case if Γ and σ are countable, provided $\varepsilon - \sigma$ is infinite, since β should be disjoint from σ. We write $\sigma_y = \{x \in \sigma \mid x \text{ adj. } y\}$, for $y \in \beta$.

Definition D3: An *ω-block design* (ω-BD) on a set σ is a bigraph $G = \langle \sigma \cup \beta, \eta \rangle$ such that (i) G is ω-semiregular and (ii) $\sigma_y \neq \sigma_z$, for $y, z \in \beta$ and $y \neq z$.

Remark R2: The interpretation of G in terms of sets is $\Gamma = \{\sigma_y \mid y \in \beta\}$, but we shall adhere to the bigraph terminology. The purpose of condition (ii) is to avoid repetitions of blocks. Note that an ω-BD $G = \langle \sigma \cup \beta, \eta \rangle$ need not be connected. For let $\sigma = (1,2,3,4)$, $\beta = (6,24)$ and η consist of can(1,6), can(2,6), can(3,24) and can(4,24). Then G is a finite ω-semiregular bigraph with $V = 4$, $B = 2$, $K = 2$, $R = 1$ which has two components.

Definition D4: An *ω-balanced* block design (ω-BBD) on a set σ is an ω-BD $G = \langle \sigma \cup \beta, \eta \rangle$ on σ such that $K \geq 2$ and

(iii) if $\beta_{xy} = \{z \in \beta \mid x, y \text{ adj. } z\}$, for $\langle x, y \rangle \in (\sigma \times \sigma)^-$, there is a positive number λ such that card $\beta_{xy} = \lambda$, for all $\langle x, y \rangle \in (\sigma \times \sigma)^-$; moreover, the function $h_{xy} = \text{can}\, \beta_{xy}$ has a partial recursive extension,

(iv) there is an effective procedure which enables us, given any two distinct elements y, z of β to decide whether $(\exists x)[x \in \sigma \;\&\; x \text{ adj. } y, z]$ and if so, to find such a number x.

Remark R3: Assume that $G = \langle \sigma \cup \beta, \eta \rangle$ is an ω-BD on the set σ with $K \geq 2$ which satisfies (iii). We claim

(α) *G is connected,*

(β) *G is an ω-graph iff G satisfies* (iv).

For let us assume that $G = \langle \sigma \cup \beta, \eta \rangle$ is an ω-BD on σ with $K \geq 2$ which satisfies (iii). Define $b_{xy} = \min \beta_{xy}$, for $x, y \in \sigma$ and $x \neq y$. Let Π_1 be the effective procedure which associates b_{xy} with x and y. Since G is ω-semiregular, there also exists an effective procedure Π_2 which associates with every vertex z in β two distinct edges through z.

Re (α). Let two distinct elements p and q of $\sigma \cup \beta$ be given. If exactly one of p, q belongs to σ, we may assume without loss of generality that it be p. We now distinguish three cases:

$$\text{(I) } p, q \in \sigma, \text{ (II) } p \in \sigma, q \in \beta, \text{ (III) } p, q \in \beta.$$

If (I) holds, $\Pi_{p,q} = \langle p, b_{pq}, q \rangle$ is a minimal path from p to q; it can be effectively found from p and q by Π_1. If (II) holds, we can decide whether p adj. q, since G is an α-graph. If p adj. q, then $\Pi_{p,q} = \langle p, q \rangle$ is the only minimal path from p to q. If not[p adj. q], we choose an edge through q, say can(s, q); then $\Pi_{p,q} = \langle p, b_{ps}, s, q \rangle$ is not only a minimal path from p to q, but it can be effectively found from p and q by Π_2. Now suppose that (III) holds. Let "$x \in \sigma \,\&\, x$ adj. p, q" be abbreviated "A_x". If A_x holds for some x, then $\Pi_{pq} = \langle p, x, q \rangle$ is a minimal path from p to q for each such x. If A_x holds for no x, we choose an edge through p, say can(s, p), and an edge through q, say can(t, q); then $s \neq t$ and $\Pi_{pq} = \langle p, s, b_{st}, t, q \rangle$ is a minimal path from p to q. Note that the distance between p and q equals 2 if $(\exists x)A_x$, but 4 if not$(\exists x)A_x$. Anyhow, we proved that G is connected and we described an effective procedure to find a minimal path Π_{pq} from p to q in cases (I) and (II).

Re (β). If G satisfies (iv) we can also effectively find a minimal path from p to q in case (III), hence G is an ω-graph. Now suppose G is an ω-graph. Then we can compute the distance between p and q, hence decide whether $(\exists x)A_x$. Moreover, if $(\exists x)A_x$ is true and $\langle p, t, q \rangle$ is a minimal path from p to q, then A_t is true and t can be computed from $\langle p, t, q \rangle$. Thus G satisfies (iii).

If $G = \langle \sigma \cup \beta, \eta \rangle$ is an ω-BBD on σ, the RETs V, B, R, K and the number λ are called the *parameters* of G. We write Ω for the collection of all RETs and Λ for the collection of all isols, i.e., of all $X \in \Omega$ such that $X \neq X + 1$. Recall that for every nonzero element $A \in \Omega$ the equation $X + 1 = A$ has exactly one solution; it is denoted by $A - 1$. We have $A - 1 = A$ iff $A \in \Omega - \Lambda$, while $A - 1 < A$ iff $A \in \Lambda$.

PROPOSITION P2: *For an ω-BBD $G = \langle \sigma \cup \beta, \eta \rangle$ with parameters V, B, K, R and λ, we have $VR = BK$ and $R(K - 1) = \lambda(V - 1)$.*

PROOF: Assume the hypothesis. $VR = BK$ holds in every ω-BD, hence in G. Put $a = \min \sigma$ and

$$\tilde{\sigma} = \sigma - (a),\ \tilde{\beta} = \{y \in \beta \mid a \text{ adj. } y\},\text{ i.e., } \tilde{\beta} = m_2(\eta_a),$$

$$\tilde{\eta} = \{\text{can}(x, y) \in \eta \mid x \in \tilde{\sigma} \,\&\, y \in \tilde{\beta}\},\ \tilde{G} = \langle \tilde{\sigma} \cup \tilde{\beta}, \tilde{\eta} \rangle.$$

Let the functions f and g be related to G as described in D1. Define

$$\tilde{f} = f|\{\langle p,q,x\rangle | \langle p,q\rangle \in \tilde{\sigma} \times \tilde{\sigma} \,\&\, x \in \eta_p\},$$
$$\tilde{g} = g|\{\langle r,s,x\rangle | \langle r,s\rangle \in \tilde{\beta} \times \tilde{\beta} \,\&\, x \in \eta_r\},$$

then $\tilde{f}_{pq}(\eta_p) = \eta_q$, for $\langle p,q\rangle \in \delta\tilde{f}$ and $\tilde{g}_{rs}(\eta_r) = \eta_s$, for $\langle r,s\rangle \in \delta\tilde{g}$. It is readily seen that $\tilde{f}$ and $\tilde{g}$ have partial recursive extensions related to $\tilde{f}$ and $\tilde{g}$ as $\bar{f}$ and $\bar{g}$ are related to f and g as described in conditions (*c*) and (*d*) of D1. Since $\tilde{G}$ is an α-graph with $\tilde{\sigma}|\tilde{\beta}$, we conclude that $\tilde{G}$ is also ω-semiregular; let its parameters be $\tilde{V}$, $\tilde{B}$, $\tilde{R}$ and $\tilde{K}$. Then $\tilde{V} = \operatorname{Req} \tilde{\sigma} = V - 1$ and $\tilde{B} = \operatorname{Req} \tilde{\beta} \underset{(*)}{=} \operatorname{Req} \eta_a = R$, where (*) is justified, since the function $\operatorname{can}(a,y)$, for $y \neq a$ is partial recursive, one-to-one and maps $\tilde{\beta}$ onto η_a. Put $\tilde{a} = \min \tilde{\sigma}$ and $\tilde{b} = \min \tilde{\beta}$, then

$$\tilde{\eta}_{\tilde{a}} = \{\operatorname{can}(\tilde{a},y) | y \in \tilde{\beta} \,\&\, \tilde{a} \operatorname{adj}. y\} = \{\operatorname{can}(\tilde{a},y) | y \in \beta \,\&\, a, \tilde{a} \operatorname{adj}. y\},$$
$$\tilde{R} = \operatorname{Req} \tilde{\eta}_{\tilde{a}} = \operatorname{card}\{\operatorname{can}(\tilde{a},y) | y \in \beta \,\&\, a, \tilde{a} \operatorname{adj}. y\} = \lambda,$$
$$\tilde{\eta}_{\tilde{b}} = \{\operatorname{can}(x,\tilde{b}) | x \in \tilde{\sigma} \,\&\, x \operatorname{adj}. \tilde{b}\},$$
$$\tilde{\eta}_{\tilde{b}} \simeq \{x \in \tilde{\sigma} | x \operatorname{adj}. \tilde{b}\} \simeq \{x \in \sigma | x \operatorname{adj}. \tilde{b}\} - (a),$$
$$\tilde{K} = \operatorname{Req} \tilde{\eta}_{\tilde{b}} = \operatorname{Req} \eta_{\tilde{b}} - 1 = K - 1.$$

Application of P1 to $\tilde{G}$ yields $\tilde{R}\tilde{V} = \tilde{B}\tilde{K}$, i.e., $R(K-1) = \lambda(V-1)$.

Definition D5: An ω-BBD $G = \langle \sigma \cup \beta, \eta\rangle$ is *isolic*, if σ and β are isolated, i.e., if V, R, B and K are isols.

Definition D6: An ω-BBD $G = \langle \sigma \cup \beta, \eta\rangle$ is *symmetric*, if (a) G is isolic and (b) $V = B$ or equivalently $R = K$ or equivalently G is ω-regular.

Remark R4: According to P2 the crucial formula $R(K-1) = \lambda(V-1)$ holds for every ω-BBD G, not only if G is isolic. However, in the non-isolic case this formula loses much of its interest by being equivalent to $RK = \lambda V$. The propositions of the remainder of this paper will therefore deal with isolic ω-BBDs.

Definition D7: Let $G_1 = \langle v_1, \eta_1\rangle$ and $G_2 = \langle v_2, \eta_2\rangle$ be ω-graphs. Then G_1 is an *induced* subgraph of G_2 (written: $G_1 \leq G_2$), if $v_1 \subset v_2$ and $\eta_1 = \{\operatorname{can}(x,y) | x,y \in v_1\} \cap \eta_2$. We write $G_1 < G_2$, if $G_1 \leq G_2$ and $G_1 \neq G_2$.

DEFINITION D8: An ω-BBD $G = \langle \sigma \cup \beta, \eta \rangle$ is *locally finite*, if given two nonempty, finite sets $\sigma_0 \subset \sigma$ and $\beta_0 \subset \beta$, there exist finite sets $\sigma^*, \beta^*, \eta^*$ such that $\sigma_0 \subset \sigma^* \subset \sigma$, $\beta_0 \subset \beta^* \subset \beta$, $\eta^* \subset \eta$ such that $G^* = \langle \sigma^* \cup \beta^*, \eta^* \rangle$ is a finite ω-BBD and $G^* \leq G$. Moreover, G is *recursively* locally finite, if given two nonempty, finite sets $\sigma_0 \subset \sigma$ and $\beta_0 \subset \beta$ such finite sets $\sigma^*, \beta^*, \eta^*$ can be effectively found.

PROPOSITION P3: *Every isolic ω-BBD is recursively locally finite.*

PROOF: Let $G = \langle \sigma \cup \beta, \eta \rangle$ be an isolic ω-BBD, then we may assume without loss of generality that σ is infinite. Suppose Π_1 and Π_2 are the effective procedures mentioned in R3, Π_3 an effective procedure which associates with every $x \in \sigma$ two distinct edges through x and Π_4 the effective procedure described in condition (iv) of D4. Let $G^* = \langle \sigma^* \cup \beta^*, \eta^* \rangle$ be the ω-BBD and induced subgraph of G which can be obtained by fully exploiting the information $\sigma_0 \subset \sigma$, $\beta_0 \subset \beta$ and the effective procedures $\Pi_1, \ldots, \Pi_4$. Then σ^*, β^*, η^* can be effectively obtained from σ_0 and β_0, hence σ^*, β^*, η^* are r.e. Since σ, β, η are isolated, we conclude that σ^*, β^*, η^* are finite. Thus G is recursively locally finite.

COROLLARY: *For every infinite, isolic ω-BBD G there exists an infinite sequence $\langle G_n \rangle$ of finite ω-BBDs such that $G_0 < G_1 < \ldots < G$ and $\bigcup_{n=0}^{\infty} G_n = G$.*

We conjecture that a considerable part of the theory of BBDs [see e.g., 11, Chs. 7, 8] can be generalized to isolic ω-BBDs. However, in the remainder of this paper we shall restrict our attention to the case $\lambda = 1$, more specifically, to Steiner triple systems ($K = 3$, $\lambda = 1$) and projective planes ($V = B$, $K = R$, $\lambda = 1$). Note that if we take $\lambda = 1$ in condition (iii) of D4 and m_{xy} is the only member of β_{xy} for $x, y \in \sigma$, $x \neq y$, then the function m_{xy} from $(\sigma \times \sigma)^-$ into ε has a partial recursive extension iff the function $h_{xy} = \text{can } \beta_{xy}$ has a partial recursive extension.

§3. Steiner triple systems

We shall generalize the notion of an STS on a finite set to that of an ω-STS on a countable set.

DEFINITION D9: An *STS on a set* σ is a class Γ of 3-subsets of σ (called *triples*) such that every two distinct elements of σ belong together to exactly one triple of Γ. The *Steiner-function* (S-function) of Γ is the function S_{xy} from $(\sigma \times \sigma)^-$ into σ such that $(x, y, S_{xy}) \in \Gamma$. An STS is an *ω-STS*, if its S-function has a partial recursive extension.

REMARK R5: The function S_{xy} has the following properties: (i) $S_{xy} \notin (x, y)$, (ii) $S_{xy} = S_{yx}$, (iii) if $S_{xy} = S_{uv}$, then (x, y) and (u, v) are equal or disjoint, (iv) $S_{xy} = z \Leftrightarrow S_{yz} = x \Leftrightarrow S_{zx} = y$. Conversely, if the function f from $(\sigma \times \sigma)^-$ into σ has these four properties, then the class $\Gamma = \{(x, y, S_{xy}) | \langle x, y \rangle \in (\sigma \times \sigma)^-\}$ is an STS with S_{xy} as its S-function. For the relations between STSs, quasigroups and loops, the reader is referred to Bruck's paper [1, pp. 63–65]; they will, however, not be used in the present paper.

An STS on a set σ is *finite* (*infinite*, *isolic*, *immune*), if the set σ is finite (infinite, isolated, immune). Since every function with a finite domain is partial recursive, every finite STS is an isolic ω-STS. The order of an ω-STS Γ on a set σ is defined by $o(\Gamma) = \text{Req}\,\sigma$. Thus $o(\Gamma)$ has the usual meaning iff σ is finite. In the remainder of this section we assume that σ is a set of cardinality ≥ 3 which consists of odd numbers; this turns out to be convenient (but not essential).

DEFINITION D10: Let Γ be an STS on σ with S_{xy} as its S-function. Then the *standard representation* of Γ is the bigraph $G_\Gamma = \langle \sigma \cup \beta, \eta \rangle$, where

$$\beta = \{\text{can}(x, y, z) | (x, y, z) \in \Gamma\} = \{\text{can}(x, y, S_{xy}) | \langle x, y \rangle \in (\sigma \times \sigma)^-\},$$

$$\eta = \{\text{can}(u, v) | u \in \sigma \;\&\; v \in \beta \;\&\; u \in \rho_v\}.$$

PROPOSITION P4: *Let Γ be an isolic STS on σ and let G_Γ be its standard representation. Then*

$$\Gamma \text{ an } \omega\text{-}STS \Leftrightarrow G_\Gamma \text{ an } \omega\text{-}BBD \text{ with } K = 3 \text{ and } \lambda = 1.$$

PROOF: Let Γ be an ω-STS and $G_\Gamma = \langle \sigma \cup \beta, \eta \rangle$. Then G_Γ is a BBD with $K = 3$ and $\lambda = 1$. It remains to be shown that

$$G_\Gamma \text{ is } \omega\text{-semiregular}, \tag{7}$$

$$\sigma_y \neq \sigma_z, \text{ for } y, z \in \beta \text{ and } y \neq z, \tag{8}$$

$$G_\Gamma \text{ satisfies conditions (iii) and (iv) of D4}. \tag{9}$$

Re (7). Since σ consists of odd numbers, we have $o \notin \sigma$. Thus the canonical index of every finite subset of σ is even, β consists of even numbers and $\sigma | \beta$. For $x \in \sigma$, $y \in \beta$, say $y = \text{can}(u, v, w)$, we have x adj. y iff

$x \in (u, v, w)$; hence G_Γ is an α-graph. Now consider condition (c) of D1. For $p \in \sigma$,

$$t \in \eta_p \Leftrightarrow (\exists u)[t = \text{can}(p, u) \,\&\, u \in \beta \,\&\, p \in \rho_u\}],$$

$$t \in \eta_p \Leftrightarrow (\exists u)[t = \text{can}(p, u) \,\&\, (\exists x)[x \in \sigma - (p) \,\&\, u = \text{can}(p, x, S_{px})]],$$

$$\text{t} \in \eta_p \Leftrightarrow (\exists u)(\exists x)[t = \text{can}(p, u) \,\&\, x \in \sigma - (p) \,\&\, u = \text{can}(p, x, S_{px})]. \quad (10)$$

A function $h(x)$ is an *involution without fixed points* (abbreviated *iwfp*) of a set τ, if h is a permutation of τ such that $h(x) \neq x$ and $h^2(x) = x$, for $x \in \tau$. The *iwfp* h of τ is an *ω-iwfp*, if h has a partial recursive one-to-one extension (or equivalently, a partial recursive extension, since h is one-to-one and $h = h^{-1}$). We may assume without loss of generality that the set σ is immune. Define $\Gamma_r = \{(r, x, y) | (r, x, y) \in \Gamma\}$, for $r \in \sigma$, and $\tau_{pq} = \sigma - (p, q, S_{pq})$, for $p, q \in \sigma$, $p \neq q$. Let $S_p(x)$ and $S_q(x)$ be the functions with domain τ_{pq} such that $S_p(x) = S(p, x)$ and $S_q(x) = S(q, x)$. A subset ρ of τ_{pq} is *closed*, if $S_p(\rho) \subset \rho$ and $S_q(\rho) \subset \rho$. The *closure* ρ^* of ρ is the intersection of all closed sets α with $\rho \subset \alpha \subset \tau_{pq}$. The closure ρ^* of a finite subset ρ of τ_{pq} can be effectively obtained from ρ as follows: test whether ρ is closed; if not, adjoin to ρ all elements x such that $x \notin \rho$, but there is a finite sequence $\langle x_0, \ldots, x_n \rangle$ such that $x_0 \in \rho$, $x_n = x$ and $x_{i+1} = S_p(x_i)$ or $x_{i+1} = S_q(x_i)$, for $0 \leq i \leq n - 1$. Since ρ is a finite subset of τ_{pq}, it follows that ρ^* is a r.e., hence (τ_{pq} being immune) a finite subset of τ_{pq}. We need the following

LEMMA: *For every ordered pair $\langle p, q \rangle \in \sigma \times \sigma$ there is a one-to-one mapping Φ_{pq} from Γ_p onto Γ_q such that from $\langle p, q \rangle$ we can find (definitions of) effective procedures which enable us* (i) *given any $T \in \Gamma_p$, to compute $\Phi_{pq}(T)$, and* (ii) *given any $T \in \Gamma_q$, to compute $\Phi_{pq}^{-1} T$.*

PROOF: Let $\langle p, q \rangle \in \sigma \times \sigma$ be given. If $p = q$ we define $\Phi_{pq}(T) = T$, for $T \in \Gamma_p$. Now assume $p \neq q$. Then we put

$$\Phi(p, q, S_{pq}) = (q, p, S_{pq}),$$

$$\Phi(p, c, d) = (q, c, d), \text{ if } q \notin (c, d), S_p(c) = d, S_q(c) = d,$$

$$\Phi(p, c, d) = (q, u, v), \text{ if } q \notin (c, d), S_p(c) = d, S_q(c) \neq d,$$

where (u, v) is obtained from (c, d) as follows. Compute the (canonical index of) the finite set $\alpha_{cd} = (c, d)^*$; then $(c, d) \subset \alpha_{cd} \subset \tau_{pq}$ and both $S_p | \alpha_{cd}$ and $S_q | \alpha_{cd}$ are *ω-iwfps* of α_{cd}. Thus card α_{cd} is even, say card $\alpha_{cd} = 2m$. Then α_{cd} can be expressed as

$$\alpha_{cd} = \bigcup_{i=1}^{m} (x_i, S_p(x_i)) = \bigcup_{i=1}^{m} (y_i, S_q(y_i)),$$

where the 2-sets $(x_i, S_p(x_i))$, $(y_i, S_q(y_i))$ can be computed from α. Then we compute the numbers $c_1, \ldots, c_m$, $d_1, \ldots, d_m$, $u_1, \ldots, u_m$, $v_1, \ldots, v_m$ such that

$$\{(x_i, S_p(x_i)) \mid 1 \leq i \leq m\} = ((c_1, d_1), \ldots, (c_m, d_m)), \text{ where}$$
$$\text{can}(c_1, d_1) < \ldots < \text{can}(c_m, d_m),$$
$$\{(y_i, S_q(y_i)) \mid 1 \leq i \leq m\} = ((u_1, v_1), \ldots, (u_m, v_m)) \text{ where}$$
$$\text{can}(u_1, v_1) < \ldots < \text{can}(u_m, v_m).$$

We can now effectively locate (c, d) in the sequence $\langle (c_1, d_1), \ldots, (c_m, d_m) \rangle$, i.e., compute the number i such that $(c, d) = (c_i, d_i)$. Then we define (u, v) as (u_i, v_i). To prove that the mapping Φ_{pq} is well-defined and one-to-one we use

$$c, d, c', d' \in \tau_{pq} \;\&\; \Phi_p(c) = d \;\&\; \Phi_p(c') = d' \Rightarrow \alpha_{cd} \text{ and } \alpha_{c'd'} \text{ are identical or disjoint.} \tag{11}$$

Let R be the relation in τ_{pq} such that zRw iff there exists a finite sequence $\langle z_0, \ldots, z_n \rangle$ of elements in τ_{pq} such that $z_0 = z$, $z_n = w$ and $z_{i+1} = S_p(z_i)$ or $z_{i+1} = S_q(z_i)$, for $0 \leq i \leq n-1$. Then R is an equivalence relation in τ_{pq}, α_{cd} is the equivalence class containing c and d [for cRd, since $S_p(c) = d$], while $\alpha_{c'd'}$ is the equivalence class containing c' and d'. Thus (11) is true. We described an effective procedure for computing $\Phi_{pq}(p, c, d)$, given any triple $(p, c, d) \in \Gamma_p$. Now assume the triple $(q, u, v) = \Phi_{pq}(p, c, d)$ is given. If $p = q$ we have $(p, c, d) = (q, u, v)$. Now assume $p \neq q$. We know that $S_q(u) = v$. We can compute the set $\alpha_{cd} = (u, v)^*$ from u and v; let card $\alpha_{cd} = 2m$. By computing the 2-sets (c_i, d_i), (u_i, v_i), for $1 \leq i \leq m$ associated with α_{cd} we can find the number i with $(u, v) = (u_i, v_i)$, hence also the 2-set $(c, d) = (c_i, d_i)$. Thus $\Phi_{pq}^{-1}(q, u, v) = (p, c, d)$ can be computed from (q, u, v). This completes the proof of the lemma.

We now continue our proof of (7). In view of (10) we can define a function $f(p, q, x)$ by

$$\delta f = \{\langle p, q, t \rangle \mid \langle p, q \rangle \in \sigma \times \sigma \;\&\; t \in \eta_p\},$$
$$f(p, q, t) = \text{can}[q, \text{can}\, \Phi_{pq}(p, x, S_p(x))], \text{ where}$$
$$t = \text{can}[p, \text{can}(p, x, S_p(x))].$$

Using the lemma it follows that the function f is related to $G_\Gamma = \langle \sigma \cup \beta, \eta \rangle$ as described in condition (c) of D1. Now consider condition (d) of D1. Let $\bar{\beta} = [\varepsilon; 3]$, i.e., let $\bar{\beta}$ be the infinite, recursive set consisting of the canonical indices of all 3-sets; put $\beta = \{x \in \bar{\beta} | \rho_x \in \Gamma\}$. Define the functions $\bar{g}$ and g by: (i) $\delta \bar{g} = \bar{\beta} \times \bar{\beta}$, (ii) if $r, s \in \bar{\beta}$, say $r = \text{can}(r_1, r_2, r_3)$, $s = \text{can}(s_1, s_2, s_3)$, where $r_1 < r_2 < r_3$, $s_1 < s_2 < s_3$, then $\bar{g}[r, s, \text{can}(r_i, r)] = \text{can}(s_i, s)$, for $1 \le i \le 3$, (iii) $g = \bar{g} | \beta \times \beta$. Then g and $\bar{g}$ are related to σ and β as described in condition (d) of D1. We have now proved that G_Γ is ω-semiregular.

Re (8). Let $y, z \in \beta$, say $y = \text{can}(a, b, c)$, $z = \text{can}(d, e, f)$. Then $y \neq z$ implies $(a, b, c) \neq (d, e, f)$, hence $\sigma_y \neq \sigma_z$.

Re (9). Let $\delta h = (\sigma \times \sigma)^-$ and $h(x, y) = \text{can}\{z \in \beta | x, y \text{ adj. } z\}$. Since S_{xy} has a partial recursive extension so has the function $h(x, y) = \text{can}(x, y, S_{xy})$. Thus G_Γ satisfies condition (iii) of D4. Assume two distinct elements $y_1, y_2 \in \beta$ are given, say $y_1 = \text{can}(u_1, v_1, w_1)$, $y_2 = \text{can}(u_2, v_2, w_2)$. Then there is an x with $x \in \sigma$ & $x \text{ adj. } y_1, y_2$ if $(u_1, v_1, w_1) \cap (u_2, v_2, w_2)$ is a singleton; moreover, if such an x exists, it is the common element of (u_1, v_1, w_1) and (u_2, v_2, w_2). Since $u_1, v_1, w_1, u_2, v_2, w_2$ can be computed from y_1 and y_2, we conclude that G_Γ satisfies condition (iv) of D4. This completes the proof of (9) and thereby of the conditional from the left to the right. The other conditional is trivial. For if the function h with $\delta h = (\sigma \times \sigma)^-$ and $h(x, y) = \text{can}\{z \in \beta | x, y \text{ adj. } z\}$ has a partial recursive extension, so has the function S_{xy} with $\rho_{h(x,y)} = (S_{xy})$.

Let Γ be an isolic ω-STS and G_Γ its standard representation. Then the parameters V, B, R of G_Γ are called the *parameters* of Γ.

PROPOSITION P5: *If the isolic ω-STS Γ has parameters V, B and R, then* $V = 2R + 1$ *and* $V(V - 1) = 6B$.

PROOF: Substitution of $K = 3$, $\lambda = 1$ in $VR = BK$ and $\lambda(V - 1) = R(K - 1)$ yields $VR = 3B$ and $V = 2R + 1$. Thus $6B = V(2R) = V(V - 1)$.

DEFINITION D11: Let $A, B \in \Lambda$, $m \in \varepsilon$ and $m \ge 1$. Then *A is congruent B modulo m* [written: $A \equiv B (\text{mod } m)$], if there exist isols X and Y such that $A + mX = B + mY$.

DEFINITION D12: An isol V is a *Steiner-isol* (S-isol), if $V = o(\Gamma)$, for some isolic ω-STS Γ.

We refer to [2, p. 116] for a list of the basic properties of the

$\equiv (\text{mod}\, m)$ relation in Λ. We write Λ_s for the collection of all S-isols and ε_S for $\varepsilon \cap \Lambda_S$. It is known [1, pp. 91–97] that

$$n \in \varepsilon_S \Leftrightarrow n\ odd\ \&\ n(n-1) \equiv 0 (mod\, 6) \Leftrightarrow n \equiv 1\ or\ 3 (mod\, 6), \quad for\ n \in \varepsilon, n \geq 3. \tag{12}$$

The isolic analogue of (12), namely

$$X \in \Lambda_S \Leftrightarrow X\ odd\ \&\ X(X-1) \equiv 0 (mod\, 6) \Leftrightarrow X \equiv 1\ or\ 3 (mod\, 6), \quad for\ X \in \Lambda, X \geq 3. \tag{13}$$

fails. More specifically, (a) $X \in \Lambda_S$ implies X odd & $X(X-1) \equiv 0(\text{mod}\, 6)$ by P5, but whether the converse holds is unknown, (b) trivially, $X \equiv 1$ or 3(mod 6) implies X odd & $X(X-1) \equiv 0(\text{mod}\, 6)$, but the converse is false.

Proposition P6: *Every isol of the form $2^N - 1$ with $N \geq 2$ is an S-isol.*

Proof: We generalize the classical construction of an STS in terms of the group $\mathbb{Z}_2^n$. Let $N \geq 2$ and $v \in N$. Denote the class of all finite subsets of v by $P_{\text{fin}}(v)$ and the symmetric difference of α and β by $\alpha \oplus \beta$. Then $G_v = \langle P_{\text{fin}}(v), \oplus \rangle$ is an Abelian group with the empty set o as unit element; thus $\alpha = -\alpha$, since $\alpha \oplus \alpha = o$. Define

$$\Gamma_v = \{(\alpha, \beta, \alpha \oplus \beta) \mid \alpha, \beta \in P_{\text{fin}}(v) - (o)\ \&\ \alpha \neq \beta\},$$

then Γ_v satisfies the four conditions listed in R5, hence Γ_v is an STS on $P_{\text{fin}}(v) - (o)$. Put

$$\delta g = (\varepsilon \times \varepsilon)^-, \ g(x,y) = \text{can}(\rho_x \oplus \rho_y), \ 2^v = \{n \in \varepsilon \mid \rho_n \subset v\},$$

$$\Gamma'_v = \{(x, y, g(x,y)) \mid x, y \in 2^v - (0)\ \&\ x \neq y\},$$

then g is a partial recursive function and Γ'_v an STS on $2^v - (0)$. Define

$$\sigma = \{2n+1 \mid n \in 2^v - (0)\}, \Gamma''_v = \{(2x+1, 2y+1, 2z+1) \mid (x,y,z) \in \Gamma'_v\},$$

then σ consists of odd numbers and Γ''_v is an STS on σ. Note that Γ''_v is an ω-STS on σ, since

$$S_{2x+1, 2y+1} = 2\,\text{can}(\rho_x \oplus \rho_y) + 1 = 2g(x,y) + 1, \text{ for } x, y \in 2^v - (0).$$

Clearly, $o(\Gamma''_v) = \text{Req}(2^v - (0)) = 2^N - 1$, hence $2^N - 1 \in \Lambda_S$.

REMARK R6: Since $\oplus$ is an associative operation, each isolic ω-STS of order $2^N - 1$ constructed in the proof of P6 is associative, i.e., has the property $S(S_{xy}, z) = S(x, S_{yz})$, for distinct elements x, y, z of σ.

Since $2^n \not\equiv 0 (\text{mod } 3)$ we have $2^n \equiv 1$ or $2 (\text{mod } 3)$. In fact, $2^n \equiv 1 (\text{mod } 3)$ if n is even, while $2^n \equiv 2 (\text{mod } 3)$ if n is odd. The two statements of the preceding sentence can be generalized to isols, but the proofs are less obvious in the isolic case, since an isol need not be even or odd nor need it be $\equiv 0$, 1 or $2 (\text{mod } 3)$.

PROPOSITION P7: *For every isol* N,
(a) $2^N \equiv 1 (\text{mod } 3) \Leftrightarrow N$ *even*, (b) $2^N \equiv 2 (\text{mod } 3) \Leftrightarrow N$ odd.

PROOF: We may assume without loss of generality that $N \geq 1$. Note that $2^N \equiv 2 (\text{mod } 3) \Leftrightarrow 2^{N-1} \equiv 1 (\textit{mod } 3)$ by relations (3.11) and (3.12) of [2]. Hence (a) $\Leftrightarrow$ (b) and it suffices to prove (a). We split up (a) into two parts:

(a_1) N even $\Rightarrow 2^N \equiv 1 (\text{mod } 3)$,

(a_2) $2^N \equiv 1 (\text{mod } 3) \Rightarrow N$ even,

and we shall use the fact that (a_1) and (a_2) hold in ε.

Re (a_1). Let the canonical extension from ε to Λ of a recursive combinatorial function $h(x)$ be denoted by $h_\Lambda(X)$. Suppose that $f(x)$ and $g(x)$ are recursive, combinatorial functions. According to a result due to Nerode [8, p. 398],

$$(\forall x)[f(x) \equiv g(x) (\text{mod } 3)] \Rightarrow (\forall x)[f_\Lambda(X) \equiv g_\Lambda(X) (\text{mod } 3)].$$

The functions $f(x) = 2^{2x}$ and $g(x) = 1$ are recursive and combinatorial, hence

$$(\forall x)[2^{2x} \equiv 1 (\text{mod } 3)] \Rightarrow (\forall X)[2^{2X} \equiv 1 (\text{mod } 3)].$$

Since the hypothesis is true, so is the conclusion.

Re (a_2). E. Ellentuck pointed out to us that this can be proved using the method employed by Nerode [10, p. 413] to show that $2^X = Y^2 \Rightarrow X$ even, for isols X and Y. Consider the formula $(\forall x)(\forall y)(\exists z)[2^x = 3y + 1 \Rightarrow x = 2z]$ which holds in ε. It is a Horn sentence in prenex form involving the recursive, combinatorial functions 2^x, $3y + 1$ and $2z$; moreover, it has a recursive Skolem function. Hence this formula holds in $\varepsilon_{\Lambda*}$ by [10, part (3.3) of Thm. (3.1)]. Let $2^X = 3Y + 1$, for $X, Y \in \Lambda$.

Then there is a $Z \in \varepsilon_{\Lambda^*}$, hence a $Z \in \Lambda^*$ such that $X = 2Z$. However, $2Z \in \Lambda$ implies $Z \in \Lambda$ by [10, (4.1)], hence X is even.

PROPOSITION P8: *There are c isols which are S-isols and c which are not. Among the c S-isols there are c which are* $\equiv 1 (\mathrm{mod}\, 6)$, *c which are* $\equiv 3$ $(\mathrm{mod}\, 6)$ *and c which are neither* $\equiv 1$ *nor* $\equiv 3 (\mathrm{mod}\, 6)$.

PROOF: Every S-isol is odd by P5. Thus, since there are c isols which are not odd, there are exactly c isols which are not S-isols. Note that for $N \in \Lambda$, $N \geq 1$,

$$2^{N-1} \equiv 1 (\mathrm{mod}\, 3) \Leftrightarrow 2^N \equiv 2 (\mathrm{mod}\, 6) \Leftrightarrow 2^N - 1 \equiv 1 (\mathrm{mod}\, 6),$$

$$2^{N-1} \equiv 2 (\mathrm{mod}\, 3) \Leftrightarrow 2^N \equiv 4 (\mathrm{mod}\, 6) \Leftrightarrow 2^N - 1 \equiv 3 (\mathrm{mod}\, 6).$$

Using P7 we see that
(i) $2^N - 1 \equiv 1 (\mathrm{mod}\, 6) \Leftrightarrow N$ odd,
(ii) $2^N - 1 \equiv 3 (\mathrm{mod}\, 6) \Leftrightarrow N$ even,
(iii) $2^N - 1 \equiv 1$ or $3 (\mathrm{mod}\, 6) \Leftrightarrow N$ even or odd.
According to P6 every isol of the form $2^N - 1$ with $N \geq 2$ is an S-isol. The remaining statements of P8 now follow from the fact that the function $F(N) = 2^N - 1$ from Λ into Λ is one-to-one and that there are c even isols, c odd isols and c isols which are neither even nor odd.

COROLLARY: *There are exactly c odd isols V such that* $V(V-1) \equiv 0 (\mathrm{mod}\, 6)$, *while V is neither* $\equiv 1$ *nor* $\equiv 3 (\mathrm{mod}\, 6)$.

PROOF: Every isol $V = 2^N - 1$, where N is neither even nor odd satisfies the requirements.

So far all infinite S-isols of which we proved the existence were of the form $2^N - 1$ with $N \in \Lambda - \varepsilon$. The direct product construction which associates with two finite STSs of orders a and b a finite STS of order ab can be generalized to ω-STSs.

PROPOSITION P9: *If A and B are S-isols, so is AB.*

PROOF: Let Γ and Δ be ω-STSs on the isolated sets α and β respectively and suppose $\gamma = j(\alpha \times \beta)$. Define the following three conditions on a triple of elements of γ, say $(j(a_1, b_1), j(a_2, b_2), j(a_3, b_3))$: (I) $(a_1, a_2, a_3) \in \Gamma$ and $b_1 = b_2 = b_3$, (II) $a_1 = a_2 = a_3$ & $(b_1, b_2, b_3) \in \Delta$, (III) $(a_1, a_2, a_3) \in \Gamma$ & $(b_1, b_2, b_3) \in \Delta$. Let θ be the class of all triples of Γ which satisfy at least one (hence exactly one) of these three conditions. Then θ is an ω-STS on the isolated set γ and $o(\theta) = o(\Gamma) \cdot o(\Delta)$.

REMARK R7: Let Γ be an isolic ω-STS of order N. While G_Γ is an ω-semiregular ω-graph associated with Γ, one can also associate an ω-regular ω-graph S_Γ with Γ, the so-called *Steiner graph* of Γ. Here $S_\Gamma = \langle \beta, \eta \rangle$, where $\beta = \{\text{can}(x,y,z) | (x,y,z) \in \Gamma\}$ and two vertices are *adjacent* if the triples they represent have exactly one element in common. Let $D = \text{Req } \eta_p$, for $p \in \beta$. It can now be proved that (i) *S_Γ is connected,* (ii) *S_Γ is an ω-graph,* (iii) $D = 3(N-3)/2$, *so that D does not depend on the choice of p,* (iv) *S_Γ is ω-regular,* (v) $E = \text{Req } \eta = N(N-1)(N-3)/8$.

§4. Projective planes

There is a well-known correspondence between finite, symmetric BBDs with $\lambda = 1$ and finite, projective planes. We shall extend this correspondence to one between isolic, symmetric BBDs with $\lambda = 1$ and the isolic projective ω-planes introduced in [4] and defined below.

An *incidence system* is an ordered triple $\Pi = \langle \sigma, \beta, \text{inc.} \rangle$, where σ and β are disjoint sets and inc. $\subset (\sigma \times \beta) \cup (\beta \times \sigma)$; the elements of σ and β are called *points* and *lines* respectively. We call Π *finite* (*infinite*, *isolic immune*), if the sets σ and β are finite (infinite, isolated, immune). In this section the word "plane" is used in the sense of a projective plane. All planes under consideration are countable. We define a *plane* as an incidence system $\Pi = \langle \sigma, \beta, \text{inc.} \rangle$ such that the classical three axioms are satisfied. Let $p, q \in \sigma$, $p \neq q$, $r, s \in \beta$, $r \neq s$, then we write L_{pq} for the line joining p and q, and P_{rs} for the point in which r and s intersect. The plane $\Pi = \langle \sigma, \beta, \text{inc.} \rangle$ is an *ω-plane*, if $\sigma | \beta$ and the functions L from $(\sigma \times \sigma)^-$ into β and P from $(\beta \times \beta)^-$ into σ have partial recursive extensions. Since every function with a finite domain is partial recursive, every finite plane is an isolic ω-plane. It was proved in [4] that (i) there are c isolic ω-planes, and (ii) for every ω-plane $\Pi = \langle \sigma, \beta, \text{inc.} \rangle$ there exists a unique RET N such that every point lies on $N+1$ lines, every line passes through $N+1$ points, while $\text{Req } \sigma = \text{Req } \beta = N^2 + N + 1$. This RET N is called the *order* $o(\Pi)$ of Π. Thus $o(\Pi)$ has the usual meaning iff Π is finite.

DEFINITION D13: The *standard representation* of the incidence system $\Pi = \langle \sigma, \beta, \text{inc.} \rangle$ is the bigraph $G_\Pi = \langle \sigma \cup \beta, \eta \rangle$, where $\eta = \{\text{can}(x,y) | x \in \sigma \;\&\; y \in \beta \;\&\; x \text{ inc. } y\}$.

PROPOSITION P10: *Let G_Π be the standard representation of the isolic incidence system Π. Then*

Π an ω-plane $\Leftrightarrow$ G_Π a symmetric ω-BBD with $\lambda = 1$.

PROOF: Let $\Pi = \langle \sigma, \beta, \text{inc.} \rangle$ and let $G_\Pi = \langle \sigma \cup \beta, \eta \rangle$ be its standard representation. Since the theorem holds if Π is finite, we may assume that σ and β are infinite.

(a) Assume that Π is an ω-plane. With Π we associate an ordered quintuple $\langle a_1, \dots, a_5 \rangle$ of points no three of which are collinear; from now on $a_1, \dots, a_5$ remain fixed. Since Π is an ω-plane, we have $\sigma | \beta$. Moreover, given a point p and a line l of Π we can decide whether p inc. l by [4, §3(e)], hence whether $\text{can}(p, l) \in \eta$; thus G_Π is an α-graph. Let the lines r and s of Π be given. If $r = s$ we define $f(r, s, x) = x$, for $x \in \eta_r$, $\bar{f}(r, s, x) = x$, for $x \in \varepsilon$. Now assume $r \neq s$. From $a_1, \dots, a_5, r, s$ we can find the first point in $\langle a_1, \dots, a_5 \rangle$ which lies neither on r nor on s, say a. Let γ and δ be the sets of points of Π which are incident with r and s respectively. Define

$$h_1(x) = P(L_{ax}, s), \ h_2(y) = P(L_{ay}, r), \ \text{for } x \in \gamma, \ y \in \delta,$$

then h_1 maps γ one-to-one onto δ and $h_2 = h_1^{-1}$. Using the (definitions of) partial recursive extensions $\bar{P}$ and $\bar{L}$ of P and L respectively, we can find (definitions of) partial recursive extensions $\bar{h}_1$ and $\bar{h}_2$ of h_1 and h_2 respectively, hence also (a definition of) the partial recursive one-to-one extension $h_1^* = \bar{h}_1 | \tau$, where $\tau = \{x \in \delta\bar{h}_1 | \bar{h}_1(x) \in \delta\bar{h}_2 \ \& \ \bar{h}_2\bar{h}_1(x) = x\}$. Write $f(r, s, x)$ for $h_1(x)$ and $\bar{f}(r, s, x)$ for $h_1^*(x)$ and define the functions $g(r, s, y)$ and $\bar{g}(r, s, y)$ by

$$\delta g = \{\langle r, s, y \rangle | r, s \in \beta \ \& \ y \in \eta_r\},$$

$$g[r, s, \text{can}(r, x)] = \text{can}[s, f(r, s, x)],$$

$$\delta\bar{g} = \{\langle r, s, y \rangle | (\exists x)[y = \text{can}(r, x) \ \& \ \langle r, s, x \rangle \in \delta\bar{f}]\},$$

$$\bar{g}[r, s, \text{can}(r, x)] = \text{can}[s, \bar{f}(r, s, x)].$$

Then $g_{rs}(\eta_r) = \eta_s$ and $\bar{g}(r, s, x)$ is a partial recursive extension of $g(r, s, x)$ such that $\bar{g}_{rs}(x)$ is a one-to-one function of x. Hence G_Π satisfies condition (d) of D1. In a similar manner or using duality it can be shown that G_Π satisfies condition (c) of D1. We have proved that G_Π is ω-semiregular. Two distinct lines of Π have only one point in common, hence, since each contains at least three points, we have $\sigma_y \neq \sigma_z$, for $y, z \in \beta$, $y \neq z$. Thus G_Π is an ω-BD on the set σ. Every two distinct points x and y of Π lie on exactly one line, namely L_{xy}; since L_{xy} has a partial recursive extension, so has the canonical index of the set with L_{xy} as its only member; thus G_Π satisfies condition (iii) of D4. Every two distinct lines y and z of Π intersect in a point P_{yz} and P_{yz} can be computed from y and z; hence G_Π also satisfies condition (iv). We have proved that G_Π is an ω-BBD on σ with $\lambda = 1$. Let $N = o(\Pi)$. Then we have Req σ = Req $\beta = N^2 + N + 1$, hence $V = B$; we conclude that the ω-BBD G_Π is symmetric.

(b) Let $G_\Pi = \langle \sigma \cup \beta, \eta \rangle$ be a symmetric ω-BBD with $\lambda = 1$. To prove that Π is an ω-plane we have to show:

(α) every two distinct points x and y of Π lie on exactly one line, say L_{xy}; moreover, given x and y, we can compute L_{xy},

(β) there are four points of Π, no three of which are collinear,

(γ_1) every two distinct lines of Π intersect, i.e., have exactly one point in common,

(γ_2) if the distinct lines x and y of Π intersect, the point of intersection can be computed from x and y.

Re (α). G_Π has $\lambda = 1$ and satisfies condition (iii) of D4.

Re (β). Let p be a point of Π and l, m distinct lines through p. Then there are points q_1, q_2, r_1, r_2 such that p, q_1, q_2 are distinct points of l, while p, r_1, r_2 are distinct points of m. By (α) no three of the four points q_1, q_2, r_1, r_2 are collinear.

Re (γ_1) and (γ_2). Note that (γ_2) holds, since G_Π satisfies condition (iv) of D4. We now prove (γ_1). Let Π have the parameters V, B, R, K, then $V - 1 = R(K - 1)$, since $\lambda = 1$. Put $N = K - 1$, then $K = R = N + 1$ and $B = V = N^2 + N + 1$. Let l, m be distinct lines of Π. In view of (α) we only need to show that l and m have at least one point in common. Suppose not. Let γ and δ be the sets of points on l and m respectively. Put $\zeta = \{L_{xy} | x \in \gamma \ \& \ y \in \delta\}$ and define the function f by: $\delta f = j(\gamma \times \delta)$, $fj(x, y) = L_{xy}$. Using (α) we see that f maps $j(\gamma \times \delta)$ one-to-one onto ζ and that f has a partial recursive extension. Moreover, $f^{-1}(z) = j(P_{lz}, P_{mz})$, for $z \in \zeta$; here both P_{lz} and P_{mz} can be computed from l, m, z by (γ_2), since we know that both l and m intersect z. Thus f^{-1} also has a partial recursive extension and $j(\gamma \times \delta) \simeq \zeta$. We conclude that Req ζ = Req $\gamma \cdot$ Req $\delta = (N + 1)^2 = N^2 + 2N + 1$. Since Req $\beta = B = N^2 + N + 1$, we have Req $\beta <$ Req ζ. This contradicts the fact that $\zeta \subset \beta$. The lines l and m must therefore intersect.

REMARK R8: Once (α) and (β) of part (b) of the proof of P10 have been established, one can also prove both (γ_1) and (γ_2) without using the hypothesis that G_Π satisfies condition (iv) of D4. We only sketch the proof, but stress the fact that it depends on the fact that G_Π is *isolic* and that every *finite* symmetric BBD with $\lambda = 1$ is a projective plane. Let two distinct lines l and m of Π be given. By fully exploiting the information that G_Π is ω-regular and satisfies condition (iii) of D4 we can effectively generate a subsystem $\bar{G}_\Pi = \langle \bar{\sigma} \cup \bar{\beta}, \bar{\eta} \rangle$ of G_Π such that: l, $m \in \bar{\beta}$, the sets $\sigma, \bar{\beta}, \bar{\eta}$ are r.e., hence finite, $\bar{G}_\Pi$ is a symmetric BBD with $\lambda = 1$. Put $\bar{\Pi} = \langle \bar{\sigma}, \bar{\beta}, \text{inc.} \rangle$. Then $\bar{\Pi}$ is a finite plane with $l \cap \bar{\beta}$ and $m \cap \bar{\beta}$ as lines; let these lines intersect in the point p of $\bar{\Pi}$. Then the lines l and m of Π also intersect in p and p can be computed, since $\bar{\Pi}$ is a finite plane.

For more information concerning isolic ω-planes the reader is referred to [4] and [7].

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