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OPEN SUBSETS OF FIBRATIONS

J.F. McClendon

Suppose $T \to B$ is a fibration and E is an open subset of T. In general, of course, $E \to B$ will not be a fibration. However under certain special circumstances it will be and the purpose of the present paper is to provide sufficient conditions for this to be the case. Actually, the main results treat a slightly more general situation. If $E \subset W \subset T$ and a map $q: T \to B$ is given then say that E is a B-retract of W if there is a map $r: W \to E$ with $ri = id$ (where $i: E \to W$) and $qr = q$. The case studied here is when E is a B-retract of an open set of T. Two results treat a general map $E \to B$ (without reference to T).

There are some results in the literature (e.g. [11, Prop. 4.1], [12, Cor. 8.4], [4, Th. 3], [6, Th. 1]), giving sufficient conditions for $E \to B$ to be a fibration. These usually put a completeness condition on E or the map $E \to B$. The result here is of a different nature (E may be open in T, so needn't be complete). The main theorem (section 2) calls for relations between the homotopy of the fibers of E and T and requires some preliminary definitions (section 1). Two useful special cases can be stated here.

3.1. COROLLARY: *Suppose $T \to B$ is a Serre fibration, E a B-retract of an open subset of T, and $\pi_i(E(b), e) \to \pi_i(T(b), e)$ is isomorphic for all i. Then $E \to B$ is a Serre fibration.*

Now suppose $E \subset B \times Z, r: B \times Z \to Z$ the projection. Define $M_i(e)$ image $r_*: \pi_i(E, e) \to \pi_i(Z, re)$.

3.2. COROLLARY: *Suppose E is a B-retract of an open subset of $B \times Z$ and $r_*: \pi_i(E(b), e) \to \pi_i(Z, re)$ is monic with image $M_i(e)$ for all b, e in $E(b), i$. Then $E \to B$ is a Serre fibration.*

The above are stated in section 3 in a slightly more general form (for N-fibrations). An example is given in section 3 showing that "Serre" cannot be replaced by "Hurewicz" in 3.2. The example also answers part of a question of R. Brown [3]. A homotopy-negligible corollary is proven generalizing part of a result of R. Wong [15].

The results of the present paper will be applied in a separate paper to the study of continuous selections and fixed points for certain multivalued functions.

1. Notation

A map will be a continuous function. $[f]$ dnotes the homotopy class of f but frequently the homotopy class will also be denoted by f. I is the unit interval $[0,1]$. A function $f: X \to Y^I$ has an adjoint $f': I \times X \to Y$, $f'(t,x) = f(x)(t)$. Usually the "$'$" is omitted and the same symbol used for f and its adjoint. $f: I \times X \to Y$ gives rise to functions $f_t = f(t): X \to Y$, $f(t)(x) = f(t,x)$. f restricted to $[t,t'] \times X$ gives a homotopy (reparamatrized) $f(t) \sim f(t')$ and by reversal one $f(t') \sim f(t)$, $(t' < t)$. These homotopies can be denoted by $f_{tt'}$ but will usually be denoted simply by f.

If functions $p: X \to B$ and $q: Y \to B$ are given then a function $f: X \to Y$ is "over B" if $qf = p$. A homotopy $H: X \times I \to Y$ is over B if each H_t is over B.

Suppose $w: \partial I^i \to X$ is given. Let $\Delta_i(X,w)$ be the set of all homotopy classes of maps $f: I^i \to X$, which extend w, by homotopies which are fixed on ∂I^i. If $w(\partial I^i) = \text{point} = x_0$, then $\Delta_i(X,w) = \pi_i(X,x_0) =$ the i'th homotopy group of X at x_0. $\Delta_i(X,w)$ may be empty. If $f \in \Delta_i(X,w)$ then there is a bijection $\pi_i(X,w(0)) \to \Delta_i(X,w)$ defined by $h \to h + f$.

A map $q: T \to B$ is an N-fibration if it has the homotopy lifting property ($=$ covering homotopy property) for CW complexes of dimension $\leq N$, or equivalently, if it has the homotopy lifting property for cubes of dimension $\leq N$. A Serre fibration is a map which is an N-fibration for all N.

Two transformations will be needed. Let $p: T \to B$ be a Serre fibration.

(1) If $v: I \to B$ is a path in B from $b = v(0)$ to $b' = v(1)$ then there is a function $h_v: \pi_0(T(b)) \to \pi_0(T(b'))$ which depends only on the homotopy class of v, rel ends, and satisfies $h_{v+v'} = h_{v'}h_v$. This can be defined as follows: if $y \in T(b)$ let $L: I \to T$ be any lifting of v such that $L(0) = y$. Then $h_v[y] = [L(1)]$.

(2) Let $w: \partial I^i \to T(b)$, $w': \partial I^i \to T(b')$, and $H: I \times \partial I^i \to T$ be a homo-

topy from w to w' with pH_t constant, all t (so $H_t(\partial I^i) \subset$ some fiber). Then there is a function $h_H : \Delta_i(T(b), w) \to \Delta_i(T(b'), w')$ which depends only on the homotopy class of H (by homotopy rel $\partial I \times \partial I^i$ and over B where $I \times \partial I^i \to B$ is $u\pi_1$, $u(t) = pH_t$). It satisfies $h_{H+H'} = h_{H'} h_H$ and may be defined as $h_H[f] = [L_1]$ where L is any map filling the following diagram

$$
\begin{tikzcd}
0 \times I^i \cup I \times \partial I^i \arrow[r, "f \cup H"] \arrow[d, hook] & T \arrow[d] \\
I \times I^i \arrow[r, "u\pi_1"'] \arrow[ur, dashed, "L"] & B
\end{tikzcd}
$$

As a particular case suppose r is a path in T from y to y', $py = b$, $py' = b'$. Then there is a function $h_r : \pi_i(T(b), y) \to \pi_i(T(b'), y')$, $h_{r+r'} = h_{r'} h_r$.

All of the above assertions are easily verified by the methods of [13, p. 379–382].

If $q : T \to B$ is given and $E \subset T$ use the notation $j : E \subset T$ and j or $j(b) : E(b) \subset T(b)$, i or $i(b) : E(b) \subset E$. Sometimes i and j are dropped from the notation.

Monic means one-to-one.

2. Fibration theorem and proof

Consider

$$
\begin{tikzcd}
E \arrow[r, phantom, "\subset"] \arrow[dr, "p"'] & T \arrow[d, "q"] \\
& B
\end{tikzcd}
\quad (j : E \subset T)
$$

2.1. Definition: Suppose $T \to B$ is a Serre fibration. E is an *N-consistent* subset if the following are satisfied, $0 \leq i \leq N$.

(0) (a) $j_* : \pi_0(E(b)) \to \pi_0(T(b))$ is monic for all b, w.

(b) $h_v(j_* \pi_0(E(b)) \subset j_* \pi_0(E(b'))$ for any path v from b to b' in B.

(i) ($i \geq 1$)

(a) $j_* : \Delta_i(E(b), w) \to \Delta_i(T(b), jw)$ is monic for all b, w.

(b) $h_{jH}(j_* \Delta_i(E(b), w)) \subset j_* \Delta_i(E(b'), w')$ for all $H : w \sim w'$ with pH_t constant.

The main theorem is the following one.

2.2. Theorem: *Suppose $T \to B$ is a Serre fibration and $E \subset T$ is an N-consistent subset and a B-retract of an open subset of T. Then $E \to B$ is an N-fibration.*

Proof: The proof is by induction on N. First take $N = 0$.

$$\begin{tikzcd}
0 \arrow[r, "g"] \arrow[d] & E \arrow[d] & U \arrow[l, "r"'] \arrow[dl, "\overset{\subset}{j'}"'] \arrow[r, phantom, "\subset"] & T \arrow[dll, "j''"'] \\
I \arrow[r, "f"'] \arrow[ur, dashed, "F"] & B & &
\end{tikzcd}
\qquad \begin{array}{l} rj = id \\ j = j''j' \end{array} \qquad U \text{ open}$$

Given that $pg = fi$, it is necessary to find F with $Fi = g$ and $pF = f$. Write $e(0) = g(0)$ and let $\mathscr{E}(0) = [e(0)]$ in $\pi_0(E(f(0)))$. Define $\mathscr{E}(t)$ in $\pi_0(E(f(t)))$ by $h_f(j\mathscr{E}(0)) = j\mathscr{E}(t)$. Then $h_f(j\mathscr{E}(t)) = j\mathscr{E}(t')$ for any t, t', where the f subscript is $f_{tt'}$.

Pick $e(t) \in \mathscr{E}(t)$ so $e(t) \in E(f(t)) \subset T(f(t))$. Because $T \to B$ is a fibration there is a $K: I \to T$ with $pK = f$ and $K(t) = je(t)$. $K^{-1}(U)$ is an open subset of I so there is an open *nbhd* $W(t)$ of t in I and, by restricting K, a map $G': W(t) \to U$, over B, with $G'(t) = j'e(t)$. Define $G = rG': W(t) \to E$. Then G is over B (since r and G' are) and $G(t) = rG'(t) = rj'e(t) = e(t)$. The restriction of G to $[t', t'']$ shows (see sect. 1) that $h_f[G(t')] = [G(t'')]$. So $[G(t')] = h_f\mathscr{E}(t) = \mathscr{E}(t')$ all t' in $W(t)$.

Since I is compact there are $0 = a_0 < a_1 < \ldots < a_n = 1$ and $F_i: [a_{i-1}, a_i] \to E$ with $pF_i = f$, $[F_i(t)] = \mathscr{E}(t)$ in $\pi_0(E(f(t)))$ (each F_i is a restriction of one of the G's).

It is necessary to modify the F_i. For this purpose here and later we need the following lemma which was proven in [8] for E open in T. The present version follows by applying the version in [8] to the containing open set U and then applying the retraction.

2.3. Lemma: *Suppose $T \to B$ a Serre fibration, E a B-retract of an open subset of T. Let (X, A) be a relative CW complex, $f: X \to E$ a map, $G: A \times I \to E$ a homotopy with $G_0 = f|A$ and $pG(a, t) = pf(a)$, all a, t. Then G extends to a homotopy $H: X \times I \to E$ with $H_0 = f$ and $pH(x, t) = pf(x)$, all x, t.*

Here let $X = [a_{i-1}, a_i]$, $A = \{a_{i-1}, a_i\}$. Note $[F_i(a_j)] = [e(a_j)]$, $j = i - 1, i$. Let G on $a_j \times I$ be the homotopy from $F_i(a_j)$ to $e(a_j)$. Then 2.3 gives an extension H of G, $H: X \times I \to E$, H is over B, $H_0 = F_i$. Define $F_i' = H_1$. Then $pF_i' = f$, $F_i'(a_j) = e(a_j)$, $j = i - 1$, i. The F_i' fit together to give the desired $F: I \to E$ completing the $N = 0$ proof.

Now *assume the theorem for* $0 \leq M < N$. To prove the theorem it will suffice, by adjointness, to prove $E^I \to B^I$ is an $(N - 1)$-fibration. Since E^I is an B^I-retract of the open U^I of T^I the induction assumption shows: it suffices to prove for $0 \leq i \leq N - 1$:

(0) $\pi_0(E^I(u)) \to \pi_0(T^I(u))$ is monic, all u in B^I, and $h_v(j\pi_0(E^I(u)) \subset j\pi_0(E^I(u'))$ for all $v: u \sim u'$.

(i) $\Delta_i(E^I(u), U) \to \Delta_i(T^I(u), U)$ is monic, all u, U in $E^I(u)$, and $h_{jH}(j\Delta_i(E^I(u), U) \subset j\Delta_i(E^I(u'), U')$ for any H path from U to U' in E^I with $p^I H_t$ constant.

Proof of monic (for 0 and i): Using the Δ_i, π_i 1–1 correspondence it is easily checked that it suffices to prove $j_*: \pi_i(E^I(u), w) \to \pi_i(T^I(u), w)$ monic. Suppose $j_*[f] = 0$ and consider:

$$\begin{tikzcd}
\partial I^{i+1} \arrow[r, "f"] \arrow[d, phantom, "\cap"] & E^I(u) \arrow[r, "j"] & T^I(u) \\
I^{i+1} \arrow[ur, dashed, "F"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & U \arrow[dr, "j''"] & \\
I \times \partial I^{i+1} \arrow[r, "f"] \arrow[d, phantom, "\cap"] & E \arrow[ur, "j'"] \arrow[rr, "j"] \arrow[dd, "p"] & & T \arrow[ddll, "q"] \\
I \times I^{i+1} \arrow[ur, dashed, "F"'] \arrow[d] & & & \\
I \arrow[r, "u"] & B & &
\end{tikzcd}$$

Given that jf extends in the first diagram we must find an extension F of f as shown. By adjointness this amounts to finding F for the second, given that jf extends over B. If $t = 0$ we get

$$\begin{tikzcd}
0 \times \partial I^{i+1} \arrow[r, "f_0"] \arrow[d, phantom, "\cap"] & E(u(0)) \arrow[r] & T(u(0) \\
0 \times I^{i+1} \arrow[ur, dashed, "e(0)"'] & &
\end{tikzcd}$$

and the hypothesis that $\pi_i(E(u(0)), e) \to \pi_i(T(u(0)), e)$ is monic gives $e(0)$ as shown. Define $\mathscr{E}(0) = [e(0)] \in \Delta_i(E(u(0)), f)$ and $\mathscr{E}(t) \in \Delta_i(E(u(t)), f)$ by $h_f(j\mathscr{E}_0) = j\mathscr{E}_t$. Then $h_f(j\mathscr{E}(t)) = j\mathscr{E}(t')$ for all t, t' where the last subscript f is $f_{tt'}$.

Pick $e(t) \in \mathscr{E}(t)$ and let $g: Q(t) \to E$ where $Q = I \times \partial I^{i+1} \cup t \times I^{i+1}$ and $g = f \cup e(t)$. Because $T \to B$ is a Serre fibration jg extends to $K: I \times I^{i+1} \to T$, over B. $K^{-1}(U)$ is an open *nbhd* of Q so we get, by restricting $K, G': W \times I^{i+1} \to U$ extending f over B where W is an open *nbhd* of t in I. Define $G = rG'$. Since r and G' are over B so is G and $Gi = rG'i = rj'(f \cup e(t)) = f \cup e(t)$. The existence of G shows (see section 1) that $h_{jf}[jG(t')] = [jG(t'')]$ for all t, t' in W where the subscript f is $f_{t't''}$. So $j_*[G(t')] = h_{jf}(j_*G(t)) = h_{jf}(j_*\mathscr{E}(t))$ (since $G(t) = e(t)) = j_*\mathscr{E}(t')$. j_* monic then gives $[G(t')] = \mathscr{E}(t')$, all $t' \in W$.

The $W = W(t)$'s cover I so we get $0 = a_0 < a_1 < \ldots < a_n = 1$ and maps $F_i: [a_{i-1}, a_i] \times I^i \to E$ (each a restriction of some G) satisfying (1) F_i extends f (where both defined) (2) F_i is over B, and (3) $[F_i(t)] = \mathscr{E}(t)$ all i, t. Now the F_i must be modified to fit together to give the desired F-but the procedure is exactly like that in the $N = 0$ case using lemma 2.3, so the details are omitted here.

Proof of $h_v j\pi_0(E^I(u)) \subset j\pi_0(E^I(u'))$: This is quite similar to the next part of the proof so the details are omitted.

Proof that $h_{jH}(j\Delta_i(E^I(u), U)) \subset j\Delta_i(E^I(u'), U')$: Let $v: T^I \to T$ and $B^I \to B$ be defined by $vh = h(0)$. Then v maps the Serre fibration $T^I(u) \to T^I \to B^I$ to the Serre fibration $T(u(0)) \to T \to B$. The bottom two maps are clearly homotopy equivalences so the fibration exact sequences show that $\pi_i(T^I(u), U) \to \pi_i(T(u(0)), U(0))$ is isomorphic for all i. If $\Delta_i(T^I(u), w)$ is non-empty then it follows that $v: \Delta_i(T^I(u), w) \to \Delta_i(T(u(0)), vw)$ is also a bijection. Now consider the following diagram.

$$\begin{tikzcd}
f \arrow[r] & jf \arrow[r] & jf(0) \\
\Delta_i(E^I(u), w) \arrow[r] & \Delta_i(T^I(u), w) \arrow[r, "v"] \arrow[d, "h_{jH}"] & \Delta_i(T(u(0)), vw) \arrow[d, "h_{vjH}"] \\
\Delta_i(E^I(u'), w') \arrow[r] & \Delta_i(T^I(u'), w') \arrow[r, "v"] & \Delta_i(T(u'(0)), vw')
\end{tikzcd}$$

In proving the assertion it may be assumed that $\Delta_i(\mathrm{T}^I(u), w) \neq \emptyset$. h_{jH} shows that $\Delta_i(T^I(u'), w') \neq \emptyset$ and hence both v's are bijection. Let [h] $= h_{vjH}(jf(0))$. By hypotheses $[h] = j_*[g]$ for $g: I^i \to E(u'(0))$, $g|\partial I^i = w'$, (since $i \leq N$, in fact $i \leq N - 1$ here). Consider

$$\begin{tikzcd}
0 \times I^i \cup I \times \partial I^i \arrow[r, "g \cup w'"] \arrow[d, phantom, "\cap"] & E \arrow[d, phantom] \\
I \times I^i \arrow[r, "u'\pi_1"'] \arrow[ur, dashed, "f'"] & B
\end{tikzcd}$$

$E \to B$ is an i-fibration (since $i \leq N - 1$) so f' exists as shown and $[jf'(0)] = [jg] = [h] = h_{vjH}[jf(0)]$ proving the assertion.

A slight modification of the proof above yields a theorem giving sufficient conditions for a map $E \to B$ to be a Serre fibration. Some definitions are needed to state the result.

2.4. Definition: Let $E \to B$ be a map.

(1) $E \to B$ has the B-CHP (B-covering homotopy property) for (X, A) if: for every map $f: X \to E$ and homotpy $G: A \times I \to E$ of $f|A$ with $pG(a, t) = pf(a)$, all a, t, there is a homotopy $H: X \times I \to E$ extending G with $H_0 = f$ and $pH(x, t) = pf(x)$, all x, t. $E \to B$ has B-CHP(n) if it has B-CHP for all relative CW complexes of dimension $\leq n$.

(2) $p: E \to B$ has the local extension property (LEP) for (X, A) if for every $u: A \to E$, $v: X \to B$ with $pu = vi$ there is an open *nbhd* U of A in X and map $F: U \to E$ extending u over B. $E \to B$ has LEP(n) if it has LEP

for all relative CW complexes (X, A) *with* A a deformation retract of X and dimension $(X, A) \leq n$.

NOTE: "$E \to B$ has B-CHP" is the conclusion of Lemma 2.3. The condition B-CHP is the same as "$E \to B$ has CHP for (X, A) in Top $(\emptyset \to B)$", see [7]. Theorem 2.6 below can be viewed as giving extra conditions on a certain Top $(\emptyset \to B)$ Serre fibration to ensure it is an ordinary Serre fibration.

2.5. DEFINITION: *A* map $E \to B$ has an i-action if:

Case 1, $i = 0$: for each path $v: I \to B$ there is a function $h(v)$: $\pi_0(E(v(0))) \to \pi_0(E(v(0)))$ satisfying (1) $h(v) = h(v')$ if $v \sim v'$ rel ends, (2) $h(\text{const}) = id.$, (3) $h(v + v') = h(v')h(v)$, and (4) $h(v)[L(0)] = [L(1)]$ if $L: I \to E$ lifts v. (Liftings are not assumed to exist.)

Case 2, $i > 0$: for each $w: \partial I^i \to E(b)$, $w': \partial I^i \to E(b')$ and homotopy $H: I \times \partial I^i \to E$ from w to w' with $pH = u\pi$ for some $u: I \to B$ there is a function $h(H): \Delta_i(E(b), w) \to \Delta_i(E(b'), w')$ satisfying (1) $h(H) = h(H')$ if $H \sim H'$ rel $\partial I \times \partial I^i$ over B, (2) $h(\text{const}) = id$, $(e)h(H + H') = h(H')h(H)$ and (4) $h(H)[L(0)] = [L(1)]$ if $L: I \times I^i \to E$ extends H and $pL = u\pi$.

2.6. THEOREM: *Suppose a map $E \to B$ has B-CHP $(N + 1)$, LEP $(N + 1)$, and i-action, $i \leq N$. Then $E \to B$ is an N-fibration.*

The proof is very similar to that of Theorem 2.2 and is omitted.

2.7. COROLLARY: *Suppose a map $E \to B$ had B-CHP $(N + 1)$, LEP $(N + 1)$, and non-empty N-connected fibers. Then $E \to B$ is an N-fibration.*

3. Special cases, example

3.1. COROLLARY: *Suppose $T \to B$ is a Serre fibration, E a B-retract of an open subset of T, and $\pi_i(E(b), e) \to \pi_i(T(b), e)$ isomorphic for $i \leq N$. Then $E \to B$ is an N-fibration.*

PROOF: Conditions 0-a and 0-b of 2.1 are immediate. For i-a note that if $[f]$ and $[g]$ are in $\Delta_i(E(b), w)$ we can form their difference (glue along ∂I^i) $[d(f, g)] \in \pi_i(E(b), w(0))$. So $j_*[f] = j_*[g]$ gives $j_*[d(f, g)] = 0$ so by our current hypothesis $[d(f, g)] = 0$ and hence $[f] = [g]$. For (i-b) note first that $\Delta_i(E(b), w) \neq \emptyset$ gives $\Delta_i(E(b'), w') \neq \emptyset$ since w is in $\pi_{i-1}(E(b), w(0)) \approx \pi_{i-1}(T(b), w(0)) \approx$ (by h_H) to $\pi_{i-1}(T(b'), w'(0)) \approx$

$\approx \pi_{i-1}(E(b'), w'(0))$ and w is sent to w'. Using this the π_i, Δ_i correspondence shows $j_*: \Delta_i(E(b), w) \to \Delta_i(T(b), w)$ is bijective and the same for b', w' so condition (i-b) follows from the hypotheses of 3.1.

Now consider

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[dr] & B \times Z \arrow[r, "r"] \arrow[d] & Z \\ & B & \end{tikzcd}$$

and let $M_i(e) = \text{Image } (rj)_*: \pi_i(E, e) \to \pi_i(Z, re)$.

3.2. Corollary: *Assume E a B-retract of an open subset of $B \times Z$ and $(rji)_*: \pi_i(E(b), e) \to \pi_i(Z, re)$ monic with image $M_i(e)$ for all b, e in $E(b)$, $0 \le i \le N$. Then $E \to B$ is an N-fibration.*

Proof: Let $T = B \times Z$ so $T(b)$ is $b \times Z$. By hypothesis, $\pi_i(E(b), e) \to \pi_i(T(b), e)$ is monic. This proves 0-a of 2.1 directly and the argument of 3.1 above shows (i-a) is also true. (0-b) is similar to, but easier than, the following. *Proof of (i-b).* Consider

$$\begin{array}{ccc} \Delta_i(E(b), w) & \xrightarrow{r_* i_*} & \Delta_i(b \times Z, rw) \\ & & \downarrow{\scriptstyle h_{rH}} \\ \Delta_i(E(b'), w') & \xrightarrow{r'_* i'_*} & \Delta_i(b' \times Z, rw') \end{array}$$

and it must be shown that $h_{rH}(\operatorname{Im}(ri)_*) \subset \operatorname{Im}(r'_* i'_*)$. The following diagram is commutative

$$\begin{array}{ccc} \Delta_i(E, w) & \xrightarrow{r_*} & \Delta_i(b \times Z, rw) \\ \downarrow{\scriptstyle h_H} & & \downarrow{\scriptstyle h_{rH}} \\ \Delta_i(E, w') & \xrightarrow{r'_*} & \Delta_i(b' \times Z, rw') \end{array}$$

so $h_{rH}(\operatorname{Im}(r_*) \subset \operatorname{Im}(r'_*)$ and *it will suffice to prove* $\operatorname{Im}(r_*) = \operatorname{Im}(r_* i_*)$.

Suppose $\operatorname{Im}(r_*) \neq \emptyset$. Then $[w] = 0$ in $\pi_{i-1}(E, w(0))$ so $[rw] = 0$ in $\pi_{i-1}(Z, rw(0))$. By hypothesis $[w] = 0$ in $\pi_{i-1}(E(b), e)$ so $\operatorname{Im}(r_* i_*) \neq \emptyset$. That is, $\operatorname{Im}(r_* i_*) = \emptyset \Rightarrow \operatorname{Im}(r_*) = \emptyset$ so we can assume $\operatorname{Im}(r_* i_*) \neq \emptyset$ and must prove $\operatorname{Im}(r_*) \subset \operatorname{Im}(r_* i_*)$.

Let $k = r_* m$ be in $\operatorname{Im}(r_*)$ and consider

$$\begin{array}{ccccc} & & m & \longrightarrow & k \\ \Delta_i(E(b), w) & \longrightarrow & \Delta_i(E, w) & \longrightarrow & \Delta_i(Z, rw) \\ \uparrow & & \uparrow & & \uparrow \\ \pi_i(E(b), w(0)) & \longrightarrow & \pi_i(E, w(0)) & \longrightarrow & \pi_i(Z, rw(0)) \end{array}$$

The vertical arrows are bijections defined by adding f_0, if_0, rif_0 (resp.) where $f_0 \in \Delta_i(E(b), w)$. Thus $m = p + if_0$, $p \in \pi_i(E, w(0))$ and hence $k = rp + rif_0$. But by hypothesis $rp = riq$ so $k = riq + rif_0 = ri(q + f_0)$ showing $k \in \mathrm{Im}(r_* i_*)$ and completing the proof.

NOTE: The conditions (i) of 2.1 can be replaced by the following:
(i′) (a) $j_* : \pi_i(E(b), e) \to \pi_i(T(b), e)$ monic, all b, e
(b) $h_r(j_* \pi_i(E(b), e)) \subset j_* \pi_i(E(b'), e')$ for any path r from e to e' in E.
(c) If $\Delta_i(E(b), w) \neq \emptyset$ and $H : w \sim w' : \partial I^i \to E$ with pH_t constant then there are f in $\Delta_i(E(b), w)$ and f' in $\Delta_i(E(b'), w')$ with $h_H(jf) = jf'$.

The fact that (i′-a) gives (i-a) was shown in the proof of 3.1. We must now deduce (i-b) from (i′-b) and (i′-c). Let $a = a(f) : \pi_i(E(b), w(0)) \to \Delta_i(E(b), w)$ be the bijection defined by $a(u) = u + f$ and similarly define $a = a(jf)$, $a' = a(f')$, and $a' = a(jf')$. Let $h = h_{jH}$ (on both π_i and Δ_i). Let $g \in \Delta_i(E(b), w)$. Then it must be proved that $hjg = jg'$ for some $g' \in \Delta_i(E(b'), w')$. Let $g = a\bar{g}$. by hypothesis (i′-b), $hj\bar{g} = j\bar{g}'$ for some $\bar{g}'$. Define $g' = a'\bar{g}'$. Then $hjg = hja\bar{g} = haj\bar{g} = a'hj\bar{g} = a'j\bar{g}' = ja'\bar{g}' = jg'$; proving (i-b).

The following result generalizes [8, lemma 1.2] in that "E open" is here replaced by "E is a B-retract of an open".

3.3. THEOREM: *Suppose E is a B-retract of an open subset of T, where $T \to B$ is a Serre fibration, and each $E(b)$ is N-connected and non-empty. Then $E \to B$ is an N-fibration.*

This could be deduced from 2.2 as follows: conditions (i′-a) and (i′-b) in the above note are immediate. (i′-c) can be proved. However, proving (i′-c) seems to require a significant part of the proof in [8]. The easiest way to prove the theorem is to make a few small changes in the proof given in [8]. The proof is omitted here.

Theorem 2.2 leads to a result on homotopy negligibility that generalises the homotopy negligible part of Wong [15, Cor. 2.3] (see also [14, Th. 3]). If $A \subset X$, say that it is w-homotopically negligible (w = weakly) if $X \backslash A \to X$ is a weak homotopy equivalence and locally w-homotopically negligible if each point has a *nbhd* system $\{U_\alpha\}$ such that $U_\alpha \backslash A \to U_\alpha$ is a weak homotopy equivalence for each U_α.

3.4. COROLLARY: *Suppose $E \to B$ is a Serre fibration and $E \backslash A$ is a nbhd retract in E. Suppose $A(b)$ is locally w-homotopically negligible in $E(b)$ for all b in B. Then A is w-homotopically negligible in E.*

PROOF: It follows from [McCord, 10] that $E(b) \backslash A(b) \to E(b)$ is isomorphic in homotopy and Cor. 3.1 shows $E\backslash A \to B$ is a Serre fibration. The homotopy sequences of $E\backslash A \to B$ and $E \to B$ show that $E\backslash A \to B$ is isomorphic on homotopy so A is w-homotopically negligible.

COMMENTS: (1) One can define A to be q-homotopy negligible to mean $X\backslash A \to X$ is isomorphic on π_i for $i \leq q$. If we assume A closed in E then [Eells–Kuiper, 5] can be used instead of [McCord, 10] to prove the above corollary with "q-homotopy negligible" replacing w-homotopy negligible everywhere.

(2) If E is an ANR then "weakly" in the conclusion of the corollary can be dropped – i.e., $E\backslash A \to E$ is then a homotopy equivalence.

It is tempting to try and replace "Serre" by "Hurewicz" in 3.2 (since $T \to B$ is a product) or to make the replacement twice in 3.1. The following example (taken from [9]) shows that some caution is required.

Let $B =$ Hilbert cube, $D = \{(b,b) | b \in B\}$, and $E = (B \times B)\backslash D \subset B \times B$. Then we have

$$\begin{tikzcd} E \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d] & B \times B \arrow[dl] \\ B & \end{tikzcd}$$

$E(b) = B\backslash\{b\}$ is contractible for all b (e.g. [2, Chap III, sect 4]) so $\pi_i(E(b)) \to \pi_i(T(b))$ is isomorphic. By Cor. 3.2, $E \to B$ is a Serre fibration. However, $E \to B$ is not a Hurewicz fibration. The reason is that if it were it would have a cross section (since B is contractible), say $s: B \to E$, $\pi_1 s = id$. But then the composite $\pi_2 s: B \to B$ would be a map without a fixed point – contradicting the classical Brouwer fixed point theorem [see, Kura towski, Topology, Vol. II, p. 344].

Note that $p: E \to B$ above is an example of a Serre fibration with B and each p^{-1} (b) an ANR but p is not a Hurewicz fibration. This answers part of a question of R. Brown [3] (see also [Allaud, 1]).

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