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A GENERAL APPROXIMATION THEOREM FOR HILBERT CUBE MANIFOLDS

T.A. Chapman*

1. Introduction

Recall that the *Hilbert cube* Q is the countable infinite product of closed intervals, and a *Q-manifold* is a separable metric manifold modeled on Q. Let $f: M \to N$ and $p: N \to B$ be maps (map $\equiv$ continuous function), where M and N are compact Q-manifolds and B is a compact metric space. In this paper we will be interested in the following approximation question: *When is there a homeomorphism $h: M \to N$ for which ph is close to pf?* The best previous infinite-dimensional result in this direction is Theorem 3 of [2], where it was additionally assumed that B is an ANR, $p: N \to B$ is an approximate fibration, and the fiber has a "nice" fundamental group (for example, free abelian). With this assumption a carefully controlled engulfing result was established so that the entire problem could be wrapped up around a torus and then unwrapped in the usual manner. The best previous finite-dimensional result in this direction is the thin h-Cobordism Theorem of [7], where it was additionally assumed that B is locally simply connected and the fibers of p have locally constant fundamental groups which are "nice". Again the problem was wrapped up around a torus and unwrapped, but this time it was done on the level of geometric groups. The main result of this paper is the Approximation Theorem which is stated below. In it we substantially generalize Theorem 3 of [2] by completely eliminating the requirement that $p: N \to B$ be an approximate fibration. The only demand made on p is a locally "nice" condition on fundamental groups which does not require B to be locally simply connected and does not

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even require the locally constant π_1-assumption as in [7]. In doing this we completely lose engulfing and therefore cannot wrap up the problem around a torus as in [2], but with a much more direct mapping torus trick we still achieve the approximation result.

Here is some notation that will be used throughout this paper. Given metric spaces X and Y, a homotopy $h_t: X \to Y$ is said to be an ε-*homotopy* provided that each set $\{h_t(x) | 0 \leq t \leq 1\}$ has diameter $\leq \varepsilon$. If B is also a metric space and $p: Y \to B$ is a map, then $h_t: X \to Y$ is said to be a $p^{-1}(\varepsilon)$-*homotopy* provided that $ph_t: X \to B$ is an ε-homotopy. A map $f: X \to Y$ is said to be a $p^{-1}(\varepsilon)$-*equivalence* provided that there is a map $g: Y \to X$ so that gf is $(pf)^{-1}(\varepsilon)$-homotopic to id_X and fg is $p^{-1}(\varepsilon)$-homotopic to id_Y. If $Y = B$ and $p = id$, then we simply call f an ε-*equivalence* (see [6]).

For the statement of our main result we will require the Whitehead group functor, Wh, which is a homotopy functor from the category of spaces and maps to the category of abelian groups and homomorphisms. For any X we define $Wh(X)$ to be the direct sum

$$\oplus \{WhZ[\pi_1(C)] \mid C \text{ is a path component of } X\},$$

where $WhZ[\pi_1(C)]$ is the usual algebraically-defined Whitehead group [4, p. 39]. It follows from work of Bass–Heller–Swan–Stallings that $Wh(X) = 0$ provided that $\pi_1(C)$ is free or free abelian, for each path component C of X (see [4, pp. 43–45] for references). For convenience we define $Wh(\emptyset) = 0$. The key to our main result is the following definition. A map $p: Y \to B$ of metric spaces is said to be (ε, δ)-*nice* if given any b in B, the inclusion-induced homomorphism,

$$Wh(p^{-1}(S_\delta(b)) \times T^n) \to Wh(p^{-1}(S_\varepsilon(b)) \times T^n),$$

is the 0-homomorphism, for any n-torus T^n. ($S_\delta(b)$ is the closed δ-neighborhood of b.) A simple example of this is provided by $p = \text{proj}: B \times F \to B$, where B and F are compact ANRs for which $\pi_1(C)$ is free abelian, for all path components C of F. From the π_1-condition we conclude that $Wh(F \times T^n) = 0$, and from the local contractibility of B (and the homotopy functorality of Wh) we see that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that p is (ε, δ)-nice.

Here is the main result of this paper.

APPROXIMATION THEOREM: *Let B be a finite-dimensional compact metric space. For every $\varepsilon > 0$ there exists a decreasing set $\{\delta_i\}_{i=1}^k$ of positive numbers so that if M, N are compact Q-manifolds, $p: N \to B$ is*

(δ_1, δ_{i+1})-nice for all i, and f is a $p^{-1}(\delta_k)$-equivalence, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism.

The scope of this result is greatly enhanced by the following addendum which clarifies the manner in which k and the δ_i are chosen.

ADDENDUM: 1. *The integer k depends only on* dim B.

2. *The set $\{\delta_i\}$ clearly depends only on ε and B, and it can be rechosen as follows: For a fixed i_0 and any $x \in (0, \delta_{i_0})$ we can find another set $\{\delta'_i\}$, fulfilling the above requirements, which is of the form $\delta'_i = \delta_i$, for $i < i_0$, and $\delta'_{i_0} = x$.*

REMARKS: 1. Regarding the proof of the above result there is an elementary trick which reduces the case of B a finite-dimensional compactum to B a polyhedron. (This trick does not seem to work for B infinite-dimensional.) Then the proof proceeds by induction on dim B. In passing from the case dim $B = n - 1$ to the case dim $B = n$ a mapping torus trick is used to first establish a splitting result, and this is then used to achieve the inductive reduction.

2. As one runs through an inductive proof on dim B there are a number of obstructions which are encountered. Some of these are the K_{-i} obstructions as detected by Quinn in [7] when one looks at the general problem of putting a boundary on a manifold with control in a parameter space, while others are $\lim^1$-type obstructions as encountered by Siebenmann in his second exact sequence for infinite-simple homotopy theory [8], and which do not arise in [7]. One cannot help but wonder what form the general theory will take.

3. There is a sharper version of the Approximation Theorem which is established in Section 7. It is called the Relative Approximation Theorem and it is needed because the statement of the Approximation Theorem does not seem to be sufficiently strong to carry out the inductive proof mentioned above. This relative version is really nothing but a statement of the functorial manner in which the δ_i can be chosen when we change the target space N to another target space via an inclusion.

4. Finally we mention that there is the expected stable PL version of the Approximation Theorem whose proof runs along the same lines. The set-up is the same except that M and N are now compact polyhedra. In the conclusion, instead of requiring that f be $p^{-1}(\varepsilon)$-homotopic to a homeomorphism we require that there exists another compact polyhedron Z and PL surjections $\alpha: Z \to M, \beta: Z \to N$ with contractible point-inverses such that $f\alpha$ is $p^{-1}(\varepsilon)$-homotopic to β. Also we point out that dim(Z) is a function of dim(M), dim(N) and dim(B).

We now make a few comments concerning the organization of the material in this paper. Section 2 contains some applications of the Approximation Theorem and Section 3 has some further definitions and notation. In Section 4 we establish some general results on mapping telescopes and mapping tori which are used in Section 5 to establish a splitting theorem. Section 6 establishes a controlled version of the Sum Theorem for Whitehead torsion and Section 7 contains a proof of the Approximation Theorem. Finally in Section 8 we establish the applications of Section 2.

2. Applications

We have stated the Approximation Theorem of Section 1 in a very broad form, and because of this it is possible that the conditions could be too general to make effective use of in certain situations. So in Theorems 2.1 and 2.2 below we derive some simpler (and weaker) forms of this result. For our first theorem we will need the following definition. A map $p: Y \to B$ of metric spaces is said to be *nice* if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that p is (ε, δ)-nice. Thus the example $p = \text{proj}: B \times F \to B$ of Section 1 is nice. Note also that p does not necessarily have to be surjective.

THEOREM 2.1: *Let B be a finite-dimensional compact metric space. For every $\varepsilon > 0$ there exists a $\delta > 0$ so that if M, N are compact Q-manifolds, $p: N \to B$ is nice, and $f: M \to N$ is a $p^{-1}(\delta)$-equivalence, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism.*

REMARKS: 1. This generalizes a similar result of [2] if B is assumed to be an ANR and p is assumed to be an approximate fibration whose fiber F satisfies $Wh(F \times T^n) = 0$, for all n. The reader who is familiar with this notion should be able to easily show that such a map p is a nice map, and therefore the approximation result of [2] is a corollary of Theorem 2.1.

2. A simple example of a nice map $p: N \to B$ which is not an approximate fibration is easily constructed by noting that all UV^1-maps are nice maps, where a map $p: N \to B$ is UV^1 if given any $b \in B$ and neighborhood U of b, there is a neighborhood $V \subset U$ of b so that any map of a 1-complex into $p^{-1}(V)$ is nullhomotopic in $p^{-1}(U)$. This follows easily from the dependence of $Wh(X)$ on $\pi_1(X)$ and the homotopy functorality of Wh.

3. Finally we mention that there is the expected non-compact version of Theorem 2.1. In this version all spaces would be locally compact, and all maps and homotopies would be proper, where a *proper* map is a map for which preimages of compacta are compact. Also ε and δ would be replaced

by open covers. There are no surprises in this extension to the non-compact case, and for this reason we omit details.

For our second version of the Approximation Theorem we will need some more definitions. A map $p: E \to B$ of compact ANRs has the *ε-lifting property for k-cells* if given maps $F: I^k \times [0,1] \to B$, $f: I^k \to E$ for which $F(\_,0) = pf(\_)$, there is a map $\tilde{F}: I^k \times [0,1] \to E$ for which $\tilde{F}(\_,0) = f(\_)$ and for which $p\tilde{F}$ is ε-close to F. The map $\tilde{F}$ is called an *ε-lift* of F. We say that p is *$(\varepsilon, 1)$-movable* if it has the ε-lifting property for 0- and 1-cells, and if it also satisfies the following property which is weaker than the ε-lifting property for 2-cells.

(*) *Given maps $F: I^2 \times [0,1] \to B$ and $f: I^2 \to E$ for which $F(\_,0) = pf(\_)$, there is a map $\tilde{F}: \partial(I^2 \times [0,1]) \to E$ for which $\tilde{F}(\_,0) = f(\_)$ and for which $\tilde{F}$ is an ε-lift of $F \mid \partial(I^2 \times [0,1])$.*

Following [5] we say that p is *1-movable* if it is $(\varepsilon, 1)$-movable, for all $\varepsilon > 0$. Intuitively this means that the fibers of p have locally constant fundamental groups.

THEOREM 2.2: *Let B be a compact ANR. For every $\varepsilon > 0$ there exists a $\delta > 0$ so that if M, N are compact Q-manifolds, $p: N \to B$ is $(\delta, 1)$-movable, and $f: M \to N$ is a $p^{-1}(\delta)$-equivalence, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism provided that $Wh(F \times T^n) = 0$, for all n, where F is the homotopy fiber of p.*

One advantage of this over Theorem 2.1 is that B no longer has to be finite-dimensional (but we now have to settle for the ANR restriction). Another seemingly apparent advantage is that the $(\delta, 1)$-movable map $p: N \to B$ which occurs in the above statement is not necessarily nice. However this is artificial because it is not hard to show that the $(\delta, 1)$-movable map p can be jiggled slightly to obtain a 1-movable map (provided that δ is small), and this 1-movable map is now nice. Details are left to the reader.

Our final application of the Approximation Theorem is of a completely different character than that of Theorems 2.1 and 2.2, yet it is difficult to ignore because it follows easily from the machinery developed in this paper. For notation let $p: X \to B$ be a map of metric spaces. We say that X is *$p^{-1}(\varepsilon)$-finitely dominated* if there exists a compact polyhedron K and maps $f: K \to X$, $g: X \to K$ such that $fg: X \to X$ is $p^{-1}(\varepsilon)$-homotopic to id_X. Similarly we say that X has *$p^{-1}(\varepsilon)$-finite type* if there exists a compact polyhedron K and a $p^{-1}(\varepsilon)$-equivalence $f: K \to X$.

THEOREM 2.3: *Let B be a finite-dimensional compact metric space. For every $\varepsilon > 0$ there exists a decreasing set $\{\delta_i\}_{i=1}^{l}$ of positive numbers such that if X is an ANR, $p: X \to B$ is (δ_i, δ_{i+1})-nice for all i, and X is $p^{-1}(\delta_l)$-finitely dominated, then X has $p^{-1}(\varepsilon)$-finite type.*

REMARKS: 1. There is also an Addendum whose statement is identical to the Addendum following the statement of the Approximation Theorem.

2. If K is the compact polyhedron which $p^{-1}(\delta_l)$-finitely dominates X, then we can construct the compact polyhedron L which is $p^{-1}(\varepsilon)$-equivalent to X so that $\dim(L)$ depends only on $\dim(K)$ and $\dim(B)$.

3. As in Theorem 2.1 there is a weaker version of this result in which the set $\{\delta_i\}$ is replaced by a single $\delta > 0$ provided that p is additionally assumed to be a nice map.

4. Finally we mention that there is a relative version of Theorem 2.3 which bears the same relationship to Theorem 2.3 as does the Relative Approximation Theorem of Section 7 to the Approximation Theorem of Section 1. The reader who has read these statements should be able to easily figure out what this realtive version should be.

3. Preliminaries

All spaces in this paper will be equipped with a metric (usually denoted by d) and for maps $f, g: X \to Y$ we define

$$d(f, g) = \text{lub}\{d(f(x), g(x)) \mid x \in X\}$$

provided that it exists. The composition of maps $f: X \to Y$ and $g: Y \to Z$ is denoted by $g \circ f: X \to Z$ (or simply gf). If $f: X \to Y$ is given and $A \subset X$, then $f|A: A \to Y$ is the restriction map. For $A \subset X$ we use $\mathring{A}$ to denote the topological interior of A and $Bd(A)$ to denote the topological boundary of A.

Expanding on the definitions of Section 1 let α be an open cover of Y. A homotopy $h_t: X \to Y$ is said to be an *α-homotopy* provided that each $\{h_t(x) \mid 0 \leq t \leq 1\}$ lies in some element of α. If α is an open cover of B and $p: Y \to B$ is a map, then $h_t: X \to Y$ is said to be a *$p^{-1}(\alpha)$-homotopy* provided that $ph_t: X \to Y$ is an α-homotopy. A map $f: X \to Y$ is said to be a *$p^{-1}(\alpha)$-equivalence* provided that there exists a map $g: Y \to X$ such that gf is $(pf)^{-1}(\alpha)$-homotopic to id_X and fg is $p^{-1}(\alpha)$-homotopic to id_Y. We call g a $p^{-1}(\alpha)$-inverse of f. If $Y = B$ and $p = id$, then we simply call f an *α-equivalence.*

Now that we have brought up the subject of controlled homotopies we should also mention that there is a controlled version of the homotopy extension theorem which goes as follows: *An α-homotopy of a closed subset of a metric space into an* ANR, *X, extends to an α-homotopy of the entire space to X provided that the* 0*-level extends* [1]. As an application of this it is easy to modify the standard proof that the notions of weak deformation retraction and strong deformation retraction are equivalent for ANRs to show that if (X, A) is a compact ANR pair and the inclusion $A \hookrightarrow X$ is an α-equivalence, then for some n there is a retraction $r: X \to A$ which is $St^n(\alpha)$-homotopic to id_X rel A [1]. Of course $St^n(\alpha)$ is the n^{th} star of α defined inductively by $St^\circ(\alpha) = \alpha$ and

$$St^{n+1}(\alpha) = \{\cup\{S \cup U \mid U \in \alpha \text{ and } S \cap U \neq \emptyset\} \mid S \in St^n(\alpha)\}.$$

We will represent Q as the countable product $[0,1] \times [0,1] \times \ldots$, and for any n we let $I^n = [0,1] \times \ldots \times [0,1]$ (n-times). This gives us a natural factorization $Q = I^n \times Q_{n+1}$ and for convenience we identify I^n with $I^n \times \{(0,0,\ldots)\}$ in Q. Numerous results from Q-manifold theory will be used in the sequel such as Z-set unknotting, the Triangulation Theorem, and the Classification Theorem which relates the study of homeomorphisms on Q-manifolds to simply homotopy theory. We refer the reader to [3] for the Q-manifold theory and to [4] for the simply homotopy theory.

A map $f: X \to Y$ of compacta is said to be CE (or *cell-like*) if it is surjective and each point inverse has the (Borsuk) shape of a point. In particular a surjection $f: X \to Y$ is CE if all the point inverses are contractible. Recall from [3] that CE maps between Q-manifolds are near homeomorphisms.

If $f: X \to Y$ is a map and A is a subset of both X and Y, then we say that $f = id$ over A when $f^{-1}(A) = A$ and $f \mid A = id$. More generally if $B \subset Y$ and $f \mid : f^{-1}(B) \to B$ is a homotopy equivalence, then we say that f is a homotopy equivalence over B. Usually we say that f has property P over B if $f \mid : f^{-1}(B) \to B$ has property P.

A subset A of a compact Q-manifold M is said to be *clean* if A is a compact Q-manifold and $Bd(A)$ is a Q-manifold which is collared in A and in $M - \mathring{A}$. Similarly a subset A of a compact polyhedron P is *clean* if A is a compact subpolyhedron and $Bd(A)$ is a compact subpolyhedron which is PL collared in A and in $P - \mathring{A}$. Finally a pair (A, B) in a compact Q-manifold is said to be clean if both A and B are clean, and $B \subset \mathring{A}$. Note that if $f: M \to P$ is a map, where M is a compact Q-manifold and P is a compact polyhedron, and $A \subset P$ is clean, it is not very likely that $f^{-1}(A)$ is going to be clean in M. However it is possible to approximate f by a map $g: M \to P$ for which $f^{-1}(A)$ is clean. This is easily done by using the Triangulation

Theorem for Q-manifolds along with the approximation of maps between polyhedra by PL maps.

If A and B are clean in a compact Q-manifold M, then we say that they are *transverse* provided that $C = Bd(A) \cap Bd(B)$ is a Q-manifold and there exists a neighborhood U of C so that the triad $(U; U \cap Bd(A), U \cap Bd(B)$ is homeomorphic to the triad $(C \times R^2; C \times R \times \{0\}, C \times \{0\} \times R)$. There is an analogous definition of what it means for clean subsets A, B of a compact polyhedron P to be transverse, where the homeomorphism of triads is required to be PL. In the same vein one can easily imagine what it means for A and C to be transverse in P, where A is clean and C is a compact subpolyhedron which is PL collared. As above if $f: M \to P$ is a map of a compact Q-manifold to a compact polyhedron and $A, B \subset P$ are transverse, then f can be approximated by a map $g: M \to P$ for which $g^{-1}(A)$, $g^{-1}(B)$ are transverse.

If X is a compact ANR, then a *splitting* of X is a decomposition $X = X_1 \cup X_2$, where X_1, X_2 and $X_0 = X_1 \cap X_2$ are also compact ANRs. If X is a Q-manifold, then we additionally require the X_i to be Q-manifolds, and if X is a polyhedron, then we additionally require the X_i to be subpolyhedra.

If $f: X \to Y$ is a map of compacta, then we define the *mapping cylinder* of f, C_f, to be $(X \times [0,1]) \amalg Y/\sim$, where $\amalg$ means disjoint union and $\sim$ is the equivalence relation generated by $(x, 1) \sim f(x)$. We will represent C_f as the union of $X \times [0, 1)$ and Y, where Y is the *base* and $X \equiv X \times \{0\}$ is the *top*. The *collapse to the base* is the retraction $c: C_f \to Y$ defined by $c|Y = id$ and $c(\{x\} \times [0, 1)) = f(x)$.

Finally we will need the following notation in Section 8. Let α_t, $0 \leq t \leq 1$ and β_t, $0 \leq t \leq 1$, be paths in a space X for which $\alpha_1 = \beta_0$. Then $\alpha_t * \beta_t$, $0 \leq t \leq 1$, is the path in X wich is α_{2t} on $[0, \frac{1}{2}]$ and β_{2t-1} on $[\frac{1}{2}, 1]$.

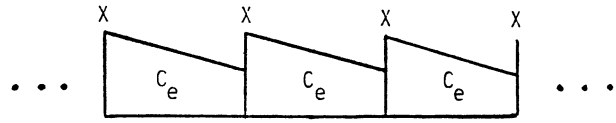
4. Constructions with mapping cylinders

As promised in Section 1 we will establish here some results concerning mapping tori and mapping telescopes which will be needed in the next section.

We begin by introducing some notation that will be used throughout this section. Let $X_0 \subset X \subset Y$ be compact ANRs so that the inclusion $X \hookrightarrow Y$ is a homotopy domination rel X_0. This means that there is a homotopy $h_t: Y \to Y$ for which $h_0 = id$, $h_1(Y) \subset X$, and $h_t|X_0 = id$. We also assume that h_t is an α-homotopy, for some open cover α of Y. Let $e = h_1|: X \to X$ and form the (direct) *mapping telescope*, S_e, which is the quotient space obtained from the disjoint union

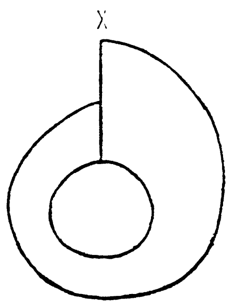
$$\ldots \amalg (X \times [-1,0]) \amalg (X \times [0,1]) \amalg (X \times [1,2]) \amalg \ldots$$

by identifying (x, n) in $X \times [n-1, n]$ with $(e(x), n)$ in $X \times [n, n+1]$. Note that S_e is just a union of countably many copies of the mapping cylinder, C_e. Here is a picture:



In a natural way S_e may be set-wise identified with $X \times R$ (non-continuously). We use $S_e[a,b]$ to denote the subset of S_e which corresponds to the subset $X \times [a,b]$ of $X \times R$. Since $e|X_0 = id$ it is clear that the subset of S_e corresponding to $X_0 \times R$ is actually homeomorphic to $X_0 \times R$.

In analogy with the above construction there is the (clockwise) *mapping torus*, T_e, which is the quotient space $C_e/\sim$, where $\sim$ is the equivalence relation generated by $(x,0) \sim x$. This is just the top of C_e sewn to its base via the map *id*. Here is a picture:



In a natural way T_e may be set-wise identified with $X \times S^1$ so that the subset of T_e corresponding to $X_0 \times S^1$ is actually homeomorphic to $X_0 \times S^1$. Let $\exp: R \to S^1$ be the covering map defined by $\exp(x) = e^{2\pi i x}$. Then it is clear that there is a covering map $\lambda: S_e \to T_e$ which makes the following rectangle commute:

$$\begin{array}{ccc} S_e \equiv X \times R \\ {\scriptstyle\lambda}\downarrow \quad\quad \downarrow{\scriptstyle id \times \exp} \\ T_e \equiv X \times S^1 \end{array}$$

(In this rectangle, $\equiv$ is used for our set identifications mentioned above.)

Now define a map $\tilde{H}: S_e \to Y \times R$ so that a typical mapping cylinder,

$S_e[n, n+1]$, is taken to $Y \times [n, n+1]$ as follows:

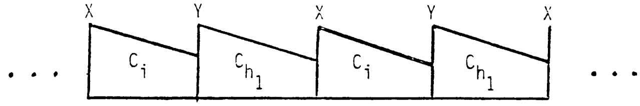
$$\tilde{H}(x,t) = \begin{cases} (h_{t-n}(x), t), & n \leq t < n+1 \\ (x,t), & t = n+1 \end{cases}$$

This map naturally wraps up to give a map $H: T_e \to Y \times S^1$ so that $\tilde{H}$ covers H.

THEOREM 4.1: *If $\pi = \mathrm{proj}: Y \times S^1 \to Y$, then there is an integer n for which $H: T_e \to Y \times S^1$ is a $\pi^{-1}(St^n(\alpha))$-equivalence. Similarly if $\tilde{\pi} = \mathrm{proj}: Y \times R \to Y$, then $\tilde{H}: S_e \to Y \times R$ is a $\tilde{\pi}^{-1}(St^n(\alpha))$-equivalence.*

PROOF: Our proof is just a variation of a trick used in Theorem 3.1 of [6], so all we will do is show that the maps H and $\tilde{H}$ are homotopy equivalences and let the reader worry about the size of the integer n. We will only treat the map $\tilde{H}$ because everything wraps up to do H simultaneously.

Let i be the inclusion $X \hookrightarrow Y$. The first step is to define a space Z which is formed by sewing together countably many copies of C_i and C_{h_1} as pictured below:



There is a natural map $r: Z \to R$ so that each $r^{-1}([n, n+\frac{1}{2}])$ is a copy of C_i and each $r^{-1}([n+\frac{1}{2}, n+1])$ is a copy of C_{h_1}. This is compatible with natural identifications

$$r^{-1}([n, n+\tfrac{1}{2})) \equiv X \times [n, n+\tfrac{1}{2}),$$

$$r^{-1}([n+\tfrac{1}{2}, n+1)) \equiv Y \times [n+\tfrac{1}{2}, n+1).$$

Our strategy is to define maps $S_e \xrightarrow{u} Z \xrightarrow{v} Y \times R$ so that u and v are homotopy equivalences and vu is homotopic to $\tilde{H}$.

We define $u: S_e \to Z$ so that it takes a typical mapping cylinder $S_e[n, n+1]$ to $r^{-1}([n, n+1])$ by $u(x,t) = (x,t)$, for all $(x,t) \in S_e[n, n+1]$. Let C_e be $S_e[0,1]$ and let $C_i \cup C_{h_1}$ be $r^{-1}([0,1])$. If $u' = u|: C_e \to C_i \cup C_{h_1}$, then u' generates u. Since $u' = id$ on $X \times \{0\}$ and $X \times \{1\}$ it will suffice to show that u' is a homotopy equivalence, for then u will also be a homotopy equivalence. To see that u' is a homotopy equivalence let $c: C_e \to X$ be the collapse to the base and let j be the inclusion, $X \hookrightarrow C_i \cup C_{h_1}$, of the base into $C_i \cup C_{h_1}$. Clearly $u' \simeq jc$ and so u' is a homotopy equivalence as desired.

Now define $v: Z \to Y \times R$ so that it takes $r^{-1}([n - \frac{1}{2}, n + \frac{1}{2}])$ to $Y \times [n - \frac{1}{2}, n + \frac{1}{2}]$ by

$$v(x,t) = \begin{cases} (h_{2(t-n)+1}(x), t), & (x,t) \in Y \times [n - \frac{1}{2}, n) \\ (x,t), & (x,t) \in r^{-1}([n, n + \frac{1}{2}]) \end{cases}$$

If $C_{h_1} \cup C_i$ is $r^{-1}([-\frac{1}{2}, \frac{1}{2}])$. then $v' = v| : C_{h_1} \cup C_i \to Y \times [-\frac{1}{2}, \frac{1}{2}]$ generates v. As above we can show that v is a homotopy equivalence simply by showing that v' is a homotopy equivalence. But v' is a homotopy equivalence because $v' \simeq kd$, where $d: C_{h_1} \cup C_i \to Y$ is the collapse to the base and k is the inclusion, $Y \hookrightarrow Y \times [-\frac{1}{2}, \frac{1}{2}]$.

Finally we have to show that vu is homotopic to $\tilde{H}$. If C_e is $S_e[0,1]$, then $vu| : C_e \to Y \times [0,1]$ is defined by

$$vu(x,t) = \begin{cases} (x,t), & 0 \le t \le \frac{1}{2} \text{ and } t = 1 \\ (h_{2t-1}(x), t), & \frac{1}{2} \le t < 1. \end{cases}$$

All we need is a homotopy of $vu|C_e$ to $\tilde{H}|C_e$ rel the left- and right-hand ends, for such a homotopy would clearly generate a homotopy of all of vu to $\tilde{H}$. But the construction of such a homotopy is easy. ■

REMARKS: 1. Note that the maps H and $\tilde{H}$ defined above satisfy $H = id$ on $X_0 \times S^1$ and $\tilde{H} = id$ on $X_0 \times R$.

2. The above proof enables us to construct a $\pi^{-1}(St^n(\alpha))$-inverse of $H, H^{-1}: Y \times S^1 \to T_e$, and a $\tilde{\pi}^{-1}(St^n(\alpha))$-inverse of $\tilde{H}, \tilde{H}^{-1}: Y \times R \to S_e$, so that $\tilde{H}^{-1}$ covers $H^{-1}, H^{-1} = id$ on $X_0 \times S^1$, and $\tilde{H}^{-1} = id$ on $X_0 \times R$.

THEOREM 4.2: *There is an integer m for which the inclusion $S_e[0, \infty) \hookrightarrow S_e$ is a $(\tilde{\pi}\tilde{H})^{-1}(St^m(\alpha))$-equivalence.*

PROOF: It is not surprising that $S_e[0, \infty) \hookrightarrow S_e$ is a homotopy equivalence, for one can easily show directly from the definition that the relative homotopy groups of the pair $(S_e, S_e[0, \infty))$ vanish. This approach does not seem to give the desired control on the homotopy equivalence, so we concoct a different proof which does give this control. The proof that we give produces and integer m which depends on n, but just as in the proof of Theorem 4.1 we will not give an explicit calculation for it.

Recall the homotopy equivalence $v: Z \to Y \times R$ of the proof of Theorem 4.1. Using v it follows that there is a homotopy $F_t: Z \to Z$ for which $F_0 = id$, $F_1(Z) \subset r^{-1}([-\frac{1}{2}, \infty))$, and $F_t|r^{-1}([-\frac{1}{2}, \infty)) = id$. This homotopy is also a $(\tilde{\pi}v)^{-1}(St^{m_1}(\alpha))$-homotopy, for some integer m_1. The homotopy F_t induces (via the map $u: S_e \to Z$) a homotopy $G_t: S_e \to S_e$ for which $G_0 = id$,

$G_1(S_e) \subset S_e[-1, \infty)$, and $G_t | S_e[0, \infty) = id$. Also, G_t is a $(\tilde{\pi}vu)^{-1}(St^{m_2}(\alpha))$-homotopy, for some integer m_2. There is a natural retraction along mapping cylinder rays of $S_e[-1, \infty)$ to $S_e[0, \infty)$. If we follow G_1 by this retraction we end up with a retraction $r: S_e \to S_e[0, \infty)$ that is $(\tilde{\pi}\tilde{H})^{-1}(St^m(\alpha))$-homotopic to id rel $S_e[0, \infty)$. ∎

5. A splitting theorem

The purpose of this section is to establish a splitting theorem (Theorem 5.2 below) which is a key ingredient in the proof of the Approximation Theorem which is given in Section 7. In Theorem 5.1 we first establish a weaker version of Theorem 5.2 whose statement lacks many of the complications of Theorem 5.2, but whose proof is substantially the same. The reason for proceeding in this manner is to make the proof more readable.

We start by setting up the notation for Theorem 5.1. Later on we will set up different notation for Theorem 5.2. Let Y be a compact polyhedron, let $p: Y \to [0, 3]$ be a PL map, and let Y be split as $Y = Y_1 \cup Y_2$, where $Y_1 = p^{-1}([1, 3])$ and $Y_2 = p^{-1}([0, 2])$. In Theorem 5.1 below we will investigate the following problem: *Given a compact polyhedron X and a $p^{-1}(\delta)$-equivalence $f: X \to Y$, when is there a "compatible" splitting of X?* One simple obstacle that we encounter to doing this is that X might not be large enough to admit such a splitting, so in our statement below we allow for a CE–PL expansion of X. Therefore what we obtain is really only a stable splitting, but it suffices for our purposes.

THEOREM 5.1: *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if X is a compact polyhedron and $f: X \to Y$ is a $p^{-1}(\delta)$-equivalence, then there exist torsions*

$$\tau_1 \in Wh(p^{-1}([1, 1.3]) \times S^1), \tau_2 \in Wh(p^{-1}([1.7, 2]) \times S^1),$$

whose vanishing implies that X admits a stable splitting which is compatible with the given splitting of Y. That is, there exists a compact polyhedron X', a CE–*PL map $r: X' \to X$, a map $f': X' \to Y$, and a splitting of X', $X' = X_1 \cup X_2$, such that f' is $p^{-1}(\varepsilon)$-homotopic to fr and $f'|: X_i \to Y_i$ is a homotopy equivalence, for $i = 0, 1, 2$.*

PROOF: We have divided the proof into several steps. In Steps I and II we identify the torsions τ_1 and τ_2, and in Step III we construct our desired splitting.

Step I

It will be more convenient to first deal with τ_2. By enlarging Y if necessary we may assume that X is a subpolyhedron of Y and we will assume that f is the inclusion. For notation let $B_t = p^{-1}([0,t])$ and $A_t = (pf)^{-1}([0,t])$, for each t. In this step we will define an element $\tau_2 \in Wh(p^{-1}([1.7,2]) \times S^1)$ whose vanishing implies that $f|: A_2 \to B_2$ can be extended to a homotopy equivalence $\tilde{f}: \tilde{A}_2 \to B_2$ such that $\tilde{A}_2$ is formed by attaching a compact polyhedron to A_2 along $(pf)^{-1}([1.6,2])$ and such that $\tilde{f}(\tilde{A}_2 - A_2) \subset p^{-1}([1.6,2])$.

Since f is a $p^{-1}(\delta)$-equivalence there exists a strong deformation retraction of Y onto X which is $p^{-1}(\delta')$-homotopic to id rel X, where δ' is a number whose size depends on δ. From this we get a PL homotopy $h_t: B_2 \to B_2$ such that $h_0 = id$, $h_1(B_2)$ lies in

$$Z = A_2 \cup p^{-1}([1.9,2]),$$

and $h_t = id$ on X_2 for all t. Also $ph_t: B_2 \to [0,3]$ is a δ'-homotopy. Thus $Z \hookrightarrow B_2$ is a homotopy domination rel A_2 and we let $e = h_1|Z: Z \to Z$, which is a homotopy idempotent on Z.

Now form the mapping torus, T_e, and note that T_e contains $A_2 \times S^1$ as a naturally-identified subset. If

$$u = p \circ \mathrm{proj}: B_2 \times S^1 \to [0,3],$$

then it follows from the constructions of Section 4 that there is a $u^{-1}(\gamma)$-equivalence $H: T_e \to B_2 \times S^1$ for which $H = id$ on $A_2 \times S^1$, where γ is a small number whose size depends on δ'. Let T_e^- be the (counter-clockwise) mapping torus, which is defined in analogy with T_e except that the rays of the mapping cylinder are identified in a counter-clockwise direction rather than in a clockwise direction. In analogy with H we obtain a $u^{-1}(\gamma)$-equivalence $H_-: T_e^- \to B_2 \times S^1$ which is the identity on $A_2 \times S^1$. Composing one with an inverse of the other yields $H_-^{-1}H: T_e \to T_e^-$, which is the identity on $A_2 \times S^1$ and which is a $(uH_-)^{-1}(6\gamma)$-equivalence.

Let $Z_1 \subset Z$ be defined by

$$Z_1 = (pf)^{-1}([1.8,2]) \cup p^{-1}([1.9,2])$$

and note that $e_1 = e|Z_1: Z_1 \to Z_1$ is a homotopy idempotent on Z_1 if δ' is sufficiently small. Also T_e contains T_{e_1}, T_e^- contains $T_{e_1}^-$, and the restriction $H_-^{-1}H|T_{e_1}$ gives a homotopy equivalence of T_{e_1} with $T_{e_1}^-$. The Whitehead torsion of this homotopy equivalence determines an element τ_2' of $Wh(T_{e_1}^-)$.

The map $H_-: T_e^- \to B_2 \times S^1$ clearly takes $T_{e_1}^-$ into $p^{-1}([1.7, 2]) \times S^1$ provided that γ is sufficiently small, and we call this map

$$\alpha: T_{e_1}^- \to p^{-1}([1.7, 2]) \times S^1.$$

It induces a homomorphism on Whitehead groups, α_*, and so our desired τ_2 is defined to be

$$\tau_2 = \alpha_*(\tau_2') \in Wh(p^{-1}([1.7, 2]) \times S^1).$$

In order to finish this step all we have to do is show that if $\tau_2 = 0$, then $\tilde{f}: \tilde{A}_2 \to B_2$ can be constructed. Let

$$T^- = (pf)^{-1}([1.65, 2]) \cup T_{e_1}^-,$$

and let $\beta: p^{-1}([1.7, 2]) \times S^1 \to T^-$ be defined by $\beta = H_-^{-1}$. (This is certainly defined if γ is small enough.) Then the maps

$$T_{e_1}^- \xrightarrow{\beta\alpha} T^-, \quad T_{e_1}^- \overset{j}{\hookrightarrow} T^-$$

are homotopic, and so $j_*(\tau_2') = 0$. But $j_*(\tau_2')$ is easily seen to be the Whitehead torsion of the homotopy equivalence $H_-^{-1}H|: T \to T^-$, where

$$T = (pf)^{-1}([1.65, 2]) \cup T_{e_1}.$$

So $h = H_-^{-1}H|: T \to T^-$ is a simple homotopy equivalence. Using the Classification Theorem of [3] we have a homotopy of $h \times id_Q: T \times Q \to T^- \times Q$ to a homeomorphism $k: T \times Q \to T^- \times Q$. We will have to lift this up to the level of mapping telescopes, so first some more notation is needed.

Recall that the mapping telescope, S_e, contains $A_2 \times R$ as a naturally-identified subset. If

$$\tilde{u} = p \circ \text{proj}: B_2 \times R \to [0, 3],$$

then from Section 4 we have a $\tilde{u}^{-1}(\gamma)$-equivalence $\tilde{H}: S_e \to B_2 \times R$ so that $\tilde{H} = id$ on $A_2 \times R$. Similarly there is a $\tilde{u}^{-1}(\gamma)$-equivalence $\tilde{H}_-: S_e^- \to B_2 \times R$, where S_e^- is the (inverse) mapping telescope which covers the (counter-clockwise) mapping torus T_e^-. Composing one with an inverse of the other yields a $(\tilde{u}\tilde{H}_-)^{-1}(6\gamma)$-equivalence $\tilde{H}_-^{-1}\tilde{H}$: $S_e \to S_e^-$. We also have a commutative rectangle

$$\begin{array}{ccc} S_e & \xrightarrow{\tilde{H}_-^{-1}\tilde{H}} & S_e^- \\ \lambda \downarrow & & \downarrow \lambda_- \\ T_e & \xrightarrow{H_-^{-1}H} & T_e^- \end{array},$$

where λ is the covering map mentioned in Section 4 and λ_- is its counterpart for S_e^- and T_e^-. If $\tilde{h}:\lambda^{-1}(T) \to \lambda_-^{-1}(T^-)$ is just a restriction of $\tilde{H}_-^{-1}\tilde{H}$, then we have a commutative rectangle

$$\begin{array}{ccc} \tilde{T} = \lambda^{-1}(T) & \xrightarrow{\tilde{h}} & \lambda_-^{-1}(T^-) = \tilde{T}^- \\ \lambda \downarrow & & \downarrow \lambda_- \\ T & \xrightarrow{h} & T^- \end{array}$$

The homotopy $h \times id_Q \simeq k$ can be covered by a homotopy $\tilde{h} \times id_Q \simeq \tilde{k}$, where it is easily seen that $\tilde{k}$ must be a homeomorphism. Moreover it is easily seen that this homotopy is bounded under the natural maps of $\tilde{T}$ and $\tilde{T}^-$ to R. Thus $\tilde{h} \times id \simeq \tilde{k}$ is a proper homotopy. We may assume that $(pf)^{-1}(1.65)$ is a Z-set in $(pf)^{-1}([1.65, 2])$, so it follows that $(pf)^{-1}(1.65) \times S^1$ is a Z-set in T and T^-. Similarly $(pf)^{-1}(1.65) \times R$ is a Z-set in $\tilde{T}$ and $\tilde{T}^-$. By a relative version of Z-set unknotting (Proposition 2.4 of [2]) or a relative version of the Classification Theorem of [3] we may assume that

$$h \times id \simeq k \operatorname{rel}(pf)^{-1}(1.65) \times S^1 \times Q,$$
$$\tilde{h} \times id \simeq \tilde{k} \operatorname{rel}(pf)^{-1}(1.65) \times R \times Q.$$

This means that the homotopy $\tilde{h} \times id \simeq \tilde{k}$ can be extended via the identity to a proper homotopy $\tilde{H}_-^{-1}\tilde{H} \times id_Q \simeq \hat{k}$, where $\hat{k}$ is a homeomorphism of $S_e \times Q$ onto $S_e^- \times Q$ which is the identity on the complement of $\tilde{T} \times Q$. The map $f \mid A_2$ factors into the composition

$$A_2 \hookrightarrow Z \xrightarrow{i} S_e \xrightarrow{\tilde{H}} B_2 \times R \xrightarrow{\text{proj}} B_2,$$

where i identifies Z with $Z \times \{0\}$ in S_e. So all we have to do is extend i to a homotopy equivalence $\tilde{i}:\tilde{Z} \to S_e$ such that $\tilde{Z}$ is formed by attaching a compact polyhedron to Z along

$$(pf)^{-1}([1.6, 2]) \cup p^{-1}([1.9, 2])$$

so that $\tilde{H}\tilde{i}(\tilde{Z} - Z)$ lies in $p^{-1}([1.6, 2]) \times R$.

Choose n a large integer and let

$$A = (S_e[0, \infty) \times Q) \cap \bar{k}^{-1}(S_e^-(-\infty, n] \times Q),$$

which is a compact Q-manifold in $S_e \times Q$. It is easy to see that the inclusion $A \hookrightarrow S_e \times Q$ is a homotopy equivalence. So all we have to do is show how to extend $Z' \hookrightarrow A$ to a homotopy equivalence in the prescribed manner, where Z' is just $i(Z) \times \{\text{point}\} \subset S_e \times Q$. Let

$$A' = A - [(pf)^{-1}([0, 1.65)) \times [0, n]],$$

which is a compact Q-manifold. Since A' is triangulable and since $A' \cap Z'$ is a Z-set in A', there is a compact polyhedron P in A' which contains $A' \cap Z'$ as a compact subpolyhedron and for which there is a strong deformation retraction of A' onto P. This implies that$Z' \cup P \hookrightarrow Z' \cup A'$ is a homotopy equivalence, and it is clear that $Z' \cup A' \hookrightarrow A$ is a homotopy equivalence. Thus $Z' \cup P \hookrightarrow A$ is a homotopy equivalence as desired. This completes Step I.

Step II

In this step we deal with τ_1. Proceeding in analogy with Step I we can find a torsion $\tau_1 \in Wh(p^{-1}([1, 1.3] \times S^1)$ whose vanishing implies that $f|: X - \mathring{A}_1 \to Y - \mathring{B}_1$ can be extended to a homotopy equivalence $\bar{f}: \overline{X - \mathring{A}_1} \to Y - \mathring{B}_1$ such that $\overline{X - \mathring{A}_1}$ is formed by attaching a compact polyhedron to $X - \mathring{A}_1$ along $(pf)^{-1}([1, 1.4])$ and such that $\bar{f}(\overline{X - \mathring{A}_1} - (X - \mathring{A}_1))$ lies in $p^{-1}([1, 1.4])$. The details are similar.

Step III

In this step we combine the results of Steps I and II to finish the proof of our theorem. Let $g_t: Y \to Y$ be a homotopy rel X for which $g_0 = id$, $g_1(Y) = X$, and pg_t is a δ'-homotopy. (See the first paragraph of Step I.) Let P_1 be the compact polyhedron of Step II which is attached to $X - \mathring{A}_1$ to form $\overline{X - \mathring{A}_1}$, and let P_2 be the compact polyhedron of Step I which is attac hed to A_2 to form $\tilde{A}_2$. Define $\hat{f}: X \cup P_1 \cup P_2 \to Y$ by

$$\hat{f} = \begin{cases} f, & \text{on } X \\ \bar{f}, & \text{on } P_1 \\ \tilde{f}, & \text{on } P_2 \end{cases}$$

Let $D^n = [-1, 1]^n$ and let X be identified with $X \times \{0\}$ in $X \times D^n$. Then $g_1\hat{f}: X \cup P_1 \cup P_2 \to X \times D^n$ is a map which is the identity on X. If n is sufficiently large, then $g_1\hat{f}$ can be approximated by a PL embedding $\alpha: X \cup P_1 \cup P_2 \to X \times D^n$ so that $\alpha|X = id$. The polyhedron X' and the CE–PL map $r: X' \to X$ that we are seeking are defined by

$$X' = X \times D^n, \quad r' = \text{proj}: X \times D^n \to X.$$

Next we define the splitting, $X' = X_1 \cup X_2$. Let

$$X_1 = [(X - \mathring{A}_{1.45}) \times D^n] \cup \alpha((X - \mathring{A}_1) \cup P_1),$$
$$X_2 = (A_{1.55} \times D^n) \cup \alpha(A_2 \cup P_2).$$

So X_1 and X_2 are compact subpolyhedra of X' for which $X' = X_1 \cup X_2$, and for δ' sufficiently small we have $X_0 = X_1 \cap X_2$ equal to

$$(A_2 - \mathring{A}_1) \cup \alpha(P_1) \cup \alpha(P_2) \cup [(A_{1.55} - \mathring{A}_{1.45}) \times D^n].$$

We are given a homotopy equivalence $\bar{f}: (X - \mathring{A}_1) \cup P_1 \to Y_1$ from Step II and we are given a homotopy equivalence $\tilde{f}: A_2 \cup P_2 \to Y_2$ from Step I. If $\beta_1: X_1 \to \alpha((X - \mathring{A}_1) \cup P_1)$ is the map obtained by projecting $(X - \mathring{A}_{1.45}) \times D^n$ to $X - \mathring{A}_{1.45}$, then we obtain a map $f_1: X_1 \to Y_1$ defined by $f_1 = \bar{f}\alpha^{-1}\beta_1$. In a similar manner we obtain $f_2 = \tilde{f}\alpha^{-1}\beta_2: X_2 \to Y_2$.

ASSERTION: *f_1 is $p^{-1}(\delta'')$-homotopic to $fr|X_1$, where δ'' is a small number whose size depends on δ'.*

PROOF: We have

$$fr\alpha|(X - \mathring{A}_1) \cup P_1 \simeq frg_1\bar{f} = g_1\bar{f} \simeq \bar{f}.$$

Thus

$$fr|\alpha((X - \mathring{A}_1) \cup P_1) \simeq \bar{f}\alpha^{-1}|\alpha((X - \mathring{A}_1) \cup P_1),$$

and so $fr\beta_1 \simeq \bar{f}\alpha^{-1}\beta_1$. Since $fr\beta_1 = fr|X_1$ we are done ■

It follows by an analogous proof that f_2 is $p^{-1}(\delta'')$-homotopic to $fr|X_2$. Moreover these homotopies $f_1 \simeq fr|X_1$ and $f_2 \simeq fr|X_2$ are rel$(A_{1.55} - \mathring{A}_{1.45}) \times D^n$. So we can homotop fr to $f': X' \to Y$, where $f' = f_1$ on $A_{1.55} \times D^n$ and $f' = f_2$ on $(X - \mathring{A}_{1.45}) \times D^n$. Moreover this is a $p^{-1}(\delta'')$-homotopy. To see that $f'|X_2: X_2 \to Y_2$ is a homotopy equivalence just observe that $f'|X_2 \simeq f_2$, and f_2 was constructed to be a homotopy equivalence. Similarly $f'|X_1: X_1 \to Y_1$ is a homotopy equivalence. Finally to see that $f'|X_0 : X_0 \to Y_0$ is a homotopy equivalence just use Proposition 2.1 of [2] which tells how to sew together partial equivalences to obtain global equivalences. This completes Step III. ■

REMARK: There is an additional useful fact which follows from the above proof, but which does not appear in the above statement. If we let $\pi : [1,3] \to [2,3]$ be the retraction which takes $[1,2]$ to $\{2\}$, then we can construct f' so that $f'| : X_1 \to Y_1$ is a $(\pi p)^{-1}(\varepsilon)$-equivalence.

We now set up the notation for Theorem 5.2. Let Y, B and K be compact polyhedra and let $Y \xrightarrow{p} B \xrightarrow{q} [0,3]$ be PL maps such that $q^{-1}([1,2]) = K \times [1,2]$ and $q|K \times [1,2] = \text{proj} : K \times [1,2] \to [1,2]$. The splitting $B = B_1 \cup B_2$, where $B_1 = q^{-1}([1,3])$ and $B_2 = q^{-1}([0,2])$, gives us a splitting $Y = Y_1 \cup Y_2$, where $Y_i = p^{-1}(B_i)$. In Theorem 5.2 we address the following problem: *Given a compact polyhedron X and a $p^{-1}(\delta)$-equivalence $f : X \to Y$, when is there a "compatible" stable splitting of X?* The essential difference now is that the meaning of "compatible" is slightly more restrictive. Just as in Theorem 5.1 we are asking for a CE–PL map $r : X' \to X$, a map $f' \to Y$, and a splitting $X' = X_1 \cup X_2$ such that $f' \simeq fr$ and $f'| : X_i \to Y_i$ is a homotopy equivalence, but now we have the additional requirement that the homotopy $f' \simeq fr$ must be a $p^{-1}(\varepsilon)$-homotopy, where the ε-control is how in B (rather than in $[0,3]$ as was the case in Theorem 5.1). Except for the notational changes the above proof works equally well to deal with this requirement, so no further proof is necessary. (See Remark 1 below for some comments on the proof.)

THEOREM 5.2: *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if X is a compact polyhedron and $f : X \to Y$ is a $p^{-1}(\delta)$-equivalence, then there exist compact Q-manifolds P_1, P_2 and homotopy equivalences*

$$h_1 : P_1 \to p^{-1}(K \times [1, 1.3]) \times S^1 \times Q,$$
$$h_2 : P_2 \to p^{-1}(K \times [1.7, 2]) \times S^1 \times Q$$

which are obstructions to constructing a stable splitting of X which is compatible with the given splitting of Y. This means that if h_1 and h_2 are homotopic to homeomorphisms, then there exists a compact polyhedron X', a CE–PL *map $r : X' \to X$, a map $f' : X' \to Y$, and a splitting of X', $X' = X_1 \cup X_2$, such that f' is $p^{-1}(\varepsilon)$-homotopic to fr and $f'| : X_i \to Y_i$ is a homotopy equivalence, for $i = 0, 1, 2$.*

REMARKS: 1. Before we completely dismiss the proof as a duplicate of the proof of Theorem 5.1 we should at least point out how the obstructions h_1 and h_2 arise. This is really the only difference, and for simplicity we only look at h_1. Recall from Theorem 5.1 that a torsion element $\tau_1 \in Wh(p^{-1}([1, 1.3]) \times S^1)$ was defined. In the notation of Theorem 5.2 this becomes a torsion element in $Wh(p^{-1}(K \times [1, 1.3]) \times S^1)$, and if we multiply by Q this is represented by a homotopy equivalence h_1 of a compact Q-manifold P_1 to $p^{-1}(K \times [1, 1.3]) \times S^1 \times Q$ so that h_1 is homotopic to a homeomorphism if and only if $\tau_1 = 0$ [3, Theorem 38.1].

2. The above statement seems to suggest that the size of δ depends on Y, B and ε, but the proof actually shows that it depends only on B and the size of ε.

3. It follows from the proof of Theorem 5.1 that the torsion element τ_1 can be chosen to be "supported" on $p^{-1}([1.1, 1.2]) \times S^1$. This means that τ_1 is represented by a PL homotopy equivalence $g_1 : L \to p^{-1}([1, 1.3]) \times S^1$ so that g_1 is a homotopy equivalence over $p^{-1}([1.1, 1.2]) \times S^1$ and a simple homotopy equivalence over the complement of $p^{-1}(1.1, 1.2]) \times S^1$. In the language of Theorem 5.2 this means that the obstruction h_1 can be chosen so that

$$h_1^{-1}(p^{-1}(K \times [1.1, 1.2]) \times S^1 \times Q)$$

is clean, h_1 is a homotopy equivalence over $p^{-1}(K \times (1.1, 1.2]) \times S^1 \times Q$, and h_1 is a homeomorphism over the complement of $p^{-1}(K \times [1.1, 1.2]) \times S^1 \times Q$. There is an analogous statement for h_2 which asserts that it can be chosen to be supported on $p^{-1}(K \times [1.8, 1.9]) \times S^1 \times Q$.

There is another somewhat similar way to further restrict the supports of h_1 and h_2 provided that we start out with more restrictions on f. Specifically suppose that we are given a compact polyhedron W which is clean in both X and Y, and suppose that W intersects each $q^{-1}([0, t])$ transversally, for $t \in [1, 2]$. (Actually we only need this assumption for the various t that arise in the proof of Theorem 5.2.) If $f| : W \to W$ is the identity, then the obstruction h_1 can be chosen so that

$$(W \times S^1 \times Q) \cap [p^{-1}(K \times [1, 1.3]) \times S^1 \times Q]$$

is clean in P_1 and h_1 is the identity on this set. Again there is an analogous statement for h_2.

4. We now make a remark which resembles the remark following the proof of Theorem 5.1. Since $q^{-1}([1, 2]) = K \times [1, 2]$ there is a retraction $\pi : q^{-1}([1, 3]) \to q^{-1}([2, 3])$ which takes each $\{x\} \times [1, 2]$ to $\{x\} \times \{2\}$. Also let $u : p^{-1}(K \times [1, 2]) \times S^1 \times Q \to K$ be the composition

$$p^{-1}(K \times [1,2]) \times S^1 \times$$
$$Q \xrightarrow{\text{proj}} p^{-1}(K \times [1,2]) \xrightarrow{p} K \times [1,2] \xrightarrow{\text{proj}} K.$$

Given any $\gamma > 0$ we can choose $\delta > 0$ small enough so that the obstructions h_1 and h_2 are $u^{-1}(\gamma)$-equivalences. Also if δ is sufficiently small, and in the statement of Theorem 5.2 we additionally assume that h_1 is $u^{-1}\left(\frac{\varepsilon}{2}\right)$-homotopic to a homeomorphism, then $f'|: X_1 \to Y_1$ is additionally a $(\pi p)^{-1}(\varepsilon)$-equivalence.

5. Finally we mention the following relative version. Suppose that we are given a compact polyhedron W which is a PL collared subpolyhedron of both X and Y such that W intersects each $q^{-1}([0,t])$ transversally. If $f|: W \to W$ is the identity, then the obstruction h_1 can be formulated so that

$$W_1 = (W \times S^1 \times Q) \cap [p^{-1}(K \times [1, 1.3]) \times S^1 \times Q]$$

is a Z-set in P_1 and h_1 is the identity on this set. There is of course an analogous statement for h_2 asserting that $h_2 | W_2 = id$. Moreover if h_1 and h_2 are homotopic to homeomorphisms rel W_1 and W_2, respectively, then the splitting of Theorem 5.2 can be constructed so that $r = id$ over W, $f' = id$ over W, and $f' \simeq fr$ rel W. Also W (in X') is transverse to X_1 and Y_1.

6. A controlled sum theorem

The purpose of this section is to establish a version of the Sum Theorem for Whitehead torsion [4, p. 76] which will be needed in Section 7. More specifically we will need a sum theorem with appropriate controls in a given parameter space, so the version of [4] does not apply.

Our main result is Theorem 6.2 below, but first it will be convenient to establish the following lemma which will be needed in its proof. For notation let (X, A) be a compact ANR pair, let Y be a compact ANR, and let $f: A \to Y$ be a map. The *adjunction space*, $X \cup_f Y$, is the quotient space $X \amalg Y/\sim$, where $\sim$ is the equivalence relation generated by $x \sim f(x)$, for all $x \in A$. It is obtained by sewing X to Y along A via the map f. Observe that any retraction $r: X \to A$ induces a retraction $r_f: X \cup_f Y \to Y$.

LEMMA 6.1: *Let (X, A) and Y be as above, let $r: X \to A$ be a retraction, let α be an open cover of Y, and let $f, g: A \to Y$ be maps which are α-homotopic.*

Then there is a homeomorphism

$$h:(X \cup_f Y) \times Q \to (X \cup_g Y) \times Q$$

for which the maps $r_f \circ \mathrm{proj}$ *and* $r_g \circ \mathrm{proj} \circ h$ *of* $(X \cup_f Y) \times Q$ *to* Y *are* α-*homotopic.*

PROOF: Assume that $A \subset Q$ is a Z-set and define $f' : A \to Y \times Q$ by $f'(a) = (f(a), a)$. Form $X \cup_{f'} (Y \times Q)$ and note that there exists a CE map s_1 of $X \cup_{f'} (Y \times Q)$ onto $X \cup_f Y$ obtained simply by collapsing out the Q-factor. Since s_1 is CE there is a homeomorphism

$$h_1 : [X \cup_{f'} (Y \times Q)] \times Q \to (X \cup_f Y) \times Q$$

which is close to $s_1 \times id_Q$. Also define $g' : A \to Y \times Q$ by $g'(a) = (g(a), a)$, form $X \cup_{g'} (Y \times Q)$, and let s_2 be the CE map of $X \cup_{g'} (Y \times Q)$ to $X \cup_g Y$ which corresponds to s_1. This gives us a homeomorphism

$$h_2 : [X \cup_{g'} (Y \times Q)] \times Q \to (X \cup_g Y) \times Q$$

which is close to $s_2 \times id_Q$.

The maps $f', g' : A \to Y \times Q$ are Z-embeddings which are $\pi^{-1}(\alpha)$-homotopic, where $\pi = \mathrm{proj} : Y \times Q \to Y$. By Z-set unknotting there exists a homeomorphism $u: Y \times Q \to Y \times Q$ such that $uf' = g'$ and such that u is $\pi^{-1}(\alpha)$-homotopic to the identity. This induces a homeomorphism

$$h_3 : X \cup_{f'} (Y \times Q) \to X \cup_{g'} (Y \times Q).$$

Then our desired homeomorphism h is the composition

$$h = h_2 \circ (h_3 \times id_Q) \circ h_1^{-1}.$$

It is easy to see that $r_f \circ \mathrm{proj}$ is α-homotopic to $r_f \circ \mathrm{proj} \circ h$ provided that h_1 is sufficiently close to $s_1 \times id$ and h_2 is sufficiently close to $s_2 \times id$. ■

We now set up some notation for our main result. Let X and Y be compact ANRs for which $X \subset Y$ and let $p: Y \to B$ be a map, where B is a compact metric space. We are also given splittings,

$$X = X_1 \cup X_2 \quad \text{and} \quad Y = Y_1 \cup Y_2,$$

so that $X_i = X \cap Y_i$, for each i.

THEOREM 6.2: *For every $\varepsilon > 0$ there exists a $\delta > 0$ so that if each $X \times Q \hookrightarrow (X \cup Y_i) \times Q$ is $(p \circ \mathrm{proj})^{-1}(\delta)$-homotopic to a homeomorphism, then $X \times Q \hookrightarrow Y \times Q$ is $(p \circ \mathrm{proj})^{-1}(\varepsilon)$-homotopic to a homeomorphism. Moreover δ depends only on B and ε.*

PROOF: For $i = 0, 1, 2$ let $h_i : (X \cup Y_i) \times Q \to X \times Q$ be a homeomorphism whose inverse is $(p \circ \mathrm{proj})^{-1}(\delta)$-homotopic to inclusion. For the pair $(Y_2, X_2 \cup Y_0) \times Q$ let $f_1 : (X_2 \cup Y_0) \times Q \to X \times Q$ be defined by $f_1 = h_1 | (X_2 \cup Y_0) \times Q$, thus giving the adjunction space

$$(Y_2 \times Q) \cup_{f_1} (X \times Q).$$

Similarly for the pair $(Y_2, X_2 \cup Y_0) \times Q$ let $f_0 : (X_2 \cup Y_0) \times Q \to X \times Q$ be defined by $f_0 = h_0 | (X_2 \cup Y_0) \times Q$, thus giving

$$(Y_2 \times Q) \cup_{f_0} (X \times Q).$$

The maps f_1 and f_0 are homotopic, so by Lemma 6.1 there is a homeomorphism

$$u : [(Y_2 \times Q) \cup_{f_1} (X \times Q)] \times Q \to [(Y_2 \times Q) \cup_{f_0} (X \times Q)] \times Q.$$

We may view $Y \times Q$ as $(Y_2 \times Q) \cup_{id} ((X \cup Y_1) \times Q)$, which is $Y_2 \times Q$ attached to $(X \cup Y_1) \times Q$ along $(X_2 \cup Y_0) \times Q$ via the identity. The homeomorphism h_1 therefore induces a homeomorphism

$$\tilde{h}_1 : Y \times Q \to (Y_2 \times Q) \cup_{f_1} (X \times Q).$$

Similarly we view $(X \cup Y_2) \times Q$ as $Y_2 \times Q$ attached to $(X \cup Y_0) \times Q$ along $(X_2 \cup Y_0) \times Q$ via the identity, so the homeomorphism h_0 induces a homeomorphism

$$\tilde{h}_0 : (X \cup Y_2) \times Q \to (Y_2 \times Q) \cup_{f_0} (X \times Q).$$

The composition h, defined by

$$X \times Q \xrightarrow{\mu} X \times Q \times Q \xrightarrow{h_2^{-1} \times id} (X \cup Y_2) \times Q \times$$

$$\times\, Q \xrightarrow{\tilde{h}_0 \times id} [(Y_2 \times Q) \cup_{f_0} (X \times Q)] \times Q$$

$$\xrightarrow{\mu^{-1}} [(Y_2 \times Q) \cup_{f_1} (X \times Q)] \times Q \xrightarrow{\tilde{h}_1^{-1} \times id} Y \times Q \times Q \xrightarrow{\tau} Y \times Q,$$

is our desired homeomorphism, where $\mu^{-1}: X \times Q \times Q \to X \times Q$ and $\tau: Y \times Q \times Q \to Y \times Q$ are homeomorphisms which are close to projection onto the first two factors. It is easy to check that h is $(p \circ \mathrm{proj})^{-1}(\varepsilon)$-homotopic to inclusion. ∎

REMARK: We have carefully chosen the notation for the above result so that it is easy to prove, but there is a trivial modification of it which will be more directly applicable in Section 7. For this modification let X and Y be compact ANRs for which $Y \subset X$ and let $p: Y \to B$ be a map, where B is a compact metric space. Also let

$$X = X_1 \cup X_2 \quad \text{and} \quad Y = Y_1 \cup Y_2$$

be splittings so that $Y_i = Y \cap X_i$. With this new notation the above result may be rephrased as follows: *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *so that if* $f: X \to Y$ *is a retraction for which each*

$$f| \times id_Q: (X_i \cup Y) \times Q \to Y \times Q$$

is $(p \circ \mathrm{proj})^{-1}(\delta)$*-homotopic to a homeomorphism, then* $f \times id_Q: X \times Q \to Y \times Q$ *is* $(p \circ \mathrm{proj})^{-1}(\varepsilon)$*-homotopic to a homeomorphism.* In Section 7 we have made choices so that the spaces X, Y and $X_i \cup Y$ are all Q-manifolds, thus the stabilizing factor Q can be dropped.

7. Proof of the approximation theorem

The purpose of this section is to prove the Approximation Theorem as stated in Section 1. The following lemma is the first step, which is a routine reduction of the case in which B is a compactum to the case in which B is a polyhedron.

LEMMA 7.1: *If the Approximation Theorem is true for B a compact polyhedron, then it is also true for B a finite-dimensional compactum.*

PROOF: Let B be a compactum as given and let $\varepsilon > 0$ be given. Choose a compact polyhedron B' which contains B as a subspace. For convenience assume that B and B' have induced metrics as subspaces of some euclidean space. For any given $p: N \to B$ we define $p': N \to B'$ to be the composition $N \xrightarrow{p} B \hookrightarrow B'$. The following observation will be useful.

ASSERTION: *If p is (τ, μ)-nice and 2λ is a Lebesgue number for the open cover of B by μ-balls, then p' is $(\tau + \mu + \lambda, \lambda)$-nice.*

PROOF: Choose any $b' \in B'$ and note that $(p')^{-1}(S_\gamma(b')) = p^{-1}(S_\gamma(b') \cap B)$, for any $\gamma > 0$. So all we have to do is show that

$$(*) \quad Wh(p^{-1}(S_\lambda(b') \cap B) \times T^n) \to Wh(p^{-1}(S_{\tau+\mu+\lambda}(b') \cap B) \times T^n)$$

is the 0-homomorphism. Note that diam $(S_\lambda(b') \cap B) \leq 2\lambda$, thus there is a $b \in B$ for which $S_\lambda(b') \cap B \subset S_\mu(b)$. This gives us

$$S_\lambda(b') \cap B \subset S_\mu(b) \subset S_\tau(b) \subset S_{\tau+\mu+\lambda}(b') \cap B,$$

so the inclusion (*) factors through the 0-homomorphism

$$Wh(p^{-1}(S_\mu(b)) \times T^n) \to Wh(p^{-1}(S_\tau(b)) \times T^n). \qquad \blacksquare$$

If k' is the integer of the Approximation Theorem for the polyhedron B', then we will show that the integer k for B is $k = 3k' - 2$. So choose $\varepsilon > 0$ and let $\{\delta_i\}_{i=1}^k$, $f: M \to N$, and $p: N \to B$ be given for which p is (δ_i, δ_{i+1})-nice and f is a $p^{-1}(\delta_k)$-equivalence. By the Assertion we can choose the δ_i so that p' is (δ_1, δ_4)-nice, (δ_4, δ_7)-nice, $\ldots$, (δ_{k-3}, δ_k)-nice. Thus $f: M \to N$ is a $(p')^{-1}(\delta'_{k'})$-equivalence and p' is $(\delta'_i, \delta'_{i+1})$-nice, where $\{\delta'_i\}_{i=1}^{k'}$ is the set $\{\delta_1, \delta_4, \ldots, \delta_k\}$. Since the Approximation Theorem is true for polyhedra we have f $(p')^{-1}(\varepsilon$-homotopic to a homeomorphism, and this implies that f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism as desired.

By using a similar proof we see that if the Addendum is true for B a compact polyhedron, then it is also true for B a finite-dimensional compactum. $\blacksquare$

Thus we are left with the problem of proving the Approximation Theorem for B a compact polyhedron. As indicated in Section 1 we will induct on dim B, and in order to carry this out it will be necessary to prove a stronger result. For this we will need the following definition. If N is a compact Q-manifold and (N_1, N_2) is a clean pair in N, then a map $p: N \to B$ is said to be (ε, δ)-*nice on* (N_1, N_2) if given any $b \in B$, the inclusion-induced homomorphism

$$Wh((p^{-1}(S_\delta(b)) \cap N_2) \times T^n) \to Wh((p^{-1}(S_\varepsilon(b)) \cap N_1) \times T^n)$$

is the 0-homomorphism, for any n. Here is the stronger result that we will prove.

RELATIVE APPROXIMATION THEOREM: *Let B be a compact polyhedron. For every $\varepsilon > 0$ there exists a decreasing set $\{\delta_i\}_{i=1}^k$, $\delta_i > 0$, so that if M, N are compact Q-manifolds, $M_k \subset M$ is clean, $\{N_i\}_{i=1}^k$ is a collection of clean subsets of N so that each (N_i, N_{i+1}) is a clean pair, $p: N \to B$ is (δ_i, δ_{i+1})-nice on (N_i, N_{i+1}), and $f: M \to N$ is a $p^{-1}(\delta_k)$-equivalence which restricts to give a $p^{-1}(\delta_k)$-equivalence of M_k onto N_k and which restricts to give a homeomorphism of $M - \mathring{M}_k$ onto $N - \mathring{N}_k$, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism.*

REMARK: For this relative version of the Approximation Theorem of Section 1 there is no change in the statement of the Addendum. In fact we will not give any details for the proof of the Addendum because it will be implicit in the proof of the above result that the Addendum is also being proved.

PROOF: We have arranged the proof in a series of steps.

Step I

To start off the induction we first treat the case dim $B = 0$. For this case we will show that $k = 2$ suffices. So choosing $\varepsilon > 0$ we seek numbers $\delta_1 > \delta_2 > 0$ so that if $p: N \to B$ is (δ_1, δ_2)-nice on (N_1, N_2) and f is a $p^{-1}(\delta_2)$*-equivalence for which* $f|: M_2 \to N_2$ is a $p^{-1}(\delta_2)$-equivalence and $f|: M - \mathring{M}_2 \to N - \mathring{N}_2$ is a homeomorphism, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism. Choose any $\delta_1 < \varepsilon$ which is less than the minimum distance between any two distinct points of B. Then the problem clearly reduces to the case $B = \{\text{point}\}$. So we are reduced to considering a homotopy equivalence $f: M \to N$ for which $f|: M_2 \to N_2$ is a homotopy equivalence and $f| : M - \mathring{M}_2 \to N - \mathring{N}_2$ is a homeomorphism. Also inclusion induces the 0-homomorphism, $Wh(N_2) \to Wh(N_1)$. By the Sum Theorem for Whitehead torsion we see that the torsion of f, $\tau(f)$, lies in the image of $Wh(N_2) \to Wh(N_1)$. Thus $\tau(f) = 0$ and so f is homotopic to a homeomorphism.

Step II

We now set up the basic notation that will be used for the inductive step. Assume that the Relative Approximation Theorem is true over all compact polyhedra of dimension $n - 1$. We will prove that the result is true over a fixed compact polyhedron B of dimension n. For a given $\varepsilon > 0$ we will show that there is a decreasing sequence $\{\delta_i\}_{i=1}^k$ of positive numbers so that if $p: N \to B$ is (δ_i, δ_{i+1})-nice and f is a $p^{-1}(\delta_k)$-equivalence, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism. This will be done in Steps III and IV below. Note that this only establishes the "absolute" version of the Relative Approximation Theorem over n-dimensional polyhedra. For the relative version the proof is essentially the same except notationally more complex. In Step V we make a few comments about what has to be done to make the arguments of Steps III and IV apply to the relative case.

Let l be the integer of the Relative Approximation Theorem over $(n-1)$-dimensional polyhedra. We will show that the integer k that we are seeking can be taken to be $k = 2l + 3$. Here is a rough description of the manner in which the δ_i are chosen.

1. For our given $\varepsilon > 0$ if we choose the "ε" of the Remark following Theorem 6.2 to be $\frac{\varepsilon}{2}$, then $2\delta_1$ is chosen to be the corresponding "δ."

2. A choice of $\delta_2 \leq \delta_1$ will do.

3. Then triangulate B so that the diameter of each n-simplex is $< \delta_2$ and let L be the $(n-1)$-skeleton of B.

4. Now applying the inductive hypothesis over L, with δ_2' playing the role of the given ε, we choose $\{\delta_3, \ldots, \delta_{l+2}\}$ to be the corresponding set of δ_i's whose existence is guaranteed by this hypothesis. (δ_2' is defined below.)

5. Similarly applying the inductive hypothesis over S^{n-1}, with $\delta_{l+2}'/2$ playing the role of the given ε, we choose $\{\delta_{l+3}, \ldots, \delta_{2l+2}\}$ to be the corresponding set of δ_i's. (δ_{l+2}' is defined below.)

6. Finally $\delta_{2l+3} = \delta_k$ is chosen small enough so that the obstructions h_1 and h_2 of Theorem 5.2 are defined.

In what follows we will assume that choices 1, 2 and 3 have been made as described above. Choices 4, 5 and 6 require some further explanation and will be dealt with in Steps III and IV. To simplify matters we assume that $B = L \cup \Delta^n$, where Δ^n is the *only* n-cell. The proof is essentially the same if B is the union of L and an aribtrary number of n-cells. If σ is the star of the barycenter of Δ^n in the 2^{nd} barycentric subdivision of B, we may write $B - \mathring{\sigma} = L \cup S^{n-1} \times [0,3]$, where $S^{n-1} \times \{0\} = Bd(\sigma)$ and $S^{n-1} \times \{3\} = Bd(\Delta^n) = S^{n-1}$. For any fixed met-
ric d on L choose a second metric on B so that

(1) the distance between any two points (x_1, y_1) and (x_2, y_2) of $S^{n-1} \times [0,3]$ is max $\{d(x_1,x_2), |y_1 - y_2|\}$,

(2) the distance between any point x_1 of L and (x_2, y_2) of $S^{n-1} \times [0,3]$ is max $[\mathrm{d}(x_1,x_2), 3 - y_2\}$.

In all of our remaining calculations concerning $\delta_3, \delta_4, \ldots, \delta_k$ we will use this second metric. We are justified in making this change of metric because we are only interested in choosing small values of the δ_i.

Step III

In this step we will be dealing with the choices of $\delta_{l+3}, \ldots, \delta_k$, where it is assumed that $\delta_3, \ldots, \delta_{l+2}$ have already been selected. The goal of this step will be to split up M and N into pieces so that, in Step IV, the inductive hypothesis can be applied to one of these pieces. We will also have to use the inductive hypothesis to show that this splitting can be carried out.

Consider the splitting of B, $B = B_1 \cup B_2$, defined by

$$B_1 = L \cup (S^{n-1} \times [3 - \delta_{l+2}, 3]) \text{ and}$$
$$B_2 = \sigma \cup (S^{n-1} \times [0, 3 - \tfrac{1}{2} \cdot \delta_{l+2}]).$$

This induces a splitting of N, $N = N_1 \cup N_2$, where $N_i = p^{-1}(B_i)$. Of course this requires N_1 and N_2 to be clean in N. In fact we lose no generality in assuming that each $p^{-1}(\sigma \cup (S^{n-1} \times [0, r]))$ is clean in N, for $r \in (0, 3)$. (We may make such assumptions by the Triangulation Theorem for Q-manifolds and the approximation of maps between polyhedra by PL maps.) We seek a splitting, $M = M_1 \cup M_2$, and a homotopy $f \simeq f'$ so that $f'| : M_i \to N_i$ and $f'| : M_i \cap M_2 \to N_1 \cap N_2$ are homotopy equivalences. Also we want $f \simeq f'$ to be a $p^{-1}\left(\dfrac{\varepsilon}{2}\right)$-homotopy and we want $f'| : M_1 \to N_1$ to be a $(\pi p)^{-1}(\delta'_{l+2})$-equivalence, where π: $B - \mathring{\sigma} \to L$ is the radially-defined retraction and δ'_{l+2} is a number whose size depends on δ_{l+2}.

If M and N were polyhedra, then the results of Section 5 could be applied directly to the above splitting problem. However we can indirectly use Section 5 by applying the Triangulation Theorem for Q-manifolds as follows: Write $M = X \times Q$ and $N = Y \times Q$, where X and Y are compact polyhedra. This can be done so that there exists a splitting, $Y = Y_1 \cup Y_2$, for which $Y_i \times Q = N_i$. For a large integer m define $\hat{p} : Y \times I^m \to B$ by the composition

$$Y \times I^m \hookrightarrow Y \times Q = N \xrightarrow{p} B,$$

and define $\hat{f} : X \times I^m \to Y \times I^m$ by the composition

$$X \times I^m \hookrightarrow X \times Q = M \xrightarrow{f} N = Y \times Q \xrightarrow{\text{proj}} Y \times I^m.$$

Then $\hat{f}$ is still a $\hat{p}^{-1}(\delta_k)$-equivalence, and f splits as desired, provided that $\hat{f}$ splits.

Recall from Theorem 5.2 that we encounter two obstructions when we attempt to carry out the above splitting. For convenience choose

$$0 < r_1 < r_2 < r_3 < r_4 < s_1 < s_2 < s_3 < s_4 < 3$$

so that $r_1 = 3 - \delta_{l+2}$ and $s_4 = 3 - \frac{1}{2}\cdot\delta_{l+2}$. If $u: p^{-1}(S^{n-1} \times [0,3]) \times S^1 \to S^{n-1}$ is the composition

$$p^{-1}(S^{n-1} \times [0,3]) \times S^1 \xrightarrow{\text{proj}} p^{-1}(S^{n-1} \times [0,3]) \xrightarrow{p} S^{n-1} \times [0,3] \xrightarrow{\text{proj}} S^{n-1},$$

then these obstructions are represented by $u^{-1}(\delta_{2l+2})$-equivalences

$$h_1 : P_1 \to p^{-1}(S^{n-1} \times [r_1, r_4]) \times S^1,$$
$$h_2 : P_2 \to p^{-1}(S^{n-1} \times [s_1, s_4]) \times S^1,$$

where P_1 and P_2 are compact Q-manifolds. In order to formulate these obstructions we must choose δ_k small with respect to δ_{2l+2}. By Theorem 5.2 (and Remark 4 following its statement) it suffices to show that h_1 and h_2 are $u^{-1}(\delta'_{l+2}/2)$-homotopic to homeomorphisms. In order to carry out the construction of these homeomorphisms we will use the inductive hypothesis, where $\delta'_{l+2}/2$ plays the role of ε and $\{\delta_{l+3}, \ldots, \delta_{2l+2}\}$ plays the role of $\{\delta_i\}$. Because of the similarity of the two cases we will only give the details for h_1.

All we have to do is exhibit a set $\{C_i\}_{i=1}^{l}$, where C_i is a clean manifold in $p^{-1}(S^{n-1} \times [r_1, r_4]) \times S^1$ so that (C_i, C_{i+1}) is a clean pair, u is $(\delta_{l+i+2}, \delta_{l+i+3})$-nice on (C_i, C_{i+1}), and we must exhibit a clean $P_l \subset P_1$ for which $h_1| : P_l \to C_l$ is a $u^{-1}(\delta_{2l+2})$-equivalence and for which we have a homeomorphism

$$h_1| : P_1 - \mathring{P}_l \to p^{-1}(S^{n-1} \times [r_1, r_4]) \times S^1 - \mathring{C}_l.$$

The construction of P_l and C_l is an easy task. Set

$$C_l = p^{-1}(S^{n-1} \times [r_2, r_3]) \times S^1$$

and recall from Remark 3 following Theorem 5.2 that h_1 can be constructed so that our desired P_l exists. To obtain arbitrary C_i, $1 \le i \le l$, just let

$$C_i = p^{-1}(S^{n-1} \times [\alpha_i, \beta_i]) \times S^1,$$

where α_i and β_i are chosen so that

$$r_1 = \alpha_1 < \ldots < \alpha_l = r_2, r_4 = \beta_1 > \ldots > \beta_l = r_3.$$

We have to make choices so that if $b \in S^{n-1}$, then

$$Wh(p^{-1}(S_{i+1} \times [\alpha_{i+1}, \beta_{i+1}]) \times S^1 \times T^n) \to Wh(p^{-1}(S_i \times [\alpha_i, \beta_i]) \times S^1 \times T^n)$$

is the 0-homomorphism, where S_{i+1} is the δ_{l+i+3}-neighborhood of b and S_i is the δ_{l+i+2}-neighbohood of b. This is easily done by choosing $\beta_l - \alpha_l = 2\delta_{2l+2}$, and for each i choosing

$$\alpha_{i+1} - \alpha_i = \beta_i - \beta_{i+1} = \delta_{l+i+2} - \delta_{l+i+3}.$$

Now to show that u is $(\delta_{l+i+2}, \delta_{l+i+3})$-nice on (C_i, C_{i+1}) choose b, S_i, and S_{i+1} as above. We must show that

$$Wh((u^{-1}(S_{i+1}) \cap C_{i+1}) \times T^n) \to Wh((u^{-1}(S_i) \cap C_i) \times T^n)$$

is 0. But we clearly have

$$u^{-1}(S_{i+1}) \cap C_{i+1} = p^{-1}(S_{i+1} \times [\alpha_{i+1}, \beta_{i+1}]) \times S^1 \text{ and}$$
$$u^{-1}(S_i) \cap C_i = p^{-1}(S_i \times [\alpha_i, \beta_i]) \times S^1,$$

and this is all we need. This completes Step III.

Step IV

In this step we will put everything together to finish the proof of the absolute version of the Relative Approximation Theorem. We are given the splittings $M = M_1 \cup M_2$, $N = N_1 \cup N_2$, and the map $f': M \to N$. All we have to do is show that f' is $p^{-1}\left(\frac{\varepsilon}{2}\right)$-homotopic to a homeomorphism. For $i = 0, 1, 2$ let $f_i = f'|: M_i \to N_i$. We may assume that $N \subset M$, $N_i = N \cap M_i$, and each N_i is a Z-set in M_i. Also, by using the homotopy extension theorem with control, we can construct the map f' to be a retraction so that $f_1: M_1 \to N_1$ is homotopic to id_{M_1} rel N_1 via a $(\pi p f_1)^{-1}(\delta_{l+2})$-homotopy. This was the reason for using δ'_{l+2} above when we first described $f'|: M_1 \to N_1$.

Extend f_1 via the identity to $\bar{f}_1|: \bar{M}_1 \to \bar{N}_1$, where

$$\bar{N}_1 = N_1 \cup p^{-1}(S^{n-1} \times [3 - \delta_3, 3])$$

and $\bar{M}_1 = M_1 \cup \bar{N}_1$. $\bar{f}_1$ is still a $(\pi p)^{-1}(\delta_{l+2})$-equivalence and $\bar{N}_1$ is a Q-manifold because we have assumed that $p^{-1}(\sigma \cup (S^{n-1} \times [0, r]))$ is clean.

Also $\bar{M}_1$ is a Q-manifold because it is easily seen to be the union of two compact Q-manifolds which meet in a Q-manifold that is a Z-set in each side. Now let δ_2' be a number whose size depends on δ_2 and apply the inductive hypothesis to $\bar{M}_1 \xrightarrow{\bar{f}_1} \bar{N}_1 \xrightarrow{\pi p} L$, where δ_2' plays the role of the given ε and $\{\delta_3, \ldots, \delta_{l+2}\}$ plays the role of the $\{\delta_i\}$. By using an argument similar to that used in Step III above we get $\bar{f}_1\ (\pi p)^{-1}(\delta_2')$-homotopic to a homeomorphism. Since $p^{-1}(S^{n-1} \times \{3 - \delta_3\})$ is a Z-set in both $\bar{M}_1$ and $\bar{N}_1$, we can use Z-set unknotting to get $\bar{f}_1\ (\pi p)^{-1}(\delta_2)$-homotopic to a homeomorphism rel $p^{-1}(S^{n-1} \times \{3 - \delta_3\})$. This was the reason that δ_2' was used above.

Extend f_i via the identity to $\tilde{f}_i: \tilde{M}_i \to N$, where $\tilde{M}_i = M_i \cup N$. Note that all the $\tilde{M}_i$ are Q-manifolds. It follows from the above paragraph that $\tilde{f}_1$ is $p^{-1}(2\delta_1)$-homotopic to a homeomorphism because $\delta_2 \leq \delta_1$. Since diam $(\Delta^n) < \delta_2$ and p is (δ_1, δ_2)-nice, it follows from ordinary simple homotopy theory that $\tilde{f}_0$ and $\tilde{f}_1$ are $p^{-1}(2\delta_1)$-homotopic to homeomorphisms. By the Remark following Theorem 6.2 we have $f'\ p^{-1}\left(\frac{\varepsilon}{2}\right)$-homotopic to a homeomorphism. This completes Step IV.

Step V

As indicated earlier, the above proof establishes only the "absolute" version of the Relative Approximation Theorem. We now outline what changes must be made in the above argument to make it work for the relative case. The difference now is that we have the clean sets $M_k \subset M$ and $N_i \subset N$ so that p is (δ_i, δ_{i+1})-nice on (N_i, N_{i+1}), $f|: M_k \to N_k$ is a $p^{-1}(\delta_k)$-equivalence, and $f|: M - \mathring{M}_k \to N - \mathring{N}_k$ is a homeomorphism. We can use this homeomorphism to make an identification between $M - \mathring{M}_k$ and $N - \mathring{N}_k$ so that $M - \mathring{M}_k = N - \mathring{N}_k$ and $f = id$ over $N - \mathring{N}_k$. Also if we set $M_i = f^{-1}(N_i)$, it is clean in M an we may assume that $f|: M_i \to N_i$ is also a $p^{-1}(\delta_k)$-equivalence. Without loss of generality we may assume that the various $p^{-1}(\sigma \cup (S^{n-1} \times [0, r]))$ that we will encounter are clean and intersect all of the N_i transversally.

The first changes in the argument come in Step III. Instead of considering the full map $f: M \to N$ we only work with $f|: M_{l+2} \to N_{l+2}$. So our given splitting of N_{l+2} is $N_{l+2} = N^1 \cup N^2$, where $N^i = p^{-1}(B_i) \cap N_{l+2}$ for $i = 0, 1, 2$. We now seek a splitting, $M_{l+2} = M^1 \cup M^2$, and a $p^{-1}\left(\frac{\varepsilon}{2}\right)$-homotopy $f|M_{l+2} \simeq f''$ so that each $f''|: M^i \to N^i$ is a homotopy equivalence and $f''|: M^1 \to N^1$ is a $(\pi p)^{-1}(\delta_{l+2}')$-equivalence. To do this we first use the fact that $f = id$ on $M - \mathring{M}_k$ to formulate the obstructions h_1 and h_2 so that (additionally) they are the identity over the complements of $[p^{-1}(S^{n-1} \times [r_1, r_4])$

$\cap N_{2l+2}] \times S^1$ and $[p^{-1}(S^{n-1} \times [s_1, s_4]) \cap N_{2l+2}] \times S^1$, respectively. This is accomplished by using Remark 3 following Theorem 5.2. Now focusing on h_1 we redefine C_i as

$$C_i = [p^{-1}(S^{n-1} \times [\alpha_i, \beta_i]) \cap N_{l+i+2}] \times S^1,$$

and the argument goes as in Step III to produce our desired splitting $f|M_{l+2} \simeq f''$. By Remark 5 following Theorem 5.2 this can be done so that, additionally, $f|M_{l+2} \simeq f''$ rel $Bd(M_{l+2})$. Thus we can extend f'' via the identity to $f': M \to N$, and we now have $f \simeq f'$ rel $M - \mathring{M}_{l+2}$. This completes the changes in Step III.

There are also changes in Step IV, but the only significant ones arise in showing that $\bar{f}_1$ is homotopic to a homeomorphism. Happily this is again a repetition of the ideas sketched above, so no further explanation is necessary. This completes Step V and the proof of the Relative Approximation Theorem. ∎

8. Proofs of the applications

As the title indicates we will establish Theorems 2.1–2.3 in this section.

PROOF OF THEOREM 2.1: This is fairly simple compared with the proof of Theorem 2.2. Using the Approximation Theorem we first choose numbers $\{\delta_i\}_{i=1}^k$ so that if f is a $p^{-1}(\delta_k)$-equivalence and p is (δ_i, δ_{i+1})-nice, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism. We call such a collection $\{\delta_i\}_{i=1}^k$ *desirable*. Using the Addendum to the Approximation Theorem, along with the fact that p is nice, we can choose another desirable collection $\{\delta_i^1\}_{i=1}^k$ such that $\delta_1^1 = \delta_1$ and p is (δ_1^1, δ_2^1)-nice. Repeating this we can choose another desirable collection $\{\delta_i^2\}_{i=1}^k$ such that $\delta_1^2 = \delta_1^1$, $\delta_2^2 = \delta_2^1$, and p is (δ_2^2, δ_3^2)-nice. Thus p is $(\delta_i^2, \delta_{i+1}^2)$-nice for $i = 1$ and 2. Iterating this procedure we can inductively select a desirable
collection $\{\delta_i^k\}_{i=1}^k$ such that p is $(\delta_i^k, \delta_{i+1}^k)$-nice, for all i. ∎

Before we begin the proof of Theorem 2.2 it will be convenient to establish a lemma. All spaces in the following statement are ANRs.

LEMMA 8.1: *Let B and $\varepsilon > 0$ be given. There exists a $\delta > 0$ so that if $p: E \to B$ is $(\delta, 1)$-movable and $q: \mathscr{E} \to B$ is a Hurewicz fibration for which $E \hookrightarrow \mathscr{E}$ is a homotopy equivalence and $q|E = p$, then*

(1) *any map* $\phi:(I^k,\partial I^k)\to(\mathscr{E},E)$ *is* $q^{-1}(\varepsilon)$*-homotopic to* ψ *rel* δI^k *so that* $\psi(I^k)\subset E$, *for* $k=0$ *and* 1,

(2) *any map* $\phi:(I^2,\partial I^2)\to(\mathscr{E},E)$ *is* $q^{-1}(\varepsilon)$*-close to* ψ *so that* $\psi(I^2)\subset E$ *and* $\psi=\phi$ *on* ∂I^2.

PROOF: We first treat the case $k=0$. Choose any $x\in\mathscr{E}$. We must find a path in $\mathscr{E}$ from x to E so that the q-image of this path is close to $q(x)$. Let $r:\mathscr{E}\to E$ be a retraction which is homotopic to *id* rel E. Let $r_t:\mathscr{E}\to E$ be a homotopy rel E for which $r_0=id$ and $r_1=r$. Define $F:I^\circ\times I\times I\to B$ by $F(s,t)=qr_t(x)$ and $f:I^\circ\to\mathscr{E}$ by $f(s)=r_1(x)$. Note that $pf(s)=qr_1(x)=F_1(s)$. Thus there is a map $\tilde{F}:I^\circ\times I\to E$ for which $\tilde{F}_1=f$ and for which $d(p\tilde{F},F)<\delta$. There is a path in $\mathscr{E}$ from x to $\tilde{F}_0(s)\in E$ defined by $r_t(x)*\tilde{F}_{1-t}(s)$. The q-image of this path is very close to $qr_t(x)*qr_{1-t}(x)$, and so by the homotopy lifting property we can deform the path $r_t(x)*\tilde{F}_{1-t}(s)$ rel the ends to a new path in $\mathscr{E}$ from x to $\tilde{F}_0(s)$ whose q-image has small diameter.

We now treat the case $k=1$. Let $\phi:(I,\partial I)\to(\mathscr{E},E)$ be a map and define $F:I^2\to B$ by $F(s,t)=qr_t(\phi(s))$. If $f:I\to E$ is defined by $f(s)=$ $=r(\phi(s))$, then we have $pf(\_)=F(\_,1)$. Choose $\tilde{F}:I^2\to E$ for which $\tilde{F}_1$ $=f$, $\tilde{F}_t(s)=\phi(s)$ for $s\varepsilon\partial I$, and $d(p\tilde{F},F)<\delta$. Our desired map $\psi:I^2\to E$ is given by $\psi=\tilde{F}_0$. As in the case $k=0$ we can get our desired homotopy $\phi\simeq\psi$ rel ∂I. This completes the proof of part (1). The proof of part (2) is similar. ∎

PROOF OF THEOREM 2.2: We have divided the proof into three cases. They are: B is a polyhedron, B is a Q-manifold, and B is an ANR.

I. B is a Polyhedron. Our strategy is to imitate the proof of Theorem 2.1, but in order to do so we will need the

ASSERTION: *If* $b\varepsilon B$ *and* $\mu>\nu>0$ *are numbers such that* $S_\nu(b)$ *contracts to* b *in* $S_\mu(b)$, *then* $\delta>0$ *can be chosen so that the* $(\delta,1)$*-movable map* $p:N\to B$ *of Theorem* 2.2 *is* $(2\mu,\nu)$*-nice. Moreover,* δ *depends only on* μ *and* ν.

PROOF: We have to show that

$$Wh(p^{-1}(S_\nu(b))\times T^n)\to Wh(p^{-1}(S_{2\mu}(b))\times T^n)$$

is the 0-homomorphism. Let $q:\mathscr{E}\to B$ be a Hurewicz fibration for which $E\hookrightarrow\mathscr{E}$ is a homotopy equivalence and $q|E=p$. Because $S_\nu(b)$ contracts to b it follows that the fibration $q:\mathscr{E}\to B$ is trivial over $S_\nu(b)$. This induces a retraction $r: q^{-1}(S_\nu(b)) \to q^{-1}(b)$, which we identify with $\mathscr{F}$. Using Lemma 8.1 there exists a homomorphism $\theta: \pi_1(\mathscr{F}) \to \pi_1 p^{-1}(S_{2\mu}(b))$. Thus we get a homomorphism $\theta r_* i_*: \pi_1(p^{-1}(S_\nu(b))) \to \pi_1(p^{-1}(S_{2\mu}(b)))$, where i is the inclusion $p^{-1}(S_\nu(b)) \hookrightarrow q^{-1}(S_\nu(b))$. Once again using Lemma 8.1 it is easy to see that $\theta r_* i_* = j_*$, where j is the inclusion $p^{-1}(S_\nu(b)) \hookrightarrow p^{-1}(S_{2\mu}(b))$. (This is the place where the fact that the contraction of $S_\nu(b)$ to b *takes place in* $S_\mu(b)$ is invoked.) Now multiplying by T^n and applying the Whitehead group functor we conclude that

$$Wh(p^{-1}(S_\nu(b)) \times T^n) \to Wh(p^{-1}(S_{2\mu}(b)) \times T^n)$$

factors through $Wh(\mathscr{F} \times T^n)$, which is 0. ■

Now returning to the proof of the polyhedral case we start with a desirable collection $\{\delta_i\}_{i=1}^k$. Since the δ_i can be chosen small we may assume that each $S_{\delta_{i+1}}(b)$ contracts to b in $S_{\delta_{i/2}}(b)$, for all $b \in B$. By the Assertion we may choose $\delta > 0$ so that p is (δ_i, δ_{i+1})-nice, thus f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism as desired. This completes the polyhedral case.

II. B is a Q-Manifold. We are given a $p^{-1}(\delta)$-equivalence $f: M \to N$, where $p: N \to B$ is $(\delta, 1)$-movable. We can choose a factorization $B = B_1 \times Q$, where B_1 is a polyhedron so that each $\{x\} \times Q$ has a small diameter. Let $q = \mathrm{proj}: B \to B_1$ and note that $qp: N \to B$ is still $(\delta, 1)$-movable. The proof of this fact is straightforward (see the proof of Assertion 1 in the proof of Theorem 1 of [2]). It is also easy to see that the homotopy fibers of p and qp are the same, so we can apply the polyhedral case to get $f(qp)^{-1}\left(\frac{\varepsilon}{2}\right)$-homotopic to a homeomorphism. Thus f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism because the sets $\{x\} \times Q$ have small diameter in B.

III. B is an ANR. We are given a $p^{-1}(\delta)$-equivalence $f: M \to N$, where $p: N \to B$ is $(\delta, 1)$-movable. Then $f \times id: M \times Q \to N \times Q$ is a $(p \times id)^{-1}(\delta)$-equivalence, where $p \times id: N \times Q \to B \times Q$ is easily seen to be $(\delta, 1)$-movable. But $B \times Q$ is now a Q-manifold, so by the Q-manifold case we have $f \times id\ (p \times id)^{-1}\left(\frac{\varepsilon}{2}\right)$-homotopic to a homeomorphism $h: M \times Q \to N \times Q$. Let $u: M \times Q \to M$ and $v: N \times Q \to N$ be homeomorphisms which are close to projections. Then $vhu^{-1}: M \to N$ is a homeomorphism which is $p^{-1}(\varepsilon)$-close to f. ■

PROOF OF THEOREM 2.3: To begin with we are given B and $\varepsilon > 0$. If k is the integer of the Approximation Theorem for this data, then we will show that our desired l may be taken to be $l = 2k + 1$. So let $\{\delta_i\}_{i=1}^{l}$ and $p: X \to B$ be given so that p is (δ_i, δ_{i+1})-nice and X is $p^{-1}(\delta_l)$-finitely dominated, i.e., there is a compact polyhedron K and maps $f: K \to X$, $g: X \to K$ for which fg is $p^{-1}(\delta_l)$-homotopic to id_X. We will show that if the δ_i are correctly chosen and $l = 2k + 1$, then X has $p^{-1}(\varepsilon)$-finite type.

Our proof is an easy modification of the proof of Theorem 5.1. So define $e = gf: K \to K$ and as in the proof of Theorem 5.1 form the commutative diagram

$$\begin{tikzcd}
S_e \arrow[r, "\tilde{H}"] \arrow[d] & X \times R \arrow[d] & S_e^- \arrow[l, "\tilde{H}_-"'] \arrow[d] \\
T_e \arrow[r, "H"] & X \times S^1 & T_e^- \arrow[l, "H_-"']
\end{tikzcd},$$

where the horizontal maps are homotopy equivalences and the vertical maps are covering maps. If $\pi = \mathrm{proj}: X \times S^1 \to X$ and $q = p\pi H_-$: $T_e^- \to B$, then δ_l can be chosen small enough so that $h = H_-^{-1}H$: $T_e \to T_e^-$ is a $q^{-1}(\delta_{2k})$-equivalence. Observe that for each $b \in B$ the inclusion $q^{-1}(S_{\delta_{i+1}}(b)) \hookrightarrow q^{-1}(S_{\delta_{i-1}}(b))$ factors into the composition

$$q^{-1}(S_{\delta_{i+1}}(b)) \xrightarrow{H_-} (p\pi)^{-1}(S_{\delta_{i+1}}(b)) \hookrightarrow$$
$$(p\pi)^{-1}(S_{\delta_i}(b)) \xrightarrow{H_-^{-1}} q^{-1}(S_{\delta_{i-1}}(b))$$

provided that δ_{2k} is chosen small enough. If $\{\delta_i'\}_{i=1}^{k}$ is the set $\{\delta_2, \delta_4, \ldots, \delta_{2k}\}$, then $q: T_e^- \to B$ is $(\delta_i', \delta_{i+1}')$-nice. Applying the Approximation Theorem there is $(q \circ \mathrm{proj})^{-1}(\delta_1)$-homotopy of

$$h \times id_Q: T_e \times Q \to T_e^- \times Q$$

to a homeomorphism $k: T_e \times Q \to T_e^- \times Q$. This is covered by a homotopy $\tilde{h} \times id_Q \simeq \tilde{k}$, where $\tilde{h} = \tilde{H}_-^{-1}\tilde{H}$ and $\tilde{k}$ is a homeomorphism. For n a large integer let

$$A = (S_e[0, \infty) \times Q) \cap \tilde{k}^{-1}(S_e^-(-\infty, n] \times Q),$$

which is a compact Q-manifold. Write $A = L \times Q$, where L is a compact polyhedron. Then for m a large integer the composition

$$L \times I^m \hookrightarrow L \times Q = A \hookrightarrow S_e \times Q \xrightarrow{\tilde{H} \times id} X \times R \times Q \xrightarrow{\text{proj}} X$$

is a $p^{-1}(\varepsilon)$-equivalence as desired. ■

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