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THE CAUCHY PROBLEM FOR LORENTZ METRICS WITH PRESCRIBED RICCI CURVATURE

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1. Introduction

Much attention has been devoted to the Ricci tensor by mathematicians and physicists, because of its importance in geometry and general relativity. A fundamental problem in geometry is to determine to what extent the curvature determines the geometry of a manifold. In the present context, one can ask which symmetric tensors are the Ricci tensors of (Lorentz) metrics. We give examples in section 3 that indicate that even the local version of this question is difficult to resolve. In physics, the Einstein equations for the gravitational field in general relativity involve the Ricci tensor in an essential way.

Both of these problems can be cast as existence problems involving quasi-linear systems of partial differential equations. If Ricci is the operator (system of second-order quasi-linear partial differential operators) that maps metrics to their Ricci curvature tensors (a precise definition is given in section 2), then the problems can be formulated as follows. For the geometry problem, we let the tensor $R_{\alpha\beta}$ be given, and ask when there exists a Lorentz metric g (signature $-, +, +, +$) that satisfies the equation

$$\mathrm{Ricc}(g) = R_{\alpha\beta} \tag{1.1}$$

In particular, we will study the circumstances under which the Cauchy

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problem for this equation (in which values of the unknown metric g and its first derivatives are given on a hypersurface) can be solved.

The formulation of the physical problem is somewhat more complicated. In this problem, we are given the (stress-energy) tensor $T_{\alpha\beta}$ as a function of physical quantities called the matter fields, which in turn are required to satisfy certain other partial differential equations. We then ask whether the Cauchy problem for the system

$$\operatorname{Ricc}(g) = T_{\alpha\beta} - \frac{1}{n-2}(\operatorname{tr} T)g_{\alpha\beta} \tag{1.2}$$

coupled with the equations of the matter fields, has a solution. The matter equations are required to satisfy certain conditions which are enumerated in section 5.

Local existence in time for the simplest case of both problems, namely $R_{\alpha\beta} \equiv T_{\alpha\beta} \equiv 0$ was first proved by Choquet-Bruhat in [1]. Her proof is based upon the observation of Lanczos [16] that the formula for the Ricci tensor simplifies considerably in harmonic coordinates (see also [6] for a discussion of harmonic coordinates and the Ricci tensor). Later, Choquet-Bruhat, Lichnerowicz and others proved the local Cauchy theorem for many cases of the physics equation (1.2) where $T_{\alpha\beta}$ is specified in terms of well-known physical fields (see [15], our section 5 also contains an example), and eventually the general theorem that we prove in section 5 was obtained (see [10] or [8]). All these proofs used harmonic coordinates in an essential way. Our proof of Theorem 5.5 has the feature that it is valid in any coordinate system (gauge). This may prove to be advantageous for physicists, since a preferred gauge is often determined by a physical problem, and such a gauge need not be harmonic.

In our discussion we will show that, although the physics and geometry problems discussed above appear quite similar, they differ significantly in several aspects. It may be surprising to note that the existence question for equation (1.1) is much more subtle that that for (1.2). The reason for this is that the physical problem is essentially homogeneous in nature while (1.1) is truly inhomogeneous. As an example of contrasting results, we note that the solution of the Cauchy problem for (1.1) is generically unique (Proposition 3.10), while that for (1.2) is only unique up to diffeomorphism (change of gauge). We also hope to clear up some misconceptions about equation (1.1) that are often implied and sometimes even stated outright in the literature.

For the geometry problem, the main theorem we prove here, Theorem 3.6, concerns the local (in "time") existence of Lorentz metrics with prescribed smooth, nonsingular Ricci tensors. The proof of this theorem is somewhat like that of Choquet-Bruhat for the homogeneous case, in that the basic degeneracy of the Ricci operator is overcome by controlling the action of the group of diffeomorphisms on it. The analogous result for Riemannian metrics with locally prescribed nonsingular Ricci tensors was presented in [4] and [5].

A few words about notation. Although all of our results are valid in Lorentz space of any dimension ($\geqq 3$, for the two-dimensional case the results of [4] can be adapted), we will usually work in four-dimensional "space-time". Occasionally (as in (1.2)), we will give the general form of an expression that is dimension-dependent, and n will always denote the dimension. Otherwise, most of our notation is standard tensor notation, and the summation convention applies throughout. Greek indices run from 0 to 3 (or $n-1$), and Latin indices from 1 to 3. Often, we will identify the coordinate x^0 as t.

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2. Geometric preliminaries

Our first task is to specify some of the operators and equations with which we will be dealing. As is well-known, the operator that maps metrics to their Ricci tensors is given by

$$\operatorname{Ricc}(g)_{\alpha\beta} \equiv \frac{\partial \Gamma^{\sigma}_{\alpha\beta}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\sigma}_{\alpha\sigma}}{\partial x^{\beta}} + \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\tau}_{\sigma\tau} - \Gamma^{\sigma}_{\alpha\tau}\Gamma^{\tau}_{\sigma\beta} \tag{2.1}$$

where

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}\left(\frac{\partial g_{\beta\sigma}}{\partial x^{\gamma}} + \frac{\partial g_{\sigma\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\sigma}}\right) \tag{2.2}$$

are the Christoffel (connection) symbols of the metric g. Even more explicitly,

$$\operatorname{Ricc}(g)_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}\left(\frac{\partial^2 g_{\alpha\mu}}{\partial x^{\beta}\,\partial x^{\nu}} + \frac{\partial^2 g_{\beta\mu}}{\partial x^{\alpha}\,\partial x^{\nu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\,\partial x^{\beta}} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^{\mu}\,\partial x^{\nu}}\right) + H(g, \nabla g) \tag{2.3}$$

where H is a rational function of g and its first derivatives. As a differential operator, Ricc is not very well behaved. In [14] it is shown that the linearization of the Ricci operator about a particular metric g is:

$$\begin{aligned}\mathrm{Ricc}'(g)h &\equiv \frac{d}{dt}\mathrm{Ricc}(g+th)\bigg|_{t=0}\\ &= \tfrac{1}{2}\Delta_{\mathrm{L}}h - \mathrm{div}^*\,\mathrm{div}\,\mathrm{G}(h)\end{aligned} \tag{2.4}$$

where Δ_L is the Lichnerowicz "Laplacian":

$$(\Delta_{\mathrm{L}}h)_{\alpha\beta} = -g^{\sigma\tau}h_{\alpha\beta;\sigma\tau} + R_{\alpha\sigma}h^{\sigma}_{\beta} + R_{\beta\sigma}h^{\sigma}_{\alpha} - 2R_{\alpha\sigma\beta\tau}h^{\sigma\tau}$$

(the covariant derivatives and curvature tensors that appear are those of g). Note that Δ_{L} is a hyperbolic operator if g is a Lorentz metric. The divergence operator $\mathrm{div}: \mathrm{S}^2\mathrm{T}^* \to \mathrm{T}^*$ and its L^2 adjoint div^* are defined as follows for $h \in \mathrm{S}^2\mathrm{T}^*$ and $v \in \mathrm{T}^*$:

$$(\mathrm{div}\,h)_\alpha = -g^{\sigma\tau}h_{\sigma\alpha;\tau} \qquad (\mathrm{div}^*\,v)_{\alpha\beta} = \tfrac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha}) \tag{2.5}$$

Finally, G is the invertible self-adjoint algebraic operator defined by

$$(\mathrm{G}(h))_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}(\mathrm{tr}\,h)g_{\alpha\beta} \tag{2.6}$$

and the trace $(= g^{\sigma\tau}h_{\sigma\tau})$ is taken with respect to g. Note that $\mathrm{G}(\mathrm{Ricc}(g))$ is the stress-energy tensor of the metric g, and that the expression on the right-hand side of (1.2) is simply $\mathrm{G}^{-1}(T)$.

From (2.4) (or, more easily, from (2.3)) it can be computed that the principal symbol of Ricc is

$$\sigma^{\mathrm{Ricc}}(h) = \tfrac{1}{2}g^{\mu\nu}[h_{\alpha\mu}\xi_\nu\xi_\beta + h_{\beta\mu}\xi_\alpha\xi_\nu - h_{\mu\nu}\xi_\alpha\xi_\beta - h_{\alpha\beta}\xi_\mu\xi_\nu]$$

and we see that if $h_{\alpha\beta} = \xi_\alpha\xi_\beta$, then $\sigma^{\mathrm{Ricc}}(h) = 0$ with $h \neq 0$. Thus, the Ricci operator is quite degenerate, since every direction is characteristic for it at every point. This degeneracy is related to the invariance of the Ricci operator under the action of the group of diffeomorphisms, i.e., that

$$\phi^*\,\mathrm{Ricc}(g) = \mathrm{Ricc}(\phi^* g) \tag{2.7}$$

for any diffeomorphism ϕ. As shown in [12], this invariance manifests itself in the Bianchi identity, which states that if $\mathrm{Ricc}(g) = R$, then

$$-\mathrm{div}\,\mathrm{G}(R) = g^{\sigma\tau}(R_{\sigma\alpha;\tau} - \tfrac{1}{2}R_{\sigma\tau;\alpha}) = 0 \tag{2.8}$$

For physicists, (2.8) is good news; it provides the "conservation of mass-energy" law and renders (1.2) underdetermined in a way that will be explained more fully in section 5. For geometers, (2.8) places a necessary condition on the tensor $R_{\alpha\beta}$ in equation (1.1): namely, that there must exist metrics with respect to which $-\operatorname{div} \mathrm{G}(R) = 0$ in order that a solution of (1.1) can be found. We will see in Example 3.4 that this condition is truly a restriction upon the set of $R_{\alpha\beta}$'s that we can consider. Equation (1.1) becomes in this way an overdetermined system.

Looking at (2.8) as a first-order differential expression in g (the first derivatives of g enter in the Christoffel symbols associated with the covariant derivatives) provides us with a way of rendering both of the equations (1.1) and (1.2) into hyperbolic form. So that we are not prejudiced by the association of the letter R with the Ricci tensor, we let M be any fixed symmetric tensor, i.e., $M \in \mathrm{S}^2\mathrm{T}^*$. Write

$$\mathrm{B}_M(g) \equiv -\operatorname{div} \mathrm{G}(M) \tag{2.9}$$

We lineairze this operator and get

$$\begin{aligned} \mathrm{B}'_M(g)h &\equiv \frac{d}{dt}\mathrm{B}_M(g+th)\Big|_{t=0} \\ &= M^\sigma_\alpha (\operatorname{div} \mathrm{G}(h))_\sigma - Q^{\sigma\tau}_\alpha h_{\sigma\tau} \end{aligned} \tag{2.10}$$

where

$$Q^{\sigma\tau}_\alpha = g^{\sigma\mu} g^{\nu\tau}(M_{\mu\alpha;\nu} - \tfrac{1}{2} M_{\mu\nu;\alpha})$$

If M is an invertible map from T to T*, so that its inverse $N \in \mathrm{S}^2\mathrm{T}$ that verifies $N^{\alpha\beta} M_{\beta\gamma} = \delta^\alpha_\gamma$ is well-defined, then we see that the principal (highest-order) part of the linearization of the operator

$$-\operatorname{div}^*(N \operatorname{div} \mathrm{G}(M))$$

is given by

$$\tfrac{1}{2} g^{\sigma\tau}\left(\frac{\partial^2 h_{\sigma\tau}}{\partial x^\alpha\,\partial x^\beta} - \frac{\partial^2 h_{\sigma\alpha}}{\partial x^\tau\,\partial x^\beta} - \frac{\partial^2 h_{\sigma\beta}}{\partial x^\tau\,\partial x^\alpha}\right) \tag{2.11}$$

Comparing (2.3) and (2.11), we see that the operator

$$\operatorname{Ricc}(g) - \operatorname{div}^* M^{-1}(\operatorname{div} \mathrm{G}(M)) \tag{2.12}$$

is hyperbolic (as long as M is invertible), since the principal part of its linearization is simply $-\tfrac{1}{2} g^{\sigma\tau} h_{\alpha\beta;\sigma\tau}$, i.e., the d'Alembertian operator. We also note that the system (2.12) is uncoupled in its highest-order terms. Needless to say, a significant step in our existence proofs for (1.1) and (1.2) will be to replace their left-hand sides by (2.12), with M chosen appropriately.

We will have several occasions to use the following lemma.

LEMMA 2.13: *Let g be a Lorentz metric. Then the operator* div G div* *is strictly hyperbolic.*

PROOF: A simple computation using (2.5), (2.6) and the Ricci identities shows that, for any covector v,

$$\begin{aligned}\operatorname{div} \mathrm{G} \operatorname{div}^*(v) &= \tfrac{1}{2} g^{\sigma\tau}(v_{\sigma;\tau\alpha} - v_{\alpha;\sigma\tau} - v_{\sigma;\alpha\tau}) \\ &= \tfrac{1}{2}(-g^{\sigma\tau} v_{\alpha;\sigma\tau} - S^{\tau}_{\alpha} v_{\tau})\end{aligned}$$

where S is the Ricci tensor of the metric g. Thus, the highest-order part of this operator is half of the usual d'Alembertian. Q.e.d.

We now turn to the setup for the Cauchy problem. Since we are given data on a spacelike hypersurface (i.e., on a surface such that $g_{\alpha\beta}\xi^{\alpha}\xi^{\beta} > 0$ for vectors ξ tangent to the surface), we assume that, via a change of coordinates, the surface is (an open subset of) the hyperplane $t = 0$. To make the hyperplane $t = 0$ spacelike, we require that the initial data $g_{\alpha\beta}(0, x^i)$ satisfy the following two hypotheses:

$$\begin{aligned}&\text{(i) the 3-by-3 matrix } g_{ij}(0, x^i) \text{ is (uniformly) positive definite} \\ &\text{(ii) } g_{00}(0, x^i) < 0.\end{aligned} \tag{2.14}$$

As is well-known to physicists, there are four compatibility conditions that the initial data $g_{\alpha\beta}(0, x^i)$, $\partial g_{\alpha\beta}/\partial t\,(0, x^i)$ must satisfy. This is because of the following:

PROPOSITION 2.15: *The values of* $\mathrm{G}(\operatorname{Ricc}(g))^0_\lambda$ $(\lambda = 0, \ldots, 3)$ *on the surface $t = 0$ do not depend upon the second t-derivatives of any $g_{\alpha\beta}$.*

PROOF: We just use (2.3) and (2.6):

$$\begin{aligned}\mathrm{G}(\operatorname{Ricc}(g))^0_\lambda &= g^{0\alpha}[\operatorname{Ricc}(g)_{\alpha\lambda} - \tfrac{1}{2} g^{\beta\gamma} g_{\alpha\lambda} \operatorname{Ricc}(g)_{\beta\gamma}] \\ &= \tfrac{1}{2} g^{0\alpha} g^{\mu\nu}\left(\frac{\partial^2 g_{\mu\alpha}}{\partial x^\nu\,\partial x^\lambda} + \frac{\partial^2 g_{\nu\lambda}}{\partial x^\mu\,\partial x^\alpha} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha\,\partial x^\lambda} - \frac{\partial^2 g_{\alpha\lambda}}{\partial x^\mu\,\partial x^\nu}\right) \\ &\qquad - \tfrac{1}{2}\delta^0_\lambda g^{\beta\gamma} g^{\mu\nu}\left(\frac{\partial^2 g_{\mu\beta}}{\partial x^\nu\,\partial x^\gamma} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\beta\,\partial x^\gamma}\right) + \begin{array}{l}\text{lower}\\ \text{order}\\ \text{terms}\end{array}\end{aligned}$$

$$\begin{aligned}&= \tfrac{1}{2}\left(\delta^0_\lambda g^{0\alpha} g^{0\mu}\frac{\partial^2 g_{\mu\alpha}}{\partial t^2} + g^{00} g^{0\nu}\frac{\partial^2 g_{\nu\lambda}}{\partial t^2} - \delta^0_\lambda g^{00} g^{\mu\nu}\frac{\partial^2 g_{\mu\nu}}{\partial t^2} - g^{0\alpha} g^{00}\frac{\partial^2 g_{\alpha\lambda}}{\partial t^2}\right) \\ &\quad - \tfrac{1}{2}\left(\delta^0_\lambda g^{\beta 0} g^{\mu 0}\frac{\partial^2 g_{\mu\beta}}{\partial t^2} - \delta^0_\lambda g^{00} g^{\mu\nu}\frac{\partial^2 g_{\mu\nu}}{\partial t^2}\right) + \text{terms that do not involve } \partial^2/\partial t^2. \\ &= \text{terms that do not involve } \partial^2/\partial t^2.\end{aligned}$$

Q.e.d.

The upshot of the proposition is that we must require our initial data to satisfy

$$(\mathrm{G}(\mathrm{Ricc}(g))^0_\lambda = \mathrm{G}(R)^0_\lambda \quad \text{on } t = 0 \tag{2.16a}$$

or

$$(\mathrm{G}(\mathrm{Ricc}(g))^0_\lambda = T^0_\lambda \quad \text{on } t = 0 \tag{2.16b}$$

depending on the problem at hand.

We conclude the section with a technical lemma about the restrictions of 1-forms to the surface $t = 0$ that we will use repeatedly.

LEMMA 2.17: *Suppose that the hypotheses* (2.14) *are satisfied, and that* $v_\alpha(0, x^i) = 0$ *and* $(\mathrm{G}(\mathrm{div}^*(v)))^0_\lambda(0, x^i) = 0$. *Then* $\partial v_\alpha/\partial t(0, x^i) = 0$.

PROOF: We have

$$\mathrm{G}(\mathrm{div}^*(v)))^0_\lambda = \tfrac{1}{2}(v_{\lambda;}{}^0 + v^0{}_{;\lambda} - \delta^0_\lambda g^{\sigma\tau} v_{\sigma;\tau})$$

Since $v = 0$ on the hypersurface $t = 0$, covariant derivatives of v reduce to ordinary derivatives there, and $\partial v_\alpha/\partial x^i(0, x^i) = 0$. Thus

$$(\mathrm{G}(\mathrm{div}^*(v)))^0_i = \tfrac{1}{2} g^{00} \frac{\partial v_i}{\partial t} = 0$$

follows immediately for $i = 1, 2, 3$, and

$$(\mathrm{G}(\mathrm{div}^*(v)))^0_0 = \tfrac{1}{2} g^{00} \frac{\partial v_0}{\partial t} = 0$$

follows from that. Since $g^{00} < 0$, the proof is complete. Q.e.d.

3. Lorentz metrics with prescribed Ricci curvature

We now address ourselves to the geometric problem outlined in the introduction. Let $R_{\mu\nu}(x^a)$ be a given symmetric tensor in a neighborhood of (an open subdomain Ω of) the hypersurface $t = 0$. Also, let $a_{\mu\nu}(x^i)$ and $b_{\mu\nu}(x^i)$ be symmetric n-by-n tensors in the $n - 1$ variables x^i, where, if $g_{\mu\nu}(0, x^i) = a_{\mu\nu}(x^i)$, then conditions (2.14) are satisfied. We would like to find a Lorentz metric $g_{\alpha\beta}$ defined on $\Omega \times [-\tau, \tau]$ for some τ so that $\mathrm{Ricc}(g) = R$ on $\Omega \times [-\tau, \tau]$, $g_{\mu\nu}(0, x^i) = a_{\mu\nu}(x^i)$ and $\partial g_{\mu\nu}/\partial t\,(0, x^i) = b_{\mu\nu}(x^i)$.

As we discussed in section 2, the Cauchy data $a_{\alpha\beta}$, $b_{\alpha\beta}$ must satisfy certain compatibility conditions imposed by the Ricci equation and the Bianchi identity. First, there are the n conditions

$$(\mathrm{G}(\mathrm{Ricc}(\bar{g})))_\lambda^0 = (\mathrm{G}(R))_\lambda^0 \qquad \text{on } t = 0 \tag{3.1}$$

of Proposition 2.15, where $\bar{g}_{\mu\nu}(x^\alpha) = a_{\mu\nu}(x^i) + tb_{\mu\nu}(x^i)$. Also, the Bianchi identity (2.8) places n more conditions on the data, because (2.8) depends only upon the first derivatives of g, which are determined completely by the Cauchy data on the initial surface. Thus, $a_{\mu\nu}$ and $b_{\mu\nu}$ must also satisfy

$$\operatorname{div} \mathrm{G}(R) = 0 \qquad \text{on } t = 0 \tag{3.2}$$

where the divergence and G operators are those of the metric $\bar{g}_{\mu\nu}(x^\alpha)$.

Now we must face the real restrictions imposed by the Bianchi identity. In section 2, we mentioned that this identity is a real obstruction to solving the equation $\mathrm{Ricc}(g) = R$. That this is so is indicated by the following examples:

Example 3.3. Let

$$R = t\,dt \otimes dt \pm dx^1 \otimes dx^1 \pm dx^2 \otimes dx^2 \pm dx^3 \otimes dx^3$$

There is no Cauchy data for R on the surface $t = 0$ that satisfies conditions (3.1) and (3.2), since Bianchi identity (2.8) could not be satisfied there. In fact, on the surface $t = 0$, the t-component of (3.2) becomes $g^{00} = 0$, in which case the surface $t = 0$ could not be spacelike.

Example 3.4. Let

$$R = \sum_{\alpha=0}^{4} (x^0 dx^\alpha \otimes dx^\alpha + \tfrac{1}{2}x^\alpha dx^0 \otimes dx^\alpha + \tfrac{1}{2}x^\alpha dx^\alpha \otimes dx^0)$$

There is no Cauchy data for R on *any* surface containing the origin, because (2.8) implies

$$g^{00}(0) = g^{01}(0) = g^{02}(0) = g^{03}(0) = 0$$

which is impossible for any metric.

The preceding examples were "borrowed" from [4], and more detail can be found there.

Example 3.5. Let

$$R = t^2\,dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 - 2dx^3 \otimes dx^3$$

If we set $a(x^i) = -dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$ and $b(x^i) \equiv 0$, then the compatibility conditions (3.1) and (3.2) are all satisfied for $t = 0$. However, there is no solution of $\mathrm{Ricc}(g) = R$ with this Cauchy data, since $\dfrac{\partial}{\partial t}(\operatorname{div} \mathrm{G}(R)) = 0$ for $t = 0$ implies $g^{00} = 0$ there, a contradiction.

The tensors R in the above examples shared one important feature in common: Each was singular on the surface $t = 0$. We now turn to the basic existence result of this section, it being for nonsingular "Ricci candidates" R.

THEOREM 3.6: *Let Ω be a bounded subdomain of the hyperplane $t = 0$, and suppose that the C^k tensor $(k \geqq 4)$ $R_{\alpha\beta}$ is given on $(-\varepsilon, \varepsilon) \times \Omega$, that R^{-1} exists there, and that the initial data $g_{\alpha\beta}(0, x^i) = a_{\alpha\beta}(x^i) \in \mathrm{C}^k(\Omega)$ and $\partial g_{\alpha\beta}/\partial t\,(0, x^i) = b_{\alpha\beta}(x^i) \in \mathrm{C}^{k-1}(\Omega)$ satisfy the hyperbolicity conditions* (2.14) *and the compatibility conditions* (3.1) *and* (3.2). *Then for some $0 < \varepsilon' \leqq \varepsilon$, and $\Omega' \subseteq \Omega$, there exists a C^k metric g that satisfies* $\mathrm{Ricc}(g) = R$ *on $(-\varepsilon', \varepsilon') \times \Omega$ and agrees with the initial data on $\{0\} \times \Omega'$.*

PROOF: As intimated in section 2, we are going to take advantage of equation (2.12), and use $M = R$. That is, we deal with the equation

$$\mathrm{Ricc}(g) - \operatorname{div}^*(R^{-1} \operatorname{div} \mathrm{G}(R)) = R \tag{3.7}$$

Because (3.7) is strictly hyperbolic, we know from standard quasilinear hyperbolic equation theory [13] that a solution of the Cauchy problem for it will exist locally in time. Let g be that solution, with the Cauchy data given in the statement of the theorem. Then g will satisfy all the claims of the theorem if we can show that the 1-form $R^{-1} \operatorname{div} \mathrm{G}(R)$ is identically zero on $(-\varepsilon', \varepsilon') \times \Omega'$. Let $u = R^{-1} \operatorname{div} \mathrm{G}(R)$. Then, taking $\operatorname{div} \mathrm{G}(\cdot)$ of both sides of (3.7) and using the Bianchi identity (2.8), we see that

$$-\operatorname{div} \mathrm{G} \operatorname{div}^*(u) = Ru \tag{3.8}$$

Thus, u is the (unique!) solution of the homogeneous linear, strictly hyperbolic system (3.8). Furthermore, $u = 0$ on the initial surface Ω, since the Cauchy data satisfies condition (3.2). If we can show that

$\dfrac{\partial u}{\partial t} = 0$ on Ω, then the uniqueness theorem for hyperbolic equations will guarantee that $u \equiv 0$ and complete the proof. To do this, we use (3.1), which implies that

$$(\mathrm{G}(\operatorname{div}^*(u)))^0_\lambda = 0 \qquad \text{on } \Omega$$

Thus, u satisfies the hypotheses of Lemma 2.17 on Ω, and therefore $\dfrac{\partial u}{\partial t} = 0$ there. Q.e.d.

REMARK 3.9: In [7], Fischer and Marsden use their theory of quasilinear symmetric hyperbolic systems to find asymptotically flat H^s solutions of $\operatorname{Ricc}(g) = 0$. Given some metric g_0 with nonsingular Ricci tensor R_0, we can use their theory to find an "asymptotically g_0" metric for an "asymptotically R_0" Ricci candidate, given appropriate, asymptotically g_0 Cauchy data.

We turn now to the question of uniqueness of solutions to the Cauchy problem for equation (1.1). From the Bianchi identity (2.8), we see that, if R^{-1} exists, then any solution of $\operatorname{Ricc}(g) = R$ is automatically a solution of equation (3.7). Since (3.7) is strictly hyperbolic, it follows from the standard uniqueness theory for hyperbolic equations that there is only one solution of the Cauchy problem for (3.7). This proves the following:

PROPOSITION 3.10: *If R^{-1} exists, then the solution of the Cauchy problem for* $\operatorname{Ricc}(g) = R$ *is unique.*

Proposition 3.10 is somewhat striking in comparison with the well-known uniqueness result for the homogeneous case. There, it is known that, given two solutions g_1 and g_2 of $\operatorname{Ricc}(g) = 0$ with the same Cauchy data, there exists a diffeomorphism ϕ such that $g_2 = \phi^*(g_1)$ (We will prove a slight generalization of this result in section 4 below). We will call this kind of uniqueness "*geometric uniqueness*" and that of Proposition 3.10 "*functional uniqueness*". It is clear that the latter implies the former.

To see where our stronger uniqueness comes from, we sketch the standard [1] proof for the homogeneous case. There, since only compatibility conditions (3.1) need to be satisfied, four conditions are imposed upon the initial data, namely, that on Ω we have $\Gamma^\alpha \equiv g^{\sigma\tau}\Gamma^\alpha_{\sigma\tau} = 0$ (i.e., harmonic coordinates are used). Then, instead of solving $\operatorname{Ricc}(g) = 0$, the hyperbolic equation $\operatorname{Ricc}^h(g) = 0$ is solved. The operator

Ricc^h is the expression for the Ricci tensor in terms of harmonic coordinates (see [6] for a proof that Ricc^h is indeed a hyperbolic operator). It is then shown that the harmonic condition $\Gamma^\alpha = 0$ is propogated off the initial surface (much as in our proof that $\mathrm{div}\, G(R) \equiv 0$) to complete the proof.

The loss of functional uniqueness in the homogeneous case comes from the facts that

(i) The zero tensor remains zero under the action of any diffeomorphism, and
(ii) $\mathrm{Ricc}(g) = 0$ does not imply $\mathrm{Ricc}^h(g) = 0$, nor vice versa in general, since the latter equation is not tensorial.

Fact (i) shows that, if g solve some Cauchy problem for $\mathrm{Ricc}(g) = 0$, then so does $\phi^*(g)$ for any diffeomorphism ϕ that preserves Ω and the Cauchy data. Fact (ii) shows why our uniqueness proof could not work in the homogeneous case. An interesting corollary of these observations is the following:

Corollary 3.11: *Let* $\mathrm{Ricc}(g) = R$ *with* R *invertible. If there is a diffeomorphism* ϕ *that preserves some spacelike hypersurface* S *and Cauchy data on* S *and such that* $\phi^*(R) = R$, *then* $\phi^*(g) = g$.

Remark 3.12: We remark that Theorem 3.6 *cannot* be proved using harmonic coordinates, because we cannot guarantee that the condition $\Gamma^a = 0$ is necessarily propogated off the initial hypersurface. We demonstrate this via the following example. Let ds^2 be the standard Riemannian Euclidean metric on $\mathbb{R}^3$, and set

$$g = -dt^2 + f(t)ds^2 \tag{3.13}$$

Then

$$\mathrm{Ricc}(g) = \left(-\frac{3f''}{2f} + \frac{3(f')^2}{4f^2}\right)dt^2 + \left(\frac{f''}{2} + \frac{(f')^2}{4f}\right)ds^2$$

Furthermore, $\Gamma^1 = \Gamma^2 = \Gamma^3 \equiv 0$, but $\Gamma^0 = (3f')/(2f)$. Thus, if $f(t) = 1 + t^2$, say, we have that $\mathrm{Ricc}(g)$ is nonsingular near $t = 0$, so (3.13) is the unique solution of the Cauchy problem for

$$\mathrm{Ricc}(g) = \frac{-3}{1+t^2}dt^2 + \frac{1+2t^2}{1+t^2}ds^2$$

with $g(0, x^i) = -dt^2 + ds^2$ and $\dfrac{\partial g}{\partial t}(0, x^i) = 0$. Furthermore, the coordinates are harmonic on the hypersurface $t = 0$, but nowhere else. Thus, Example 3.5, Proposition 3.10 and this result show that the geometry problem for $\mathrm{Ricc}(g) = R$ exhibits many features that are different from those of the corresponding physics problem.

4. Existence of Lorentz-Einstein metrics

This section provides a transition from the geometry material of the previous section to the physics of the next. An essential modification of the proof of Theorem 3.6 will be motivated here in a simple situation so that we can use it later.

As is well-known, an Einstein metric g is one for which

$$\mathrm{Ricc}(g) = c \cdot g \tag{4.1}$$

for some constant c. We fix c and examine the Cauchy problem for equation (4.1), where as before we must assume that our Cauchy data satisfies certain compatibility conditions. In fact, since $\mathrm{G}(g) = \dfrac{2-n}{2} g$, we have the following analog of condition (3.1):

$$(\mathrm{G}(\mathrm{Ricc}(g)))^0_\lambda = c \frac{2-n}{2} \delta^0_\lambda \tag{4.2}$$

on $t = 0$. The Bianchi identity does not constrain the initial data because $-\operatorname{div} \mathrm{G}(g) = 0$ for any metric g.

To solve (4.1), assuming (4.2) is satisfied, we would like to use the same approach as in section 3. However, we cannot use cg in place of R on the left side of equation (3.7), since the "correction terms" would then vanish a priori, leaving us with the original nonhyperbolic Ricci operator. Rather, we will introduce a "surrogate" Ricci tensor M, that is, any invertible tensor M that satisfies

$$\operatorname{div} \mathrm{G}(M) = 0 \qquad \text{on } t = 0 \tag{4.3}$$

where the operators in (4.3) are those of the "initial" metric $\bar{g}_{\mu\nu} = a_{\mu\nu}(x^i) + t b_{\mu\nu}(x^i)$, as discussed in section 3. The simplest way to pick such an M is to set

$$M_{\mu\nu} = a_{\mu\nu}(x^i) + t b_{\mu\nu}(x^i) + t^2 L_{\mu\nu}(t, x^i) \tag{4.4}$$

for any $L_{\mu\nu}$. It is then easily verified that this choice of $M_{\mu\nu}$ satisfies (4.3). We will have more to say about specific choices of M later. Once we have chosen M, we let g be the solution of the hyperbolic equation

$$\mathrm{Ricc}(g) - \mathrm{div}^*(M^{-1}\,\mathrm{div}\,\mathrm{G}(M)) = cg \tag{4.5}$$

on $(-\varepsilon', \varepsilon') \times \Omega'$ (notation of section 3). Then, just as in the proof of Theorem 3.6, the covector $u = M^{-1}\,\mathrm{div}\,\mathrm{G}(M)$ satisfies

$$\mathrm{div}\,\mathrm{G}(\mathrm{div}^*(u)) = 0$$

with $u_\alpha = 0$ and $\dfrac{\partial u_\alpha}{\partial t} = 0$ for $t = 0$. This proves

THEOREM 4.6: *Let Ω be a bounded subdomain of $t = 0$, and suppose that the initial data $g_{\alpha\beta}(0, x^i) = a_{\alpha\beta}(x^i) \in \mathrm{C}^k(\Omega)$ and $\partial g_{\alpha\beta}/\partial t(0, x^i) = b_{\alpha\beta}(x^i) \in \mathrm{C}^{k-1}(\Omega)$ satisfy the hyperbolicity conditions* (2.14) *and the compatibility conditions* (4.2) *for some constant c, and $k \geqq 4$. Then for some $\varepsilon' > 0$ and $\Omega' \subseteq \Omega$, there exists a C^k metric g that satisfies* $\mathrm{Ricc}(g) = cg$ *on $(-\varepsilon', \varepsilon') \times \Omega'$ and agrees with the initial data on $\{0\} \times \Omega'$.*

As before, we can state an H^s version of this result. However, there is no functional uniqueness for Einstein metrics as we had in Proposition 3.10. One would expect this since we had a lot of freedom to choose M in the proof of Theorem 4.6. In fact, if g solves the Cauchy problem for $\mathrm{Ricc}(g) = cg$, and ϕ is any diffeomorphism that leaves $t = 0$ and the Cauchy data fixed, then $\phi^*(g)$ is another solution of the problem. We can, however, state the following.

PROPOSITION 4.7: *The solution of the Cauchy problem for* $\mathrm{Ricc}(g) = cg$ *is geometrically unique.*

PROOF: This proof is based on the standard proof of geometric uniqueness in the homogeneous case (see [1] or [7]), and will carry over essentially without change to the physics problem discussed in section 5.

Let g_1 and g_2 be two solutions of the same Cauchy problem. Then as is well-known (for a proof see [7]), unique diffeomorphisms ϕ_1 and ϕ_2 can be found that preserve the surface $t = 0$ and for which the new coordinates for $\bar{g}_1 = \phi_1^*(g_1)$ and $\bar{g}_2 = \phi_2^*(g_2)$ are harmonic. That is, new coordinates $\bar{x}_1^\alpha$ and $\bar{x}_2^\alpha$ can be found with

$$\bar{x}_i^j = x^j \qquad \frac{\partial \bar{x}_i^j}{\partial t} = 0 \qquad \begin{array}{l} j = 1, 2, 3, \ldots, n-1 \\ i = 1, 2. \end{array} \tag{4.8}$$

on $t = 0$, and

$$\bar{t}_i = 0 \quad \frac{\partial \bar{t}_i}{\partial t} = 1 \quad i = 1, 2 \tag{4.9}$$

on $t = 0$, with $\bar{x}_1$ harmonic with respect to $\bar{g}_1$ and $\bar{x}_2$ harmonic with respect to $\bar{g}_2$. The initial data for the values of $g_{\alpha\beta}$ and for $\partial g_{ij}/\partial t$ remains unchanged when passing to barred coordinates, and from the transformation rule for Christoffel symbols and the fact that $\bar{\Gamma}^\alpha = 0$ (the harmonic condition), we have

$$\frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{t}} = \frac{\partial g_{\alpha\beta}}{\partial t} - \frac{g_{\sigma\tau}}{g^{00}} [\delta^0_\alpha \Gamma^\sigma \delta^\tau_\beta - \delta^0_\beta \Gamma^\tau \delta^\sigma_\alpha]$$

on Ω. Consequently, the Cauchy data in the new coordinates is completely determined by the original Cauchy data and by conditions (4.8) and (4.9). But then $\bar{g}_1$ and $\bar{g}_2$ are both solutions of the hyperbolic problem $\mathrm{Ricc}^h(g) = cg$ with the same Cauchy data, thus $\bar{g}_1 = \bar{g}_2$. If $\psi = \phi_2^{-1} \cdot \phi_1$, then $\psi^*(g_1) = g_2$. Q.e.d.

In the sense of this proposition, Theorem 4.6 is a more-or-less direct generalization of the Choquet-Bruhat homogeneous theory. Also, the geometric uniqueness of Lorentz–Einstein metrics in the natural analog of the analytic continuation theorem for Riemannian Einstein metrics proved in [6].

We remark that all the global results that apply to the homogeneous Ricci equation also apply to (4.1) (cf. [3]). For example, using a Zorn's Lemma argument, it can be shown that every set of Cauchy data satisfying (4.2) has a maximal development, and that such a maximal development is maximal for every spacelike hypersurface within it.

5. The Cauchy problem in general relativity

We give a very brief outline of the postulates at the foundation of general relativity, both in order to motivate our existence theorem, and also for the sake of briefly collecting all the relevant facts in a few paragraphs. The relevant physical background that we only sketch, can be found in [10], [18], [9], etc... To a certain extent, we follow the treatment in [10].

Two ingredients form the basis of the theory. First is the realization that "the world might not be flat", i.e., that the geometry of space-time might be different from that endowed by the standard Minkowskian metric of special relativity. Thus, the equations that define all matter and energy fields should be written in general covariant (coordinate invariant) form. Second is the fact that the metric of spacetime is not flat precisely because it is influenced by the fields present in the spacetime, i.e., the metric satisfies certain equations into which the fields (actually, the stress-energy tensor determined by the fields) must enter. Thus, the metric and the fields together are required to satisfy some coupled system of equations. So, the situation is as follows:

We have the values of a Lorentz metric $g_{\alpha\beta}$ and its normal derivative on an initial hypersurface in our spacetime manifold M (actually, the geometry of M will be determined by the evolution of the initial data), along with the various matter (energy) fields (and possibly their normal derivatives), which we will label $\psi_1, \psi_2, \ldots, \psi_r$. The functions ψ form a section in some tensor bundle over M, and are governed by a system of (partial differential) equations. We also have a rule for associating a stress-energy tensor $T(\psi_1, \psi_2, \ldots, \psi_r; g)$ to the matter fields, usually derived from a variational principle (see e.g. [10, Chapter 3]). The tensors T, g and the fields ψ_i must satisfy the following postulates:

(i) "Local causality". The hypothesis of local causality is an assumption about the form of the equations satisfied by the fields. Its content is that signals can propogate through space only at a finite speed, at most the speed of light. Essentially, we require that the matter fields satisfy partial differential equations that (with respect to a fixed metric) are well-determined hyperbolic differential equations whose characteristics lie on or within the light cone of the metric.

(ii) "Conservation of mass-energy". The fact that mass-energy is conserved is summarized in the equation

$$\operatorname{div}(T) = 0, \tag{5.1}$$

where T is the stress-energy tensor determined by the fields and the metric. This is again an assumption about the matter equations, namely that they should imply (5.1). Also, we must assume (with the physicists) that the stress-energy tensor is zero in any open set where all the fields are zero, i.e., that $T(0, 0, \ldots, 0; g) = 0$ for any metric g. Sometimes, the converse of this last statement is assumed (denying the existence of fields with "negative energy"), but this hypothesis is not required in our study of the Cauchy problem.

(iii) "The field equations". The final hypothesis concerns the way the metric depends on the matter fields; in many ways this is the centerpiece of general relativity. Einstein showed that the only tensorial expression that involves only derivatives of g up to second-order and is consistent with (5.1) is

$$\mathrm{G}(\mathrm{Ricc}(g)) + \Lambda g = T \tag{5.2}$$

where Λ is the "cosmological constant". In this section we will assume that Λ is zero (the discussion of the next section shows how to deal with nonzero Λ, and it will be clear that Theorem 5.5 is valid for any choice of Λ). For the purpose of proving existence for the Cauchy problem, we rewrite (5.2) as follows:

$$\mathrm{Ricc}(g) = \mathrm{G}^{-1}(T) = T - \frac{1}{n-2}(\mathrm{tr}\, T)g \tag{5.3}$$

(compare with (1.2)).

The hypotheses of relativity as given above make it plain how to set the stage for the Cauchy problem. Let

$$\Psi(\psi_1, \psi_2, \ldots, \psi_r; g) = 0 \tag{5.4}$$

be the matter equations, and assume that they imply (5.1). We must solve the Cauchy problem for the coupled system (5.3), (5.4), given appropriate initial data on some initial hypersurface we will call $t = 0$. The hypersurface should be rendered spacelike by the initial values of g. If (5.4) is second-order, we require values for $\psi_i(0, x^j)$ and $\partial\psi_i/\partial t\,(0, x^j)$, and we couple (5.4) with (5.3) (and T may depend on the first derivatives of the ψ_i as well as on the ψ_i). If (5.4) is first-order (symmetric hyperbolic), we require values only for $\psi_i(0, x^j)$, and recast (5.3) as a symmetric first-order system (it will become hyperbolic later) as in [7]. For the purpose of proving Theorem 5.5, we assume that (5.4) is second-order.

THEOREM 5.5: *Let Ω be a bounded subdomain of $t = 0$, and suppose that the initial data $g_{\alpha\beta}(0, x^i) = a_{\alpha\beta}(x^i) \in \mathrm{C}^k(\Omega)$ and $\partial g_{\alpha\beta}/\partial t\,(0, x^i) = b_{\alpha\beta}(x^i) \in \mathrm{C}^{k-1}(\Omega)$ satisfy the hyperbolicity conditions* (2.14), *and that the initial data $\psi_j(0, x^i) \in \mathrm{C}^k(\Omega)$ and $\partial\psi_j/\partial t\,(0, x^i) \in \mathrm{C}^{k-1}(\Omega)$ are given with $k \geqq 4$. The initial data for g and ψ must satisfy the compatibility condition*

$$(\mathrm{G}(\mathrm{Ricc}(g)))^0_\lambda = T^0_\lambda(\psi; g) \qquad \text{on } t = 0.$$

Also, suppose that the matter equations (5.4) *satisfy the postulates* (i) *and* (ii) *above. Then for some* $\varepsilon > 0$ *and* $\Omega' \subseteq \Omega$, *there exists a* C^k *metric* g *and fields* ψ_j *that satisfy the system* (5.3) *and* (5.4) *on* $(-\varepsilon, \varepsilon) \times \Omega'$ *and agree with the initial data on* $\{0\} \times \Omega'$.

PROOF: The technique is that of Theorem 4.6. We pick a "surrogate" Ricci tensor M as in the proof of Theorem 4.6 that satisfies $\operatorname{div} G(M) = 0$ on $t = 0$. Once again, the tensor (4.4) will do nicely for the purpose. We then solve the system (5.4) coupled with

$$\operatorname{Ricc}(g) - \operatorname{div}^*(M^{-1}(\operatorname{div} G(M))) = G^{-1}(T(\psi; g)) \tag{5.6}$$

This hyperbolic system has a solution by standard theory (i.e., see [11], [13], or at worst, the Cauchy–Kovalevski theorem for analytic data), and the conservation postulate (ii) guarantees that we can apply Lemma 2.17 to show that $u = M^{-1} \operatorname{div} G(M)$ is identically zero, since when we take $\operatorname{div} G(\cdot)$ of both sides of (5.6), we will get $\operatorname{div} G \operatorname{div}^*(u) = 0$. Q.e.d.

REMARKS: (1) Once again, we can use standard H^s theories [7] to produce an H^s result, and the usual maximality theorems [3] hold for the solution.

(2) By appealing to harmonic coordinates as in the proof of Proposition 4.7, it is easily shown that the solution of the Cauchy problem for (5.3), (5.4) is geometrically unique. The fact that the solution is only geometrically unique, and the freedom to choose M before constructing the solution are the reasons for referring to the physics problem as underdetermined.

(3) It is interesting to compare the formulation and proof of the theorem above to the usual one which uses harmonic coordinates. The above proof has the advantage of not placing extra constraints upon the initial data, a feature that might be desirable in perturbation theory or in a numerical analysis of the problem. In fact, the system (5.4)–(5.6) is strictly hyperbolic so that more-or-less standard numerical methods might be applicable. Smarr [17] has noted that harmonic coordinates are not particularly well-suited to numerical computations.

(4) Finally, we note that, given any initial data, we can "force" it to evolve so that the background coordinates become harmonic after an arbitrary time τ, provided that the problem has a solution for $t > \tau$ as follows. First, we note that from equations (2.8) and (2.9), we have

$$\begin{aligned}(M^{-1}(\operatorname{div} G(M))) &= \tfrac{1}{2} g_{\alpha\lambda} g^{\mu\nu} (M^{-1})^{\alpha\beta} \left(\frac{\partial M_{\mu\beta}}{\partial x^\nu} + \frac{\partial M_{\nu\beta}}{\partial x^\mu} - \frac{\partial M_{\mu\nu}}{\partial x^\beta} \right) - g_{\alpha\lambda} \Gamma^\alpha \\ &= g_{\alpha\lambda} g^{\mu\nu} [{}_{(M)}\Gamma^\alpha_{\mu\nu} - {}_{(g)}\Gamma^\alpha_{\mu\nu}] \end{aligned} \tag{5.7}$$

where the latter Γ-notation indicates from which "metric" the Christoffel symbols come (clearly, M can be considered a pseudo-Riemannian metric). If we choose M so that $\operatorname{div} G(M) = 0$ for $t = 0$ with respect to the initial data, and also so that M is flat (as a metric) for all $t > \tau$, then (5.7) will imply that ${}_{(g)}\Gamma^\alpha = 0$ for $t > \tau$, i.e., that the coordinates are harmonic. To construct such an M, we let $\delta_{\alpha\beta}$ be the Lorentz–Kronecker delta, and let $\kappa(t)$ be any function such that $\kappa(t) \equiv 0$ if $t < \frac{1}{4}\tau$ and $\kappa(t) \equiv 1$ if $t > \frac{1}{2}\tau$. Following (4.4), we set

$$M_{\alpha\beta}(t, x^i) = a_{\alpha\beta}(x^i) + tb_{\alpha\beta}(x^i) + \kappa(t)[\delta_{\alpha\beta} - a_{\alpha\beta}(x^i) - tb_{\alpha\beta}(x^i)]$$

Clearly, this M satisfies all the necessary requirements.

We conclude this section with an illustration of the Theorem applied to the simplest example of an Einstein system system coupled to a matter equation, namely that of a single scalar field. For other examples, the reader may refer to [10], [15] or [2].

The matter equation that should be satisfied by our scalar field ψ is

$$g^{\alpha\beta}\psi_{;\alpha\beta} - \frac{m^2}{h^2}\psi = 0 \tag{5.8}$$

and the stress-energy tensor is

$$T_{\alpha\beta} = \psi_{;\alpha}\psi_{;\beta} - \tfrac{1}{2}g_{\alpha\beta}\left(\psi_{;\mu}\psi_{;\nu}g^{\mu\nu} + \frac{m^2}{h^2}\psi^2\right) \tag{5.9}$$

(see [10, p. 67]). Clearly, (5.8) and (5.9) satisfy postulates (i) and (ii) of relativity, since $\operatorname{div}(T)_\lambda = -(g^{\alpha\beta}\psi_{;\alpha\beta} - m^2\psi/h^2)\psi_{;\lambda} = 0$. If we substitute (5.9) into (5.6), and couple the resulting equation with (5.8), we have a standard, second-order quasilinear hyperbolic system, whose principal part is uncoupled. Local existence in time of a solution is then guaranteed by any of the standard quasilinear hyperbolic theories.

6. Other related equations

Using the techniques of the previous sections, we can discuss existence and uniqueness of solutions to the Cauchy problem for several equations that are closely related to the geometric equation (1.1) or to the physical equation (1.2). We begin with an example that is, in a sense, halfway between. We suppose that the invertible tensor $S_{\alpha\beta}(t, x)$ and the scalar function $f(t, x)$ are given in a neighborhood of $t = 0$ and try to solve the Cauchy problem for

$$\mathrm{Ricc}(g) = S_{\alpha\beta} + f(t, x)g_{\alpha\beta} \tag{6.1}$$

Since we will have to take $\operatorname{div} \mathrm{G}(\cdot)$ of both sides of (some variation of) equation (6.1) in our existence proof, the following calculation will be useful:

$$\operatorname{div} \mathrm{G}(f(t, x)g) = \frac{n-2}{2} df \tag{6.2}$$

where d denotes exterior differentiation (This equation has as a consequence the familiar fact that $\mathrm{Ricc}(g) = fg$ implies that f is a constant for $n \geqq 3$, by the Bianchi identity). The important thing about (6.2) is that the right-hand side does not involve g, since exterior differentiation does not depend upon the metric structure of the manifold. So, we consider the auxiliary system

$$\mathrm{Ricc}(g) - \operatorname{div}^* \mathrm{S}^{-1}[\operatorname{div} \mathrm{G}(S) + \frac{n-2}{2} df] = S + fg \tag{6.3}$$

This system is hyperbolic, and if our initial data for g satisfies the hyperbolicity conditions (2.14) and the compatibility condition:

$$(\mathrm{G}(\mathrm{Ricc}(g)))^0_\lambda = (\mathrm{G}(fg + S))^0_\lambda \quad \text{on } t = 0 \tag{6.4}$$

and

$$\operatorname{div} \mathrm{G}(fg + S) = 0 \quad \text{on } t = 0 \tag{6.5}$$

then the solution of the Cauchy problem for (6.3) whose existence is guaranteed by standard theory is indeed the (functionally unique!) solution of (6.1). In fact, if $u = S^{-1}\left[\operatorname{div} \mathrm{G}(S) + \frac{n-2}{2} df\right]$, then taking $\operatorname{div} \mathrm{G}(\cdot)$ of both sides of (6.3) yields $\operatorname{div} \mathrm{G} \operatorname{div}^*(u) = -Su$. As usual, conditions (6.4) and (6.5) allow us to use Lemma 2.17 to conclude that $u \equiv 0$. We have proved

THEOREM 6.6: *If the initial data for g satisfy the usual conditions* (2.14), (6.4) *and* (6.5), *then the Cauchy problem for* $\mathrm{Ricc}(g) = S + fg$ *has a unique solution, if S is invertible.*

We remark that functional uniqueness for (6.1) is proved in exactly the same manner as Proposition 3.10.

As a second example, we consider a truly inhomogeneous version of the physics problem of section 5. We consider the equations of the matter fields (5.4), and form the stress-energy tensor T from them as usual so as to satisfy local causality and conservation of mass-energy. However, we then assume the existence of some "external" force, that is independent of the fields in (5.4), and that sets up a nonsingular Ricci tensor field $R_{\alpha\beta}$. We consider the coupled system that consists of (5.4) and

$$\mathrm{Ricc}(g) = R_{\alpha\beta} + \mathrm{G}^{-1}[T_{\alpha\beta}(\psi_1, \ldots, \psi_r; g)] \tag{6.7}$$

As usual, we replace (6.7) with a hyperbolic system:

$$\mathrm{Ricc}(g) - \operatorname{div}^* R^{-1}(\operatorname{div}\mathrm{G}(R)) = R + \mathrm{G}^{-1}(T) \tag{6.8}$$

The coupled system (5.4), (6.8) is a well-determined hyperbolic system. If the initial data satisfy

$$\left.\begin{aligned} &\operatorname{div}\mathrm{G}(R) = 0 \\ &(\mathrm{G}(\mathrm{Ricc}(g)))^0_\lambda = \mathrm{G}(R)^0_\lambda + T^0_\lambda \end{aligned}\right\} \quad \text{on } t = 0$$

then, as usual, the solution of the auxiliary hyperbolic system (5.4), (6.8) will satisfy the original system (5.4), (6.7). Thus, we have an existence theorem for an inhomogeneous Einstein equation. The solution is functionally unique.

Our final example is related to the truly inhomogeneous Einstein equation, i.e., we consider the system

$$\mathrm{Ricc}(g) = T_{\alpha\beta} + f(t, x, \operatorname{tr} T)g_{\alpha\beta} \tag{6.9}$$

where $T_{\alpha\beta}$ is some given invertible tensor, and f is a scalar function of its $n+1$ variables. In the actual Einstein equation, $f(t, x, \operatorname{tr} T) = -\frac{1}{2}\operatorname{tr} T$. We will demonstrate that the auxiliary system

$$\mathrm{Ricc}(g) - \operatorname{div}^* T^{-1}\left(\operatorname{div}\mathrm{G}(T) + \frac{n-2}{2}\,df\right) = T + fg \tag{6.10}$$

is strictly hyperbolic. To do this is more complicated than for (6.3), since now g is involved in tr T; so the correction terms contain second-order derivatives of g other than those needed to cancel the bad terms in the linearization of Ricc(g). Another complication is that the system (6.10), although hyperbolic, does not appear to be symmetric hyperbolic (in the sense of [11], say), and we have not been able to apply any C^k or H^s theory to it. Thus, our existence theory for (6.9) is limited to the analytic case (i.e., T, f, and the initial data must be real analytic). Of course, given that (6.10) is hyperbolic, and that its characteristics are precisely the null vectors of the metric g, it is straightforward to use our technique to show that any solution of (6.10) is also a solution of (6.9), provided the initial data satisfies

$$\left.\begin{aligned} &\operatorname{div} G(T) + \frac{n-2}{2} df = 0 \\ &(G(\operatorname{Ricc}(g)))^0_\lambda = G(T + df)^0_\lambda \end{aligned}\right\} \quad \text{on } t = 0$$

Thus, we concentrate on the proof of

PROPOSITION 6.11: *The system* (6.10) *is strictly hyperbolic, with characteristics coinciding with the null vectors of g, provided $\partial f/\partial(\operatorname{tr} T) < \dfrac{1}{n-2}$.*

PROOF: We already know from the discussion surrounding equation (2.12) that the principal part of the linearization of $\operatorname{Ricc}(g) - \operatorname{div}^* T^{-1}(\operatorname{div} G(T))$ is $-\frac{1}{2}g^{\sigma\tau}h_{\alpha\beta;\sigma\tau}$, so we need only compute the linearization of

$$-\operatorname{div}^* T^{-1}\left(\frac{n-2}{2} df\right) \tag{6.12}$$

The terms with second derivatives of g come only from the dependence of f upon $\operatorname{tr} T = g^{\sigma\tau}T_{\sigma\tau}$. Since the linearization of $\operatorname{tr} T$ is $-g^{\sigma\mu}h_{\mu\nu}g^{\nu\tau}T_{\sigma\tau}$, the principal part of the linearization of (6.12) at t_0, x_0 and the metric g is

$$\frac{n-2}{2}\Lambda\left(S^\gamma_\alpha g^{\sigma\mu}g^{\nu\tau}T_{\sigma\tau}\frac{\partial^2 h_{\mu\nu}}{\partial x^\beta \partial x^\gamma} + S^\gamma_\beta g^{\sigma\mu}g^{\nu\tau}T_{\sigma\tau}\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\gamma}\right)$$

where S is T^{-1}, i.e., $S^{\alpha\beta}T_{\beta\gamma} = \delta^\alpha_\gamma$, and Λ is the value of $\dfrac{\partial f}{\partial(\operatorname{tr} T)}$ at t_0, x_0, and g. Thus, the principal symbol of (6.10) at t_0, x_0 and g is

$$\begin{aligned} \sigma^{(6.10)}(\xi)h &= -\tfrac{1}{2}g^{\sigma\tau}h_{\alpha\beta}\xi_\sigma\xi_\tau + \frac{n-2}{2}\Lambda T^{\mu\nu}(S^\gamma_\alpha h_{\mu\nu}\xi_\beta\xi_\gamma + S^\gamma_\beta h_{\mu\nu}\xi_\alpha\xi_\gamma) \\ &= -\tfrac{1}{2}(g^{\sigma\tau}\xi_\sigma\xi_\tau)h_{\alpha\beta} + \frac{n-2}{2}\Lambda[(S^\gamma_\alpha\xi_\beta\xi_\gamma + S^\gamma_\beta\xi_\alpha\xi_\gamma)\otimes T^{\mu\nu}]h_{\mu\nu} \end{aligned} \tag{6.13}$$

From this expression, we see that $\sigma^{(6.10)}(\xi)$ is a linear operator of the form $c\mathrm{I} + \frac{n-2}{2} \Lambda p \otimes q$, where I is the identity mapping of S^2T^*, $c = -\frac{1}{2}|\xi|^2$, $p = S^{\gamma}_{\alpha}\xi_{\beta}\xi_{\gamma} + S^{\gamma}_{\beta}\xi_{\alpha}\xi_{\gamma} \in S^2T^*$, and $q = T^{\mu\nu} \in (S^2T^*)^*$. In other words, $\sigma^{(6.10)}(\xi)$ is just a rank-one perturbation of (the constant $c = -\frac{1}{2}|\xi|^2$ times) the identity. We will use the following lemma from linear algebra.

LEMMA 6.14: *An operator of the form $M = c\mathrm{I} + kp \otimes q$ is invertible if and only if $c \neq 0$ and $k(\operatorname{tr} p \otimes q) \neq -c$.*

PROOF: We calculate

$$M^2 = c^2\mathrm{I} + 2ckp \otimes q + k^2(\operatorname{tr} p \otimes q)p \otimes q$$

Let $t = \operatorname{tr} p \otimes q$, then

$$\begin{aligned} M^2 &= c^2\mathrm{I} + (2c + kt)kp \otimes q \\ &= (2c + kt)M - (c^2 + ckt)\mathrm{I} \end{aligned}$$

Thus,

$$\frac{((2c + kt)\mathrm{I} - M)}{c^2 + ckt} \cdot M = \mathrm{I}$$

provided $c^2 + ckt \neq 0$, i.e., $c \neq 0$ and $kt \neq -c$. Q.e.d. lemma.

We relate the lemma to (6.13). There, we have an operator in the form discussed in the lemma with $c = -\frac{1}{2}|\xi|^2$, which is zero only on the light cone of the metric (i.e., on the characteristics). The trace of $p \otimes q$ is just their inner product:

$$t = (S^{\gamma}_{\mu}\xi_{\nu}\xi_{\gamma} + S^{\gamma}_{\nu}\xi_{\mu}\xi_{\gamma})T^{\mu\nu} = |\xi|^2$$

So $kt \neq -c$ means

$$\frac{n-2}{2} \Lambda |\xi|^2 \neq \tfrac{1}{2}|\xi|^2$$

that is, $\Lambda \neq \frac{1}{n-2}$. Thus, if $\Lambda \neq \frac{1}{n-2}$, the symbol (6.13) is singular only in the characteristic (light cone) directions. This completes the proof of the propostion. Q.e.d.

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