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THE NON-ARCHIMEDEAN SPACE $\mathscr{C}^\infty(X)$

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Abstract

The non-archimedean space $\mathscr{C}^\infty(X)$ is a non-archimedean nuclear space.

Introduction

The classical nuclear space $\mathscr{C}^\infty(X)$, of infinitely differentiable functions on X (X open subset of $\mathbb{R}$) has been studied for many years. In the case of functions $f: X \to K$, where X is a subset of a non-archimedean (n.a.) valued field K the definition and investigation of such spaces present some problems which have only recently been overcome (see [4]).

In this paper we show that the n.a. space $\mathscr{C}^\infty(X)$ as defined in [4] is nuclear in the n.a. sense. We also pay attention to the case where $X = \mathbb{Z}_p$ (the p-adic integers). The paper starts with some additional information on compactoïd subsets of n.a. locally convex spaces, and a n.a. version of the Ascoli theorem.

The following notions and notations will be used:

- K is a complete field with a (non trivial) n.a. valuation and X is a subset of K ($X \neq \phi$) without isolated points.
- $\mathscr{C}(X)$ is the space of continuous functions

$$f: X \to K.$$

If X is compact then $\mathscr{C}(X)$ is a n.a. Banach space for the norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)|, f \in \mathscr{C}(X)$$

- If E is a n.a. locally convex space over K then $\mathscr{U}_E$ is a fundamental system of zero-neighborhoods in E and $\mathscr{P}_E$ is a family of n.a. semi-norms determining the topology of E.
- For $p \in \mathscr{P}_E$ we denote by E, p the space E semi-normed by p, by E_p the associated n.a. normed space, $E_p = E, p/p^{-1}(0)$ and by $\hat{E}_p$ its completion.
  By $\varphi_{p,q}$ we denote the canonical (continuous linear) mapping

$$\varphi_{p,q} : \hat{E}_q \to \hat{E}_p, \ (q \geq p).$$

- If A is a subset of E then

$$C(A) = \{\sum_{i=1}^{n} \lambda_i a_i \ \ a_i \in A, |\lambda_i| \leq 1\}$$

$$= K\text{-convex hull of } A.$$

§1. Preliminaries on compactoïd sets

1.1. Definition: A subset B of a n.a. locally convex space $E, \mathscr{U}_E$ is called compactoïd if for every $U \in \mathscr{U}_E$ there exists a finite set $S \subset E$ such that $B \subset C(S) + U$.

1.2. Proposition: *Let E be a n.a. Banach space and F a closed subspace of E. Suppose B is a subset of F which is compactoïd as a subset of E. Then B is compactoïd in F.*

Proof: This follows from [6] Theorem 4.37 ($\alpha) \Rightarrow \eta$)).

1.3. Lemma: *Let E be a n.a. normed space and B a subset of E. Then B is compactoïd in E if and only if B is compactoïd in $\hat{E}$ (the completion of E).*

Proof: The "only if" part is immediate from the definition. Suppose now that B is compactoïd in $\hat{E}$ and let

$$B_\varepsilon \in \mathscr{U}_E (B_\varepsilon = B(0, \varepsilon) = \{x \in E \ \ \|x\| \leq \varepsilon\})$$

Then $\bar{B}_\varepsilon^{\hat{E}}$ (the closure of B_ε in $\hat{E}$) is a zero-neighbourhood in $\hat{E}$. Hence

$\exists x_1, x_2, \ldots, x_n \in \hat{E}$ such that

$$B \subset C(\{x_1, x_2, \ldots, x_n\}) + \bar{B}_\varepsilon^{\hat{E}}$$

Choose $y_1, y_2, \ldots, y_n \in E$ such that

$$\|y_i - x_i\|_{\hat{E}} < \varepsilon,\ i = 1, 2, \ldots, n.$$

Take $a \in B$. Then a can be written as

$$a = \sum_{i=1}^{n} \lambda_i x_i + \bar{b},\ \text{with}\ |\lambda_i| \leq 1, i = 1, 2, \ldots, n$$

and $\bar{b} \in \bar{B}_\varepsilon^{\hat{E}}$. Put $b_1 = \sum_{i=1}^n \lambda_i(x_i - y_i) + \bar{b}$. Then

$$\|b_1\|_{\hat{E}} \leq \max\{\max_{i=1,\ldots,n} |\lambda_i|\, \|x_i - y_i\|_{\hat{E}}, \|\bar{b}\|_{\hat{E}}\} \leq \varepsilon$$

But $b_1 = a - \sum_{i=1}^n \lambda_i y_i \in E$. So $\|b_1\|_{\hat{E}} = \|b_1\|_E \leq \varepsilon$, or $b_1 \in B_\varepsilon$.

Finally $a = \sum_{i=1}^n \lambda_i y_i + b_1$, so $B \subset C(\{y_1, \ldots, y_n\}) + B_\varepsilon$, which shows that B is compactoïd in E.

1.4. Proposition: *Let E be a n.a. normed space, F a subspace of E and E a subset of F.*
Then B is compactoïd in F if and only if B is compactoïd in E.

Proof: The "only if" part is trivial
Suppose now that B is compactoïd in E.
Then B is compactoïd in $\hat{E}$ (1.3). Now $\hat{F}$ is closed in $\hat{E}$ and $B \subset \hat{F}$. Hence (1.2) B is compactoïd in $\hat{F}$ and therefore in F (1.3).

1.5. Lemma: *Let E, p be a n.a. semi-normed space and E_p the associated normed space. Denote by π the canonical surjection $\pi: E, p \to E_p$ and let B be a subset of E.*
Then B is compactoïd in E, p if and only if B is compactoïd in E_p.

Proof: If B is compactoïd in E, p then $\pi(B)$ is compactoïd in E_p by the continuity of π.
Suppose now that $\pi(B)$ is compactoïd in E_p.
The unit ball in E_p is given by $\pi(A)$ where

$$A = \{x \in E \ \ p(x) \leq 1\}$$

Choose $\varepsilon > 0$, then there exist $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \in E_p$ $(\bar{x}_i = \pi(x_i))$ such that

$$\pi(B) \subset C(\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}) + \pi(\varepsilon A)$$

For $a \in B$ we can write $\bar{a} = \pi(a)$ as $\bar{a} = \sum_{i=1}^{n} \lambda_i \bar{x}_i + \bar{b}$, with $|\lambda_i| \leq 1$, $i = 1, 2, \ldots, n$; $\bar{b} \in \pi(\varepsilon A)$ or $\bar{a} = \pi(\sum_{i=1}^{n} \lambda_i x_i + b)$, $b \in \varepsilon A$, $\pi(b) = \bar{b}$. Hence $p(a - \sum_{i=1}^{n} \lambda_i x_i - b) = 0$, which implies that $a - \sum_{i=1}^{n} \lambda_i x_i - b = C \in \varepsilon A$ or $a = \sum_{i=1}^{n} \lambda_i x_i + d$, with $d \in \varepsilon A$. Hence $B \subset C(\{x_1, x_2, \ldots, x_n\}) + \varepsilon A$.

1.6. Proposition: *Let E, $\mathscr{P}_E$ be a n.a. locally convex space, F a subspace of E and B a subset of F.*
Then B is compactoïd in F if and only if B is compactoïd in E.

Proof: The "only if" part is trivial.
Suppose now that B is compactoïd in E. Then B is compactoïd in E, p for all $p \in \mathscr{P}_E$, and $\pi_p(B)$ is compactoïd in $E_p(Vp \in \mathscr{P}_E)$, where π_p: $E, p \to E_p$ is the canonical surjection. (See (1.5)). Denote still by p the restriction of p to F. Then F_p can be identified with a subspace of E_p. Since $\pi_p(B) \subset F_p$, we have by (1.4) that $\pi_p(B)$ is compactoïd in F_p. By (1.5), B is compactoïd in F, p. Since this holds for all $p \in \mathscr{P}_E$, we conclude that B is compactoïd in F.

1.7. Proposition: *Let $(E_i)_{i \in I}$ be a family of n.a. normed spaces and let A_i be a subset of $E_i (i \in I)$. Then $A = \prod_{i \in I} A_i$ is compactoïd in $E = \prod_{i \in I} E_i$ if and only if A_i is compactoïd in E_i for all $i \in I$.*

Proof: If A is compactoïd in E then A_i is compactoïd in E_i by the continuity of the projections. Denote by $\mathscr{U}^i$ a fundamental system of zero-neighbourhoods in E_i and put

$$\mathscr{U}_{i_1, \ldots, i_n} = \prod_{i \in I} V_i \text{ where } V_i \in \mathscr{U}^i \ i \in \{i_1, i_2, \ldots, i_n\}$$
$$V_i = E_i \text{ elsewhere.}$$

Then $\mathscr{U} i_1, \ldots, i_n$ is a zero-neighbourhood in E. Now there exist $y_1^{i_j}, y_2^{i_j}, \ldots, y_m^{i_j} \in E_{i_j}$ $(j = 1, 2, \ldots, n)$ such that

$$A_{i_j} \subset C(\{y_1^{i_j}, y_2^{i}, \ldots, y_n^{i_j}\}) + V_{i_j}.$$

Put $X_k^{i_j} = (x_i)_{i \in I} \in E$ with $x_{i_j} = y_k^{i_j}, x_i = 0 \ \ (i \neq i_j)$, $k = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Then $A \subset C(S) + \mathscr{U}_{i_1, i_2, \ldots, i_n}$ where $S = \{X_k^{i_j}\ k = 1,2,\ldots,m; j = 1,2,\ldots,n\}$, which shows that A is compactoïd in E.

1.8. THEOREM (Ascoli): *Let X be a compact subset of K, and let A be an equicontinuous, pointwise bounded subset of $\mathscr{C}(X)$.*
Then A is compactoïd in the n.a. Banach space $\mathscr{C}(X)$, $\|.\|_\infty$.

PROOF: Take $\varepsilon > 0$.
Then for all $x \in X$, there exists an open neighbourhood U_x of x in X such that $|f(x) - f(y)| < \varepsilon$, $\forall y \in U_x$, $\forall f \in A$.
The family $(U_x)_{x \in X}$ covers X and since X is compact there is a finite subcover of X, say $X \subset \bigcup_{i=1,2,\ldots,n} U_{x_i}$.
Now $A(x_i) = \{f(x_i)\ f \in A\}$ is bounded and therefore compactoïd in K. $(i = 1,2,\ldots,n)$.
Hence there exist $\alpha_1^i, \alpha_2^i, \ldots, \alpha_m^i \in K$ such that

$$A(x_i) \subset C(\{\alpha_1^i, \alpha_2^i, \ldots, \alpha_m^i\}) + B(0,\varepsilon),\ i = 1,2,\ldots,n$$

Let $\xi_{U_{x_i}}$ be the characteristic function of $\mathscr{U}_{x_i}$ and put $f_j^i = \alpha_j^i \xi_{U_{x_i}} \in C(X)$, $i = 1,\ldots,n; j = 1,\ldots,m$.
Let $S = \{f_j^i\ i = 1,\ldots,n; j = 1,\ldots,m\}$. We shall prove that

$$A \subset C(S) + B_\varepsilon, \text{ with } B_\varepsilon = \{g \in \mathscr{C}(X)\ \|g\|_\infty \leq \varepsilon\} \tag{1}$$

Take $f \in A$ and $x \in X$.
Then $x \in U_{x_i}$ for some $i \in \{1,2,\ldots,n\}$. Now $f(x) = f(x_i) + [f(x) - f(x_i)]$, where $f(x_i)$ can be written as $f(x_i) = \sum_{j=1}^m \lambda_j^i \alpha_j^i + \mu$ with $|\lambda_j^i| \leq 1$ and $|\mu| \leq \varepsilon$.
Moreover $|f(x) - f(x_i)| < \varepsilon$ and $\alpha_j^i = f_j^i(x)$. So $|f(x) - \sum_{j=1}^m \lambda_j^i f_j^i(x)| \leq \varepsilon$, $x \in U_{x_i}$.
Finally, putting $g(x) = f(x) - \sum_{j=1}^m \lambda_j^i f_j^i(x)$, $x \in U_{x_i}$. We have $g \in \mathscr{C}(X)$ and

$$\|g\|_\infty = \sup_{x \in X} |g(x)| = \sup_{i=1,\ldots,n} \sup_{x \in U_{x_i}} |g(x)| \leq \varepsilon$$

So $g \in B_\varepsilon$ and $f(x) = \sum_{j=1}^m \lambda_j^i f_j^i(x)$, $x \in X$, which proves (1).

§2. The spaces $\mathscr{C}^n(X)$ (X compact)

2.1. Definitions and notations (see [4]): Let X be a compact subset of K. We put $\mathscr{C}^0(X) = C(X)$, $\|.\|_\infty$. For $n \geq 1$ let

$$\nabla^n X = \{(x_1, x_2, \ldots, x_n) \in X^n \;\; x_i \neq x_j \text{ whenever } i \neq j\}$$

For $f: X \to K$ define $\Phi_n(f): \nabla^{n+1} X \to K$ by induction as follows: $\phi \circ f = f$ and for $n \geq 1$:

$$\Phi_n f(x_1, \ldots, x_{n+1}) = \frac{\Phi_{n-1} f(x_1, x_3, x_4, \ldots, x_{n+1}) - \Phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}$$

$\phi_n f$ is called the n^{th} difference quotient of f.

The function f is said to be n times continuously differentiable ($f \in \mathscr{C}^n(X)$) if the function $\phi_n f$ can (uniquely) be extended to a continuous function $\bar{\phi}_n f$ on X^{n+1}.

2.2. Theorem:
(i) $C(X) \supset \mathscr{C}^1(X) \supset \mathscr{C}^2(X) \supset \ldots$ ([4] *p.* 78).
(ii) *For every n the space* $\mathscr{C}^n(X)$ *is a n.a. Banach space for the norm*

$$\|f\|_n = \max_{k=0,1,\ldots,n} \|\bar{\phi}_k(f)\|_\infty, \; f \in \mathscr{C}^n(X)$$

([4] p. 79).

2.3. Definition: Let E and F be n.a. Banach spaces. A continuous linear mapping $T: E \to F$ is called compact if the image $T(B_E)$, of the unit ball in E, is compactoïd in F.

2.4. Theorem: *The canonical injections*
$\mathscr{C}^n(X) \to \mathscr{C}^{n-1}(X)$, $n = 1, 2, 3, \ldots$,
are compact.

Proof: Let A be the unit ball in $\mathscr{C}^n(X)$. We prove that A is compactoïd in $\mathscr{C}^{n-1}(X)$.
For $i = 0, 1, \ldots, (n-1)$ we consider the mapping $T_i: \mathscr{C}^{n-1}(X) \to$ $\to \mathscr{C}(X^{i+1}): f \to \bar{\phi}_i f$. Then:
(1) $T_i(A)$ *is pointwise bounded* ($i = 0, 1, \ldots, (n-1)$).

Indeed: for $p \in X^{i+1}$ we have

$$\sup_{f\in A} |\bar{\phi}_i f(p)| \le \sup_{f\in A} \|\bar{\phi}_i f\|_\infty \le \sup_{f\in A} \|f\|_{n-1} \le 1$$

(2) *$T_i(A)$ is equicontinuous* $(i = 0, 1, \ldots, (n-1))$
Indeed:
For $f \in A$, $(x_1, x_2, \ldots, x_{i+1}, a_1, a_2, \ldots, a_{i+1}) \in \nabla^{2i+2}$

$$p = (x_1, x_2, \ldots, x_{i+1}), q = (a_1, a_2, \ldots, a_{i+1}) \tag{*}$$

We have ([4] p. 76)

$$\phi_i f(p) - \phi_i f(q) = \sum_{j=1}^{i+1} (x_j - a_j)\phi_{i+1} f(a_1, \ldots, a_j, x_j, \ldots, x_{i+1})$$

So

$$\begin{aligned} &|\phi_i f(p) - \phi_i f(q)| \\ &\le \max_{j=1,\ldots,(i+1)} \{|x_j - a_j| . |\phi_{i+1} f(a_1, \ldots, a_j, x_j, \ldots, x_{i+1})|\} \\ &\le \max_{j=1,\ldots,(i+1)} |x_j - a_j| \cdot \max_{j=1,\ldots,(i+1)} |\phi_i f(a_1, \ldots, a_j, x_j, \ldots, x_{i+1})| \end{aligned}$$

Now $\max_{j=1,\ldots,(i+1)} |x_j - a_j| = \|p - q\|$ in X^{i+1} and for $j = 1, \ldots, (i+1)$

$$|\phi_{i+1} f(a_1, \ldots, a_j, x_j, \ldots, x_{i+1})| \le \|\bar{\phi}_{i+1} f\|_\infty \le \|f\|_n \le 1$$

This gives

$$|\phi_i f(p) - \phi_i f(q)| \le \|p - q\| \text{ for } p \text{ and } q \text{ as in } (*)$$

and for all $f \in A$.
By the definition of $\bar{\phi}_i$ we finally obtain

$$(\bar{\phi}_i f(p) - \bar{\phi}_i f(q)| \le \|p - q\|, \forall p, q \in X^{i+1}, \forall f \in A,$$

from which (2) follows.
Applying (1.8) we conclude that $T_i(A)$ is compactoïd in $\mathscr{C}(X^{i+1})$, $i = 0, 1, \ldots, (n-1)$.
By (1.7) we then have that the set $\prod_{i=0,1,\ldots,n-1} T_i(A)$ is compactoïd in $\mathscr{C}(X) \times \mathscr{C}(X^2) \times \ldots \times \mathscr{C}(X^n) = E$.

We now define $T:\mathscr{C}^{n-1}(X) \to E$ by $T(f) = (T_i f)_{i=0,1,\ldots,(n-1)}$.
Then $T(A) \subset \prod_{i=0,1,\ldots,(n-1)} T_i(A)$, which implies that $T(A)$ is compactoïd in E.
Since each T_i is linear ([4] p. 76), T is linear as well. So $T(\mathscr{C}^{n-1}(X))$ is a subspace of E.
Since $T(A) \subset T(\mathscr{C}^{n-1}(X))$ we have by (1.6) that $T(A)$ is a compactoïd subset of $T(\mathscr{C}^{n-1}(X))$, with the topology induced by E.

The proof will be complete if we show that $T(\mathscr{C}^n(X))$ and $\mathscr{C}^{n-1}(X)$ are isometric. This is immediate from

$$\|T(f)\|_E = \max_{i=0,1,\ldots,(n-1)} \|T_i(f)\|_\infty = \max_{i=0,1,\ldots,(n-1)} \|\bar{\Phi}_i f\|_\infty = \|f\|_{n-1}.$$

§3. Nuclearity of $\mathscr{C}^\infty(X)$ (X compact)

3.1. Remark: According to the classical definition, a n.a. nuclear space is defined starting from the notion of nuclear mapping between n.a. Banach spaces.
The n.a. "translation" of the definition of a nuclear mapping reads:
(1) A linear mapping T from a n.a. Banach space E into a n.a. Banach space F is nuclear if there exist sequences $(y_n) \subset F$ and $(a_n) \subset E'$ such that $\lim_n \|a_n\|_E, \|y_n\|_F = 0$ and $T(x) = \sum_{n=1}^\infty a_n(x).\ y_n$, $\forall x \in E$.
Now it is proved in [6] (Theorem 4.40) that a continuous linear mapping $T: E \to F$ is compact (in the sense of (2.3)) if and only if it has property (1).
Therefore we give the following definition of a n.a. nuclear space.

3.2. Definition: A n.a. locally convex space $E, \mathscr{P}_E$ is nuclear if for all $p \in \mathscr{P}_E$ there exists $q \in \mathscr{P}_E$, $q > p$, such that the canonical mapping $\varphi_{pq}: \hat{E}_q \to \hat{E}_p$ is compact.

3.3. Remark: N.a. nuclear spaces have been studied in [1] and [2] under the condition that K is a spherically complete field. In these papers they were called Schwartz spaces. We shall from now on use the term "nuclear space" to point out that there is no restriction on K.

3.4. Definition: The space $\mathscr{C}^\infty(X)$ is defined by $\mathscr{C}^\infty(X) = \bigcap_{n=1}^\infty \mathscr{C}^n(X)$ ([4] p. 75]. A n.a. locally convex topology on $\mathscr{C}^\infty(X)$ is defined by the sequence of n.a. norms

$$(\|f\|_n)_n, f \in \mathscr{C}^\infty(X), n = 1, 2, \ldots \text{([4] p. 119).}$$

With this topology the space $\mathscr{C}^\infty(X)$ is a n.a. Frechet space ([4] p. 119).

3.5. Theorem: *The space $\mathscr{C}^\infty(X)$ is nuclear.*

Proof: For $p_n = \|.\|_n$ we denote by E_n the n.a. normed space $\mathscr{C}^\infty(X)$, $p_n = (\mathscr{C}^\infty(X))_{p_n}$ and by $\hat{E}_n$ its completion. We first describe $\hat{E}_n$, $n = 1, 2, \ldots$ Since every $\|.\|_n$ is a n.a. norm, the spaces E_n are set-theoretically equal to $\mathscr{C}^\infty(X)$ and the topology on E_n is the n.a. norm topology induced by $\mathscr{C}^n(X)$. Now $\mathscr{C}^n(X)$ is complete ((2.2) ii)) and $\mathscr{C}^\infty(X)$ is dense in $\mathscr{C}^n(X)$, $n = 1, 2, \ldots$ ([4] p. 95). Consequently $\hat{E}_n = \mathscr{C}^n(X)$, $n = 1, 2, \ldots$ and both spaces are isometric for each n. For $n = 1, 2, \ldots$ the canonical mapping $\varphi_{n,n-1} : \hat{E}_n \to \hat{E}_{n-1}$ is the canonical injection $\mathscr{C}^n(X) \to \mathscr{C}^{n-1}(X)$. By (2.4) each mapping $\varphi_{n,n-1}$ is compact. Hence $\mathscr{C}^\infty(X)$ is nuclear.

3.6. Corollaries: *The results obtained in* [1] *and* [2] *for n.a. Schwartz spaces* (*see* (3.3)) *obviously do not all remain true in our general case. However some of them are still valid, either with the same proof either replacing the word "c-compact" by the word "compactoïd" and applying the results obtained in* §1 *instead of the well-known properties of c-compact sets.*
In this way we obtain
(i) *Every bounded subset of $\mathscr{C}^\infty(X)$ is compactoïd.*
Hence

(a) *the topology of $\mathscr{C}^\infty(X)$ is not normable.*
(b) *if K is spherically complete, then $\mathscr{C}^\infty(X)$ is reflexive.*

(ii) *Let E be any n.a. Banach space.*
Then every continuous linear mapping from $\mathscr{C}^\infty(X)$ to E is compact. (*I.e. there exists a zero-neighbourhood in $\mathscr{C}^\infty(X)$ whose image is compactoïd in E*).
(iii) *$\mathscr{C}^\infty(X)$ can be identified with a subspace of the space c_0^N* (*countable product of spaces c_0*).

§4. The space $\mathscr{C}^\infty(\mathbb{Z}_p)$

In this section we take $K = \mathbb{Q}_p$ and $X = \mathbb{Z}_p$ (which is a compact subset of $\mathbb{Q}_p$).

4.1. Definition: The sequence space

$$s(K) = \{(a_k)\ a_k \in K \text{ and } \lim_k |a_k|\, k^n = 0, n = 0, 1, 2, \dots\}$$

equipped with the topology given by the n.a. norms

$$\|(a_k)\|_p = \sup_k |a_k|\, k^n, n = 0, 1, 2,$$

will be called the space of rapidly decreasing sequences over K. (compare with the "classical" definition)

4.2. Proposition: *The space $s(K)$ is a n.a. nuclear Frechet space.*

Proof: $s(K)$ is a n.a. Köthe space corresponding to the matrix $(b_k^n) = (k^n)$ (see [3] Def. 2.1). The result then follows from the criterium for nuclearity of n.a. Köthe sequence spaces. ([3], Prop. 3.5).

4.3. Theorem: *The space $\mathscr{C}^\infty(\mathbb{Z}_p)$ is (linearly and topologically) isomorphic to the space $s(\mathbb{Q}_p)$.*

Proof: Let $f \in \mathscr{C}^\infty(\mathbb{Z}_p)$. Then $f \in \mathscr{C}(\mathbb{Z}_p)$ and so f can be written in its expansion with respect to the Mahler basis

$$f(x) = \sum_{k=0}^{\infty} a_k \binom{x}{k}, a_k \in \mathbb{Q}_p, \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!} \qquad (*)$$

The proof is based on the following results, proved in [5]:
Let f be written as $(*)$, then

1°) $f \in \mathscr{C}^n(\mathbb{Z}_p)$ if and only if $\lim_k |a_k|\, k^n = 0$
and

2°) the norms $\max_{0 \le j \le n} \|\Phi_j f\|_\infty$ (see 2.2) and $|a_0| \vee \sup_{k \ge 1} |a_k|\, k^n$ are equivalent.

The desired result is now obtained by identifying $f \in \mathscr{C}^\infty(\mathbb{Z}_p)$ with the sequence (a_k) of coëfficients in its Mahler expansion.

§5. Nuclearity of $\mathscr{C}^\infty(X)$ when X is not (necessarily) compact

5.1. DEFINITIONS: On the space $\mathscr{C}^n(X)$, defined as in (2.1), a n.a. locally convex topology is defined as follows. (The notations are as in (2.1.)). For $f \in \mathscr{C}^n(X)$ and $B \subset X$, B compact, let

$$\|\Phi_i f\|_B = \sup_{p \in B^{i+1}} |\bar{\phi}_i f(p)|, \quad i = 0, 1, \ldots,$$

and

$$\|f\|_{B,n} = \max_{i=0,1,\ldots,n} \|\phi_i f\|_B \quad \text{(see [4])}$$

Then every $\|.\|_{B,n}$ is a n.a. semi-norm on $\mathscr{C}^n(X)$. We consider on $\mathscr{C}^n(X)$ the n.a. locally convex topology determined by the family of n.a. semi-norms.

$$\{\|.\|_{B,n} \; B \text{ compact subset of } X\}$$

On the space $\mathscr{C}^\infty(X) = \bigcap_n \mathscr{C}^n(X)$ a n.a. locally convex topology is then determined by the family of n.a. semi-norms

$$\{\|.\|_{B,n} \; n = 0, 1, 2, \ldots; B \text{ compact subset of } X\}$$

5.2. LEMMA: *Let E, F be n.a. normed spaces with completions $\hat{E}$ and $\hat{F}$, and let $f: E \to F$ be a compact mapping. Then the extension $\hat{f}: \hat{E} \to \hat{F}$ is compact.*

PROOF: Let B be the unit ball in E. Then $\bar{B}^{\hat{E}}$ is the unit ball in $\hat{E}$. Now $\hat{f}(\bar{B})^{\hat{E}} \subset \overline{\hat{f}(B)}^{\hat{F}} = \overline{f(B)}^{\hat{F}}$ and $f(B)$ is compactoïd in F and hence in $\hat{F}$. Its closure $\overline{f(B)}^{\hat{F}}$ is then still compactoïd in $\hat{F}$ and so is the subset $\hat{f}(\bar{B})^{\hat{E}}$.

5.3. LEMMA: *Let $f \in \mathscr{C}^n(X)$ and B a (compact) subset of X. Denote by f^B the restriction of f to B. Then $(\phi_i f)^{B^{i+1}} = \phi_i(f^B)$ and $f^B \in \mathscr{C}^n(B)$.*

PROOF: This follows immediately from the definitions.

5.4. THEOREM: *The space $\mathscr{C}^\infty(X)$ is nuclear.*

PROOF: Put $\mathscr{C}^\infty(X) = E$. Further, for $B \subset X$, B compact, and $n \in \{0, 1, 2, \ldots\}$ we put $F_{B,n} = \{f \in E \, \|f\|_{B,n} = 0\}$. Then $E_{B,n} = E/F_{B,n}$ is a n.a. normed space, normed by

$$\|[f]_{B,n}\|_{E_{B,n}} = \|f\|_{B,n}, [f]_{B,n} \in E_{B,n}, f \in E.$$

($[f]$ is the equivalence class to which f belongs). For all n and all B, the canonical mapping

$$\varphi_{B,n} : E_{B,n} \to E_{B,n-1} \text{ is defined by}$$
$$\varphi_{B,n}([f]_{B,n}) = [f]_{B,n-1}$$

For the nuclearity of $\mathscr{C}^\infty(X)$ it is now (see (3.2) and (5.2)) sufficient to prove that $\forall B, \forall n$; $\varphi_{B,n} : E_{B,n} \to E_{B,n-1}$ is compact. Let $[A]_{B,n}$ be the unit ball in $E_{B,n}$. $[A]_{B,n} = \{[f]_{B,n} \in E_{B,n} \, \|f\|_{B,n} \leq 1\}$ and put $\varphi_{B,n}([A]_{B,n}) = [A]_{B,n-1}$. We have to prove that $[A]_{B,n-1}$ is compactoïd in $E_{B,n-1}$. With the notations of (5.3) we have

$$f \in \mathscr{C}^\infty(X) \Rightarrow f^B \in \mathscr{C}^\infty(B) \text{ and } \bar{\Phi}_i f^B \in \mathscr{C}(B^{i+1})$$

Moreover, if $[g]_{B,n-1} = [f]_{B,n-1}$, then

$$\|f - g\|_{B,n-1} = 0 \text{ or } \|\phi_i(f - g)\|_B = 0, \; i = 0, 1, \ldots, n-1$$

It follows that the next definition is meaningful. Define $T_i : E_{B,n-1} \to C(B^{i+1})$, $i = 0, 1, \ldots, (n-1)$. By $T_i([f]_{B,n-1}) = \bar{\Phi}_i f^B$.
It can be shown, just as in the proof of proposition (2.4), that $T_i([A]_{B,n-1})$ is a compactoïd subset of $\mathscr{C}(B^{i+1})$. The proof then proceeds exactly as in (2.4).

5.5. REMARK: The corollaries (3.6) (i), (ii), (iii)) are still valid in this general case. Corollary (3.6) (iv) has to be replaced by "$C^\infty(X)$ can be identified with a subset of c_0^I, I some index set".

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