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LOCAL COEFFICIENTS AND NORMALIZATION OF INTERTWINING OPERATORS FOR *GL(n)*

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Introduction

The purpose of this paper is to use the theory of automorphic forms to answer several questions in the representation theory of the group GL_r over a non-archimedean local field F. More precisely, in this paper, the following results are obtained.

In Appendix II of [10], R.P. Langlands has suggested a normalizing factor for the intertwining operators, coming from the theory of automorphic forms. The normalized operators are then conjectured to be unitary on the unitary line and to satisfy a functional equation (see Theorem 3.1 here). For real groups these are verified by J. Arthur in [1]. When the group is non-archimedean the problem is more complicated. In Theorem 3.1 of the present paper, we shall show that, if a unitary representation of a Levi factor of a standard parabolic subgroup of $GL_r(F)$ is a component of a cusp form, then the corresponding normalized operators satisfy the above conditions, and therefore, in this case, we provide a positive answer to the above Langlands' conjecture.

More precisely, let $P_\theta = MN$, $\theta \subset \Delta$, be a standard parabolic subgroup of $G = GL_r$ generated by the partition $(r_1, \ldots, r_p)$ of r. Let π be an irreducible unitary representation of $M(F)$ which furthermore is a component of a cusp form on $M(\mathbb{A})$. Let w be an element of the Weyl group of G which permutes the blocks of M. Given $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$, let $A(\nu, \pi, w)$ be the corresponding intertwining operator defined by relation (1.1) of Section 1. There is a decomposition

$$A(\nu, \pi, w) = A(\nu_{N-1}, \pi_{N-1}, w_{N-1}) \cdot \ldots \cdot A(\nu_1, \pi_1, w_1),$$

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where w_i, $1 \leq i \leq N-1$, are simple reflections and the decomposition $w = w_{N-1} \cdot \ldots \cdot w_1$ is reduced.

For each i, let $\pi_{i,1}$ and $\pi_{i,2}$ be the representations of the adjacent blocks of the standard Levi factor of $P_{\theta_i}(F)$ which are interchanged by w_i. Let m_i and n_i be the dimensions of the corresponding blocks, respectively. Denote by $L(s, \pi_{i,1} \times \pi_{i,2})$ the Langlands' L-function attached to the pair $(\pi_{i,1}, \pi_{i,2})$ by H. Jacquet, I.I. Piatetski–Shapiro, and J.A. Shalika [6].

Finally, if $C_\chi(-2s/(m_i+n_i)\cdot \rho_{\theta_i}, \pi_{i,1} \otimes \tilde{\pi}_{i,2}, \theta_i, w_{\theta_i})$ is the corresponding local coefficient, defined in [15], then by Proposition 3.1 of this paper, the product

$$C_\chi(-2s/(m_i+n_i)\cdot \rho_{\theta_i}, \pi_{i,1} \otimes \tilde{\pi}_{i,2}, \theta_i, w_{\theta_i})\cdot \frac{L(s, \pi_{i,1} \times \pi_{i,2})}{L(1-s, \tilde{\pi}_{i,1} \times \tilde{\pi}_{i,2})} \tag{1}$$

is a monomial in q^s. We denote this monomial by $\varepsilon(s, \pi_{i,1} \times \pi_{i,2}, \chi)$. It is justified by functional equation (2.2.1) that we call $\varepsilon(s, \pi_{i,1} \times \pi_{i,2}, \chi)$ the Langlands' root number attached to the pair $(\pi_{i,1}, \pi_{i,2})$ (it is the subject matter of a subsequent paper to show that this root number is the same as the one defined in [6]). Now, write

$$r(\pi, w, \chi) = \prod_{i=1}^{N-1} \varepsilon(0, \pi_{i,1} \times \tilde{\pi}_{i,2}, \chi) L(1, \pi_{i,1} \times \tilde{\pi}_{i,2})/L(0, \pi_{i,1} \times \tilde{\pi}_{i,2}).$$

This is the normalizing factor which was conjectured by Langlands in [10]. Then in Theorem 3.1 of this paper, we show that the normalized operator $r(\pi, w, \chi)A(0, \pi, w)$ satisfies the properties listed in [10].

Beyond the results of [15], Theorem 3.1 is a routine exercise, unless it is complemented by Proposition 3.1 which shows that (1) is in fact a monomial. Its proof uses two different versions of the functional equation satisfied by these L-functions [6, 15]. This is an example of our general method of using global functional equations to compute the local coefficients for GL_r in this paper (defined in general in [15]). By product formula (1.5) and Jacquet's quotient theorem, it is enough to compute them for pairs of supercuspidal representations. This is the subject of Section 2 of this paper (Theorems 2.1 and 2.2). The reader must observe that when π is supercuspidal, Theorem 2.2 becomes a special case of Proposition 3.1 (which is true in this case by Lemma 2.5), and consequently, one could conclude it from the proposition. But the proof given here for Theorem 2.1 (and consequently Theorem 2.2) is independent of the results of [6] and requires only the functional equation proved in [15]. Because of the detailed proof of Theorem 2.1, we have only sketched the proof of Proposition 3.1 which uses several similar lemmas and arguments. It would be desirable to use local methods to obtain similar formulas for any pair of non-degenerate representations.

As our first application of these results, in Theorem 4.1 and 4.2, we obtain a formula for the Plancherel measure $\mu(\pi; \nu)$ (cf. [12]) for the group $GL_r(F)$. We should mention that a more general formula for $\mu(\pi; \nu)$ (π supercuspidal), and consequently less accurate in the case of GL_r, has been obtained by A.J. Silberger in [13].

Next, in Theorem 4.3, we give a new proof of a result of I.N. Bernstein and A.V. Zelevinskii (cf. [2]) on reducibility of the induced representations of $GL_r(F)$. More precisely, let $P = MN$ be a parabolic subgroup of $G = GL_r$, and assume $M = GL_{r_1} \times \ldots \times GL_{r_p}$. Let $\pi = \pi_1 \otimes \ldots \otimes \pi_p$ be an irreducible supercuspidal representation of $M(F)$. Suppose for some i and j, $1 \leq i \neq j \leq p$, $\pi_i \cong \pi_j \otimes |\det|$. Then $\operatorname{Ind}_{P(F)\uparrow G(F)} \pi$ is reducible. More generally, in Theorem 4.4, we obtain a similar reducibility criterion in terms of Jacquet–Piatetski-Shapiro–Shalika's L-functions, if π is any irreducible unitary representation of $M(F)$ which is a component of a cusp form.

Finally, let $\mathfrak{a}$ be the real Lie algebra of the center of M, and suppose $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ is in the positive Weyl chamber. Assume $I(\nu, \pi, \theta)$ (π unitary and supercuspidal) is reducible. Then in Theorem 4.5 and its corollary, we prove that the image of the intertwining operator $A(-\nu, \pi, w_\theta)$ is degenerate (see Appendix of [14]). In particular if P is maximal and ν is in the positive Weyl chamber, then the unique nondegenerate subquotient of $I(-\nu, \pi, \theta)$ is in fact a subrepresentation. A similar result for real groups is proved in [16].

We conclude our introduction by remarking that the fact that the reducibility of the induced representations and the poles of intertwining operators are governed by the same kind of law (i.e. if P is maximal and π_1 and π_2 are supercuspidal, then $I(\nu(0, s'), \pi_1 \otimes \pi_2, \theta)$ is reducible if and only if $\pi_1 \cong \pi_2 \otimes |\det|^{\pm 1}$ and $A(\nu(0, s'), \pi_1 \otimes \pi_2, w_\theta)$ has a pole if and only if $\pi_1 \cong \pi_2$) is not a coincidence but a consequence of the fact that they respectively determine the poles and the zeros of the local coefficients, which are simply related by Proposition 2.2.

1. Preliminaries

1.0. Fix a positive integer r. Let F be a non-archimedean local field containing $\mathbb{Q}$, and denote by G the group GL_r. Let B be the subgroup of upper triangular elements of G. We say a parabolic subgroup P of G is standard if $P \supset B$. Then there exists a partition $(r_1, r_2, \ldots, r_p)$ of r such that $M(F) = GL_{r_1}(F) \times \ldots \times GL_{r_p}(F)$ is the Levi factor of P in the usual manner. In fact if Δ denotes the set of simple roots of B, then there exists a subset of Δ such that M is generated by Δ. Consequently, for every $\theta \subset \Delta$, we shall use P_θ, M_θ, and N_θ to denote the corresponding P, M, and N, respectively.

Now, let A_θ be the center of M_θ, and denote by $\mathfrak{a}_\theta$, the real Lie algebra of A_θ. Then $\mathfrak{a}_\theta \cong \mathbb{R}^p$ and $(\mathfrak{a}_\theta)^*_{\mathbb{C}} \cong \mathbb{C}^p$. Finally, define

$$H_\theta : M_\theta \to \mathrm{Hom}(X(M_\theta), \mathbb{Z})$$

by

$$\langle H_\theta(m), \chi \rangle = \log_q |\chi(m)| \qquad \forall \chi \in X(M_\theta),$$

where $X(M_\theta)$ denotes the group of F-rational characters of M_θ, and q is the number of elements in the residue field of F. Observe that $X(M_\theta) \cong \mathbb{Z}^p$ and therefore $H_\theta(M_\theta) \cong \mathbb{Z}^p \subseteq \mathfrak{a}_\theta$. Extend H_θ to P_θ by $H_\theta(mn) = H_\theta(m)$. Finally, let $\langle\, , \rangle$ denote the pairing between $(\mathfrak{a}_\theta)^*_{\mathbb{C}}$ and $\mathfrak{a}_\theta$. It may be considered, as we in fact do, to be the one induced from the standard pairing of $\mathfrak{a}_\phi$, the real Lie algebra of the subgroup of diagonals (ϕ denotes the empty set).

Let $\mathcal{O}$ be the ring of integers of F, and let $K = GL_r(\mathcal{O})$. Then for every $\theta \subset \Delta$, $G(F) = KP_\theta(F)$.

Now, let (π, V) be an irreducible admissible representation of $M(F)$. Fix $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$ and set

$$I(\nu, \pi, \theta) = \operatorname*{Ind}_{P_\theta(F) \uparrow G(F)} (\pi \otimes q^{\langle \nu, H_\theta(\ ) \rangle})$$

as in [15]. Let $V(\nu, \pi, \theta)$ denote the space of $I(\nu, \pi, \theta)$. Then for $n \in N_\theta(F)$, $m \in M_\theta(F)$, $f \in V(\nu, \pi, \theta)$

$$f(gnm) = \pi(m^{-1}) q^{\langle -\nu - \rho_\theta, H_\theta(m) \rangle} f(g),$$

where ρ_θ denotes half of the sum of the positive roots which generate N_θ.

Let W denote the Weyl group of G with respect to the diagonal elements. Then W is isomorphic to the permutation group in n elements.

Now, let θ and θ' be two subsets of Δ, and set

$$W(\theta, \theta') = \{w \in W \mid w(\theta) = \theta'\}.$$

We say θ and θ' are associate if $W(\theta, \theta')$ is not empty. Let $\{\theta\}$ denote the set of all the associates of θ. Finally, let ψ^+ and ψ^- be the positive and negative roots spanned by Δ, respectively, and denote by Σ_θ^+ and Σ_θ^- the similar sets generated by θ.

1.1. *Intertwining operators.* Fix two associated subsets θ and θ' of Δ, and let w be in $W(\theta, \theta')$. Let N_θ^- be the unipotent subgroup generated by the roots in $\psi^- - \Sigma_\theta^-$, and set $N_w = U \cap wN_\theta^- w^{-1}$.

Given $f \in V(\nu, \pi, \theta)$, define:

$$A(\nu, \pi, w)f(g) = \int_{N_w} f(gnw)\,dn \tag{1.1}$$

To study the convergence of (1.1), choose a standard parabolic subgroup $P_* = M_* N_*$ of M_θ, and a supercuspidal representation σ of M_* such that $\pi \subset \operatorname{Ind}_{P_*(F)\uparrow M_\theta(F)} \sigma$. Suppose $P_* = P_{\theta_*}$ for some $\theta_* \subset \theta \subset \Delta$. Choose $\tilde{\nu} \in (\mathfrak{a}_{\theta_*})_{\mathbb{C}}^*$ such that

$$\langle \tilde{\nu}, H_{\theta_*}(a) \rangle = \langle \nu, H_\theta(a) \rangle \qquad (\forall a \in A_{\theta_*}).$$

Finally, take $\nu_0 \in \mathfrak{a}_{\theta_*}^*$ and an irreducible unitary supercuspidal representation σ_0 such that $\sigma = \sigma_0 \otimes q^{\langle \nu_0, H_{\theta_*}(\ ) \rangle}$. Then $I(\nu_0, \pi, \theta) \hookrightarrow I(\tilde{\nu} + \nu_0, \sigma_0, \theta_*)$, and the convergence of (1.1) reduces to that of $A(\tilde{\nu} + \nu_0, \sigma_0, w)$. Now it follows from [15] that the integral representation of $A(\tilde{\nu} + \nu_0, \sigma_0, w)$ converges absolutely if for every $\alpha \in \Sigma^+(\theta_*)$ with $w(\alpha) \in \psi^-$

$$\langle \operatorname{Re}(\tilde{\nu}), H_\alpha \rangle \ll 0 \tag{1.2}$$

where notation is as in [15]. More precisely

$$H_\alpha = \operatorname{diag}(0, \ldots, 0, \underset{\substack{\uparrow \\ R_i'+1}}{1}, \ldots, \underset{\substack{\uparrow \\ R_i'+r_i'}}{1}, 0, \ldots, 0, \underset{\substack{\uparrow \\ R_j'+1}}{-1}, \ldots, \underset{\substack{\uparrow \\ R_j'+r_j'}}{-1}, 0, \ldots, 0),$$

where $R_i' = \sum_{t=1}^{i-1} r_t'$, $R_j' = \sum_{t=1}^{j-1} r_t'$, $1 \leq i \leq p'$, and if $(i,j) \in \psi^+$, then $\alpha = [(i,j)] \in \Sigma^+(\theta_*)$. Here p' is the number of blocks in M_* and $\Sigma^+(\theta_*)$ denotes the set of positive roots of (G, A_{θ_*}).

Now, observe that $w \in W(\theta_*, w(\theta_*))$. Let $\theta_1, \ldots, \theta_n$, and $w_i \in W(\theta_i, \theta_{i+1})$ be as in Lemma 2.1.2 of [15] (replacing θ by θ_* in the statement of the lemma). Then from Theorem 2.1.1 of [15], it follows that

$$\begin{aligned} A(\nu, \pi, w) &= A(\tilde{\nu}, \sigma, w) \mid V(\nu, \pi, \theta) \\ &= A(\tilde{\nu}_{N-1}, \sigma_{N-1}, w_{N-1}) \cdot \ldots \cdot A(\tilde{\nu}_1, \sigma_1, w_1) \mid V(\nu, \pi, \theta) \end{aligned} \tag{1.3}$$

where $\tilde{\nu}_i = w_{i-1}(\tilde{\nu}_{i-1})$, $\sigma_i = w_{i-1}(\sigma_{i-1})$, $2 \le i \le N-1$, $\tilde{\nu}_1 = \tilde{\nu}$, and $\sigma_1 = \sigma$. Here $N = \mathrm{card}(\Sigma_r(\theta_*, w(\theta_*), w))$ (cf. [15]). Observe that, by analytic continuation (cf. [12, 15]) this holds for all ν.

Now, from Proposition 2.4.2 of [15], it follows that

$$A(w(\nu), w(\pi), w^{-1})A(\nu, \pi, w) = \prod_{i=1}^{N-1} \gamma^2(M_i'/P_i)\mu^{-1}(\sigma_i : \tilde{\nu}_i) \tag{1.4}$$

where the notation is as in [15] (also cf. [12]).

Finally observe that (1.3) holds, even if σ_i's are not necessarily supercuspidal.

1.2. *Local coefficients.* Let ψ be a non-trivial additive character of F. Denote by $U = N_\phi$ the subgroup of upper triangular elements whose diagonals are all ones. Define:

$$\chi(u) = \psi(a_1 u_{12} + \ldots + a_{n-1} u_{n-1,n}),$$

where $a_1, \ldots, a_{n-1} \in F^*$ are fixed. Then χ defines a non-degenerate character of U, and every non-degenerate character of U is of this form for some $a_1, \ldots, a_{n-1} \in F^*$. Throughout this paper we shall fix $a_1, \ldots, a_{n-1} \in F^*$ and define χ as above.

Since σ is supercuspidal, so will be $\sigma_1, \ldots, \sigma_{N-1}$ as representations of $M_1(F), \ldots, M_{N-1}(F)$, respectively. Consequently $\sigma, \sigma_1, \ldots, \sigma_{N-1}$ are all non-degenerate. Let $C_\chi(\tilde{\nu}, \sigma, \theta_*, w)$ be the local coefficient attached to $\tilde{\nu}$, σ, θ_*, and w as in [15] (also see Lemma 2.2 and Proposition 2.1 of the present paper). Denote the same coefficients for $\tilde{\nu}_i$, σ_i, θ_i, and w_i, $1 \le i \le N-1$, by $C_\chi(\tilde{\nu}_i, \sigma_i, \theta_i, w_i)$. Furthermore suppose π is non-degenerate. Then by Proposition 3.2.1 of [15]

$$C_\chi(\nu, \pi, \theta, w) = C_\chi(\tilde{\nu}, \sigma, \theta_*, w) = \prod_{i=1}^{N-1} C_\chi(\tilde{\nu}_i, \sigma_i, \theta_i, w_i) \tag{1.5}$$

Observe that (1.5) is still true if $\theta_* = \theta$ and $\sigma = \pi$ is not necessarily supercuspidal.

As in [15], define:

$$\mathscr{A}(\nu, \pi, w) = C_\chi(\nu, \pi, \theta, w)A(\nu, \pi, w) \tag{1.6}$$

Then

$$\mathscr{A}(\nu, \pi, w) = \mathscr{A}(\tilde{\nu}_{N-1}, \sigma_{N-1}, w_{N-1}) \ldots \mathscr{A}(\tilde{\nu}_1, \sigma_1, w_1) \mid V(\nu, \pi, \theta) \tag{1.7}$$

Now suppose π is unitary; then by Proposition 3.1.4 of [15], $\mathscr{A}(\nu, \pi, w)$ is normalized, i.e.

a) $\mathscr{A}(w(\nu), w(\pi), w^{-1})\mathscr{A}(\nu, \pi, w) = I$

b) $\mathscr{A}(\nu, \pi, w)^* = \mathscr{A}(-w(\bar{\nu}), w(\pi), w^{-1})$

c) $\mathscr{A}(\nu, \pi, w)$ is unitary if $-\bar{\nu} = \nu$.

Finally Proposition 3.1.3 of [15] states that

$$C_\chi(w(\nu), w(\pi), w(\theta), w^{-1}) = \overline{C_\chi(-\bar{\nu}, \pi, \theta, w)} \tag{1.8}$$

2. A formula for local coefficients

As it is clear from relation (1.5), to compute local coefficients, we need only to compute $C_\chi(\nu, \pi, \theta, w)$ for maximal parabolic subgroups and supercuspidal representations. Thus, let $M_\theta \cong GL_m \times GL_n$, $m + n = r$, and $\pi = \pi_1 \otimes \pi_2$, where π_1 and π_2 are supercuspidal representations of $GL_m(F)$ and $GL_n(F)$, respectively. Observe that then $w = w_\theta = w_l \cdot w_{l,\theta}$, where w_l and $w_{l,\theta}$ are the longest elements in the Weyl groups of G and M_θ, respectively.

Let α be the simple root $(m, m+1)$, and denote by β, a base for the one dimensional complex dual of the real Lie algebra $\mathfrak{z}$ of the center of G. Then given a pair of complex numbers s and s', $\nu = -mn/(m+n) \cdot s \cdot \alpha + s' \cdot \beta$ is an element of $(\mathfrak{a}_\theta)^*_{\mathbb{C}}$. In fact with respect to the basis (α, β), every $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$ can uniquely be represented in this way for some s and s'. Let $\mu(g) = |\det(g)|$, and use ω_1 and ω_2 to denote the central characters of π_1 and π_2, respectively. We need the following lemma:

LEMMA 2.1: (a) *Suppose $m \neq n$, then as a function of ν, $C_\chi(\nu, \pi, \theta, w_\theta)$ never vanishes* ($\forall \nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$).

(b) *Suppose $m = n$, but for no complex number s, $\pi_1 \cong \pi_2 \otimes \mu^s$. Then for all $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$, $C_\chi(\nu, \pi, \theta, w_\theta)$ is non-zero.*

(c) *Suppose $m = n$, and for some $s \in \mathbb{C}$, $\pi_1 \cong \pi_2 \otimes \mu^s$. Then the zeros of $C_\chi(-m/2 \cdot s \cdot \alpha + s' \cdot \beta, \pi, \theta, w_\theta)$ are same as those of the polynomial $1 - \omega_1\omega_2^{-1}(\varpi)q^{-ms}$.*

PROOF: From property c) of $\mathscr{A}(\nu, \pi, w_\theta)$, it follows that the zeros of $C_\chi(\nu, \pi, \theta, w_\theta)$ are exactly the poles of $A(\nu, \pi, w_\theta)$. Now part (a) follows from the fact that in this case $A(\nu, \pi, w_\theta)$ is entire (cf. [11]). To prove parts (b) and (c), we argue as follows. From the results of [11], it follows that if $m = n$, $A(-m/2 \cdot s \cdot \alpha + s' \cdot \beta, \pi, w_\theta)$ has a pole if and only if $\pi_1 \cong \pi_2 \otimes \mu^s$ in which case it is simple and is given by the polynomial $1 - \omega_1\omega_2^{-1}(\bar{\omega})q^{-ms}$. This completes the lemma.

Now, let $\lambda_{s,s'}$ be the Whittaker functional

$$\lambda(-mn/(m+n) \cdot s \cdot \alpha + s' \cdot \beta, \pi, \theta, \chi)$$

attached to the representation

$I(-mn/(m+n) \cdot s \cdot \alpha + s' \cdot \beta, \pi, \theta) = I(s, s', \pi)$.

More precisely, for every $f \in I(s, s', \pi)$, there exists an open compact subgroup $N_f \subset N_{w(\theta)}$, $w = w_\theta$, such that the following principal value integral (cf. [5])

$$\int_{N_{w(\theta)}} (f(nw), e)\overline{\chi(n)}\, dn$$

is given by

$$\lambda_{s,s'}(f) = \int_{N_f} (f(nw), e)\overline{\chi(n)}\, dn.$$

Here the notation is as in [15], i.e. $(f(g), e)$ denotes the value of the Whittaker function $f(g)$ at the identity element of M_θ. We need the following lemma:

LEMMA 2.2: *For every $f \in I(s, s', \pi)$, $\lambda_{s,s'}(f)$ is a polynomial in q^s and q^{-s}.*

PROOF: Let K be a compact open subgroup of G, and denote by $I(\nu, \pi, \theta)^K$ the subspace of the elements in $I(\nu, \pi, \theta)$ which are left invariant under K, $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$. Given an open compact subgroup $N_0 \subset U$, define

$$P_{\chi, N_0}\phi(g) = (\text{meas}(N_0))^{-1} \int_{N_0} \phi(ng)\overline{\chi(n)}\, dn.$$

Then from Lemma 2.2 of [5], it follows that there exists a compact open subgroup $N_0 \subset U$ such that for every $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$ and every $\phi \in I(\nu, \pi, \theta)^K$ the function $P_{\chi, N_0}\phi$ has support in $P_{w(\theta)}wP_\theta$. Choose a larger compact open subgroup $N_* \subset U$ such that $N_* \supset N_0$ and $P_{\chi, N_0}\phi$ has support in $N_* wP_\theta$ for all $\phi \in I(\nu, \pi, \theta)^K$. Then:

$$\begin{aligned}\lambda(\nu, \pi, \theta, \chi)(\phi) &= \lambda(\nu, \pi, \theta, \chi)(P_{\chi, N_*}\phi) \\ &= \int_{N_* \cap N_{w(\theta)}} (\phi(nw), e)\overline{\chi(n)}\, dn\end{aligned}$$

(notation as in [5, 15]).

Now, given $\phi \in I(\nu, \pi, \theta)$, choose $f \in C_c^\infty(G) \otimes V$, bi-invariant under K, such that $\phi = P_{\nu,\pi} f$ is given by

$$P_{\nu,\pi} f(g) = \int_{P_\theta} q^{\langle \nu + \rho_\theta, H_\theta(p)\rangle} \pi(m) f(gp) d_l(p).$$

Then

$$\lambda(\nu, \pi, \theta, \chi)(\phi) = \int_{P_\theta \times N_*} q^{\langle \nu + \rho_\theta, H_\theta(P)\rangle} (\pi(m) f(nwp), e)\overline{\chi(n)}\, dn\, d_l(p).$$

Consequently P itself will range over a compact subset of P_θ. $H_\theta(M_\theta)$ is a lattice in $\mathfrak{a}_\theta$, and $\langle \nu + \rho_\theta, H_\theta(p)\rangle$ (for $\nu = -mn/(m+n)\cdot s \cdot \alpha$) takes only non-zero values for the image of $H_\theta(M_\theta)$ in $\mathfrak{a}_\theta/\mathfrak{z}$. This image is generated by $t_{m,n} = (1/m + 1/n, 0)$ as a lattice in $\mathbb{R}^1$. Consequently $\langle \nu + \rho_\theta, H_\theta(P)\rangle$ can take only a finite number of values when p ranges over this compact set, and furthermore its values are integral multiples of $\langle -mn/(m + + n)\cdot s\cdot \alpha, t_{m,n}\rangle = -s$. This completes the proof.

LEMMA 2.3: *For every $f \in I(\nu, \pi, \theta)$, $\nu = -mn/(m+n)\cdot s\cdot\alpha + s'\cdot\beta$, and for every $g \in G$, $A(\nu, \pi, \theta) f(g)$ is a rational function of q^s.*

PROOF: (notation as in Section 2 of [15]) With an argument similar to that of Lemmas 1.4 and 1.5 of [13] one can conclude that, for every $\psi \in L(\pi, P)$, $P = P_\theta$, $\tau = {}^0\tau$, $c_{P|P}(1, \pi, \nu)\psi(m)$ is a rational function of q^s, $\forall m \in M_\theta = M$ (using the Riemann sphere attached to $\mathfrak{a}_{\mathbb{C}}^*/L^* + \mathfrak{z}_{\mathbb{C}}^*$, $L = H_\theta(M_\theta)$). Then by Lemma 2.2.4 of [15]

$$c_{P|P}(1, \pi, \nu)\psi_T = \gamma(G/P)^{-1} \psi_{j_{P^-|P}(\nu,\pi)T}$$

for every $T \in F_\theta \cdot \mathrm{End}^\circ(\mathscr{H}(\pi)) \cdot F_\theta$. Choose $F \subset \mathscr{E}(K)$ such that $\psi = \psi_T$ for some $T \in F_\theta \cdot \mathrm{End}(\mathscr{H}_F) \cdot F_\theta$. Then $\psi_{jT}(m)$ is a rational function of q^s for all $m \in M$, where j denotes $j_{P^-|P}(\nu, \pi)$. But for $(k_1, k_2) \in K^2$

$$\psi_{jT}(m)(k_1 : k_2) = \mathrm{tr}(\pi(m)\kappa_{jT}(k_2 : k_1)).$$

Now, let $\tilde{F} \subset \mathscr{E}(K_M)$, $K_M = K \cap M$, be a finite subset. Then for every

$$f \in \alpha_{\tilde{F}} \underset{K_M}{*} C_c^\infty(M) \underset{K_M}{*} \alpha_{\tilde{F}}$$

$$\mathrm{tr}(\pi(f)\kappa_{jT}(k_2 : k_1))$$

is a rational function of q^s. But now, since π is irreducible, by Lemma 1.11.3 of [12]

$$\pi(\alpha_{\bar{F}} * C_c^\infty(M) * \alpha_{\bar{F}}) = \text{End } V_{\bar{F}}$$

(V is the space of π). Consequently, letting $\pi(f)$ be suitable projections or permutation elements of End $V_{\bar{F}}$, we conclude that for every $(k_1, k_2) \in K^2$, all the matrix coefficients of $\kappa_{jT}(k_2 : k_1)$ are rational functions of q^s. Now for every $f \in V(\nu, \pi, \theta)$

$$(jTf)(k_2) = \int_K \kappa_{jT}(k_2 : k_1) f(k_1^{-1}) dk_1$$

is itself a rational function of q^s, and the lemma follows if we choose T to be the identity of $F_\theta \cdot \text{End}(\mathscr{H}_F) \cdot F_\theta$.

We now prove:

PROPOSITION 2.1: (a) *Suppose* $m \neq n$; *then* $C_\chi^{-1}(\nu, \pi, \theta, w_\theta)$, $\nu = -mn/(m+n) \cdot s \cdot \alpha + s' \cdot \beta$, *is a polynomial in* $\mathbb{C}[q^s, q^{-s}]$.

(b) *Suppose* $m = n$, *but for no complex number* s, $\pi_1 \cong \pi_2 \otimes \mu^s$. *Then* $C_\chi^{-1}(\nu, \pi, \theta, w_\theta) \in \mathbb{C}[q^s, q^{-s}]$, *where* $\nu = -m/2 \cdot s \cdot \alpha + s' \cdot \beta$.

(c) *Suppose* $m = n$, *and for some* $s \in \mathbb{C}$, $\pi_1 \cong \pi_2 \otimes \mu^s$. *Then*

$$C_\chi^{-1}(\nu, \pi, \theta, w_\theta) = Q(q^s, q^{-s})/(1 - \omega_1 \omega_2^{-1}(\bar{\omega}) q^{-ms}),$$

$\nu = -m/2 \cdot s \cdot \alpha + s' \cdot \beta$, *where* $Q(q^s, q^{-s}) \in \mathbb{C}[q^s, q^{-s}]$ *is relatively prime to* $1 - \omega_1 \omega_2^{-1}(\bar{\omega}) q^{-ms}$.

PROOF: $C_\chi(\nu, \pi, \theta, w_\theta)$ is defined by

$$C_\chi(\nu, \pi, \theta, w_\theta) \lambda(w_\theta(\nu), w_\theta(\pi), w_\theta(\theta), \chi) \cdot A(\nu, \pi, w_\theta) = \lambda(\nu, \pi, \theta, \chi).$$

Now the proposition follows from combining Lemmas 2.1, 2.2, and 2.3.

Finally we need:

LEMMA 2.4: *The values of* $C_\chi(\nu, \pi, \theta, w_\theta)$, $\nu = -mn/(m+n) \cdot s \cdot \alpha + s' \cdot \beta$ *are independent of* s'.

PROOF: Let N_θ^- be the opposite to the unipotent group N_θ. Then for Re(s) large

$$A(\nu, \pi, w_\theta) f(w_\theta^{-1}) = \int_{N_\theta^-} f(n^-) dn^-.$$

But now it is clear from the decomposition of n^- that $\langle s'\cdot\beta, H_\theta(m(n^-))\rangle = 0$. This proves the independence of $A(v,\pi,w_\theta)f$ from s', for any $f\in I(v,\pi,\theta)$. The same argument applies to

$$\lambda_{s,s'}(f) = \int_{N_{w(\theta)}} (f(nw),e)\overline{\chi(n)}\,dn,$$

and the lemma follows.

The main result of this section is the following theorem.

THEOREM 2.1: (a) *Suppose $m\neq n$; then there exists a non-zero complex number $c(\pi)$ and an integer $n(\pi)$, both depending only on π such that*

$$C_\chi(-mn/(m+n)\cdot s\cdot\alpha + s'\cdot\beta,\pi,\theta,w_\theta) = c(\pi)q^{n(\pi)s}$$

(b) *Suppose $m = n$, but for no complex number s, $\pi_1\cong\pi_2\otimes\mu^s$. Then*

$$C_\chi(-m/2\cdot s\cdot\alpha + s'\cdot\beta,\pi,\theta,w_\theta) = c(\pi)q^{n(\pi)s},$$

where $c(\pi)\in\mathbb{C}-\{0\}$ and $n(\pi)\in\mathbb{Z}$.

(c) *Suppose $m = n$, and for some $s\in\mathbb{C}$, $\pi_1\cong\pi_2\otimes\mu^s$. Then*

$$C_\chi(-m/2\cdot s\cdot\alpha + s'\cdot\beta,\pi,\theta,w_\theta) = c(\pi)q^{(\pi)s}\frac{1-\omega_1\omega_2^{-1}(\bar\omega)q^{-ms}}{1-\omega_1^{-1}\omega_2(\bar\omega)q^{-m(1-s)}}$$

where $c(\pi)\in\mathbb{C}-\{0\}$ and $n(\pi)\in\mathbb{Z}$.

At this time we can only prove this theorem by global methods. We need some preparation:

We fix a number field k whose v_0 completion k_{v_0} is F. Let $\mathbb{A}$ be the ring of adeles of k, and denote by $G(\mathbb{A})$, the group $GL_r(\mathbb{A})$. We then set $P(\mathbb{A})$, $M(\mathbb{A})$, and $N(\mathbb{A})$ to denote the groups $P_\theta(\mathbb{A})$, $M_\theta(\mathbb{A})$, and $N_\theta(\mathbb{A})$. We again assume $M(\mathbb{A}) = GL_m(\mathbb{A})\times GL_n(\mathbb{A})$, $m+n=r$, i.e. P is maximal. We need the following lemma:

LEMMA 2.5: *Let π be an irreducible unitary supercuspidal representation of $GL_r(F)$. Denote by ρ_0 the central character of π. Suppose $\rho_0(\bar\omega)=1$. Then there exists a cuspidal representation $\sigma=\bigotimes_v\sigma_v$ of $GL_r(\mathbb{A})$ such that $\sigma_{v_0}=\pi$.*

PROOF: Fix a character ρ of $\mathbb{A}^*/k^*$, $\rho=\bigotimes_v\rho_v$, such that $\rho_{v_0}=\rho_0$. This is certainly possible by, for example, considering $\rho_0\otimes\bigotimes_{v\neq v_0}\mathbb{1}_v$ as a character of $\mathcal{O}_{v_0}^* \cdot \prod_{v \neq v_0} \mathcal{O}_v^* \cdot k_\infty^+ \cong A^*/k^*$, and defining ρ through the decomposition $A^* \cong k^* \cdot \prod_v \mathcal{O}_v^* \cdot k_\infty^+$.

Let V be the space of π, and denote by $(\tilde{\pi}, \tilde{V})$ its contragredient. Fix $\tilde{u} \in \tilde{V}$, $\tilde{u} \neq 0$, and consider the map $u \to f_u$, defined by $f_u(g) = \langle \pi(g)u, \tilde{u} \rangle$. Denote by $\mathscr{H}$ the image of V under this mapping into the space of smooth functions on $GL_r(k_{v_0})$ which are of compact support modulo the center $Z_r(k_{v_0})$.

Now, let f be a function on $G(\mathbb{A})/G(k)$, $G = GL_r$, which has compact support modulo $Z(\mathbb{A})$, $Z = Z_r$, and furthermore for every $z \in Z(\mathbb{A})/Z(k)$, $f(gz) = \rho(z)f(g)$. Define

$$\tilde{f}(g) = \sum_{\gamma \in G(k)/Z(k)} f(g\gamma)$$

which for a fixed g is a finite sum. Then clearly $\tilde{f} \in L_2(G(\mathbb{A})/G(k), \rho)$. Now observe that f may in fact be chosen to be of the form $f = \bigotimes_v f_v$, where for almost all finite v, f_v is the characteristic function of $K_v = GL_r(\mathcal{O}_v)$. In fact, given $v \neq v_0$ finite, suppose $\rho_v | Z_r(\mathcal{O}_v^*) = K_v \cap Z_r(k_v) \equiv 1$. Then we choose f_v to be the characteristic function of K_v. Next, choose $K_v' \subset K_v$, a compact open subgroup such that $\rho_v | K_v' \cap Z_r(k_v) \equiv 1$. Now, we let f_v be the characteristic function of K_v'. Finally for $v = v_0$, fix a function $f_{v_0} \in \mathscr{H}$, such that $f_{v_0}(e_{v_0}) \neq 0$. Take a small open compact subgroup K_{v_0}' such that $\rho_{v_0} | Z_r(k_{v_0}) \cap K_{v_0}' \equiv 1$ and $f_{v_0}(k_0') = f_{v_0}(e_{v_0}) \neq 0$ for all $k_0' \in K_{v_0}'$. Now let

$$C = G(k) \cap (\Pi K_v' \cdot \Pi K_v \cdot K_{v_0}')$$

and let f_∞ be any C^∞-function of compact support satisfying the following conditions:

1) $\operatorname{supp}(f_\infty) \cap G(k) \subset K_{v_0}'$
2) $\operatorname{supp}(f_\infty) \cap C \neq \phi$
3) $f_\infty | C = 1$.

Then

$$\begin{aligned} \tilde{f}(e) &= \sum_{\gamma \in G(k)/Z(k)} \prod_{v \neq v_0} f_v(\gamma) \cdot f_{v_0}(\gamma) \cdot f_\infty(\gamma) \\ &= \operatorname{card}(\operatorname{supp}(f_\infty) \cap C) \\ &\neq 0. \end{aligned}$$

Consequently, the map $f \to \tilde{f}$ defines a non-zero $G(\mathbb{A})$-morphism from $\bigotimes_{v \neq v_0} C_c^\infty(G(k_v), \rho_v) \otimes \mathscr{H}$ into $L_2(G(\mathbb{A})/G(k), \rho)$.

To prove the cuspidality of $\tilde{f}$, we only need to check that for every N, unipotent radical of an arbitrary standard parabolic subgroup of G,

$$\int_{N(\mathbb{A})/N(k)} f(gn)\,dn = 0,$$

for every $g \in G(\mathbb{A})$, and $f \in \bigotimes_{v \neq v_0} C_c^\infty(G(k_v), \rho_v) \otimes \mathscr{H}$. Take $f = \bigotimes_v f_v$ with $f_{v_0} \in \mathscr{H}$. Let Q be an open compact subgroup of $\prod_{v < \infty} N(k_v) \cap N(\mathbb{A})$. Fix a fundamental domain $\mathscr{F}$ of $N(k_\infty)/N(k)$. Then $Q \times \mathscr{F}$ is a fundamental domain of $N(k_\infty) \times Q/N(k) \cap (N(k_\infty) \times Q)$, which by approximation theorem is isomorphic to $N(\mathbb{A})/N(k)$. Consequently we only need to prove that for an appropriate Q

$$\prod_v \int_{Q \cap N(k_v)} f_v(g_v n_v)\,dn_v = 0.$$

But for a suitable Q this follows from

$$\int_{Q \cap N(k_{v_0})} f_{v_0}(g_{v_0} n_{v_0})\,dn_{v_0} = 0$$

which is a consequence of the fact that f_{v_0} is a matrix coefficient of a supercuspidal representation. This completes the lemma.

Now, let the notation be as in Theorem 2.1. Set

$$v(s, s') = -mn/(m+n) \cdot s \cdot \alpha + s' \cdot \beta.$$

Define $\omega_i' \in \mathcal{O}_F^*$ and $s_i \in \mathbb{C}$ by $\omega_i(\varepsilon \bar{\omega}^r) = \omega_i'(\varepsilon) q^{rs_i}$, $r \in \mathbb{Z}$, $\varepsilon \in \mathcal{O}_F^*$, $i = 1, 2$. Here $\bar{\omega}$ is the fixed uniformizing parameter of F explained before. Set $\pi_1' = \pi_1 \otimes \mu^{s_1/m}$ and $\pi_2' = \pi_2 \otimes \mu^{s_2/n}$. Then π_i' has $\omega_i' \in \mathcal{O}_F^*$, $i = 1, 2$, as its central character. By Lemma 2.5, let $\sigma_1 = \bigotimes_v \sigma_{1,v}$ and $\sigma_2 = \bigotimes_v \sigma_{2,v}$ be two cuspidal representations of $GL_m(\mathbb{A})$ and $GL_n(\mathbb{A})$, respectively, such that $\sigma_{i,v_0} = \pi_i'$, $i = 1, 2$. We also fix a non-degenerate character $\chi' = \bigotimes_v \chi_v'$ of $U(\mathbb{A})/U(k)$ such that $\chi_{v_0}' = \chi$. Let S be a finite set of places of k, including the archimedean ones, such that $v \notin S$ implies that $\sigma_{1,v}$, $\sigma_{2,v}$, and χ_v' are all unramified; clearly $v_0 \in S$. Let S' be the subset of all the finite places in S different from v_0. Finally we extend S and S' to contain all the finite places of k which lie over the same rational prime of Q as v_0 ($v_0 \notin S'$).

We need:

LEMMA 2.6: *There exists a character* $\rho_1 = \bigotimes_v \rho_{1,v}$ *of* $\mathbb{A}^*/k^*$ *such that*

(1) $\rho_{1,v} = 1$ *for* $v = v_0$, *and*

(2) *for all* $v \in S'$, $C_{\chi'_v}(\nu(s,s'), (\sigma_{1,v} \otimes \rho_{1,v}) \otimes \sigma_{2,v}, \theta, w_\theta)$ *is a monomial in* q_v^s.

PROOF: By Lemma 12.5 of [7], choose ρ_1 satisfying (1) such that for all $v \in S'$ the order of $\rho_{1,v}$ is arbitrarily large. Now fix $v \in S'$, choose $P^*_{i,v} = M^*_{i,v} N^*_{i,v}$, $i = 1,2$, parabolic subgroups of GL_m and GL_n, respectively; and irreducible supercuspidal representation $\tau_{i,v}$ of $M^*_{i,v}$ such that

$$\sigma_{1,v} \hookrightarrow \operatorname*{Ind}_{P^*_{1,v}(k_v) \uparrow GL_m(k_v)} \tau_{1,v} \text{ and } \sigma_{2,v} \hookrightarrow \operatorname*{Ind}_{P^*_{2,v}(k_v) \uparrow GL_n(k_v)} \tau_{2,v}.$$

Then

$$\sigma_{1,v} \otimes \rho_{1,v} \hookrightarrow \operatorname*{Ind}_{P^*_{1,v}(k_v) \uparrow GL_m(k_v)} (\tau_{1,v} \otimes \rho_{1,v}).$$

Suppose $M^*_{i,v} \cong GL_{r_{i,1}} \times \ldots \times GL_{r_{i,t_i}}$, $i = 1,2$, $t_i \in \mathbb{Z}^+$, $r_{i,1} + \ldots + r_{i,t_i} = m$ or n according as $i = 1$ or 2; and assume $\tau_{i,v} = \tau_{i,1} \otimes \ldots \otimes \tau_{i,t_i}$. Then by relation (1.5) we only need to prove (2) for

$$C_{\chi'_v}(\nu(s,s'), (\tau_{1,j} \otimes \rho_{1,v}) \otimes \tau_{2,k}, \theta_{jk}, w_{\theta_{jk}})$$

$1 \leq j \leq t_1$, $1 \leq k \leq t_2$. But now if we choose the orders of $\rho_{1,v}$ sufficiently large, we will have $\tau_{1,j} \otimes \rho_{1,v} \not\cong \tau_{2,k} \otimes \mu^s$ for all $s \in \mathbb{C}$, $1 \leq j \leq t_1$, and $1 \leq k \leq t_2$. Consequently $A(\nu(s,s'), (\tau_{1,j} \otimes \rho_{1,v}) \otimes \otimes \tau_{2,k}, w_{\theta_{jk}})$ is holomorphic for all s and $s' \in \mathbb{C}$. It also follows from Theorem 4.2 of [3] that for all s and $s' \in \mathbb{C}$, $I(\nu(s,s'), (\tau_{1,j} \otimes \rho_{1,v}) \otimes \otimes \tau_{2,k}, \theta_{jk})$ is irreducible. Now (2) follows if we use the definition of the local coefficients and their rationality (cf. Proposition 2.1). This completes the lemma.

We now prove:

PROPOSITION 2.2: *There exist a non-zero complex number* $c_1(\pi)$ *and an integer* $n_1(\pi)$ *such that*

$$C_\chi(\nu(s,s'), \pi, \theta, w_\theta) C_\chi(\nu(1-s,s'), \tilde{\pi}, \theta, w_\theta) = c_1(\pi) q^{n_1(\pi)s} \quad (2.1)$$

PROOF: For every $v \notin S$, let $L(s, \sigma_{1,v} \times \sigma_{2,v})$ be the local unramified Langlands' L-function attached to the pair $(\sigma_{1,v}, \sigma_{2,v})$, which is defined in general by H. Jacquet, I.I. Piatetski–Shapiro, and J.A. Shalika in [6]. Choose ρ_1 as in Lemma 2.6. Let $S_1 \supset S$ be a finite set of places such that $v \notin S_1$ implies that $\sigma_{1,v} \otimes \rho_{1,v}$ is unramified. Set:

$$L_{S_1}(s, (\sigma_1 \otimes \rho_1) \times \sigma_2) = \prod_{v \notin S_1} L(s, (\sigma_{1,v} \otimes \rho_{1,v}) \times \sigma_{2,v}).$$

Then by Theorem 4.1 of [15] we have

$$\begin{aligned} &L_{S_1}(s, (\sigma_1 \otimes \rho_1) \times \tilde{\sigma}_2) \\ &\quad = \prod_{v \in S_1} C_{\chi'_v}(\nu(s, s'), (\sigma_{1,v} \otimes \rho_{1,v}) \otimes \sigma_{2,v}, \theta, w_\theta) \\ &\quad \cdot L_{S_1}(1 - s, (\tilde{\sigma}_1 \otimes \rho_1^{-1}) \times \sigma_2) \end{aligned} \tag{2.2.1}$$

Changing s to $1-s$, σ_1 to $\tilde{\sigma}_1$, σ_2 to $\tilde{\sigma}_2$, and ρ_1 to ρ_1^{-1}, we obtain

$$\begin{aligned} &L_{S_1}(1 - s, (\tilde{\sigma}_1 \otimes \rho_1^{-1}) \times \sigma_2) \\ &\quad = \prod_{v \in S_1} C_{\chi'_v}(\nu(s, s'), (\tilde{\sigma}_{1,v} \otimes \rho_{1,v}^{-1}) \otimes \tilde{\sigma}_{2,v}, \theta, w_\theta) \\ &\quad \cdot L_{S_1}(s, (\sigma_1 \otimes \rho_1) \times \tilde{\sigma}_2) \end{aligned} \tag{2.2.2}$$

Finally comparing (2.2.1) and (2.2.2) we have

$$\begin{aligned} &\prod_{v \in S_1} C_{\chi'_v}(\nu(s, s'), (\sigma_{1,v} \otimes \rho_{1,v}) \otimes \sigma_{2,v}, \theta, w_\theta) \\ &\qquad \cdot C_{\chi'_v}(\nu(1 - s, s'), (\tilde{\sigma}_{1,v} \otimes \rho_{1,v}^{-1}) \otimes \tilde{\sigma}_{2,v}, \theta, w_\theta) = 1 \end{aligned} \tag{2.2.3}$$

First suppose $v = \infty$. Then by Theorem 3.2.2 of [15], the corresponding factor in (2.2.3) is $+1$ or -1.

Next suppose $v \in S_1$ is finite. First assume $v \in S$ and $v \neq v_0$. Then for every $v \neq v_0$, the corresponding factor in (2.2.3) is only a monomial of q_v^s (Lemma 2.6). Now assume $v = v_0$ or $v \in S_1 - S$. Since no $v \in S_1 - S$ lies over the same rational prime as v_0, we conclude that no two polynomials in $\mathbb{C}[q^s, q^{-s}]$, $q = q_{v_0}$, and $\mathbb{C}[q_v^s, q_v^{-s}]$, $v \in S_1 - S$, can have a non-trivial factor in common. Consequently, the corresponding factor for v_0 in (2.2.3) must be a monomial in q^s, i.e. there exist $c_1(\pi') \in \mathbb{C} - \{0\}$ and $n_1(\pi') \in \mathbb{Z}$, $\pi' = \pi'_1 \otimes \pi'_2$, such that

$$C_\chi(\nu(s, s'), \pi', \theta, w_\theta) C_\chi(\nu(1 - s, s'), \tilde{\pi}', \theta, w_\theta) = c_1(\pi') q^{n_1(\pi')s} \tag{2.2.4}$$

Now, set

$$t = s + (m+n)/2mn \cdot (s_1 - s_2).$$

Then

$$\pi' \otimes q^{\langle \nu(s,s'), H_\theta(\ )\rangle} \cong \pi \otimes q^{\langle \nu(t,s'), H_\theta(\ )\rangle}$$

and (2.2.4) reduces to

$$C_\chi(\nu(t,s'), \pi, \theta, w_\theta) C_\chi(\nu(1-t,s'), \tilde{\pi}, \theta, w_\theta) = c_1(\pi) q^{n_1(\pi')t}$$

$c_1(\pi) \in \mathbb{C} - \{0\}$. The proposition now follows if we set $n_1(\pi) = n_1(\pi')$ and change t to s.

REMARK 2.1: Lemma 2.6 would not be necessary if we knew that v_0 is the only place of k lying over a rational prime.

PROOF OF THEOREM 2.1: For parts (a) and (b), observe that by the same parts of Proposition 2.1, $C_\chi^{-1}(\nu(s,s'), \pi, \theta, w_\theta)$ and $C_\chi^{-1}(\nu(1-s,s'), \tilde{\pi}, \theta, w_\theta)$ must both be polynomials in $\mathbb{C}[q^s, q^{-s}]$. Consequently by Proposition 2.2, $C_\chi(\nu(s,s'), \pi, \theta, w_\theta)$ must be a monomial as desired (independent of s' by Lemma 2.4).

For part (c), we again observe by (2.1) that the poles of $C_\chi(\nu(s,s'), \pi, \theta, w_\theta)$ must be the same as the zeros of $C_\chi(\nu(1-s,s'), \tilde{\pi}, \theta, w_\theta)$. But the zeros of this last function are given by $1 - \omega_1^{-1}\omega_2(\varpi) q^{-m(1-s)}$, and therefore $C_\chi(\nu(s,s'), \pi, \theta, w_\theta)$ has the required form. This completes the theorem.

Now, let

$$L(s, \pi_1 \times \pi_2) = (1 - \omega_1\omega_2(\varpi) q^{-ms})^{-1}$$

if $m = n$, $\omega_1\omega_2$ unramified, and there exists some $s \in \mathbb{C}$ such that $\pi_1 \cong \pi_2 \otimes \mu^s$, and

$$L(s, \pi_1 \times \pi_2) = 1$$

otherwise. In each case define (cf. Theorem 2.1)

$$\varepsilon(s, \pi_1 \times \pi_2, \chi) = c(\pi_1 \otimes \tilde{\pi}_2) q^{n(\pi_1 \otimes \tilde{\pi}_2)s}.$$

Then

$$C_\chi(-mn/(m+n) \cdot s \cdot \alpha, \pi_1 \otimes \tilde{\pi}_2, \theta, w_\theta) = \varepsilon(s, \pi_1 \times \pi_2, \chi) \frac{L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2)}{L(s, \pi_1 \times \pi_2)}.$$

In [6], using a completely different method, similar factors are defined for any pair of irreducible representations (cf. Section 3 here), and in this particular case their L-functions are in agreement with ours (cf. [15]). It is the subject of a subsequent paper to show that the two root numbers are also equal.

We now reformulate Theorem 2.1 as follows:

THEOREM 2.2: *Let π_1 and π_2 be two irreducible supercuspidal representations of $GL_m(F)$ and $GL_n(F)$, respectively. Define $\varepsilon(s, \pi_1 \times \pi_2, \chi)$ and $L(s, \pi_1 \times \pi_2)$ as above. Then*

$$C_\chi(-mn/(m+n)\cdot s\cdot\alpha + s'\cdot\beta, \pi_1 \otimes \pi_2, \theta, w_\theta) = \varepsilon(s, \pi_1 \times \tilde{\pi}_2, \chi)\frac{L(1-s, \tilde{\pi}_1 \times \pi_2)}{L(s, \pi_1 \times \tilde{\pi}_2)}.$$

We now observe the following properties of $L(s, \pi_1 \times \pi_2)$, $c(\pi)$, and $n(\pi)$.

PROPOSITION 2.3: *Let π_1 and π_2 be two irreducible unitary supercuspidal representations of $GL_m(F)$ and $GL_n(F)$, respectively. Define $\varepsilon(s, \pi_1 \times \pi_2, \chi)$ and $L(s, \pi_1 \times \pi_2)$ as before. Write*

$$\varepsilon(s, \pi_1 \times \pi_2, \chi) = c(\pi_1 \otimes \tilde{\pi}_2)q^{n(\pi_1 \otimes \tilde{\pi}_2)s}.$$

Then

$$\overline{L(\bar{s}, \pi_1 \times \tilde{\pi}_2)} = L(s, \tilde{\pi}_1 \times \pi_2) \tag{2.2}$$

$$c(\pi_1 \otimes \pi_2) = \overline{c(\pi_2 \otimes \pi_1)} \tag{2.3}$$

and

$$n(\pi_1 \otimes \pi_2) \equiv n(\pi_2 \otimes \pi_1) \bmod (2\pi i(\log q)^{-1}) \tag{2.4}$$

PROOF: (2.2) is a consequence of $\bar{\omega}_i = \omega_i^{-1}$. To prove (2.3) and (2.4), we use relations (1.8) and (2.2), together with Theorem 2.1, to conclude

$$\overline{c(\pi_1 \otimes \pi_2)}q^{-n(\pi_1 \otimes \pi_2)s} = c(\pi_2 \otimes \pi_1)q^{-n(\pi_2 \otimes \pi_1)s}.$$

The desired relations are now trivial.

3. Unitary representations and normalization of intertwining operators

We let the notation be as in Section 1, i.e. $P = P_\theta$ is any standard parabolic subgroup of $G = GL_r$ generated by a partition $(r_1, \ldots, r_p)$ of r. We fix an irreducible unitary non-degenerate representation π of $M(F) = M_\theta(F)$, where $M = M_\theta$ is the standard Levi factor of $P_\theta = M_\theta \cdot N_\theta$. Fix $w \in W(\theta, \theta')$, where $\theta' \in \{\theta\}$.

Choose $\theta = \theta_1, \ldots, \theta_N \subset \Delta$, $N = \mathrm{card}(\Sigma_r(\theta, w(\theta), w))$, as in Theorem 2.1.1 of [15] such that

$$A(\nu, \pi, w) = A(\nu_{N-1}, \pi_{N-1}, w_{N-1}) \ldots A(\nu_1, \pi_1, w_1),$$

$w = w_{N-1} \ldots w_1$, $\nu_i = w_{i-1}(\nu_{i-1})$, $\pi_i = w_{i-1}(\pi_{i-1})$, $2 \leq i \leq N-1$, $\nu_1 = \nu$, and $\pi_1 = \pi$. The representations π_i are all unitary.

For each i, let $\pi_{i,1}$ and $\pi_{i,2}$ be two adjacent representations of the blocks of the standard Levi factor of $P_i(F)$ which are interchanged by w_i. Let m_i and n_i be the dimensions of the corresponding blocks, respectively. Denote by $L(s, \pi_{i,1} \times \tilde{\pi}_{i,2})$ the Langlands' L-function attached to the pair $(\pi_{i,1}, \tilde{\pi}_{i,2})$ by H. Jacquet, I.I. Piatetski–Shapiro, and J.A. Shalika [6].

The following lemma is clear.

Lemma 3.1: *The product*

$$\prod_{i=1}^{N-1} L(s, \pi_{i,1} \times \tilde{\pi}_{i,2})$$

is independent of the decomposition of w.

Now, in view of Lemma 3.1, we define

$$L(s, \pi, w) = \prod_{i=1}^{N-1} L(s, \pi_{i,1} \times \tilde{\pi}_{i,2}) \tag{3.1}$$

From now on we assume $\nu = 0$. Then $I(\pi, \theta) = I(0, \pi, \theta)$ is the unitarily induced representation of $G(F)$ by π. We now introduce a new normalizing factor for $A(0, \pi, w)$ as follows. We set

$$r(\pi, w, \chi) = C_\chi(0, \pi, \theta, w) \frac{L(1, \pi, w)}{L(1, \tilde{\pi}, w)} \tag{3.2}$$

and we define

$$\alpha(0, \pi, w) = r(\pi, w, \chi)A(0, \pi, w) \tag{3.3}$$

We then have:

THEOREM 3.1: *The operators* $\alpha(0, \pi, w)$ *have the following properties*:
(a) $\alpha(0, w(\pi), w^{-1})\alpha(0, \pi, w) = 1$
(b) $\alpha(0, \pi, w)^* = \alpha(0, w(\pi), w^{-1})$, *and consequently* $\alpha(0, \pi, w)$ *is unitary.*
(c) *Given* $v \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$, *define* $\pi_v = \pi \otimes q^{\langle v, H_\theta(\ )\rangle}$, *and set*

$$\alpha(v, \pi, w) = \alpha(0, \pi_v, w).$$

Then $\alpha(v, \pi, w)$ *is a meromorphic function of* v *on* $(\mathfrak{a}_\theta)^*_{\mathbb{C}}$, *and furthermore for every decomposition* $w = w_1 w_2$ *the functional equation*

$$\alpha(v, \pi, w) = \alpha(w_2(v), w_2(\pi), w_1)\alpha(v, \pi, w_2) \tag{3.4}$$

is satisfied.

PROOF: We only need to show (a) and (b) as (c) follows from the results in [15]. To prove (a), observe that by part (a) of Proposition 3.1.4 of [15], the same identity holds for $\mathscr{A}(0, \pi, w)$ (cf. Section 1.2 here). But now by relation (3.1)

$$L(s, w(\pi), w^{-1}) = \prod_{i=1}^{N-1} L(s, \tilde{\pi}_{i,1} \times \pi_{i,2}),$$

and consequently

$$L(s, w(\pi), w^{-1}) = L(s, \tilde{\pi}, w) \tag{3.1.1}$$

Part (a) now follows from relations (3.2) and (3.1.1), together with the above observation.

To prove (b), we observe that if π_1 and π_2 are unitary, then

$$\begin{aligned}\overline{L(s, \pi_1 \times \pi_2)} &= L(\bar{s}, \bar{\pi}_1 \times \bar{\pi}_2)\\ &= L(\bar{s}, \tilde{\pi}_1 \times \tilde{\pi}_2).\end{aligned}$$

Consequently

$$\overline{L(1, \pi, w)}/\overline{L(1, \tilde{\pi}, w)} = L(1, \tilde{\pi}, w)/L(1, \pi, w)$$

which is the normalizing factor for $\mathscr{A}(0, w(\pi), w^{-1})$ by (3.1.1). Now (b) follows from part (b) of Proposition 3.1.4 of [15].

Now, let π_1 and π_2 be two irreducible unitary representations of $GL_m(F)$ and $GL_n(F)$, which furthermore are respectively the v_0 local components of the cuspidal representations $\sigma_1 = \bigotimes_v \sigma_{1,v}$ and $\sigma_2 = \bigotimes_v \sigma_{2,v}$ of $GL_m(\mathbb{A})$ and $GL_n(\mathbb{A})$ as in Lemma 2.5, i.e. $\pi_i = \sigma_{i,v_0}$, $i = 1, 2$. As before $\mathbb{A}$ is the ring of adeles of a number field k for which $k_{v_0} = F$. The following proposition will allow us to define the Langlands' root number attached to a pair of unitary representations satisfying the above condition. (We assume the existence of the global functional equations attached to the pairs which is the subject of a work in progress of H. Jacquet, I.I. Piatetski–Shapiro, and J.A. Shalika [6]).

Proposition 3.1: *Let π_1 and π_2 be two irreducible unitary representations of $GL_m(F)$ and $GL_n(F)$, which furthermore are local components of some cuspidal representations of $GL_m(A)$ and $GL_n(A)$, respectively. Then there exists a non-zero complex number $c(\pi_1 \otimes \pi_2)$ and an integer $n(\pi_1 \otimes \pi_2)$ such that*

$$C_\chi(-2s/(m+n)\cdot \rho_\theta, \pi_1 \otimes \tilde{\pi}_2, \theta, w_\theta)\frac{L(s, \pi_1 \times \pi_2)}{L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2)}$$

is equal to

$$c(\pi_1 \otimes \pi_2) q^{n(\pi_1 \otimes \pi_2)s}.$$

We set

$$\varepsilon(s, \pi_1 \times \pi_2, \chi) = c(\pi_1 \otimes \pi_2) q^{n(\pi_1 \otimes \pi_2)s}$$

and we call $\varepsilon(s, \pi_1 \times \pi_2, \chi)$ the Langlands root number attached to the pair (π_1, π_2). Furthermore $c(\pi_1 \otimes \pi_2)$ and $n(\pi_1 \otimes \pi_2)$ satisfy relations (2.3) *and* (2.4), *respectively.*

Proof: This can be proved by comparing functional equation (2.2.1) with that of [6], and making use of the lemmas proved in Section 2, the following lemma, and finally an argument similar to that of Proposition 2.2 and the fact that Γ-functions (L-functions at ∞) are not rational functions of q^s.

Lemma 3.2: *Let π_1 and π_2 be two irreducible admissible non-degenerate representations of $GL_m(k_v)$ and $GL_n(k_v)$, respectively* (v *finite*). *Denote by $L(s, \pi_1 \times \pi_2)$ the corresponding Jacquet–Shalika L-function attached to* *the pair* (π_1, π_2). *Let* ρ *be a character of* $\mathcal{O}_v^*$. *Then there exists an integer* $m_v \geq 0$ *such that* $L(s, (\pi_1 \otimes \rho) \times \pi_2) = 1$ *for every* ρ *whose order is greater than* m_v.

PROOF: Given $w_1 \in W(\pi_1)$ and $w_2 \in W(\pi_2)$; let $\psi(s, w_1, w_2)(\psi(s, w_1, w_2, \phi)$ if $m = n)$ be as in [6] (as in [9], resp.). Then $L(s, \pi_1 \times \pi_2)$ is the G.C.D. of all these $\psi(s, w_1, w_2)(\psi(s, w_1, w_2, \phi)$, resp.) when w_1 and w_2 range over $W(\pi_1)$ and $W(\pi_2)$ (ϕ over $\mathscr{S}(F^m)$, resp.), respectively. But now a glance at the integral representations of ψ's and making use of Proposition 2.2 of [8] will show that, when $w_1 \in W(\pi_1 \otimes \rho)$, $w_2 \in W(\pi_2)$, and the order of ρ is sufficiently large, $\psi(s, w_1, w_2)(\psi(s, w_1, w_2, \phi)$, resp.) are all holomorphic and therefore $L(s, (\pi_1 \otimes \rho) \times \pi_2) = 1$. This completes the lemma.

REMARK 3.1: This is a result similar to Theorem 2.2.

REMARK 3.2: We in fact believe that our root numbers are equal to those of H. Jacquet, I.I. Piatetski–Shapiro, and J.A. Shalika.

We now set

$$\varepsilon(s, \pi, w, \chi) = \prod_{i=1}^{N-1} \varepsilon(s, \pi_{i,1} \times \tilde{\pi}_{i,2}, \chi)$$

which is clearly independent of the decomposition of w since $L(s, \pi, w)$ is. Then

$$r(\pi, w, \chi) = \varepsilon(0, \pi, w, \chi) \cdot L(1, \pi, w)/L(0, \pi, w)$$
$$= \prod_{i=1}^{N-1} \varepsilon(0, \pi_{i,1} \times \tilde{\pi}_{i,2}, \chi) \cdot L(1, \pi_{i,1} \times \tilde{\pi}_{i,2})/L(0, \pi_{i,1} \times \tilde{\pi}_{i,2})$$

is the normalizing factor conjectured by Langlands in Appendix II of [10]. Consequently in this case Theorem 3.1 provides a positive answer for the questions posed there.

4. Applications

4.1. *Plancherel measures.* From relation (1.4), it is clear that to compute the Plancherel measures, one only needs to compute them for maximal parabolic subgroups and supercuspidal representations. For this reason, let us take the notation as in Section 2, i.e. take $M = GL_m \times GL_n$ and $\pi = \pi_1 \otimes \pi_2$, where π_1 and π_2 are irreducible supercuspidal representations of $GL_m(F)$ and $GL_n(F)$, respectively. We then have

THEOREM 4.1: *Let π_1 and π_2 be two irreducible supercuspidal representations of $GL_m(F)$ and $GL_n(F)$, respectively. Set $M_\theta = GL_m \times GL_n$, and $\nu = -mn/(m+n)\cdot s\cdot\alpha + s'\cdot\beta$. Then*

$$\mu(\pi,\nu) = \gamma^2(G/P_\theta)|c(\pi)|^2 \frac{1-\omega_1\omega_2^{-1}(\bar{\omega})q^{-ms}}{1-\omega_1^{-1}\omega_2(\bar{\omega})q^{-m(1-s)}} \frac{1-\omega_1^{-1}\omega_2(\bar{\omega})q^{ms}}{1-\omega_1\omega_2^{-1}(\bar{\omega})q^{-m(1+s)}}$$

if $m = n$ and $\pi_1 \cong \pi_2 \otimes \mu^s$ for some $s \in \mathbb{C}$, and

$$\mu(\pi,\nu) = \gamma^2(G/P_\theta)|c(\pi)|^2$$

otherwise. In other words:

$$\mu(\pi,\nu) = \gamma^2(G/P_\theta)\varepsilon(s,\pi_1\times\tilde{\pi}_2,\chi)\varepsilon(-s,\tilde{\pi}_1\times\pi_2,\chi) \cdot\frac{L(1+s,\pi_1\times\tilde{\pi}_2)}{L(s,\pi_1\times\tilde{\pi}_2)}\cdot\frac{L(1-s,\tilde{\pi}_1\times\pi_2)}{L(-s,\tilde{\pi}_1\times\pi_2)}.$$

PROOF: This is a consequence of the relation

$$C_\chi(w_\theta(\nu), w_\theta(\pi), w_\theta(\theta), w_\theta^{-1})C_\chi(\nu,\pi,\theta,w_\theta) = \gamma^{-2}(G/P_\theta)\mu(\pi;\nu)$$

(Proposition 3.1.1 of [15]), and Theorems 2.1 and 2.2.

More generally, we have:

THEOREM 4.2: *Let π_1 and π_2 be two irreducible unitary representations of $GL_m(F)$ and $GL_n(F)$ which furthermore are local components of some cuspidal representations of $GL_m(\mathbb{A})$ and $GL_n(\mathbb{A})$, respectively. Set $M_\theta = GL_m \times GL_n$, $\nu = -mn/(m+n)\cdot s\cdot\alpha + s'\cdot\beta$, and $\pi = \pi_1\otimes\pi_2$. Then*

$$\mu(\pi,\nu) = \gamma^2(G/P_\theta)\varepsilon(s,\pi_1\times\tilde{\pi}_2,\chi)\varepsilon(-s,\pi_1\times\tilde{\pi}_2,\chi) \cdot\frac{L(1+s,\pi_1\times\tilde{\pi}_2)}{L(s,\pi_1\times\tilde{\pi}_2)}\cdot\frac{L(1-s,\tilde{\pi}_1\times\pi_2)}{L(-s,\tilde{\pi}_1\times\pi_2)}.$$

4.2. *Irreducibility Criteria.* We shall now give a new proof of a result of I.N. Bernstein and A.V. Zelevinskii (if part of Theorem 3 of [2]) on irreducibility of the representations induced from supercuspidal representations. The proof is now very short; in fact not much besides Theorem 2.1 of this paper is necessary to prove this converse to their Theorem 4.2 in [3] (if part of Theorem 3 of [2]).

We again choose a partition $(r_1, \ldots, r_p)$ of r and we let $P = P_\theta = M_\theta N_\theta$ be the corresponding parabolic subgroup. Then $M_\theta(F) = GL_{r_1}(F) \times \times \ldots \times GL_{r_p}(F)$. Let $\pi = \pi_1 \otimes \ldots \otimes \pi_p$ be an irreducible supercuspidal representation of $M_\theta(F)$.

THEOREM 4.3: (I.N. Bernstein and A.V. Zelevinskii). *Suppose for some i and j, $1 \leq i \neq j \leq p$, $\pi_i \cong \pi_j \otimes \mu$. Then the representation $I(0, \pi, \theta)$ is reducible.*

PROOF: Suppose for some i and j as above, $\pi_i \cong \pi_j \otimes \mu$. Then by part (c) of Theorem 2.1 of this paper and Proposition 3.2.1 of [15], there exists a factor $C_\chi(\nu_k, \pi_k, \theta_k, w_k)$ in $C_\chi(\nu, \pi, \theta, w_\theta)$ which has a pole at $\nu_k = 0$. Now let M'_k denote the centralizer of A_{Ω_k} in G, $\Omega_k = \theta_k \cup \{\alpha_k\}$, and use $P'_k = M'_k \cdot N'_k$ for its corresponding standard parabolic subgroup. Then by Theorem 3.3.1 of [15], the representation $\operatorname{Ind}_{P^*_k(F) \uparrow M'_k(F)} \pi_k$ is reducible, where $P^*_k = M_{\theta_k} \cdot (M'_k \cap N_{\theta_k})$. Consequently

$$I(0, \pi_k, \theta_k) = \operatorname*{Ind}_{P'_k(F) \uparrow G(F)} \big(\operatorname*{Ind}_{P^*_k(F) \uparrow M'_k(F)} \pi_k \big)$$

is reducible. But now by Theorem 6.4.1 of [4], so is $I(0, \pi, \theta)$. This proves the theorem.

Theorem 4.3 is in fact true for certain unitary representations. For the sake of simplicity we restrict ourselves to maximal parabolics, i.e. we assume $r = 2$. Then we have

THEOREM 4.4: *Let π_1 and π_2 be two irreducible unitary representations of $GL_m(F)$ and $GL_n(F)$ which furthermore are local components of some cuspidal representations of $GL_m(\mathbb{A})$ and $GL_n(\mathbb{A})$, respectively. Let $L(s, \pi_1 \times \pi_2)$ be the corresponding Jacquet–Piatetski–Shapiro–Shalika L-function. Suppose s_0 is a pole of $L(1 - s, \tilde{\pi}_1 \times \pi_2)/L(s, \pi_1 \times \tilde{\pi}_2)$. Then $I(-mn/(m+n) \cdot s_0 \cdot \alpha, \pi, \theta)$ is reducible, where $\pi = \pi_1 \otimes \pi_2$.*

PROOF: This is a consequence of the definition of the local coefficients and Proposition 3.1.

4.3. *Non-degeneracy of subrepresentations.* It is proved in [16] that for a principal series representation of a real group, the unique (topologically) non-degenerate subquotient appears in fact as a subrepresentation, if the parameter $\nu \in (\mathfrak{a}_\phi)^*_{\mathbb{C}}$ is in the positive Weyl chamber. Here we shall prove a similar result for $G(F) = GL_r(F)$, F non-archimedean, when $P = MN$ is maximal, $M = GL_m \times GL_n$, and $\pi = \pi_1 \times \pi_2$ is supercuspidal and unitary.

THEOREM 4.5: *Suppose* $\mathrm{Re}(s) \geq 0$. *Set* $\nu(s, s') = -mn/(m+n) \cdot s \cdot \alpha + s' \cdot \beta$. *Then* $I(\nu(s,s'), \pi, \theta)$ *contains a unique non-degenerate irreducible subrepresentation. Furthermore suppose* $I(\nu(s,s'), \pi, \theta)$ *is reducible; then the image of* $A(\nu(s,s'), \pi, \theta)$ *in* $I(\nu(-s,s'), w_\theta(\pi), w_\theta(\theta))$ *is degenerate.*

PROOF: We only need to prove this for reducible $I(\nu(s,s'), \pi, \theta)$. By Theorem 4.3, $I(\nu(s,s'), \pi, \theta)$ is reducible only when $s = 1$, $m = n$, and $\pi_1 \cong \pi_2$. Consequently by part (c) of Theorem 2.1, $C_\chi(\nu(s,s'), \pi, \theta, w_\theta)$ has a pole at $s = 1$. Now from the corollary of Theorem 3.3.1 of [15], it follows that the image of $A(\nu(1,s'), \pi, \theta)$ in $I(\nu(-1,s'), w_\theta(\pi), w_\theta(\theta))$ is degenerate, and consequently the kernel of $A(\nu(1,s'), \pi, \theta)$ which is irreducible (the length of $I(\nu(1,s'), \pi, \theta)$ is 2) must be in fact non-degenerate. This completes the theorem.

COROLLARY: *Suppose* $P_\theta = M_\theta N_\theta$ *is a standard parabolic subgroup of* G, *and assume* $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$ *is in the positive Weyl chamber of* (P_θ, A_θ), *i.e.* $\mathrm{Re}\langle \nu, H_\alpha \rangle > 0$ *for* $\forall \alpha \in \Sigma^+(\theta)$. *Let* π *be an irreducible unitary supercuspidal representation of* $M_\theta(F)$. *Suppose* $I(-\nu, \pi, \theta)$ *is reducible. Then the well-defined image of* $A(-\nu, \pi, w_\theta)$ *in* $I(-w_\theta(\nu), w_\theta(\pi), w_\theta(\theta))$, *i.e. the Langlands' quotient, is degenerate.*

PROOF: Since ν is in the positive Weyl chamber and π is unitary; the operator $A(-\nu, \pi, w_\theta)$ is well defined (even convergent). But now suppose $I(-\nu, \pi, \theta)$ is reducible. Then by Theorem 3.3.1 and Proposition 3.2.1 of [15], one of the rank one local coefficients must have a pole. Now Theorem 4.5 implies that the image of the corresponding rank one intertwining operator is degenerate, which proves the corollary.

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