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REAL PROJECTIVE STRUCTURES ON RIEMANN SURFACES

Gerd Faltings

§1. Introduction

Suppose X is a compact Riemann surface of genus $g \geq 0$, $S \subset X$ a finite set, and that for each $x \in S$ there is given an integer $\mathbf{e}(x) \in \{2, 3, \ldots, \infty\}$. We use the notations

$$S^f = \{x \in S \mid \mathbf{e}(x) < \infty\},$$
$$S^\infty = S - S^f,$$
$$X' = X - S^\infty,$$

and

$$\chi = 2g - 2 + \sum_{x \in S} (1 - 1/\mathbf{e}(x)).$$

Classical uniformization-theory then tells us that for $\chi > 0$ there exists a discrete subgroup

$$\bar{\Gamma} \subset \mathbf{PSL}(2, \mathbb{R}) = \mathrm{Aut}(\mathbb{H})$$

($\mathbb{H}$ is the upper halfplane) and a projection

$$\pi : \mathbb{H} \to X',$$

which induces an isomorphism

$$_{\bar{\Gamma}}\backslash\mathbb{H} \xrightarrow{\sim} X',$$

such that for any $z \in H$ the stabilizer $\bar{\Gamma}_z \subset \bar{\Gamma}$ has order $\mathbf{e}(\pi(z))$, and such

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that the points of S^∞ correspond to the cusps of $\bar{\Gamma}$. A proof of these facts can be found in [2], Ch. IV. 9., for example. Furthermore $\bar{\Gamma}$ can be generated by elements

$$\bar{A}_1, \ldots, \bar{A}_g, \bar{B}_1, \ldots, \bar{B}_g, \bar{C}_1, \ldots, \bar{C}_n,$$

where n is the number of elements of S, and the C_i's correspond to certain $x_i \in S$. The defining relations between these generators are:

$$\bar{C}_i^{e(x_i)} = 1, \text{ for } x_i \in S^f,$$
$$\bar{A}_1\bar{B}_1\bar{A}_1^{-1}\bar{B}_1^{-1}\bar{A}_2\bar{B}_2\bar{A}_2^{-1}\bar{B}_2^{-1}\ldots\bar{A}_g\bar{B}_g\bar{A}_g^{-1}\bar{B}_g^{-1}\bar{C}_1\bar{C}_2\ldots\bar{C}_n = 1.$$

Unfortunately this beautiful result does not tell us how to construct this covering of X', if for example X is given as a complex submanifold of some projective space. Thus, it should be interesting to look for a more concrete construction, especially since some open problems in algebraic geometry and number-theory can be seen as problems in uniformization-theory. For example the Weil-conjecture about elliptic curves over $\mathbb{Q}$ asks for uniformizations with $\bar{\Gamma}$ contained in $\mathbf{PSL}(2, \mathbb{Z})$.

The approach of this paper is as follows: Suppose $\bar{\Gamma}$ can be lifted to a subgroup Γ of $\mathbf{SL}(2, \mathbb{R})$. (This happens precisely if all the finite $\mathbf{e}(x)$ are odd). The natural representation of Γ on $\mathbb{R}^2$ defines a locally constant sheaf $\mathbf{V}$ on $X - S$, whose second exterior power is constant. Γ is then conjugate in $\mathbf{SL}(2, \mathbb{C})$ to the monodromy-group of a connection on the holomorphic vectorbundle $\mathscr{E} = \mathbf{V} \otimes_{\mathbb{R}} \mathcal{O}_X$. $\mathscr{E}$ can be extended to a vectorbundle on X such that the connection has regular singular points in S. Furthermore the singularities can be described.

We call a connection on $\mathscr{E}$ permissible if it fits into that description. Up to a certain equivalence-relation the permissible connections form an affine complex linear space of dimension $3g - 3 + n$, and we want to characterize the uniformizing connection among them. There is one further property of this connection besides being permissible, namely that its monodromy-group is conjugate in $\mathbf{SL}(2, \mathbb{C})$ to a subgroup of $\mathbf{SL}(2, \mathbb{R})$. Unfortunately this does not determine it uniquely, contrary to some people's (and originally the author's) belief. (See for example [6]). We even conjecture that there always exist infinitely many such connections, and we give some reasons for this conjecture.

More precisely we denote by $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{R}))$ respectively $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{C}))$ the spaces of representations of Γ in $\mathbf{SL}(2, \mathbb{R})$ respectively $\mathbf{SL}(2, \mathbb{C})$, where the representations must satisfy certain additional conditions which are fulfilled by the monodromy-representations of permissible connections. We thus obtain a mapping from the space of permissible connections into $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{C}))$, the monodromy-mapping. We show that $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{C}))$ is a complex manifold of dimension $6g - 6 + 2n$ near the image of a permissible connection, and that our mapping is an immersion of complex manifolds there. Furthermore if the permissible connection has real monodromy, so that its image is contained in $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{R}))$, $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{R}))$ is a real submanifold of dimension $6g - 6 + 2n$ intersecting transversally the image of our mapping. Finally if we form a universal family of our pairs (X, S), with base a suitable Teichmüller-space $\mathbf{T}$ of dimension $3g - 3 + n$, the permissible connections form a complex vectorbundle of rank $3g - 3 + n$ over $\mathbf{T}$.

The total space of this vectorbundle is a complex manifold of dimension $6g - 6 + 2n$, and the monodromy-mapping is a local isomorphism of this manifold into $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{C}))$. The connections with real monodromy form a real submanifold of dimension $6g - 6 + 2n$ such that the projection onto $\mathbf{T}$ becomes a local isomorphism if we restrict it to this real manifold. I hope, but cannot prove, that this local isomorphism is a covering.

In any case we see that permissible connections with real monodromy are isolated in the space of all permissible connections over a fixed X, and that they survive under small deformations of X. If the projection from the space of permissible connections onto the Teichmüller-space is a covering we can construct connections with real monodromy on a given special X by first constructing them on a X' which is of the same type as X, and then deforming X' into X. There is a certain topological invariant associated with a permissible connection with real monodromy, and this invariant is fixed under deformations. Thus we should be able to get infinitely many real connections on any X, by constructing them on various X''s in such a way that these invariants are different. We shall show examples of such constructions, which should make it clear that the only real difficulty in this approach is the deformation-process.

The topological invariant mentioned above is a decomposition of X into certain pieces which makes X look locally like the double of a Riemann-surface with boundary. If the permissible connection satisfies a certain property we can even give a rough classification of these connections, which shows that X must be related to some real curve, i.e., to a one-dimensional complex algebraic manifold which can be defined over $\mathbb{R}$. A simple dimension-count in the various moduli-spaces shows that this cannot happen for the general curve X. Unfortunately in concrete examples it usually seems to be impossible to verify this condition.

The paper is organized as follows: In the next three chapters we fix notations and list usually well-known results about uniformization and connections, real algebraic curves, and the spaces $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{C}))$. Often short proofs are given, since many results are not so easily accessible in the literature, or not in the generality we need.

After that we define uniformization-data and permissible connections. We here generalize mildly some results of Gunning in [4], but the main purpose of this part of the paper is a rephrasing of the classical theory in a different terminology, better suited for our purposes, which I could not find in the literature. For example it becomes rather clear why quadratic differentials are important in uniformization-theory.

In any case I do not claim that this language was invented by myself, nor that it constitutes an enormous progress.

In the next two chapters the reality-conditions come into play. We first define the decomposition of X mentioned above, and use it to classify a certain subclass of connections with real monodromy. After that we give an Eichler–Shimura theorem, and as consequence of that the deformation-theory mentioned above.

As far as I know these results are new, and they partly verify and partly show to be wrong some conjectures of Prof. I. Morrison. At the end we show how to construct connections with real monodromy. The structure theory of these connections tells us precisely how to do this.

I conclude this introduction with some remarks of a more personal nature: I am not a specialist of this field, and I ask the experts for their pardon for the inconveniences caused by this fact. My interest in this matter stems from my attempts to describe the uniformization-process, which is analytic in its nature, in terms of algebraic geometry, for example if X is given as a curve in some projective space. The permissible connections have such a description, but we cannot say more, so that I did not succeed with my original goals. Today I even think that an algebraic-geometric description of the uniformizing connection is impossible, the main reason being that it does not depend holomorphically on the moduli of the curve X. Nevertheless I hope that the reader finds some interest in the by-products of my fruitless efforts.

The referee has pointed out to me that Prof. Mandelbaum has obtained some results about ramified projective structures and the spaces $\mathbf{REP}(\Gamma, \mathbf{SL}(2, \mathbb{C}))$. His results are in *Trans. Amer. Math. Soc.*, vol. *163* (1972), 261–275, and vol. *183* (1973), 37–58; *Math. Annalen* vol. *214* (1975), 49–59.

§2. Uniformization and connections

The following results are standard: (Compare [1], [2]). Let X be a compact Riemann surface of genus g,

$$\mathbf{e}: X \to \mathbb{N} \cup \{\infty\} = \{1, 2, 3, \ldots, \infty\}$$

a function which takes the value one for almost all $x \in X$,

$$S^f = \{x \in X \mid 2 \le \mathbf{e}(x) < \infty\},$$
$$S^\infty = \{x \in X \mid \mathbf{e}(x) = \infty\},$$
$$S = S^f \cup S^\infty.$$

If

$$\chi = 2g - 2 + \sum_{x \in X} (1 - 1/\mathbf{e}(x))$$

is positive we can write as in the introduction

$$X' = X - S^\infty = {}_{\bar{\Gamma}}\backslash \mathbb{H},$$

where $\mathbb{H}$ denotes the upper half-plane, and $\bar{\Gamma} \subset \mathbf{PSL}(2, \mathbb{R})$ is a discrete subgroup such that the mapping from $\mathbb{H}$ to X' has the ramification prescribed by $\mathbf{e}$.

$\mathbb{H}^*$ is the union of $\mathbb{H}$ and the set of fixed-points of non-trivial parabolic elements of $\bar{\Gamma}$, with the usual topology. If ∞ is such a cusp a fundamental system of neighbourhoods of ∞ in $\mathbb{H}^*$ is given by the unions of ∞ itself and the sets $\{z \in \mathbb{H} \mid \mathrm{Im}(z) > c\}$, for c a positive number. This picture is transformed to the other cusps by conjugating $\bar{\Gamma}$ in $\mathbf{PSL}(2, \mathbb{R})$, and there is an isomorphism of topological spaces

$$X \cong {}_{\bar{\Gamma}}\backslash \mathbb{H}^*.$$

$\bar{\Gamma}$ is uniquely determined up to conjugation in $\mathbf{PSL}(2, \mathbb{R})$ by X and the function $\mathbf{e}$, and from a canonical dissection of X we obtain generators

$$\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_g, \bar{B}_1, \bar{B}_2, \ldots, \bar{B}_g, \bar{C}_1, \ldots, \bar{C}_n$$

of $\bar{\Gamma}$, where $S = \{x_1, \ldots, x_n\}$, and the $\bar{C}_j$ correspond to counter-clockwise paths around x_j.

$\bar{\Gamma}$ is defined by the relations

(i) $\bar{C}_j^{e_j} = 1$, for $x_j \in S^f$, $e_j = \mathbf{e}(x_j)$

(ii) $\bar{A}_1\bar{B}_2\bar{A}_1^{-1}\bar{B}_2^{-1}\ldots\bar{A}_g\bar{B}_g\bar{A}_g^{-1}\bar{B}_g^{-1}\bar{C}_1\bar{C}_2\ldots\bar{C}_n = 1$.

Following the introduction we ask whether there exists a subgroup $\Gamma \subset \mathbf{SL}(2,\mathbb{R})$ mapping isomorphically to $\bar{\Gamma}$ under the projection from $\mathbf{SL}(2,\mathbb{R})$ onto $\mathbf{PSL}(2,\mathbb{R})$. Such a lifting is obtained by choosing elements $A_1,\ldots,A_g$, $B_1,\ldots,B_g$, $C_1,\ldots,C_n$ of $\mathbf{SL}(2,\mathbb{R})$ projecting onto the corresponding elements of $\bar{\Gamma}$, and such that these elements fulfill the relations (i) and (ii).

In any case these relations are fulfilled up to a sign, and the A, B, C's define a lifting if and only if -1 is not contained in the group they generate. This implies that all finite e_i's are odd: For if $e_j = 2f$, with an integer f, the element C_j^f has order two in $\mathbf{SL}(2,\mathbb{R})$ (relation (i)!) and thus is equal to -1. Thus we have found a necessary condition for the existence of a lifting, and this condition turns out to be also sufficient:

THEOREM 1: *A lifting Γ exists if and only if all finite e_j are odd. Moreover any lifting can be obtained from a special one by*

(a) *Changing arbitrarily the signs of the A's and B's.*

(b) *Changing the signs of an even number of C_j's, for $x_j \in S^\infty$.*

PROOF: The second part of the theorem is trivial, and we only have to show the existence of a lifting, provided all the finite e_j are odd. In this case for $x_j \in S^f$ there exists a unique element C_j projecting to $\bar{C}_j$ and fulfilling relation (i). If S^∞ is not void we can fulfill relation (ii) by changing the sign of one $C_j, x_j \in S^\infty$, if necessary. We therefore assume that $S^\infty = 0$.

The C_j's are now given by relation (i), and we show that for arbitrary liftings $A_1,\ldots,A_g$, $B_1,\ldots,B_g$ of $\bar{A}_1,\ldots,\bar{A}_g$, $\bar{B}_1,\ldots,\bar{B}_g$ the group generated by these liftings and the C's does not contain -1. X is now a compact Riemann surface, and its canonical bundle $\mathscr{K}_X$ is even degree $2g-2$, so that there exists a theta-characteristic, that is a holomorphic linebundle $\mathscr{L}$ on X with

$$\mathscr{L}^{\otimes 2} \cong \mathscr{K}_X.$$

$\mathscr{L}$ has a non-trivial meromorphic section. Its square corresponds under the isomorphism above to a meromorphic differential on X which has even order in any point of X. The pullback of this differential under the projection of $\mathbb{H}$ onto $X = {}_{\bar{\Gamma}}\backslash\mathbb{H}$ is a $\bar{\Gamma}$-invariant meromorphic differential on $\mathbb{H}$ which also has even order everywhere: The orders are increased by the numbers $e_j - 1$ which are even. We write this differential as $f(z)dz$, with f a meromorphic function on $\mathbb{H}$ having even orders. As $\mathbb{H}$ is simply connected there exists a meromorphic function g on $\mathbb{H}$ with $f = g^2$.

The $\bar{\Gamma}$-invariance of $f(z)dz$ is equivalent to the following identity for any element

$$\gamma = \begin{pmatrix} ab \\ cd \end{pmatrix} \in \Gamma:$$

$$f(\gamma(z)) = f((az + b)/(cz + d)) = (cz + d)^2 \cdot f(z).$$

Taking square-roots we see that g satisfies a relation

$$g(\gamma(z)) = \varepsilon(Z)(cz + d)g(z),$$

where

$$\varepsilon : \Gamma \to \{\pm 1\}$$

is a character.

By relation (i) the C_j lie in the kernel of ε, and if -1 is an element of Γ the relation (ii) cannot be true, so that then

$$-1 = \prod_{j=1}^{g} ([A_j, B_j]) \prod_{j=1}^{n} C_j.$$

As all the factors on the right side are in the kernel of ε, $\varepsilon(-1)$ is equal to $+1$, and the functional equation for g, applied to $\gamma = -1$, reads

$$g(z) = -g(z).$$

But f and g do not vanish, and we have the contradiction we were looking for.

Remark: Usually this sort of theorem is proved via connectedness of a suitable Teichmüller-space. Theta characteristics are also used for this purpose in [3].

Once we have chosen a particular lifting Γ or $\bar{\Gamma}$ we may distinguish between regular and irregular cusps: For any $x_j \in S^\infty$, the cusp $z \in \mathbb{P}^1(\mathbb{R})$ defined by C_j projects onto x_j, and C_j generates the stabilizer Γ_Z of z in Γ. The eigen-values of C_j are equal and both have the value $+1$ or -1, and C_j is conjugate in $\mathbf{SL}(2, \mathbb{R})$ to

$$\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

(C_j is not conjugate to the inverse of one of the elements above, since it corresponds to a path counter-clockwise around x_j).

If the eigenvalues of C_j are equal to $+1$ we call x_j a regular cusp, otherwise an irregular cusp. S_r^∞ respectively S_i^∞ denotes the set of regular respectively irregular cusps.

Of course the decomposition $S^\infty = S_r^\infty \cup S_i^\infty$ depends on the choice of the lifting Γ of $\bar{\Gamma}$, and since we may change in sign an even number of the C_j's the only thing which is canonical is the parity of the number of elements in S_r^∞. We shall see later that this parity is always even, so that there exist liftings with $S_r^\infty = 0$, but sometimes no liftings with $S_i^\infty = 0$. Thus in contrast to its name the case of a regular cusp is somehow exceptional. We shall see other examples for this, and as a general rule it can be said that the cusps in S_i^∞ behave like the limits of points of S^f, if the ramification-index e approaches infinity.

For example it is not difficult to see that for $x_j \in S^f$ the element C_j is conjugate in $\mathbf{SL}(2, \mathbb{R})$ to

$$-\begin{pmatrix}\cos(\pi/e_j) & \sin(\pi/e_j)\\ -\sin(\pi/e_j) & \cos(\pi/e_j)\end{pmatrix}$$

and hence also to

$$-\begin{pmatrix}\cos(\pi/e_j) & 1\\ -\sin^2(\pi/e_j) & \cos(\pi/e_j)\end{pmatrix}$$

which converges for $\mathbf{e}(x_j) = e_j \to \infty$ to

$$-\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

We also have to deal with meromorphic connections. If $\mathscr{E}$ is a vector-bundle (= locally free sheaf) on X we recall that a holomorphic connection on $\mathscr{E}$ is a mapping of sheafs

$$\nabla : \mathscr{E} \to \mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{K}_X$$

satisfying

$$\nabla(f \cdot g) = f \cdot \nabla(g) + g \otimes df$$

for local sections f of $\mathcal{O}_X$ and g of $\mathscr{E}$. Such connections exist if and only if every direct summand of $\mathscr{E}$ has degree zero (compare [11]).

Meromorphic connections are defined in the same way, with the only difference that $\nabla(g)$ has only to be a meromorphic section of $\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{K}_X$. Meromorphic connections exist always , since $\mathscr{E}$ is free as a meromorphic bundle.

If g is a holomorphic section of $\mathscr{E}$ near some point $x \in X$, and ∇ a meromorphic connection on $\mathscr{E}$, the meromorphic part of $\nabla(g)$ is a well-defined element of $\mathscr{E}_x \otimes_{\mathcal{O}_{X,x}} (\mathscr{M}(\mathscr{K}_X)/\mathscr{K}_X)_x$, ($\mathscr{M}(\mathscr{K}_X)$ = sheaf of meromorphic differentials), which depends $\mathcal{O}_{X,x}$-linearly on g. Thus ∇ has a meromorphic part in x which is an element of $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})_x \otimes_{\mathcal{O}_{X,x}} (\mathscr{M}(\mathscr{K}_X)/\mathscr{K}_X)_x$, and which vanishes for almost all $x \in X$.

The term of degree one in this meromorphic part defines a residue

$$\mathrm{Res}_x(\nabla) \in \mathrm{End}_{\mathbb{C}}(\mathscr{E}(x), \mathscr{E}(x)),$$

where $\mathscr{E}(x) = E_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$.

A meromorphic connection ∇ is said to have a regular singular point in x if its meromorphic part has at most a pole of first order. If $\lambda, \lambda_2, \ldots, \lambda_r$ are the eigenvalues of $\mathrm{Res}_x(\nabla)$ it is then classical that the monodromy-transformation for a small loop counter-clockwise around x has eigenvalues $\exp(-2\pi i \lambda_j), 1 \leq j \leq r$. In general holomorphic connections, meromorphic connections and meromorphic connections with regular singular points allow the usual operations: If $\mathscr{E}$ and $\mathscr{F}$ have such connections, there are canonical ones on $\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}), \Lambda^{\nu}\mathscr{E}$ etc. Furthermore if $Y \to X$ is a holomorphic mapping, the pullback of any bundle with connection on X has a canonically defined connection.

If $\mathscr{L}$ is a line-bundle on X with $\mathscr{L}^{\otimes 2} \cong \mathcal{O}_X$, $\mathscr{L}$ has degree zero and thus a holomorphic connection. This connection is defined up to a global holomorphic differential form ω, and changing it by ω changes the corresponding connection on $\mathscr{L}^{\otimes 2} \cong \mathcal{O}_X$ by 2ω. Therefore $\mathscr{L}$ has a unique holomorphic connection for which the isomorphism $\mathscr{L}^{\otimes 2} \cong \mathcal{O}_X$ is horizontal, if $\mathcal{O}_X$ has the connection given by ordinary differentiation. We call this connection the canonical connection on $\mathscr{L}$, and we know that its monodromy is contained in $\{\pm 1\}$.

§3. Real algebraic curves

Any Riemann surface may be considered as an algebraic curve defined over $\mathbb{C}$. Sometimes this algebraic variety is already definable over the real numbers. This happens precisely if there exist an antiholomorphic involution on the surface, and these involutions correspond bijectively to the different real models of the curve.

The basix example here is the double of a Riemann surface with boundary, which has a canonical real structure. The real points of this real curve are the fixed-points of the involution, hence the points in the boundary of our original Riemann surface.

Not every real curve is of this form, since for example there exist curves X over $\mathbb{R}$ for which $X(\mathbb{C}) - X(\mathbb{R})$ is connected. ($X(\mathbb{C}), X(\mathbb{R})$ denote the $\mathbb{C}$-respectively $\mathbb{R}$-valued points of a real algebraic curve X.) We shall see that all counterexamples are of this form.

So let us assume that X is an irreducible nonsingular algebraic curve which is defined over $\mathbb{R}$. We identify X with the Riemann surface $X(\mathbb{C})$, together with the involution τ of $X(\mathbb{C})$ defined by complex conjugation. $X(\mathbb{R}) \subset X(\mathbb{C})$ consists of the fixed-points of τ, and it is a compact one-dimensional real manifold, hence a union of circles. Let g be the genus of X.

THEOREM 2:

(i) *$X(\mathbb{R})$ has at most $g + 1$ connected components. (Harnack's theorem).*

(ii) *$X(\mathbb{C}) - X(\mathbb{R})$ has at most two components, and exactly two if $X(\mathbb{R})$ has $g + 1$ (= maximal number) components.*

(iii) *$X(\mathbb{C}) - X(\mathbb{R})$ has two components if and only if X is the double of a Riemann surface with boundary.*

PROOF: We may assume that $X(\mathbb{R})$ is not void. Following Prof. Geyer we introduce the Picard-variety P and the Jacobian J of X. This are algebraic groups defined over $\mathbb{R}$, and $P = J \times \mathbb{Z}$ as $\mathbb{R}$-variety.

Any real algebraic line-bundle defines a continuous real linebundle on $X(\mathbb{R})$.

If this continuous bundle is trivial an easy argument à la Stone–Weierstraß shows that our algebraic bundle has a real meromorphic section which does not have poles or zeroes in $X(\mathbb{R})$. It is then isomorphic to the real bundle defined by a divisor D on X with

$$D = \tau(D)$$

and

$$\text{support}(D) \subset X(\mathbb{C}) - X(\mathbb{R}).$$

D is of the form

$$D = D_1 + \tau(D_1),$$

with a divisor D_1, and our original algebraic bundle is of the form $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{M}^\tau$, where $\mathscr{M}$ is an algebraic bundle defined over $X(\mathbb{C})$, and $\mathscr{M}^\tau$ denotes the pullback under τ of the bundle complex conjugate to $\mathscr{M}$. Conversely any real bundle of the form $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{M}^\tau$ induces the trivial continuous bundle on $X(\mathbb{R})$.

Finally any continuous bundle on $X(\mathbb{R})$ can be obtained this way, for example by starting with the real algebraic bundle defined by a certain divisor with support in $X(\mathbb{R})$, which contains exactly one point of each connected component of $X(\mathbb{R})$ on which the continuous bundle is non-trivial.

As the continuous real linebundles on $X(\mathbb{R})$ are classified by $H^1(X(\mathbb{R}), \mathbb{F}_2)$ ($\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$) we have defined an isomorphism

$$H^1(X(\mathbb{R}), \mathbb{F}_2) \cong P(\mathbb{R})/(1 + \tau)(P(\mathbb{C})) = J(\mathbb{R})/(1 + \tau)(J(\mathbb{C})) \oplus \mathbb{F}_2.$$

Let

$$V = H^1(X, \mathcal{O}_X), L = H^1(X(\mathbb{C}), \mathbb{Z}).$$

τ operates on V and L, and:

$$J(\mathbb{C}) = V/2\pi i L.$$

If $L^\pm$ denotes the $\pm$-eigenspaces of τ on L, the mapping induced by multiplication with πi,

$$L/2L \hookrightarrow V/2\pi i L = J(\mathbb{C}),$$

is easily seen to induce an isomorphism

$$(L^+ + L^-)/(L^- + 2L) \cong J(\mathbb{R})/(1 + \tau)(J(\mathbb{C})).$$

(The calculation is easy, if one does not overlook the fact that because of the factor πi the $L^\pm$ are sent into the opposite eigenspaces on V.)

We thus obtain an injection

$$(L^+ + L^-)/(L^- + 2L) \hookrightarrow H^1(X(\mathbb{R}), \mathbb{F}_2),$$

with cokernel of dimension one over $\mathbb{F}_2$. As the domain of this injection is a quotient of $L^+/2L^+$, and as L^+ is a free abelian group of rank g, $H^1(X(\mathbb{R}), \mathbb{F}_2)$ has dimension at most $g + 1$, and assertion (i) follows. On the other hand we have a commutative diagram

$$\begin{array}{ccc} L = H^1(X(\mathbb{C}),\mathbb{Z}) & \to & H^1(X(\mathbb{R}),\mathbb{Z}) \\ \downarrow & & \downarrow \\ L/2L = H^1(X(\mathbb{C}),\mathbb{F}_2) & \to & H^1(X(\mathbb{R}),\mathbb{F}_2). \end{array}$$

As $H^1(X(\mathbb{R}),\mathbb{Z})$ is invariant under τ the upper horizontal mapping annihilates L^-, and we obtain a mapping

$$L/(L^- + 2L) \to H^1(X(\mathbb{R}),\mathbb{F}_2).$$

On the submodule $(L^+ + L^-)/(L^- + 2L) \subseteq L/(L^- + 2L)$ this mapping is easily seen to coincide with the injection defined before, so that the lower horizontal mapping in the diagram above is either surjective, or its cokernel has dimension one over $\mathbb{F}_2$.

On the other hand, if $H_c^2(X(\mathbb{C}) - X(\mathbb{R}),\mathbb{F}_2)$ denotes the cohomology with compact support of $X(\mathbb{C}) - X(\mathbb{R})$, this cokernel is isomorphic to the kernel of the surjection of $H_c^2(X(\mathbb{C}) - X(\mathbb{R}),\mathbb{F}_2)$ onto $H^2(X(\mathbb{C}),\mathbb{F}_2) = \mathbb{F}_2$.

As the dimension of $H_c^2(X(\mathbb{C}) - X(\mathbb{R}),\mathbb{F}_2)$ is equal to the number of connected components of $X(\mathbb{C}) - X(\mathbb{R})$ we see that this number is one or two, and precisely two if the cokernel is not trivial. This happens for example if $X(\mathbb{R})$ has $g + 1$ components, since $L/(L^- + 2L)$ has dimension g. This shows assertion (ii), and (iii) is trivial.

§4. Representation-spaces

Let X, S, $\mathbf{e}$, Γ etc. have the meanings attributed to them in §2, with all finite $\mathbf{e}(x)$ odd. Let $\mathbf{HOM}(\Gamma,\mathbf{SL}(2,\mathbb{C}))$ denote the set of homomorphisms of Γ in $\mathbf{SL}(2,\mathbb{C})$, and $\mathbf{HOM}_P(\Gamma,\mathbf{SL}(2,\mathbb{C}))$ the subset consisting of those homomorphisms such that for $j = 1,\dots,n$ the image of C_j is conjugate to C_j itself. If F is the free group in the generators $A_1,\dots,A_g$, $B_1,\dots,B_g$, $C_1,\dots,C_n$, and $Z \in F$ denotes the element

$$A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1}\dots A_gB_gA_g^{-1}B_g^{-1}C_1C_2\dots C_n,$$

$\mathbf{HOM}_P(\Gamma,\mathbf{SL}(2,\mathbb{C}))$ has naturally the structure of a complex space: We define a mapping from $\mathbf{SL}(2,\mathbb{C})^{2g+n}$ to $\mathbf{SL}(2,\mathbb{C}) \times \mathbb{C}^n$ by sending a tupel $(X_1,\dots,X_g,\ Y_1,\dots,Y_g,\ Z_1,\dots,Z_n)$ onto

$$X_1Y_1X_1^{-1}Y_1^{-1}X_2Y_2X_2^{-1}Y_2^{-1}\dots X_nY_nX_n^{-1}Y_n^{-1}Z_1Z_2\dots Z_n,$$

$$\mathrm{tr}(Z_1),\dots,\mathrm{tr}(Z_n).$$

If for $x_j \in S$ we call t_j the number

$$t_j = -2\cos(\pi/e_j), \text{ for } x_j \in S^f,$$
$$t_j = 2, \text{ for } x_j \in S_r^\infty, \text{ and}$$
$$t_j = -2, \text{ for } x_j \in S_i^\infty,$$

$\mathbf{HOM}_P(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ is naturally an open subset of the fibre of this mapping over $(1, t_1, \ldots, t_n)$, and inherits the complex structure from this fibre.

Any element ρ of $\mathbf{HOM}_P(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ defines a representation of Γ on $\mathbb{C}^2$. We denote by $\mathbf{HOM}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ the open subset consisting of those ρ's for which $\mathbb{C}^2$ is an irreducible Γ-module.

Lemma: *$\mathbf{HOM}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ is a complex manifold of dimension $6g - 3 + 2n$, and the complex tangent-space to this manifold can be identified with $Z_P^1(\Gamma, \mathfrak{sl}(2,\mathbb{C}))$, the set of parabolic crossed homomorphisms of Γ into $\mathfrak{sl}(2,\mathbb{C})$.*

Here Γ operates via conjugation on $W = \mathfrak{sl}(2,\mathbb{C})$, and a crossed homomorphism c is called parabolic if there exists elements $w_1, \ldots, w_n \in W$ with $c(C_j) = \rho(C_j)w_j - w_j$.

Proof: $\mathbf{HOM}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ is an open subset of the fibre of a complex mapping, and it is a complex manifold if this mapping is a submersion. The tangential-space of $\mathbf{SL}(2,\mathbb{C})^{2g+n}$ in a point $\rho \in \mathbf{HOM}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ can be identified with the set of crossed homomorphisms of F into W, and the tangential-mapping can be identified with the natural mapping

$$Z^1(F, W) \to Z^1(\{Z\}, W) \oplus \bigoplus_{j=1}^{n} H^1(\{C_j\}, W).$$

If M denotes the manifold $X - S$, W defines a locally constant sheaf $\mathbf{W}$ on M, and the cokernel of the mapping of $Z^1(F, W)$ into $Z^1(\{Z\}, W)$ can be identified with $H^2(M, \mathbf{W})$. As $\mathbf{W}$ is self-dual this is dual to

$$H_c^0(M, \mathbf{W}) \subseteq H^0(M, \mathbf{W}) = \{\Gamma\text{-invariants of } W\} = 0$$

If $\tilde{\Gamma}$ denotes the quotient of Γ by the normal subgroup generated by Z, the kernel of the mapping

$$Z^1(F, W) \to Z^1(\{Z\}, W)$$

is $Z^1(\tilde{\Gamma}, W)$, and we have to consider the mapping

$$Z^1(\tilde{\Gamma}, W) \to \bigoplus_{j=1}^{n} H^1(\{C_j\}, W).$$

For this let $M = X - S$ as before, and choose for each $x_j \in S_j$ a small disk $x \in U_x \subset X$. If $B^1(\tilde{\Gamma}, W)$ denotes the crossed homomorphisms of the form $\rho(\gamma)w - w$, for an element $w \in W$, our mapping annihilates $B^1(\tilde{\Gamma}, W)$, and the induced mapping from

$$Z^1(\tilde{\Gamma}, W)/B^1(\tilde{\Gamma}, W) = H^1(\tilde{\Gamma}, W)$$

onto the direct sum of the $H^1(\{C_j\}, W)$ may be identified with the mapping

$$H^1(M, \mathbf{W}) \to \bigoplus H^1(U_x - \{x\}, \mathbf{W}).$$

The cokernel of this mapping injects into $H_c^2(M, \mathbf{W})$, which is dual to

$$H^0(M, \mathbf{W}) = 0.$$

All in all we now know that our mapping is a submersion, that the tangent-space of the fibre at a point $\rho \in \mathbf{HOM}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C}))$ can be identified with $Z_P^1(\tilde{\Gamma}, W)$, and that the fibre has dimension $6g + 3n - 3 - n = 6g - 3 + 2n$.

It remains to show that $Z_P^1(\tilde{\Gamma}, W)$ is equal to its subspace $Z_P^1(\Gamma, W)$. For this we show that $\dim(H_P^1(\tilde{\Gamma}, W)) = \dim(H_P^1(\Gamma, W))$, which can be derived as follows: W defines a sheaf $\mathbf{W}$ on $X - S^\infty$, which is locally constant on $M = X - S$, and induces there the locally constant sheaf $\mathbf{W}$ defined above. $H_P^1(\Gamma, W)$ and $H_P^1(\tilde{\Gamma}, W)$ can be identified with $H_P^1(X - S^\infty, \mathbf{W})$ and $H_P^1(X - S, \mathbf{W})$, where H_P^1 stands for parabolic cohomology, which is the image of the cohomology with compact support in the ordinary cohomology. It is a wellknown property of parabolic cohomology that $H_P^1(X - S^\infty, \mathbf{W})$ and $H_P^1(X - S, \mathbf{W})$ coincide, and we are through.

The group $\mathbf{PGL}(2, \mathbb{C})$ acts by conjugation on $\mathbf{Hom}(\Gamma, \mathbf{SL}(2, \mathbb{C}))$, $\mathbf{Hom}_P(\Gamma, \mathbf{SL}(2, \mathbb{C}))$, and $\mathbf{Hom}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C}))$, and the action on the last space is proper and free. The quotient of $\mathbf{Hom}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C}))$ under $\mathbf{PGL}(2, \mathbb{C})$ is called $\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C}))$ (parabolic representations), and its tangent-space at a representation ρ is canonically isomorphic to the quotient $Z_P^1(\Gamma, W)/B^1(\Gamma, W) = H_P^1(\Gamma, W)$, since $B^1(\Gamma, W)$ is the tangent-space of the $\mathbf{PGL}(2,\mathbb{C})$-orbit through ρ. Hence

THEOREM 3: $\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C})) = \mathbf{Hom}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))/\mathbf{PGL}(2,\mathbb{C})$ is naturally a complex manifold of dimension $6g-6+2n$, whose tangent-space in a point can be canonically identified with $H_P^1(\Gamma, W)$, where $W = \mathfrak{sl}(2,\mathbb{C})$ is a Γ-module via the adjoint representation. $\mathbf{Hom}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ is a principal $\mathbf{PGL}(2,\mathbb{C})$-bundle over $\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))$.

REMARK: If we consider absolutely irreducible real representations, we get a real manifold $\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{R}))$ of dimension $6g-6+2n$ which is a submanifold of $\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2,\mathbb{C}))$, since two real representations of Γ with trivial commuting algebras are isomorphic if their complexifications are.

§5. Uniformization data and permissible connections

As before X denotes a compact Riemann surface of genus g, $\mathscr{K}_X$ the canonical bundle on X.

DEFINITION:

(i) A uniformization datum on X is a tuple $\mathbf{D} = (\mathbf{e}, S_r^\infty, \mathscr{L}, \varphi)$, where $\mathbf{e}$ is a function on X with values in $\{1, 3, \ldots, \infty\}$, which is equal to one in almost all points of X, such that the quantity $\chi(D) = 2g - 2 + \sum_{x \in X}(1 - 1/\mathbf{e}(x))$ is positive. S_r^∞ is a subset of even cardinality of $S^\infty = \mathbf{e}^{-1}(\{\infty\})$. $\mathscr{L}$ is a linebundle on X. φ is an isomorphism $\mathscr{L}^{\otimes 2} \cong \mathrm{K}_X(S_r^\infty)$.

(ii) If $\mathscr{M}$ is a linebundle on X, together with an isomorphism $\mathscr{M}^{\otimes 2} = \mathscr{O}_X$, the twist of $\mathbf{D}$ by $\mathscr{M}$ is the uniformization datum $(\mathbf{e}, S_r^\infty, \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}, \varphi_1)$, where φ_1 is the obvious isomorphism derived from φ:

$$\varphi_1 : (\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M})^{\otimes 2} \cong \mathscr{L}^{\otimes 2} \otimes_{\mathscr{O}_X} \mathscr{M}^{\otimes 2} \cong \mathscr{L}^{\otimes 2} \cong \mathscr{K}_X(S_r^\infty).$$

Suppose $\mathbf{D} = (e, S_r^\infty, \mathscr{L}, \varphi)$ is a uniformization datum on X, and

$$p : \tilde{X} \to X$$

a finite mapping of Riemann surfaces.

For any $\tilde{x} \in \tilde{X}$ $\mathbf{f}(\tilde{x})$ denotes the ramification index of p in $\tilde{x}$, and we further assume that $\mathbf{f}(\tilde{x})$ divides $\mathbf{e}(p(\tilde{x}))$ for all $\tilde{x} \in \tilde{X}$. The canonical bundle of $\tilde{X}$ is then isomorphic to

$$p^*(\mathscr{K}_X)(\sum_{\tilde{x}\in\tilde{X}} (\mathbf{f}(\tilde{x}) - 1)\tilde{x}).$$

DEFINITION: Under the assumptions above the pullback $p^*(\mathbf{D}) = (\tilde{\mathbf{e}}, \tilde{S}_r^\infty, \tilde{\mathscr{L}}, \tilde{\varphi})$ *of* $\mathbf{D}$ is the uniformization datum defined as follows:

$\tilde{\mathbf{e}}(\tilde{x}) = \mathbf{e}(p(\tilde{x}))/\mathbf{f}(\tilde{x})$, for $\tilde{x} \in \tilde{X}$.

$\tilde{\mathscr{L}} = p^*(\mathscr{L})(D)$, where D is the divisor $D = \sum_{\tilde{x}\in\tilde{X}} d(\tilde{x})\tilde{x}$, the function d defined by:

$d(\tilde{x}) = (\mathbf{f}(\tilde{x}) - 1)/2$, for $p(\tilde{x}) \notin S_r^\infty$ and $\mathbf{f}(x)$ *odd*,

$d(\tilde{x}) = \mathbf{f}(\tilde{x})/2$, for $p(\tilde{x}) \notin S_r^\infty$ and $\mathbf{f}(x)$ even,

$d(\tilde{x}) = 0$, for $p(\tilde{x}) \in S_r^\infty$.

$\tilde{S}_r^\infty$ consists of those $\tilde{x} \in X$ for which either $p(\tilde{x}) \in S_r^\infty$, or $p(\tilde{x}) \in S_i^\infty = S^\infty - S_r^\infty$ and $\mathbf{f}(x)$ is even.

$\tilde{\varphi}$ is the obvious isomorphism derived from φ by pullback:

$\tilde{\varphi} : \tilde{\mathscr{L}}^{\otimes 2} \cong p^*(\mathscr{L})^{\otimes 2}(2D) \cong p^*(\mathscr{K}_X(S_r^\infty))(2D) \cong p^*(\mathscr{K}_X)(2D + p^*(S_r^\infty)) \cong \mathscr{K}_{\tilde{X}}(\tilde{S}_r^\infty)$.

REMARK: It is not quite obvious that $\tilde{S}_r^\infty$ has even cardinality, but this is a consequence of the existence of the isomorphism $\tilde{\varphi}$, since $\mathscr{L}^{\otimes 2}$ and $\mathscr{K}_{\tilde{X}}$ have even degree.

Furthermore if n is the degree of the mapping p an easy calculation gives the formula

$$\chi(p^*(\mathbf{D})) = n \cdot \chi(\mathbf{D}).$$

Let us now fix a $\mathbf{D} = (\mathbf{e}, S_r^\infty, \mathscr{L}, \varphi)$, and we look for extensions

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1}(S_r^\infty) \to 0.$$

These are classified by

$$H^1(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}^{-1}(S_r^\infty), \mathscr{L})) \cong H^1(X, \mathscr{L}^{\otimes 2}(-S_r^\infty)) \cong H^1(X, \mathscr{K}_X) \xrightarrow{\sim} \mathbb{C},$$

where the precise nature of the last isomorphism will be explained below.

DEFINITION:

(i) The uniformization-bundle belonging to $\mathbf{D}$ is the extension $\mathscr{E}$ whose class is given by $\chi(\mathbf{D})/2 \in \mathbb{C}$.

(ii) A permissible connection on the uniformization bundle $\mathscr{E}$ is a meromorphic connection ∇, with poles of first order in $S = e^{-1}(\{3, 5, \ldots, \infty\})$ as only singularities, which satisfies the following three conditions:

(a) The obvious isomorphism

$$\Lambda^2 \mathscr{E} \cong \mathcal{O}_X(S_r^\infty)$$

is horizontal, if $\mathcal{O}_X(S_r^\infty)$ has its natural connection defined by differentiation.

(b) For any $x \in S$ the residue of the connection in x respects the line $\mathscr{L}(x) \subset \mathscr{E}(x)$, and it is on this line equal to multiplication with $(1 - 1/e(x))/2$, for $x \in S - S_r^\infty$, 0, for $x \in S_r^\infty$.

(c) The composition

$$\mathscr{L} \hookrightarrow \mathscr{E} \xrightarrow{\nabla} \mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{K}_X(S) \to \mathscr{L}^{-1}(S_r^\infty) \otimes_{\mathcal{O}_X} \mathscr{K}_X(S) \cong \mathscr{L}(S)$$

is equal to the natural inclusion

$$\mathscr{L} \hookrightarrow \mathscr{L}(S).$$

Remarks:

(i) If an extension $\mathscr{E}$ of $\mathscr{L}^{-1}(S_r^\infty)$ by $\mathscr{L}$ has a connection with properties (a), (b), (c), it is isomorphic to the uniformization-bundle: We only sketch a proof. The extension $\mathscr{E}$ has a C^∞-section

$$r : \mathscr{L}^{-1}(S_r^\infty) \to \mathscr{E}.$$

The ∂-derivation of r can be seen as a $(0,1)$-form with values in $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{L}^{-1}(S_r^\infty), \mathscr{L}) \cong \mathscr{K}_X$, hence as a $(1,1)$-form. The class of $\mathscr{E}$ is $(1/2\pi i) \cdot$(integral of this form over X).

On the other hand the composition

$$\mathscr{L}^{-1}(S_r^\infty) \xrightarrow{r} \mathscr{E} \xrightarrow{\nabla} \mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{K}_X(S) \to \mathscr{L}^{-1}(S_r^\infty) \otimes_{\mathcal{O}_X} \mathscr{K}_X(S)$$

is a C^∞-connection, except for the fact that we have to add some correction terms in the points of S to compensate for the poles of ∇. These correction-terms are determined by the residue of ∇ in the points of S, and can be read of from conditions (a) and (b).

The curvature of the C^∞-connection thus obtained is computed by applying the $\bar{\partial}$-operator. As $\bar{\partial}(r)$ takes its values in $\mathscr{L}$ condition (c) shows that the curvature form is equal to the form used above for the computation of the class of $\mathscr{E}$, except for some correction-terms in the points of S. The (known) degree of $\mathscr{L}^{-1}(S_r^\infty)$ being given by the integral over this curvature form we derive the formula for the class of $\mathscr{E}$.

In the sequel we only need the fact that it is determined by conditions (a), (b), (c), and not what its precise value is.

(ii) If we twist $\mathbf{D}$ with a bundle $\mathscr{M}$ such that $\mathscr{M}^{\otimes 2} \cong \mathcal{O}_X$ the new $\mathscr{E}$ is derived from the old one by tensoring with $\mathscr{M}$, and the permissible connections on it correspond bijectively to the permissible connections on the old one, via tensoring with the canonical connection on $\mathscr{M}$.

(iii) Permissible connections behave well under pullback: Assume that

$$p : \tilde{X} \to X$$

is a mapping for which the pullback $p^*(\mathbf{D}) = \tilde{\mathbf{D}}$ can be defined. The bundle $p^*(\mathscr{E})(D)$ ($D = \sum_{\tilde{x} \in \tilde{X}} d(\tilde{x})\tilde{x}$ is the divisor used in the definition of the pullback $p^*(\mathbf{D})$) is an extension of

$$p^*(\mathscr{L}^{-1})(D + p^*(S_r^\infty)) = \tilde{\mathscr{L}}^{-1}(2D + p^*(S_r^\infty))$$

by

$$p^*(\mathscr{L})(D) = \tilde{\mathscr{L}}.$$

$\tilde{\mathscr{L}}^{-1}(\tilde{S}_r^\infty)$ is naturally a subsheaf of $\tilde{\mathscr{L}}^{-1}(2D + p^*(S_r^\infty))$, and its preimage in $p^*(\mathscr{E})(D)$ is an extension of the sort used in the definition of the uniformization-bundle $\mathscr{E}$. Its class is easily computed to have the right value, and we have obtained an inclusion

$$\tilde{\mathscr{E}} \hookrightarrow p^*(\mathscr{E})(D).$$

The pullback of a permissible connection on $\mathscr{E}$ induces a meromorphic connection on $\tilde{\mathscr{E}}$, and routine-calculations show that this connection is permissible again.

Definition: Two permissible connections are equivalent if they can be obtained from each other by an automorphism of the extension

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1}(S_r^\infty) \to 0.$$

The automorphisms of this extension are given by the global sections of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{L}^{-1}(S_r^\infty), \mathscr{L}) \cong \mathscr{K}_X$, and condition (c) implies that twisting a permissible ∇ with the automorphism belonging to a holomorphic one-form ω changes the restriction of ∇ to $\mathscr{L}$ by multiplication with ω. Conversely condition (b) implies that for any two permissible connections ∇ and ∇' their restrictions to $\mathscr{L}$ differ by such a form ω. Thus by replacing one of them by an equivalent one we may assume that they coincide on $\mathscr{L}$. They then differ by a mapping from $\mathscr{L}^{-1}(S_r^\infty) = \mathscr{E}/\mathscr{L}$ into $\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{K}_X(S)$, and condition (a) implies that this mapping has its image contained in $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{K}_X(S)$.

Finally adding such a mapping to a permissible connection does not affect permissibility, and the space of permissible connections up to equivalence is a complex affine space under

$$Q = \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{L}^{-1}(S_r^\infty), \mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{K}_X(S)) = \Gamma(X, \mathscr{K}_X^{\otimes 2}(S)),$$

the space of quadratic differentials with simple poles in S, of dimension $3g - 3 + n$. Q plays a prominent role in the theory of Teichmüller-spaces, and we shall see that its appearance here is connected to this.

As a motivation for all our definition we now show that the uniformization theorem provides us with a natural base-point of our space of permissible connections.

For this we fix a uniformization datum $\mathbf{D} = (\mathbf{e}, S_r^\infty, \mathscr{M}, \varphi)$, define S^f as the preimage of $\{3, 5, \ldots\}$ by $\mathbf{e}$, and S^∞ as the preimage of $\{\infty\}$. According to §2 we can find a discrete subgroup $\Gamma \subset \mathbf{PSL}(2, \mathbb{R})$ and a mapping π from $\mathbb{H}$ to $X - S^\infty$, which induce an isomorphism

$$_{\bar{\Gamma}}\backslash\mathbb{H} \cong X - S^\infty. \ (S = S^f \cup S^\infty)$$

Theorem 1 says that we can lift $\bar{\Gamma}$ to a subgroup $\Gamma \subset \mathbf{SL}(2, \mathbb{R})$, and any such lifting determines a decomposition $S^\infty = S_r^\infty \cup S_i^\infty$. The constant sheaf $\mathbb{C}^2$ on $\mathbb{H}$ is $\mathbf{SL}(2, \mathbb{R})$-homogeneous via the natural action of $\mathbf{SL}(2, \mathbb{R})$ on $\mathbb{C}^2$, and its restriction to $\pi^{-1}(X - S)$ defines a locally constant sheaf $\mathbf{V}$ on $X - S$, by taking the Γ-quotient. The line generated by the section ${}^t(z, 1)$ of $\mathcal{O}_{\mathbb{H}}^2$ is invariant under the $\mathbf{SL}(2, \mathbb{R})$-action, and defines a sub-linebundle $\mathscr{L}$ of $\mathbf{V} \otimes_{\mathbb{C}} \mathcal{O}_{X-S}$. The square of $\mathscr{L}$ is naturally isomorphic to $\mathscr{K}_X$, since sending ${}^t(z, 1)^{\otimes 2}$ to dz defines a $\mathbf{SL}(2, \mathbb{R})$-invariant isomorphism of the square of the line above with the canonical sheaf of $\mathbb{H}$. More generally this line is the restriction of the canonical subbundle $\mathcal{O}_P(-1) \subset \mathcal{O}_P^2$ on $P = \mathbb{P}^1(\mathbb{C})$, and the isomorphism comes from the wellknown identity

$$\mathscr{K}_P \cong \mathcal{O}_P(-2).$$

Everything here is $\mathbf{SL}(2, \mathbb{C})$-invariant, and the previous considerations apply to any quotient of a domain $\mathscr{D} \subset \mathbb{P}^1(\mathbb{C})$ by a group $\Gamma \subset \mathbf{SL}(2, \mathbb{C})$, which operates without fixed-points on $\mathscr{D}$.

As Γ is contained in $\mathbf{SL}(2, \mathbb{C})$ the second exterior power of the locally constant sheaf $\mathbf{V}$ is constant, and the connection ∇_0 on the holomorphic bundle associated to $\mathbf{V}$ satisfies conditions (a) and (c) in the definition of permissible connections: (a) is a clear if we use the isomorphism $\Lambda^2\mathscr{E} \cong \mathscr{O}_{\Gamma\backslash\mathscr{D}}$ coming from $\Lambda^2\mathbf{V} = \mathbb{C}$ for the identification of $\mathscr{E}/\mathscr{L}$ with $\mathscr{L}^{-1}$, and (c) results from the fact that the derivative of ${}^t(z,1)$ is ${}^t(1,0)\cdot dz$, whose pairing with ${}^t(z,1)$ is equal to dz, and that dz conversely corresponds to ${}^t(z,1)^{\otimes 2}$. We thus obtain a permissible connection on X itself if we find suitable extensions of the objects defined on $X - S$ so far. For this we analyse the behaviour of these objects near a point $x \in S$, and distinguish three cases depending on whether the point lies in S^f, S_r^∞ or S_i^∞. The analysis will also apply to any quotient of a domain in $\mathbb{P}^1(\mathbb{C})$ by a Kleinian group which can be lifted to $\mathbf{SL}(2, \mathbb{C})$.

Case A: $x \in S^f$

The basic example we have to consider is the quotient of the unit disk $\mathbb{D}$ by the group $\Gamma_0 \subset \mathbf{SL}(2, \mathbb{C})$ generated by the diagonal-matrix γ_0 with entries $\exp(\pm 2\pi i/e)$, where e is an odd number. For simplicity, we denote $\exp(2\pi i/e)$ by ρ.

The function $\zeta = z^e$ identifies ${}_{\Gamma}\backslash\mathbb{D}$ with $\mathbb{D}$. If $\dot{\mathbb{D}}$ denotes the punctured disk sections of the bundle $\mathscr{E}$ over $\dot{\mathbb{D}}$ can be identified with pairs of functions

$${}^t(f, g) : \mathbb{D} \to \mathbb{C}^2$$

satisfying

$$f(\rho^2 \cdot z) = \rho \cdot f(z), g(\rho^2 \cdot z) = \rho^{-1} \cdot g(z),$$

and the connection ∇_0 is given by

$$\nabla_0({}^t(f, g)) = {}^t(df, dg) = {}^t(df/dz, dg/dz) \cdot dz = {}^t(df/dz, dg/dz) \cdot d\zeta/(e \cdot z^{e-1}).$$

The functional equation is equivalent to a condition on the Laurent-expansions of f and g: We have

$$f(z) = \sum_{\mu \in \mathbb{Z}} a_\mu z^\mu,\ g(z) = \sum_{\mu \in \mathbb{Z}} b_\mu z^\mu,$$

with complex numbers a_μ, b_μ, and the condition means

$$a_\mu = 0, \mu \not\equiv -(e-1)/2 \bmod(e),\ \ b_\mu = 0, \mu \not\equiv +(e-1)/2 \bmod(e).$$

To define an extension of $\mathscr{E}$ to $\mathbb{D}$ we have to give a basis of $\mathscr{E}$ over $\dot{\mathbb{D}}$. For this we take the sections defined by

$$z^{-\frac{e-1}{2}}\,{}^t(1,0) \quad \text{and} \quad z^{+\frac{e-1}{2}}\,{}^t(z,1).$$

The second basis-element generates $\mathscr{L}$ over $\dot{\mathbb{D}}$, and thus the extended $\mathscr{E}$ has an extension of $\mathscr{L}$ as subbundle, with $\mathscr{E}/\mathscr{L} = \mathscr{L}^{-1}$ on $\mathbb{D}$. Also ∇_0 extends as a meromorphic connection with the required meromorphic part in 0. Finally the generator $z^{e-1}\cdot{}^t(z,1)^{\otimes 2}$ of $\mathscr{L}^{\otimes 2}$ corresponds to $z^{e-1}\cdot dz = d\zeta/e$, a generator of $\mathscr{K}_{\mathbb{D}}$.

Case B: $x \in S_r^\infty$

We have to consider the quotient $\dot{\mathbb{D}} = {}_{\Gamma_\infty}\backslash\mathbb{H}$ of the upper halfplane under the group Γ_∞ generated by the element

$$\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The sections of $\mathscr{E}$ over $\dot{\mathbb{D}}$ are again given by pairs of functions ${}^t(f,g)$, which now have to satisfy

$$f(z+1) = f(z) + g(z),\ g(z+1) = g(z),$$

and ∇_0 is given by

$$\nabla_0({}^t(f,g)) = {}^t(df, dg) = {}^t(df/dz, dg/dz)\cdot d\zeta/2\pi i\zeta,$$

where $\zeta = \exp(2\pi i z)$ is a coordinate on $\mathbb{D}$. The functional-equation is equivalent to the existence of expansions

$$f(z) = \sum_{\mu\in\mathbb{Z}} (a_\mu + b_\mu\cdot z)\zeta^\mu,\ g(z) = \sum_{\mu\in\mathbb{Z}} b_\mu\zeta^\mu,$$

and we define the extension of $\mathscr{E}$ to $\mathbb{D}$ via its basis

$${}^t(\zeta^{-1},0),\ {}^t(z,1).$$

The second element again generates an extension of $\mathscr{L}$, and its square corresponds to $dz = d\zeta/2\pi i\zeta$, which generates $\mathscr{K}_{\mathbb{D}}(0)$. The rest of the necessary verifications is trivial as well.

Case C: $x \in S_i^\infty$

This is similar to case B, the difference being that the functional equation of ${}^t(f, g)$ becomes

$$f(z+1) = -(f(z) + g(z)),\ g(z+1) = -g(z).$$

The extension of $\mathscr{E}$ is now locally generated by the elements

$${}^t(\zeta^{-1/2}, 0),\ \zeta^{1/2} \cdot {}^t(z, 1).\ (\zeta^{1/2} = \exp(\pi i z))$$

All in all we have constructed on X a linebundle $\mathscr{L}$ together with an isomorphism $\mathscr{L}^{\otimes 2} \cong \mathscr{K}_X(S_r^\infty)$, an extension $\mathscr{E}$ of $\mathscr{L}^{-1}(S_r^\infty)$ by $\mathscr{L}$ and a connection ∇_0 on $\mathscr{E}$ satisfying the requirements of a permissible connection. The tuple ($\mathbf{e}, S_r^\infty, \mathscr{L}$, the isomorphism above) is then a uniformization datum on X, and we already have remarked that the properties of ∇_0 imply that the extension $\mathscr{E}$ is the corresponding uniformization-bundle, with ∇_0 a permissible connection. Finally $\mathscr{K}_X(S_r^\infty)$ has even degree, so that S_r^∞ has even cardinality. What happens if we change the lifting Γ of $\tilde{\Gamma}$?

Two liftings differ by a homomorphism of Γ into $\{\pm 1\}$. If we fix the set $S_r^\infty \subseteq S^\infty$ we restrict ourselves to homomorphisms which factor over the surjection of Γ onto the fundamental group of X, and in general we may obtain any subset of even cardinality in S^∞ as S_r^∞.

The 2^{2g} homomorphisms of the fundamental group of X into $\{\pm 1\}$ correspond to the 2^{2g} linebundles on X with trivial square, and changing the lifting Γ by such a homomorphism amounts to twisting the uniformization datum defined above by the corresponding bundle, together with its canonical connection. We thus obtain all bundles $\mathscr{L}$ with $\mathscr{L}^{\otimes 2} = \mathscr{K}_X(S_r^\infty)$ exactly once, which means that our construction gives us all uniformization data. We thus have shown:

THEOREM 4: *The uniformization-theorem defines a unique permissible connection ∇_0 for any uniformization datum.*

DEFINITION: ∇_0 is the uniformizing connection of the uniformization datum.

REMARK: If $\mathbf{D}$ is a uniformization datum on X, and $p: \tilde{X} \to X$ a finite map such that the pullback $p^*(\mathbf{D})$ is defined, the projection π of $\mathbb{H}$ on $X = {}_\Gamma\backslash\mathbb{H}$ factors through $\tilde{X}$, so that $\tilde{X}$ is equal to ${}_{\tilde{\Gamma}}\backslash\mathbb{H}$ with a subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index. From this one easily derives that the pullback of the uniformizing connection for $\mathbf{D}$ is the uniformizing connection for $p^*(\mathbf{D})$.

Conversely for any subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index $\mathbf{D}$ can be pulled back to $\tilde{X} = {}_{\tilde{\Gamma}}\backslash\mathbb{H}^*$, and thus many questions can be reduced to the corresponding problems on $\tilde{X}$.

As there exist always normal subgroups of finite index in Γ which do not contain elements of finite order or parabolic elements with eigenvalues -1 we may often assume that S is equal to S_r^∞.

For any uniformization datum $\mathbf{D} = (\mathbf{e}, S_r^\infty, \mathscr{L}, \varphi)$ we now consider arbitrary permissible connections ∇. As before we have $X - S^\infty = {}_\Gamma\backslash\mathbb{H}$, with the group $\Gamma \subset \mathbf{SL}(2, \mathbb{R})$ being the monodromy-group of the uniformizing connection ∇_0. Up to equivalence ∇_0 and ∇ differ by a quadratic differential $q \in \Gamma(X, \mathscr{K}_X^{\otimes 2}(S))$, and we write

$$\nabla = \nabla_0 + q.$$

Condition (b) in the definition of permissible connections implies that for $x_j \in S$ the monodromy-transformations of ∇ and ∇_0 belonging to a small loop around x_j have the same eigenvalues. This means that for $x_j \in S^f$ these transformations are conjugate in $\mathbf{SL}(2, \mathbb{C})$, so that the transformation for ∇ has finite order $e_j = e(x_j)$, and that for $x_j \in S^\infty$ the corresponding ∇-monodromy is either equal to ± 1, or conjugate to the element $C_j \in \mathbf{SL}(2, \mathbb{R})$. That the first case does not occur will be the consequence of proposition 5 below.

In any case ∇ defines a locally constant sheaf on $X - S$, and as the monodromy of this locally constant sheaf in the points of S^f has the right order the pullback of this sheaf under the projection

$$\pi: \mathbb{H} - \pi^{-1}(S^f) \to X - S = {}_\Gamma\backslash(\mathbb{H} - \pi^{-1}(S^f))$$

extends to $\mathbb{H}$, and is constant there.

On the other hand $\pi^*(\mathscr{E})$ is isomorphic to $\mathscr{O}_{\mathbb{H}}^2$ if we restrict it to $\mathbb{H} - \pi^{-1}(S^f)$, with the pullback of the connection ∇_0 given by ordinary differentiation. If we write $\pi^*(q)$ as $q(z)dz^2$ with a holomorphic function $q(z)$ on $\mathbb{H} - \pi^{-1}(S^f)$ (and actually on $\mathbb{H}$), the pullback of ∇ is given by

$$\nabla({}^t(f_1, f_2)) = \left({}^t(df_1/dz, df_2/dz) + q(z)\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}{}^t(f_1, f_2)\right) \cdot dz.$$

We know that we can find a global horizontal basis of $\mathscr{O}_{\mathbb{H}}^2 \mid \mathbb{H} - \pi^{-1}(S^f)$ for this connection. The pullback of $\mathscr{L}$ is then a sub-linebundle of the free bundle of rank 2 on $\mathbb{H} - \pi^{-1}(S^f)$, and as these subbundles are classified by $\mathbb{P}^1(\mathbb{C})$ we obtain a holomorphic mapping

$$\rho: \mathbb{H} - \pi^{-1}(S^f) \to \mathbb{P}^1(\mathbb{C}).$$

If

$$\sigma : \Gamma \to \mathbf{SL}(2, \mathbb{C})$$

is the monodromy-representation of ∇ (it factors over the quotient Γ of the fundamental group of $X - S$, because the $\sigma(C_j)$ have order $\mathbf{e}(x_j)$ for $x_j \in S^f$) belonging to the global horizontal basis above, the Γ-invariance of the pullback of $\mathscr{L}$ implies the following functional equation, where $\mathbf{SL}(2, \mathbb{C})$ acts by Moebius-transformations on $\mathbb{P}^1(\mathbb{C})$:

$$\rho(\gamma(z)) = \sigma(\gamma)(\rho(z)),$$

for $z \in \mathbb{H} - \pi^{-1}(S^f)$, $\gamma \in \Gamma$.

If we change our global horizontal basis ρ is transformed by a Moebius-transformation of $\mathbb{P}^1(\mathbb{C})$, and σ is changed by conjugating with the corresponding matrix. If $\tilde{\Gamma} \subseteq \Gamma$ is a subgroup of finite index the mapping ρ remains the same for $(X, \mathbf{D}, \nabla)$ as for the pullbacks of $\mathbf{D}$ and ∇ to $\tilde{X} = {}_{\tilde{\Gamma}}\backslash\mathbb{H}^*$. If we choose $\tilde{\Gamma}$ in such a way that it is torsion-free $\tilde{S}^f$ becomes void, and we thus see that ρ extends as a holomorphic mapping to $\mathbb{H}$.

Finally from the explicit form of ∇ given above an easy calculation shows that we can recover q from ρ by the formula

$$q(z) = -\theta_2(\rho)/2,$$

where θ_2 denotes the Schwarzian derivative:

$$\theta_2(h) = (2 \cdot h' \cdot h''' - 3 \cdot (h'')^2)/(2 \cdot (h')^2).$$

At also can be seen that the derivative $d\rho/dz$ does not vanish on $\mathbb{H}$. Our theory thus fits into the classical Schwarzian differential equation, and the following result is well known:

PROPOSITION 5: *If* $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is an element of* Γ, *the mapping* ρ *is given in a suitable* ∇*-horizontal basis by an expression*

$$\rho(z) = z + \sum_{\mu \geq 0} a_\mu \exp(2\pi i \mu z),$$

with a convergent Fourier-series on $\mathbb{H}$.

Furthermore the pullback of the triple $(\mathscr{E}, \mathscr{L}, \nabla)$ on $\mathbb{H} - \pi^{-1}(S^f)$ is equal to the pullback by ρ of the triple consisting of the trivial bundle $\mathcal{O}_P^2$ on $P = \mathbb{P}^1(\mathbb{C})$, its subbundle $\mathcal{O}_P(-1)$, and the trivial connection given by d/dz. Proposition 5 together with the fact that $\sigma(C_j)$ is conjugate to C_j for $x_j \in S$ now easily implies that any point $x \in S$ has small neighbourhoods U_x, V_x in X, together with an isomorphism of U_x and V_x which respects x, such that under this isomorphism the restriction of the triple $(\mathscr{E}, \mathscr{L}, \nabla_0)$ to $U_x - \{x\}$ can be identified to the restriction of $(\mathscr{E}, \mathscr{L}, \nabla)$ to $V_x - \{x\}$. Local considerations then show that this identification extends to U_x and V_x, and thus we have:

THEOREM 6: *If ∇ is any permissible connection, the triple $(\mathscr{E}, \mathscr{L}, \nabla)$ is locally in X isomorphic to $(\mathscr{E}, \mathscr{L}, \nabla_0)$.*

This theorem allows us to reduce all local questions to explicit calculations. We give an example for this: Let $\mathscr{F} \subset \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})$ denote the endomorphisms of trace 0. ∇ defines a connection on $\mathscr{F}$, and thus a locally constant sheaf $\mathbf{W}$ on $X - S$, corresponding to the representation of Γ on $W = \mathfrak{sl}(2, \mathbb{C})$ derived from σ.

On $X - S$ the de Rham-complex is a resolution of $\mathbf{W}$:

$$0 \to \mathbf{W} \to \mathscr{F} \xrightarrow{d} \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X \to 0, \quad \text{on} \quad X - S.$$

There is a mapping from $\mathscr{F}$ to

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{L}, \mathscr{E}/\mathscr{L}) \cong \mathscr{T}_X$$

where $\mathscr{T}_X = \mathscr{K}_X^{-1}$ is the tangentbundle of X, and a subbundle of $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X$ corresponding to the endomorphisms which annihilate $\mathscr{L}$. This subbundle is isomorphic to

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}/\mathscr{L}, \mathscr{L}) \otimes_{\mathcal{O}_X} \mathscr{K} \cong \mathscr{K}_X^{\otimes 2}.$$

The subsheaf of $\mathscr{F}$ consisting of the elements sent into $\mathscr{K}_X^{\otimes 2}$ by d becomes isomorphic to $\mathscr{T}_X$, and we obtain a resolution

$$0 \to \mathbf{W} \to \mathscr{T}_X \to \mathscr{K}_X^{\otimes 2} \to 0, \quad \text{on} \quad X - S.$$

(Compare [8], Ch. V.)

How can we extend this to X? Let j denote the inclusion of $X - S$ into X. The following result is purely local, and thus Theorem 6 reduces us to some easy explicit calculations for the connection ∇_0:

COROLLARY: *The above resolution extends to an exact sequnce on X*:

$$0 \to j_*(\mathbf{W}) \to \mathscr{T}_X(-S) \to \mathscr{K}_X^{\otimes 2}(S) \to 0.$$

The degree of $\mathscr{T}_X(-S)$ being equal to $-(2g-2+n) \leq -\chi(\mathbf{D}) < 0$, and the cohomology of $\mathscr{K}_X^{\otimes 2}(S)$ being dual to the cohomology of $\mathscr{T}_X(-S)$, we can compute the cohomology of $j_*(\mathbf{W})$. Here the H^0 is equal to the $\sigma(\Gamma)$-invariants of W, and H^1 is wellknown to be the parabolic cohomology of $\mathbf{W}$. Hence: W has no $\sigma(\Gamma)$-invariants, and there is an exact sequence

$$0 \to \Gamma(X, \mathscr{K}_X^{\otimes 2}(S)) \to H_P^1(\Gamma, W) \to H^1(X, \mathscr{T}_X(-S)) \to 0.$$

THEOREM 7: *Let ∇ be a permissible connection belonging to the uniformization-datum $\mathbf{D} = (\mathbf{e}, S_r^\infty, \mathscr{L}, \varphi)$,*

$$\sigma: \Gamma \to \mathbf{SL}(2, \mathbb{C})$$

its monodromy-representation.

(i) *$\sigma(C_j)$ is conjugate in $\mathbf{SL}(2, \mathbb{C})$ to C_j, for any $x_j \in S$.*

(ii) *$\sigma(\Gamma)$ is Zariski-dense in $\mathbf{SL}(2, \mathbb{C})$.*

(iii) *The monodromy-representations define a holomorphic mapping $Q = \{$permissible connections modulo equivalence$\} \to \mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C}))$. The differential of this mapping is equal to the inclusion $Q \hookrightarrow H_P^1(\Gamma, W)$ in the exact sequence above.*

PROOF: Part (i) has been previously shown, and for part (ii) we may pass to a subgroup of finite index in Γ and assume that $S = S_r^\infty$, and that the Zariski-closure of $\sigma(\Gamma)$ is a connected algebraic group. If this group is different from $\mathbf{SL}(2, \mathbb{C})$ it is solvable, and there exists a Γ-invariant subspace $W_1 \subset W$, corresponding to a locally constant subsheaf $\mathbf{W}_1 \subset \mathbf{W}$ of dimension 1.

As the $\sigma(C_j)$ are unipotent and therefore annihilate W_1, $\mathbf{W}_1$ extends as a locally constant sheaf to X, and by the corollary to Theorem 6 this locally constant sheaf of rank 1 on X injects into the linebundle $\mathscr{T}_X(-S)$, of negative degree. This is a contradiction.

Part (iii) is rather trivial and left to the reader. We denote by $\mathscr{D} \subseteq \mathbb{P}^1(\mathbb{C})$ the image of ρ.

THEOREM 8: *Suppose $\mathscr{D}$ is different from $\mathbb{P}^1(\mathbb{C})$.*

(i) *$\mathbb{P}^1(\mathbb{C}) - \mathscr{D}$ has infinitely many points.*

(ii) *$\sigma(\Gamma) \subset \mathbf{SL}(2, \mathbb{C})$ is discrete.*

(iii) *If S^∞ is void ρ is the universal covering-mapping of $\mathscr{D}$.*

PROOF: Part (i) follows from the fact that $\mathbb{P}^1(\mathbb{C}) - \mathscr{D}$ is stable under $\sigma(\Gamma)$, and that $\sigma(\Gamma)$ is Zariski-dense in $\mathbf{SL}(2, \mathbb{C})$ (Theorem 7 (ii)). For part (ii) we may assume that the normalizer of $\mathscr{D}$ in $\mathbf{SL}(2, \mathbb{C})$ is not discrete itself. In any case this normalizer is Zariski-dense in $\mathbf{SL}(2, \mathbb{C})$, and the only $\mathscr{D}$'s having this property are the transforms of the upper halfplane under $\mathbf{SL}(2, \mathbb{C})$. In these cases the assertion will be shown in the next chapter. In the last part we may pass to a finite covering of X and assume that S is void. It is then proved by Gunning in [4], page 170.

REMARK: Gunning has also shown the first two assertions in case S is void. I do not know what happens with part (iii) in general. In any case it remains true for real monodromy.

§6. Connections with real monodromy

Let $X, \mathbf{D} = (\mathbf{e}, S_r^\infty, \mathscr{L}, \varphi)$, Γ, etc. have their previous meanings, and suppose that ∇ is a connection with real monodromy, in the following sense:

DEFINITION: A permissible connection ∇ has real monodromy if $\sigma(\Gamma)$ is conjugate in $\mathbf{SL}(2, \mathbb{C})$ to a subgroup of $\mathbf{SL}(2, \mathbb{R})$.

REMARKS: By suitably adjusting ∇ we may assume that $\sigma(\Gamma) \subset \mathbf{SL}(2, \mathbb{R})$. After this we are still allowed to compose σ with an element from the normaliter of $\mathbf{SL}(2, \mathbb{R})$ in $\mathbf{SL}(2, \mathbb{C})$. $\mathbf{SL}(2, \mathbb{R})$ has index two in this normaliter, and the normalizer is generated by it and the diagonal matrix with entries $+i$ and $-i$, which represents the mapping

$z \mapsto -z$ of $\mathbb{P}^1(\mathbb{C})$.

For $x_j \in S$ the conjugacy class of C_j in $\mathbf{SL}(2, \mathbb{C})$ splits into two classes in $\mathbf{SL}(2, \mathbb{R})$ which are interchanged by the normalizer, namely the classes of C_j and of its inverse. If σ takes its values in $\mathbf{SL}(2, \mathbb{R})$ the elements $\sigma(C_j)$ are therefore conjugate either to C_j or to C_j^{-1}.

We now fix ρ in such a way that σ takes real values.

DEFINITION:

(i) $U^+, U^- \subseteq \mathbb{H}$ are the preimages of the upper resp. lower halfplane by ρ.

(ii) $L \subset \mathbb{H}$ is the ρ-preimage of the real projective line $\mathbb{P}^1(\mathbb{R})$.

(iii) X'^+, X'^-, X^0 are the images of U^+, U^-, L under the projection π onto X.

(iv) $\{U_i | i \in I\}$ denotes the set of connected components of $\mathbb{H} - L = U^+ \cup U^-$, where I is an index set, $I = I^+ \cup I^-$, such that $i \in I^\pm \Leftrightarrow U_i \in U^\pm$.

As the differential of ρ does not vanish L is a closed real analytic submanifold of $\mathbb{H}$, which is Γ-stable since $\sigma(\Gamma)$ respects $\mathbb{P}^1(\mathbb{R})$. If an element $z \in \mathbb{H}$ lies over an $x_j \in S^f$ $\rho(z)$ is one of the fixed-points of $\sigma(C_j)$ and does not lie on $\mathbb{P}^1(\mathbb{R})$. The closure of X^0 therefore does not meet S^f. It also avoids S^∞: Suppose a point $x_j \in S^\infty$ lies in the closure of X^0. Transforming ρ and σ with a suitable element from the normalizer of $\mathbf{SL}(2, \mathbb{R})$, if necessary, we may assume that $\sigma(C_j)$ and C_j are both equal to the element $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We claim that ρ has now the form given by Proposition 5: In any case this is true after transformation with an element from $\mathbf{SL}(2, \mathbb{C})$. But the functional equation for ρ implies that this element has to centralize C_j, and is therefore upper-triangular with ± 1 in the diagonal.

If $\mathrm{Im}(z)$ is sufficiently large $\rho(z)$ lies in the upper halfplane, and our assertion follows.

All this shows that X^0 is a one-dimensional real analytic closed submanifold of X, hence compact and isomorphic to a union of circles. Also any point of S^∞ has a punctured neighbourhood totally contained in X'^+ or X'^-. We denote by $X^\pm$ the union of $X'^\pm$ and those $x_j \in S^\infty$ which are contained in the closure of $X'^\pm$.

If $V_i \subseteq \mathrm{X}^\pm$ denotes the closure of $\mathring{V}_i = \pi(U_i)$, for an index $i \in I^\pm$, the V_i are the connected components of $X - X^0$, and any arc in X^0 separates one component contained in X^+ from one contained in X^-. Thus X looks like a crazy quilt, with the circles of X^0 as seams. The whole picture remains unchanged if we compose ρ with an element from $\mathbf{SL}(2, \mathbb{R})$, and X^+ and X^- are interchanged if we use an element from the normalizer of $\mathbf{SL}(2, \mathbb{R})$ which is not contained in $\mathbf{SL}(2, \mathbb{R})$. On $(\mathbb{H} \cup -\mathbb{H}) = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ we have the hyperbolic metric given by $ds = |dw|/|\mathrm{Im}(w)|$. Pulling back by ρ leads to hyperbolic metrics on the U_i, such that $\rho | U_i$ becomes a local isometry.

THEOREM 9:

(i) *The above hyperbolic metrics on the U_i are complete.* (*Geodesics may be extended indefinitely.*)

(ii) $\mathrm{p} | U_i : U_i \to \pm\mathbb{H}$ *is an isomorphism for any* $i \in I$.

(iii) $\pi | U_i : U_i \to \mathring{V}_i$ *is the "universal cover of V_i, with ramification prescribed by* $\mathbf{e}$". (U_i *is for* V_i *what* $\mathbb{H}$ *is for* X.)

(iv) *If $\Gamma_i \subseteq \Gamma$ denotes the stabilizer of U_i in Γ, then Γ_i is the group of deck-transformations of the covering in* (iii), *and $\mathring{V}_i = {}_{\Gamma}\backslash U_i$. Furthermore, $\sigma | \Gamma_i$ is injective, and $\sigma(\Gamma_i) \subset \mathbf{SL}(2, \mathbb{R})$ is discrete.*

(v) *If $\mathfrak{L}_i = \mathfrak{L}(\sigma(\Gamma_i)) \subseteq \mathbb{P}^1(\mathbb{C})$ denotes the limit-points of $\sigma(\Gamma_i)$, ρ extends to a Γ_i-linear isomorphism $\rho : \xrightarrow{\sim} \pm\mathbb{H} \cup (\mathbb{P}^1(\mathbb{R}) - \mathfrak{L}_i)$. (This means that we have a homeomorphism which is given locally by the restriction of a power-series with non-zero first order term to its domain of definition, as well as its inverse.)*

(vi) *Either $\mathscr{D}$ is equal to $\pm\mathbb{H}$, or to $\mathbb{P}^1(\mathbb{C}) - \bigcap_{i \in I} \mathfrak{L}_i$. In the first case I has just one element, ∇ is equivalent to the uniformizing connection ∇_0, and ρ is a Moebius-transformation. In the second case I^+ and I^- are both non-void.*

(vii) *ρ induces an unramified covering*

$$\rho : \rho^{-1}(\mathbb{P}^1(\mathbb{C}) - \overline{\bigcup_{i \in I} \mathfrak{L}_i}) \to \mathbb{P}^1(\mathbb{C}) - \overline{\bigcup_{i \in I} \mathfrak{L}_i}.$$

PROOF:

(i) If a geodesic in U_i cannot be extended indefinitely its image in V_i must approach the boundary of $\mathring{V}_i$, since it cannot stay in compact subset of $\mathring{V}_i$. We therefore have to show that any point in this boundary has infinite distance from any point in the interior of $\mathring{V}_i$, for the (singular) metric induced on $\mathring{V}_i$.

For points in S^∞ we may assume that they correspond to the cusp ∞ of Γ, and that ρ has the form of Proposition 5. ρ is then sufficiently close to the identity so that near out point the metric looks very much like the hyperbolic metric on $X - S^\infty$, and we are through. On the other hand since the derivative of ρ does not vanish, for points in X^0 our metric looks like the metric on $(\mathbb{H} \cup -\mathbb{H})$ near the real projective line, and thus is complete there too.

(ii) $\rho | U_i$ is a local isometry of complete hyperbolic spaces, hence an unramified covering.

(iii) $\pi | U_i$ has the prescribed ramification, and U_i is simply connected.

(iv) The first assertion is derived from the Γ-invariance of the decomposition $\mathbb{H} - L = \bigcup_{i \in I} U_i$, and the rest follows from the fact that $\sigma(\Gamma_i)$ is the group of deck-transformation for the covering

$$(\pi | U_i) \circ (\rho | U_i)^{-1} : \pm\mathbb{H} \to \mathring{V}_i,$$

isomorphic to the covering in (iii).

(v) It is well known (compare [1], pgs. 44/45) that the mapping

$$(\pi | U_i) \circ (\rho | U_i)^{-1} : \pm \mathbb{H} \to \mathring{V}_i$$

extends to a holomorphic Γ_i-linear mapping

$$\pm \mathbb{H} \cup (\mathbb{P}^1(\mathbb{R}) - \mathfrak{L}_i) \to \bar{V}_i \subseteq X,$$

sending $\mathbb{P}^1(\mathbb{R}) - \mathfrak{L}_i$ onto the boundary $\bar{V}_i \cap X^0$, which lifts to a holomorphic mapping

$$\kappa : \pm \mathbb{H} \cup (\mathbb{P}^1(\mathbb{R}) - \mathfrak{L}_i) \to \bar{\mathrm{U}}_i,$$

with $\rho \circ \kappa = \mathrm{id}$.

We are finished if the image of κ is equal to $\bar{U}_i$. We already know that U_i is contained in this image, which is Γ_i-stable, and as $\bar{U}_i - U_i \subseteq L$ maps onto $\bar{V}_i \cap X^0$ is the quotient of $\bar{U}_i - U_i$ under the Γ_i-action.

This comes down to the fact that any $\gamma \in \Gamma$ with $\gamma(\bar{U}_i - U_i) \cap (\bar{U}_i - - U_i) \neq 0$ is already contained in Γ_i, and this assertion is correct since i is the only element $j \in I^{\pm}$, of the same parity as i, such that $\bar{U}_j \cap (\bar{U}_i - - U_i)$ is not void.

(vi) If I has only one element this is a corollary to (ii), and of the characterization of q in $\nabla = \nabla_0 + q$ via $q = -\theta_2(\rho)/2$. If I has at least two elements L is not void, and neither are I^+ and I^-. The rest follows from (v).

(vii) We assume that I has more than one element. It is already clear from (ii) that ρ is a covering over $\mathbb{H} \cup -\mathbb{H}$. So take a $w_0 \in \overline{\mathbb{P}^1(\mathbb{R}) - \bigcup_{i \in I} \mathfrak{L}_i}$, and let $K \subset \mathbb{P}^1(\mathbb{R})$ be a maximal open interval with $w_0 \in K$ and $K \cap \overline{\bigcup_{i \in I} \mathfrak{L}_i} = 0$.

Any element $z_0 \in \mathbb{H}$ with $\rho(z_0) = w_0$ is contained in L, so that there exist unique $i \in I^+$ and $j \in I^-$ with $z \in \bar{U}_i$, $z \in \bar{U}_j$.

We have seen in (v) that $\rho^{-1} : \mathbb{H} \to U_i$ and $\rho^{-1} : -\mathbb{H} \to U_j$ can be extended to K, and $(\mathbb{H} \cup -\mathbb{H}) \cup K$ being simply connected we obtain a well-defined holomorphic mapping

$$\kappa : (\mathbb{H} \cup -\mathbb{H}) \cup K \to \bar{U}_i \cup \bar{U}_j \subseteq \mathbb{H},$$

with $\rho \circ \kappa = \mathrm{id}$, $\kappa(\mathrm{w}_0) = z_0$.

Furthermore for different choices of $z_0 \in \rho^{-1}(w_0)$ the images of the corresponding κ's are disjoint, and their union is $((\mathbb{H} \cup -\mathbb{H}) \cup K)$: If for elements $z_0, z_1 \in \rho^{-1}(w_0)$ a $z \in \mathbb{H}$ is contained in the images of both the corresponding κ's, it is both times the image of $w = \rho(z)$, and as two sections of ρ are equal if they coincide at one point the corresponding κ's are equal, with common value $z_0 = z_1$ at w_0. Conversely for any $z \in \mathbb{H}$ with $w = \rho(z) \in (\mathbb{H} \cup -\mathbb{H}) \cup K$ choose an $i \in I$ with $z \in \bar{U}_i$, and let z_0 be the unique element of $\bar{U}_i$ with $\rho(z_0) = w_0$. Then z is in the image of the section corresponding to z_0. This shows our assertion, or more precisely that ρ has a lot of sections. Part (vii) is a direct consequence of this fact.

Remarks:

(a) Part (iv) implies that any simply connected V_i contains at least two elements from S: Otherwise $\sigma(\Gamma_i)$ would be cyclic, generated by a parabolic or elliptic element. Furthermore $\bar{V}_i \cap X^0$ would consist of just one arc. If V_j is the other component of $X - X^0$ bounded by this arc the element in $\sigma(\Gamma_j)$ belonging to this boundary-arc of V_j would be parabolic or elliptic and V_j would be of the same type as V_i. But then X is equal to $\mathbb{P}^1(\mathbb{C})$ with S consisting of at most two points, and that is a contradiction.

(b) Part (vi) finishes the proof of Theorem 8 (ii): If for an arbitrary connection ∇ the image $\mathscr{D}$ of ρ is equal to $\mathbb{H}, \sigma(\Gamma)$ is contained in the normalizer of $\mathscr{D}$, hence in $\mathbf{SL}(2, \mathbb{R})$, and ∇ has real monodromy. $\sigma(\Gamma)$ is then conjugate to Γ.

Theorem 10: *The following assertions are equivalent:*

(a) $\mathscr{D} \neq \mathbb{P}^1(\mathbb{C})$.

(b) $\mathfrak{L}_i = \mathfrak{L}_j$, *for all* $i, j \in I$.

They imply:

(i) ρ *is the universal covering-mapping for* D.

(ii) $\sigma(\Gamma_i)$ *has finite index in* $\sigma(\Gamma)$, *for any* $i \in I$.

(iii) *Either* $\mathscr{D}$ *is simply connected, and then* $\mathscr{D} = \pm\mathbb{H}$, *or* $\mathscr{D} = \mathbb{P}^1(\mathbb{C}) - \mathfrak{L}_i$, *for an arbitrary element* $i \in I$.

Proof: We may assume that I has more than one element. As the $\mathfrak{L}_i$ are non-void Theorem 9 (vi) shows that (b) implies (a). Conversely if (a) is true $\mathbb{P}^1(\mathbb{C}) - \mathscr{D}$ has at least three points. If $1 \neq \gamma \in \sigma(\Gamma_i)$ is a hyperbolic or parabolic element there exists therefore a point $p \in \mathbb{P}^1(\mathbb{C}) - \mathscr{D}$ which is not fixed by γ, and the transforms $\sigma(\gamma^n)$ (p) converge for $n \to \pm\infty$ to the fixed-points of γ. As $\mathbb{P}^1(\mathbb{C}) - \mathscr{D}$ is closed it contains the closure of the set of fixed-points of non-trivial hyperbolic or parabolic elements of $\sigma(\Gamma_i)$, and it is well known that this set is equal to $\mathfrak{L}_i$. Thus

$$\mathfrak{L}_i \subseteq \mathbb{P}(\mathbb{C}) - \mathscr{D} = \bigcap_{j \in I} \mathfrak{L}_j,$$

and (b) follows.

The assertions (i) and (iii) are now easy consequences of Theorem 9 (vii) and (vi), except for the fact that $\mathscr{D}$ is not simply connected. For (ii) we use the fact that $\mathfrak{L}_i = \mathbb{P}^1(\mathbb{C}) - \mathscr{D}$ is $\sigma(\Gamma)$-stable. The connected components of $\mathbb{P}^1(\mathbb{R}) - \mathfrak{L}_i$ consist of finitely many classes under $\sigma(\Gamma_i)$-conjugation, each such class corresponding to an arc bounding V_i. As $\sigma(\Gamma)$ permutes these connected components it suffices to proof that for any such component its normalizer in $\sigma(\Gamma_i)$ has finite index in its normalizer in $\sigma(\Gamma)$.

The component is an open interval with endpoints a and b, and as any element of $\mathbf{SL}(2, \mathbb{R})$ normalizing this interval permutes a and b we are ready if we show that the intersection of $\sigma(\Gamma_i)$ with the subgroup of $\mathbf{SL}(2, \mathbb{R})$ fixing a and b has finite index in the intersection of $\sigma(\Gamma)$ with this subgroup.

But this subgroup is a real Lie-group isomorphic to $\mathbb{R}^*$, and its intersection with $\sigma(\Gamma_i)$ is infinite because the quotient of our interval under this intersection is the corresponding boundary-arc of V_i, and thus compact. As $\sigma(\Gamma)$ is discrete (Theorem 8 (ii)) we are finished.

If $\mathscr{D}$ would be simply connected $\mathbb{P}^1(\mathbb{C}) - \mathscr{D}$ would be a real interval whose endpoints are permuted by $\sigma(\Gamma)$, which contradicts the Zariski-density of $\sigma(\Gamma)$ in $\mathbf{SL}(2, \mathbb{C})$.

COROLLARY: *If $\mathscr{D} \neq \mathbb{P}^1(\mathbb{C})$, each of the following conditions implies that ∇ is equivalent to ∇_0:*

(i) *$\mathscr{D}$ is simply connected.*

(ii) *$\sigma: \Gamma \to \mathbf{SL}(2, \mathbb{R})$ is injective.*

(iii) *The fixed-points of the non-trivial hyperbolic and parabolic elements are dense in $\mathbb{P}^1(\mathbb{R})$.*

PROOF: (i) is clear from the previous two theorems, (ii) implies that each Γ_i has finite index in Γ, and simple topological considerations then show that I has only one element, and (iii) means that each $\mathfrak{L}_i$ is equal to $\mathbb{P}^1(\mathbb{R})$, so that $\mathscr{D}$ is one of the connected components of $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$.

The next result will provide us with examples of connections with real monodromy different from ∇_0 which satisfy the conditions of Theorem 10.

THEOREM 11: *Let $\mathscr{D} = (\mathbf{e}, S_r^\infty, \mathscr{L}, \varphi)$ be a uniformization-datum on X. There exists on X a permissible connection with real monodromy such that $\mathscr{D} \neq \mathbb{P}^1(\mathbb{C}) \neq \mathbb{H}$, if and only if there exists a finite morphism of Riemann-surfaces*

$$p: X \to Y$$

with the following properties:

(i) *Y has a real structure such that $Y(\mathbb{C}) - Y(\mathbb{R})$ has two connected components.*

(ii) *There exists a real algebraic curve Y and a finite real morphism*

$$q: \tilde{Y} \to Y,$$

such that $\tilde{Y}(\mathbb{R}) = q^{-1}(Y(\mathbb{R}))$, and such that q can be inserted in a commutative diagram of mappings of compact Riemann-surfaces

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{q}"] & \tilde{Y} \arrow[d, "q"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

with:

(a) *p unramified.*

(b) *$q, \tilde{q}$ are Galois-coverings, with $\tilde{q}$ only ramified over the points of S, and ramification-indices there equal to $\mathbf{e}(x)$ for $x \in S^f$, and even for $x \in S_i^\infty$.*

(c) *$S = p^{-1}(T)$, for a subset $T \subset Y(\mathbb{C}) - Y(\mathbb{R})$ stable under complex conjugation.*

PROOF: If we have a connection ∇ with real monodromy such that $\mathscr{D} \neq \mathbb{P}^1(\mathbb{C})$, $\pm\mathbb{H}$, we denote by Δ the discrete group $\sigma(\Gamma) \subset \mathbf{SL}(2, \mathbb{C})$, and choose a normal subgroup $\tilde{\Delta} \subseteq \Delta$ of finite index which is torsion-free and does not contain any parabolic element with trace -2. It is well known that ${}_{\Delta}\backslash\mathscr{D}$ and ${}_{\tilde{\Delta}}\backslash\mathscr{D}$ may be imbedded with finite complements into compact Riemann-surfaces Y and $\tilde{Y}$, such that ρ induces holomorphic mappings from $X = {}_{\Gamma}\backslash\mathbb{H}$ to Y and from $\tilde{X} = {}_{\tilde{\Gamma}}\backslash\mathbb{H}$ to $\tilde{Y}$, where $\tilde{\Gamma}$ is the preimage of $\tilde{\Delta}$ under σ.

Together with the obvious projections from $\tilde{X}$ to X and $\tilde{Y}$ to Y we obtain a diagram as above, with the complex conjugation on Y and $\tilde{Y}$ induced from the complex conjugation on $\mathscr{D}$.

For the converse we start with such a diagram. The hypotheses imply that Y and $\tilde{Y}$ are doubles of Riemann surfaces with boundary V and $\tilde{V}$, and that q is induced from a Galois-covering $\tilde{V} \to V$. We let $\tilde{T} = q^{-1}(T)$, and remark that there exist discrete subgroups $\tilde{\Delta} \subseteq \Delta \subset \mathbf{SL}(2, \mathbb{R})$ such that

(a) $\tilde{\Delta}$ has finite index in Δ, Δ contains -1, and $\tilde{\Delta} \cdot \{\pm 1\}$ is normal in Δ.

(b) $V - (V \cap T) = {}_{\Delta}\backslash\mathbb{H}$, $\tilde{V} - (\tilde{V} \cap \tilde{T}) = {}_{\tilde{\Delta}}\backslash\mathbb{H}$, with $(V \cap T)$ respectively $(\tilde{V} \cap T)$ corresponding to the cusps of Δ respectively $\tilde{\Delta}$, as usual.

(c) $\tilde{\Delta}$ is torsion-free, and contains no parabolic elements of trace -2. If $\mathscr{D}$ denotes the complement in $\mathbb{P}^1(\mathbb{C})$ of the limit-points of Δ or $\tilde{\Delta}$ we then know that

$$Y - T = {}_{\Delta}\backslash\mathscr{D}, \; \tilde{Y} - \tilde{T} = {}_{\tilde{\Delta}}\backslash\mathscr{D},$$

with the complex conjugation corresponding to complex conjugation on $\mathscr{D}$.

This representation defines a uniformization-datum on $\tilde{Y}$, with the set S being equal to $\tilde{T}$ and consisting only of regular cusps. Its pullback under $\tilde{p}$ is nearly isomorphic to the pullback $\tilde{\mathbf{D}} = (\tilde{\mathbf{e}}, \tilde{S}_r^{\infty}, \tilde{\mathscr{L}}, \tilde{\varphi})$ of our given datum $\mathbf{D}$ on X, except for the fact that the two $\mathscr{L}$'s may differ by a line-bundle of order two. But this bundle becomes trivial if we pass to a suitable larger covering of X which is unramified over $\tilde{X}$, and thus we may assume that also the $\mathscr{L}$'s are equal. Finally we have on $\tilde{Y}$ a canonical permissible connection with $\tilde{\Delta}$ as monodromy-group, and its pullback gives us a permissible connection $\tilde{\nabla}$ on $\tilde{X}$ with monodromy-group contained in $\tilde{\Delta}$, hence real, and the mapping ρ belonging to $\tilde{\nabla}$ inserts into a commutative diagram:

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "\rho"] \arrow[d] & \mathscr{D} \arrow[d] \\
\tilde{X} \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{q}"] & \tilde{Y} \arrow[d, "q"] \\
X \arrow[r, "p"] & Y,
\end{tikzcd}$$

such that the second and third line are quotients of the upper line under actions of groups which fit into a diagram

$$\begin{array}{ccc} \tilde{\Gamma} & \xrightarrow{\sigma} & \tilde{\Delta} \\ \cap & & \cap \\ \Gamma & \xrightarrow{\sigma} & \Delta/\{\pm 1\} \end{array}$$

Especially the map

$$\rho : \mathbb{H} \to \mathscr{D}$$

is equivariant with respect to the morphism

$$\sigma : \Gamma \to \Delta/\{\pm 1\}.$$

Replacing $\tilde{\nabla}$ by an equivalent permissible connection we may assume that $\tilde{\nabla} = \tilde{\nabla}_i + q$, with $\tilde{\nabla}_i$ the uniformizing connection on $\tilde{X}$, and q an element of $\Gamma(\tilde{X}, \mathscr{K}_X^{\otimes 2}(\tilde{S}))$. If we show that q is invariant under the group $G = \Gamma/\tilde{\Gamma}$ of the covering q it can be seen without difficulty that $\tilde{\nabla}$ is the pullback of a permissible connection ∇ on X, whose monodromy is contained in Δ, hence real. The G-invariance of q can be seen as follows: The pullback of q on $\mathbb{H}$ is of the form $q(z)dz^2$, with $q(z) = -\theta_2(\rho)/2$. As $\rho(\gamma(z)) = \sigma(\gamma)(\rho(z))$, for $\gamma \in \Gamma$, the well known properties of the Schwarzian-derivative imply that

$$q(\gamma(z)) \cdot (d\gamma(z)/dz)^2 = q(z),$$

hence that $q(z)dz^2$ is Γ-invariant.

Remark: The proof shows that the connections with real monodromy different from the uniformizing connection are classified by the diagrams modulo a suitable equivalence relation.

Furthermore the theorem allows us to construct a lot of examples, the simpliest being given by setting e equal to 1 everywhere, and choosing for $X = \tilde{X} = Y = \tilde{Y}$ a real Riemann surface such that $X(\mathbb{C}) - X(\mathbb{R})$ has two components, for example such that $X(\mathbb{R})$ has $g + 1$ connected components.

On the other hand the theorem is not of so much use in practice since it is very difficult to find manageable conditions which assure us that $\mathscr{D} \neq \mathbb{P}^1(\mathbb{C})$, for a given connection.

§7. Infinitesimal theory and estimates

We start again with a compact Riemann surface X and a uniformization-datum $\mathbf{D} = (\mathbf{e}, S_r^\infty, \mathscr{L}, \varphi)$ on X. The permissible connections up to equivalence may be identified with $Q = \Gamma(X, \mathscr{K}_X^{\otimes 2}(S))$, with the uniformizing connection as origin. The monodromy of a connection defines a monodromy-mapping

$$\mathrm{mon}: Q \to \mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C})).$$

THEOREM 12: The mapping mon has injective differential everywhere, and it is transversal to the submanifold

$$\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{R})) \subset \mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C})).$$

PROOF: If ∇ is a permissible connection corresponding to a representation σ of Γ on $\mathbf{SL}(2, \mathbb{C})$, and on $W = \mathfrak{sl}(2, \mathbb{C})$, we have already seen that the differential $d(\mathrm{mon})$ is given by the injection of Q into $H_P^1(\Gamma, W)$ in the short exact sequence

$$0 \to Q \to H_P^1(\Gamma, W) \to H^1(X, \mathscr{T}_X(-S)) \to 0.$$

We thus may assume that ∇ has real monodromy, and have to show that Q does not intersect the real subspace $H_P^1(\Gamma, \mathfrak{sl}(2, \mathbb{R}))$ of $H_P^1(\Gamma, \mathfrak{sl}(2, \mathbb{C}))$.

Poincaré-duality induces a non-degenerate alternate form on $H_P^1(\Gamma, W)$ derived from the symmetric form $(A, B) \mapsto \mathrm{tr}(AB)$ on $W = \mathfrak{sl}(2, \mathbb{C})$, for which Q is a maximally isotropic subspace. We thus have to show that for any non-zero $q \in Q$ the cup-product with the conjugate class $\bar{q}$ does not vanish.

For this we write the pullback $\pi^*(q)$ on $\mathbb{H}$ as $q(z)dz^2$. If we choose the mapping $\rho: \mathbb{H} \to \mathbb{P}^1(\mathbb{C})$ belonging to ∇ such that σ takes real values, the class of q in $H_P^1(\Gamma, W) = H_P^1(X - S, \mathbf{W})$ is represented by the Γ-invariant W-valued $(1,0)$-form

$$(q(z)/\rho'(z)) \cdot \begin{pmatrix} \rho(z) & -\rho^2(z) \\ 1 & -\rho(z) \end{pmatrix} dz.$$

To form the cup-product with $\bar{q}$ we first have to find for each $x \in S$ $\mathbf{W}$-valued functions ψ_x defined and supported in a small punctured neighbourhood of x whose derivative is equal to the form defined by the expression above, at least near x. That this is possible is a consequence of Theorem 6 together with some easy direct calculations for the connection ∇_0.

We then substract the sum of the derivatives of the ψ_x from q, form the product with the $(0,1)$-form defined by $\bar{q}$, and integrate over X. Some easy local estimates show that the integrals over the correction-terms defined by the ψ_x converge and become zero, so that we have to calculate the integral over a fundamental-domain of Γ in $\mathbb{H}$ of

$$|q(z)/\rho'(z)|^2 \cdot \mathrm{Tr}\left[\begin{pmatrix} \rho(z) - \rho^2(z) \\ 1 - \rho(z) \end{pmatrix} \cdot \overline{\begin{pmatrix} \rho(z) - \rho^2(z) \\ 1 - \rho(z) \end{pmatrix}}\right] dz \wedge d\bar{z},$$

and we know that this integral converges. Up to constant factors it is equal to

$$\int_{\Gamma\backslash\mathbb{H}} |q(x+iy)/\rho'(x+iy)|^2 \cdot (\mathrm{Im}(\rho(x+iy)))^2 \cdot dx \cdot dy,$$

which is strictly positive for $q \neq 0$. This finishes the proof of our theorem.

COROLLARY: *The connections with real monodromy form discrete subset of Q.*

For a more complete picture we have to deform X. Let $\mathbf{T}$ denote the Teichmüller-space of marked Riemann surfaces of genus g, together with $n =$ (cardinality of S) points. There exists a universal family $\mathscr{X}$ over $\mathbf{T}$, together with the universal uniformizing connection ∇_0. The permissible connections up to equivalence form a complex manifold over $\mathbf{T}$, isomorphic to the complex vectorbundle defined by the direct image of the relative quadratic differentials on $\mathscr{X}$, with poles along $\mathscr{S}$.

We denote this complex manifold by $\mathscr{P}\mathscr{C}$. It has complex dimension $6g - 6 + 2n$, and its tangent-space in the permissible connection ∇ on X is an extension of $H^1(X, \mathscr{T}_X(-S))$, the tangent-space of $\mathbf{T}$, by $Q = \Gamma(X, \mathscr{K}_X^{\otimes 2}(S))$. The monodromy-mappings define a global analytic mapping

$$\mathit{mon}: \mathscr{P}\mathscr{C} \to \mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C})).$$

As the tangent-space $H_P^1(\Gamma, W)$ of $\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C}))$ in the image of ∇ by mon is also an extension of $H^1(X, \mathscr{T}_X(-S))$ by Q, and as the differential $d(\mathit{mon})$ induces the identity on Q, we obtain an endomorphism of

$H^1(X, \mathscr{T}_X(-S))$, and the next result says that this endomorphism is precisely what it should be:

THEOREM 13: *The induced map on* $H^1(X, \mathscr{T}_X(-S))$ *is the identity.*

PROOF: Start with a local section s of the projection $\mathscr{P}\mathscr{C} \to \mathbf{T}$, which defines a family of connections on $\mathscr{X}$. $\mathscr{mon} \circ s$ is a mapping from $\mathbf{T}$ to $\mathbf{Rep}_P^0(\Gamma, \mathbf{SL}(2, \mathbb{C})$ and we have to consider its differential.

Classes in $H^1(X, \mathscr{T}_X(-S))$ are represented by $C^\infty - (-1,1)$-forms μ on X with zeroes in S. Such a μ defines a deformation of the complex structure on X, where for the deformed structure a C^∞-function f is holomorphic if $\bar{\partial}(f) + \mu \cdot \partial(f) = 0$.

For a parameter t we may define a continuous family of deformations X_t by calling f holomorphic on X_t if $\bar{\partial}(f) + t\mu\partial(f) = 0$. This family corresponds to a small curve in $\mathbf{T}$, whose tangent-vector in the origin is given by the class represented by μ in $H^1(X, \mathscr{T}_X(-S))$, the tangent-space of $\mathbf{T}$.

By pullback via $\pi: \mathbb{H} \to X$ we get a family of complex structures on $\mathbb{H}$, defined by $\pi^*(\mu) = \mu(z) \cdot (d\bar{z}/dz)$. The family of permissible connections ∇ defines a family of C^∞-mappings

$$\rho(t, z): \mathbb{H} \to \mathbb{P}^1(\mathbb{C}),$$

such that $\rho(t, z)$ is holomorphic in the complex structure belonging to t, i.e.

$$\partial\rho(t, z)/\partial\bar{z} + t \cdot \mu(z) \cdot \partial\rho(t, z)/\partial z = 0,$$

and such that $\rho(t, z)$ satisfies a functional equation

$$\rho(t, \gamma(z)) = \sigma(t, \gamma)(\rho(t, z)),$$

with $\sigma(t, \gamma)$ a C^∞-family of morphisms from Γ to $\mathbf{SL}(2, \mathbb{C})$. The derivative of this family in $t = 0$ is essentially the image of the class defined by μ in $H^1(X, \mathscr{T}_X(-S))$ by the differential of $\mathscr{mon} \circ s$. The derivative is given by the crossed homomorphism $A \in Z^1(\Gamma, \mathfrak{sl}(2, \mathbb{C}))$ given by

$$A(\gamma) = d/dt(\sigma(t, \gamma) \circ \sigma(0, \gamma)^{-1})_{t=0}, \text{ for } \gamma \in \Gamma.$$

A defines a class in $H_P^1(\Gamma, W) = H_P^1(X - S, \mathbf{W})$. Its image in $H^1(X, \mathscr{T}_X(-S))$ is given as follows: For $\gamma \in \Gamma$ consider the $(-1,0)$-form on $\mathbb{H}$

$$(d\rho/dz)^{-1} \cdot \langle A(\gamma)^t(\rho(z), 1), {}^t(\rho(z), 1)\rangle \cdot dz^{-1}, \quad (\rho(z) = \rho(0, z))$$

where $\langle , \rangle$ denotes the natural $\mathbf{SL}(2,\mathbb{C})$-invariant alternating form on $\mathbb{C}^2 : \langle u, v\rangle = \det(u,v)$. There exists a $C^\infty-(-1,0)$-form $\psi = \psi(z)\cdot dz^{-1}$ on $\mathbb{H}$, with $\psi(z)$ holomorphic outside the inverse image by π of a compact set in $X - S$, such that $\gamma^*(\psi) - \psi$ is for any $\gamma \in \Gamma$ the $(-1,0)$-form defined above. This has to be interpreted as

$$\psi(\gamma^{-1}(z))\cdot(d\gamma^{-1}(z)/dz)^{-1} - \psi(z) = (d\rho/dz)^{-1}\cdot\langle A(\gamma)^t(\rho(z),1), {}^t(\rho(z),1)\rangle.$$

The derivative $\bar{\partial}(\psi)$ is a Γ-invariant $(-1,1)$-form, hence defines such a form on X which vanishes near the points of S. The cohomology-class of this form in $H^1(X, \mathscr{T}_X(-S))$ is independant of the choice of ψ, and we have to show that it is equal to the class defined by μ. For this we may assume that μ vanishes near the points of S, since we may represent any class in $H^1(X, \mathscr{T}_X(-S))$ by such a $(-1,1)$-form. We claim that the function

$$\psi(z) = -(\partial\rho/\partial t(0,z))/(d\rho/dz)$$

satisfies our conditions: In fact,

$$\begin{aligned} d\psi/d\bar{z} &= -(\partial^2/\partial t\partial\bar{z}(0,z))/(d\rho/dz) = \\ &\qquad -(\partial/\partial t(-t\cdot\mu\cdot\partial\rho/\partial z)(0,z))/(d\rho/dz) = \mu(z), \end{aligned}$$

so that ψ is holomorphic near the preimages of the points of S, and its $\bar{\partial}$-derivative defines the same class as μ. It remains to show the functional equation. This is done by taking the t-derivatives in

$$\rho(t,z) = \sigma(t,\gamma)(\rho(t,\gamma^{-1}(z)),$$

if we make use of the obvious formula

$$\langle A(\gamma)^t(\rho(z),1), (\rho(z),1)\rangle = d/dt(\sigma(t,\gamma)\circ\sigma(0,\gamma)^{-1}(\rho(z)))_{t=0}.$$

COROLLARY: *$\mathscr{mon}:\mathscr{PC} \to \mathscr{Rep}^0_{\mathrm{P}}(\Gamma, \mathbf{SL}(2,\mathbb{C}))$ is a local isomorphism. $\mathscr{RPC} = \mathscr{mon}^{-1}(\mathbf{Rep}^0_{\mathrm{P}}(\Gamma, \mathbf{SL}(2,\mathbb{R})))$ is a real submanifold of $\mathscr{PC}$ of dimension $6g - 6 + 2n$, whose projection onto $\mathbf{T}$ is a local diffeomorphism, as well as the mapping induced by $\mathscr{mon}$ of $\mathscr{RPC}$ into $\mathbf{Rep}^0_{\mathrm{P}}(\Gamma, \mathbf{SL}(2,\mathbb{R}))$.*

REMARK: Permissible connections with real monodromy can thus be deformed together with X. Under such a deformation the topological type of the decomposition of X into X^+, X^- and X^0 does not change. It would be nice if the real permissible connections with a given topological type of this decomposition were proper over $\mathbf{T}$. The theory of univalent mappings can be applied to the effect that the decompositions must behave rather wild if this should not be the case, but otherwise I have no convincing argument for this conjecture. Furthermore the considerations above lead to a differential equation for a family of permissible connections with real monodromy, which determines the extent to which such a family does not depend holomorphically on its parameters. For the uniformizing connection this differential equation is due to Ahlfors (*Ann. Math. 74* (1961), 171–191). That such a family never depends holomorphically on the parameters is also a consequence of the fact that the tangentspace of $\mathscr{RPC}$ does not contain any complex line of the tangentspace of $\mathscr{PC}$

To show that univalent mappings may be important for our problems we finally sketch the proof of a result which says how much ∇_0 is isolated in the set of connections with real monodromy. For this we remark that for such a permissible ∇ X^0 is not void if and only if ∇ is not equivalent to ∇_0. Also for any quadratic differential q the absolute value $|q|$ can be seen as a positive definite $(1,1)$-form, which may be compared to the hyperbolic metric ds^2 on $X - S$, deduced from $\pi : \mathbb{H} \to X$.

THEOREM 14: *Let $\nabla = \nabla_0 + q$ be a permissible connection with real monodromy on X, with $q \in \Gamma(X, \mathscr{K}_X^{\otimes 2}(S))$. Then* $|q| \geq 3/4 \cdot ds^2$ on X^0.

PROOF: Choose a $z_0 \in L = \pi^{-1}(X^0) \subset \mathbb{H}$, and let $i \in I^+$, $j \in I^-$ be the indices with $z_0 \in \bar{U}_i$, $z_0 \in \bar{U}_j$. If $L_0 \subset L$ denotes the connected component of L containing z_0, ρ maps L_0 onto an interval in $\mathbb{P}^1(\mathbb{R})$ which we may assume to be the negative real axis. ρ then defines an isomorphism of $U_i \cup U_j \cup L_0$ with the complement of the positive real axis in $\mathbb{P}^1(\mathbb{R})$. The mapping

$$w \mapsto \rho^{-1}(w^2)$$

is then an univalent mapping from $\mathbb{H}$ into $\mathbb{H}$. The theory of univalent mappings bounds the θ_2 of such a function, and this θ_2 is essentially the sum of q and the θ_2 of the mapping $w \mapsto w^2$, which is known. From this we derive the theorem, and leave the details to the reader. (For univalent mappings compare [9].)

§8. Constructions and examples

We first want to show how to construct Riemann surfaces with permissible connections with real monodromy, and then demonstrate by some examples that permissible connections can be described algebraically.

In fact our construction of permissible connections gives all such connections with real monodromy, and it is accomplished by going backwards the steps of §6. For this let us start with a Riemann surface X and a connection with real monodromy on X. We then have the decomposition $X = X^+ \cup X^- \cup X^0$, and we may form a finite graph whose vertices are the connected components of $X - X^0$ and whose edges correspond to the connected components of X^0, such that the endpoints of an edge are the connected components bounded by the corresponding arc in X^0. Furthermore we may associate to any vertex a sign $+$ or $-$, such that two vertices connected by an edge have different parity.

Conversely let us start with such a graph $\mathscr{G}$. Let vert($\mathscr{G}$) and edge($\mathscr{G}$) denote the vertices and edges of $\mathscr{G}$. We suppose that any $p \in \mathrm{vert}(\mathscr{G})$ has a parity $\pm$ as above, and that there corresponds to p a Riemann surface with boundary V_p together with a function

$$\mathbf{e}_p : V_p \to \{1, 3, 5, \ldots, \infty\},$$

which is equal to 1 for almost all $x \in V_p$.

The boundary-components of V_p should correspond bijectively to the edges of $\mathscr{G}$ with endpoint p, and furthermore V_p should be either an annulus, or the number

$$\chi(V_p) = g_p - 1 + \sum_{x \in V_p} (1 - 1/e_p(x))$$

should be positive.

Here g_p is the genus of the double of V_p, or if V_p has no boundary, of V_p itself.

Finally we suppose that not all the $\chi(V_p)$ vanish.

If we glue together the V_p along their boundaries as prescribed by $\mathscr{G}$ we obtain at least topologically a compact Riemann surface X, provided the graph was connected. On X the $\mathbf{e}_p$ glue together to a function $\mathbf{e}$, and we suppose that we have given a subset S_r^∞ of even cardinality in $S^\infty = \mathbf{e}^{-1}(\{\infty\})$.

We want to show that the glueing can be done such that we get a conformal structure on X, if we assume the following condition:

For any edge $a \in \mathrm{edge}(\mathscr{G})$ with endpoints $p, q \in \mathrm{vert}(\mathscr{G})$, the intrinsic lengths of the boundary-components of V_p and V_q corresponding to a coincide.

The intrinsic length of such a component is essentially given by the eigenvalues of the corresponding element in $\mathbf{PSL}(2,\mathbb{R})$, in the group of deck-transformations belonging to the following covering of V_p by $\pm\mathbb{H}$:

There exists a discrete subgroup $\Gamma_p \subset \mathbf{PSL}(2,\mathbb{R})$ with $\mathring{V}_p = V_p - (V_p \cap S^\infty)$ isomorphic to ${}_{\Gamma_p}\backslash \pm\mathbb{H}$ (sign according to the parity of p), and such that the ramification is given by the function $\mathbf{e}_p$, as always. The conditions on V_p imply that the element in Γ_p corresponding to the boundary-component belonging to a is hyperbolic, and the compatability-condition means that it is conjugate in $\mathbf{PSL}(2,\mathbb{R})$ to the corresponding element of Γ_q. If we define Γ_a to be the cyclic group $\mathbb{Z}$ we thus obtain injections of Γ_a into Γ_p and Γ_q conjugate under $\mathbf{PSL}(2,\mathbb{R})$, if we choose orientations on the boundary-components of V_p and V_q belonging to a which are compatable with the glueing in X.

So far X has been constructed as a topological space, and we can define a ramified topological covering of X with the ramification defined by the function $\mathbf{e}$. The group Γ of deck-transformations is by topological reasons isomorphic to the fundamental-group of the graph of groups $\mathscr{G}, \{\Gamma_p\}, \{\Gamma_a\}$, which is defined as follows ([10], Ch. I, §5):

Choose a maximal subtree $\mathscr{T} \subseteq \mathscr{G}$. Γ is the quotient of the free product of the groups Γ_p for $p \in \mathrm{vert}(\mathscr{T}) = \mathrm{vert}(\mathscr{G})$, and copies of $\mathbb{Z}$ for each $a \in \mathrm{edge}(\mathscr{G}) - \mathrm{edge}(\mathscr{T})$, by the relations defined as follows:

For $a \in \mathrm{edge}(\mathscr{T})$ the injections of Γ_a into Γ_p and Γ_q are equal, where p and q are the end-points of a.

For $a \in \mathrm{edge}(\mathscr{G}) - \mathrm{edge}(\mathscr{T})$ the two injections are conjugate under the generator 1 of the copy of $\mathbb{Z}$ belonging to a.

It is known that the group thus defined is independent of the choice of $\mathscr{T}$, and that the groups Γ_p inject into it.

We next define a homomorphism

$$\bar{\sigma} : \Gamma \to \mathbf{PSL}(2,\mathbb{R}),$$

such that the restriction of $\bar{\sigma}$ to the groups Γ_p is given by conjugation with an element from $\mathbf{PSL}(2,\mathbb{R})$:

We have to choose the conjugating elements in such a way that the relations corresponding to the different edges a of $\mathscr{G}$ are fulfilled, and for this we are still free to choose the images under σ of the elements of Γ corresponding to the edges not in $\mathscr{T}$.

As $\mathscr{T}$ is a tree the relations for $a \in \mathrm{edge}(\mathscr{T})$ can be fulfilled easily, say by induction over the number of vertices of $\mathscr{T}$, since the injections of Γ_a

into the groups $\Gamma_p, \Gamma_q \subset \mathbf{PSL}(2,\mathbb{R})$ belonging to the endpoints p, q of a are conjugate. If a is not an edge of $\mathcal{T}$ this is still true, and we can fulfill the relation by taking an appropriate element of $\mathbf{PSL}(2,\mathbb{R})$ as the image of the generator corresponding to a. The next step is the construction of a simply connected complex manifold M on which Γ operates, such that M realizes the topological ramified covering of X constructed above. X will then become a compact Riemann surface, with the conformal structure induced from M, and the projection of M onto X will be a covering of the type consider in this paper.

Furthermore if g is the genus of X we will have the formula

$$2g - 2 + \sum_{x\in X}(1 - 1/\mathbf{e}(x)) = \sum_p \chi(V_p),$$

and this is positive so that M must be isomorphic to the upper halfplane.

For the construction of M we recall from [10] that there exists a tree $\tilde{\mathcal{G}}$ on which Γ operates, such that ${}_\Gamma\backslash\tilde{\mathcal{G}}$ is isomorphic to $\mathcal{G}$, and such that for any $p\in\operatorname{vert}(\mathcal{G})$ or $a\in\operatorname{edge}(\mathcal{G})$ Γ_p respectively Γ_a is the stabilizer of a $\tilde{p}$ respectively $\tilde{a}$ in the preimage of p or a. $\mathcal{G}$ is called the universal covering of $\mathcal{G}$. M is constructed as follows:

For any $\tilde{p}\in\operatorname{vert}(\tilde{\mathcal{G}})$ we take a copy of $\pm\mathbb{H}$, with the sign corresponding to the parity of the image p of $\tilde{p}$ in $\mathcal{G}$. If $\tilde{a}\in\operatorname{edge}(\tilde{\mathcal{G}})$ with endpoints $\tilde{p}$ and $\tilde{q}$, projecting to p and q in $\mathcal{G}$, and such that p has positive and q negative parity, the copy of $\mathbb{H}$ corresponding to $\tilde{p}$ and the copy of $-\mathbb{H}$ corresponding to $\tilde{q}$ are glued together along an interval of $\mathbb{P}^1(\mathbb{R})$.

To define this interval we remark that the stabilizer $\Gamma_{\tilde{a}}$ of $\tilde{a}$ is cyclic, and that the image C under $\bar{\sigma}$ of a generator of $\Gamma_{\tilde{a}}$ is hyperbolic. One of the intervals in $\mathbb{P}^1(\mathbb{C})$ bounded by its fixed-points maps onto the boundary-components belonging to a in $\mathring{V}_p = \bar{\sigma}(\Gamma_{\tilde{p}})\backslash\mathbb{H}$ and $\tilde{V}_q = \bar{\sigma}(\Gamma_{\tilde{q}})\backslash -\mathbb{H}$. (That it is the same interval for p and q is a consequence of our choice of the orientations of these boundary-arcs, namely that they are compatable with the glueing.) We glue $\mathbb{H}$ and $-\mathbb{H}$ along this (open) interval. It is elementary that this leads to a complex manifold M, and as $\tilde{\mathcal{G}}$ is a tree M is simply connected. Finally Γ operates on M:

If $\tilde{p}\in\operatorname{vert}(\tilde{\mathcal{G}})$ and $z\in\pm\mathbb{H}$ is an element of the corresponding copy of $\pm\mathbb{H}$, γ sends z to the element $\bar{\sigma}(\gamma)(z)$ in the copy corresponding to $\gamma(\tilde{p})$.

It is clear that ${}_\Gamma\backslash M$ is homeomorphic to X, and can be glued together from the $\mathring{V}_p$. We already know that M is isomorphic to $\mathbb{H}$. The injections of $\pm\mathbb{H}$ into $\mathbb{P}^1(\mathbb{C})$ define a mapping

$$\rho: M \to \mathbb{P}^1(\mathbb{C})$$

with $\rho(\gamma(z)) = \bar{\sigma}(\gamma)(\rho(z))$, for $\gamma\in\Gamma$, $z\in M$.

If we could lift $\bar{\sigma}$ to a morphism of Γ into $\mathbf{SL}(2,\mathbb{R})$ such that for $x \in S_r^\infty$ or $x \in S_i^\infty = S^\infty - S_r^\infty$ the image of the corresponding element has the right eigenvalues, the pullback by ρ of the constant sheaf $\mathbb{C}^2$ on $\mathbb{P}^1(\mathbb{C}) = P$, together with $\mathcal{O}_P(-1) \subset \mathcal{O}_P^2$, would define a permissible connection on X with the given lifting of $\bar{\sigma}$ as monodromy-mapping. Unfortunately we cannot show this directly, and we use a trick: The lifting is a problem only if S^∞ is void. Furthermore the conditions on the traces of parabolic elements may be satisfied always since S_r^∞ has even cardinality. (Make an argument similar to the remark on page 17.)

We thus may assume that X' is compact. The obstruction to the lifting problem lies in $H^2(\Gamma, \mathbb{Z}/_{2\mathbb{Z}}) = H^2(X, \mathbb{Z}/_{2\mathbb{Z}})$, and there exists a finite unramified Galois-covering $\tilde{X}$ of X such that the obstruction vanishes on $\tilde{X}$. $\tilde{X}$ corresponds to a normal subgroup of finite index $\tilde{\Gamma} \subseteq \Gamma$, and the vanishing of the obstruction on $\tilde{X}$ means that we can lift the restriction of σ to $\tilde{\Gamma}$.

We thus obtain a permissible connection on X, with the morphism ρ being the ρ defined above on $M \cong \mathbb{H}$. As in the proof of Theorem 11 we may assume that this connection is invariant under $\Gamma/\tilde{\Gamma}$ since ρ is Γ-linear, and then descend it to X.

We therefore obtain a permissible connection with real monodromy on X whose monodromy-mapping is a lifting of $\bar{\sigma}$. Furthermore our ρ is equal to the ρ defined previously, and the decomposition of $X - X^0$ into its components is the one given by the V_p.

This finishes our construction. It shows that up to trivial topological restraints each decomposition of a Riemann-surface can be realized by a permissible connection.

To conclude this paper we give some explicit descriptions of permissible connections, to show that this is no problem at all:

Example 1: Let $X = \mathbb{P}^1(\mathbb{C})$, with $S = S_i^\infty$ consisting of $r+1$ points, $r \geq 2$. We suppose that one of those points is ∞, and denote the others by $a_1, \ldots, a_r$. If x denotes the usual coordinate on X, dx defines an isomorphism

$$\mathscr{K}_X \cong \mathcal{O}_X(-2\,\infty),$$

so that $\mathscr{L} = \mathcal{O}_X(-\infty)$ is the unique theta-characteristic. Let e_1 be a meromorphic section of $\mathscr{L}$ regular on $X - \{\infty\}$, and with a simple pole at ∞, such that $e_1^{\otimes 2}$ corresponds to dx under the isomorphism above.

If $\mathscr{E}$ denotes the uniformization-bundle e_1 forms part of a basis of $\mathscr{E}$ on $X - S$, and a second basis-element e_2 can be defined there by $\nabla_0(e_1) = e_2 \cdot dx$, where ∇_0 is the uniformizing connection on X, or any connection coinciding with it on $\mathscr{L}$. (Every permissible ∇ is equivalent to a unique connection with this property.)

e_2 is regular on $X - S$ since ∇_0 is holomorphic there, and $e_1 \wedge e_2$ corresponds to 1 under the isomorphism

$$\Lambda^2 \mathscr{E} = \mathcal{O}_X.$$

As this isomorphism is horizontal $\nabla_0(e_2)$ must be of the form $P(x) \cdot e_1 \cdot dx$, with P a rational function on X with poles only in S.

More information upon P can be obtained from a study of the singularities of ∇_0 in S:

For example, in ∞ a local coordinate is given by $z = 1/x$, and $\mathscr{E}$ has a local basis given by $f_1 = z \cdot e_1$, $f_2 = (\nabla_0(f_1) - f_1 \cdot dz/(2z))/dz$. (These are regular sections, and $f_1 \wedge f_2$ corresponds to -1.) An easy calculation shows that

$$e_2 = f_1/2 - z \cdot f_2,\ \nabla_0(e_2) = f_1 \cdot dz/(4z) + (\text{terms holomorphic in } z).$$

Comparing this with the expression above we obtain that

$$P(x) = P(1/z) = -z^2/4 + (\text{terms of order } \geq 3 \text{ in } z).$$

Similar calculations in the a_j lead to the conclusion that $P(x)$ must have the form

$$P(x) = -1/4 \cdot \sum_{j=1}^{r} 1/(x - a_j)^2 + Q(x)/(\sum_{j=1}^{r} (x - a_j)),$$

with Q a polynomial of degree $r - 2$, and leading term $(r - 1)/4 \cdot x^{r-2}$. Conversely any such P defines a permissible connection ∇, coinciding with ∇_0 on $\mathscr{L}$. We thus can parametrize the permissible connections up to equivalence by the coefficients of Q of degree $< (r - 2)$. This are $r - 2$ parameters as it should be.

Usually it seems to be impossible to find the uniformizing connection ∇_0, except in the case that there are so many automorphism of $\mathbb{P}^1(\mathbb{C})$ fixing ∞, S such that ∇_0 is uniquely determined by its property of being invariant under these automorphisms. In this case the unique choice of Q is

$$1/(4r) \cdot d^2/dx^2 \left(\prod_{j=1}^{r} (x - a_j)\right),$$

and we obtain a hypergeometric differential equation (compare [7]).

Example 2: Let X be the elliptic curve defined by $y^2 = \prod_{j=1}^{3}(x - a_j)$, $S = S_i^{\infty}$ being the unique point over $x = \infty$. We may take $\mathscr{L} = \mathcal{O}_X$ as theta-characteristic, with

$$\mathscr{L}^{\otimes 2} \cong \mathcal{O}_X \cong \mathscr{K}_X$$

given by the differential dx/y, and we obtain a basis e_1, e_2 of the uniformization-bundle $\mathscr{E}$ over $X - S$ with

$$\nabla_0(e_1) = e_2 \cdot dx/y,\ \nabla_0(e_2) = P(x) \cdot e_1 \cdot dx/y,$$

where $P(x)$ is a polynomial of degree 1, with leading term $-x/16$. Again, the symmetric choice of P is

$$P(x) = -\left(\sum_{j=1}^{3}(x - a_j)\right)/48,$$

and this gives us the uniformizing connection ∇_0 in some special cases, for examples for the curves

$$y^2 = x^3 - x,\ y^2 = x^3 - 1.$$

Of course this leads again to the hypergeometric function.

Example 3: Let X be the hyperelliptic curve $y^2 = \prod_{j=1}^{2g+1}(x - a_j)$, of genus $g \geq 2$, and choose S to be void. If p denotes the unique point of X over $x = \infty$ we take

$$\mathscr{L} = \mathcal{O}_X((g - 1)p),$$

and we see that a permissible connection ∇ is given for a suitable basis e_1, e_2 of $\mathscr{E}$ on $X - \{p\}$ by

$$\nabla(e_1) = e_2 \cdot dx/y,\ \nabla(e_2) = P(x, y) \cdot e_1 \cdot dx/y,$$

where

$$P(x, y) = Q(x) + y \cdot R(x),$$

with $Q(x)$ is a polynomial in x of degree $2g - 1$ with leading term $-g \cdot (g - 1)/4 \cdot x^{2g-1}$, and $R(x)$ has degree $\leq g - 2$ in x.
The symmetric choice is

$$P(x) = -(g-1)/(8(2g+1)) \cdot d^2/dx^2 \Big(\sum_{j=1}^{2g+1} (x - a_j)\Big),$$

$$Q = 0,$$

and gives the uniformizing connection for the curves

$$y^2 = x^{2g+1} - x, \; y^2 = x^{2g+1} - 1.$$

Again we obtain a hypergeometric differential equation.

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