COMPOSITIO MATHEMATICA

G. KOUMOULLIS

On the almost Lindelöf property in products of separable metric spaces

Compositio Mathematica, tome 48, nº 1 (1983), p. 89-100

<http://www.numdam.org/item?id=CM_1983__48_1_89_0>

COMPOSITIO MATHEMATICA, Vol. 48, Fasc. 1, 1983, pag. 89–100

ON THE ALMOST LINDELÖF PROPERTY IN PRODUCTS OF SEPARABLE METRIC SPACES

G. Koumoullis

§0. Introduction

The class of almost Lindelöf, also called "measure-compact", spaces has been studied by several authors. In particular, the almost Lindelöf property in powers of $\mathbb{R}$ is studied by Moran [11], Kemperman and Maharam [7], Hechler [6] and Fremlin [3]. In [11], and independently in [7], it is proved that $\mathbb{R}^c$ is not almost Lindelöf. Moreover, it is possible that $\aleph_1 < c$ and $\mathbb{R}^{\aleph_1}$ is not almost Lindelöf ([6]). On the other hand, if Martin's axiom is true, then $\mathbb{R}^\varkappa$ is almost Lindelöf for every $\varkappa < c$ ([3]).

This paper is concerned with the almost Lindelöf property in products of separable metric spaces. If all factors of the product space are complete, the situation remains unchanged. The general case is very different, even if only one factor is arbitrary. It is proved that if c is real-valued measurable, then a) $\mathbb{R}^\varkappa$ is almost Lindelöf for every $\varkappa < c$ and b) there exists a separable metric space Y such that $\mathbb{R}^\varkappa \times Y$ is not almost Lindelöf for any uncountable cardinal $\varkappa$. This result, at least for $\varkappa = \aleph_1$, remains valid under weaker assumptions of set theory and yields a negative answer to a question of Gardner [5]. Moreover, assuming Martin's Axiom, it is proved that $\mathbb{R}^\varkappa \times Y$ is almost Lindelöf for every separable metric space Y and every $\varkappa < c$. The above are based on the results of sections 2 and 3 which do not depend on axioms of set theory.

§1. Definitions and preliminaries

Let X be a completely regular (Hausdorff) space. The Baire (resp. Borel) sets in X are the members of the least σ-algebra $\mathscr{B}(X)$ (resp. $\mathscr{B}_0(X)$) generated by the zero (resp. closed) sets in X. We say that X is

0010-437X/83/01/0089-12$0.20

almost Lindelöf if for every (non-negative, finite, countably additive) Baire measure μ on X and every cozero covering $\mathscr{U}$ of X there is a countable $\mathscr{U}' \subset \mathscr{U}$ such that $X - \cup \mathscr{U}'$ is of μ-measure zero. An equivalent definition is that every Baire measure on X is τ-additive (see [12]).

We recall that every closed subspace of an almost Lindelöf space is almost Lindelöf ([8]). Also, if a space X is not almost Lindelöf then there exists a nonzero Baire measure μ on X and a cozero covering $\mathscr{U}$ of X such that $\mu(U) = 0$ for every $U \in \mathscr{U}$ ([11, Theorem 2.1]).

Finally, we mention a few results concerning the Baire sets in products of separable metric spaces. If $\{X_\alpha\}_{\alpha<\varkappa}$ is a family of separable metric spaces and $X = \prod_{\alpha<\varkappa} X_\alpha$ with the product topology, then $\mathscr{B}(X)$ coincides with the least σ-algebra of subsets of X such that the canonical projections $\mathrm{pr}_{X_\alpha}: X \to (X_\alpha, \mathscr{B}(X_\alpha))$, $\alpha < \varkappa$, are measurable (see [3, Proposition 4] and [15, Theorem 4]). Since every uncountable Polish (i.e. separable complete metric) space is Baire isomorphic to the Cantor set $\{0,1\}^{\aleph_0}$ ([9]), it follows from the previous result that the spaces $\mathbb{R}^\varkappa$, $\mathbb{N}^\varkappa$ and $\{0,1\}^\varkappa$ are Baire isomorphic for every infinite cardinal $\varkappa$. We also note that the weight (i.e. the least cardinal m such that there exists a base for the topology of cardinality m) of these spaces is equal to $\varkappa$.

§2. Powers of separable metric spaces

We begin with a refinement of the construction of Kemperman and Maharam [7] as modified by Hechler [6].

THEOREM 2.1: *Let λ be a probability Baire (resp. regular Borel) measure on a completely regular space Y such that Y is the union of $\leq \varkappa$ sets of λ-measure zero for some cardinal $\varkappa$. Then there is a probability measure μ defined on the product σ-algebra $\mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}(Y)$ (resp. on $\mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}_0(Y)$) such that $\mathbb{N}^\varkappa \times Y$ is the union of $\leq \varkappa$ sets of μ-measure zero, each of the form $U \times V$, where U is cozero in $\mathbb{N}^\varkappa$ and V is cozero (resp. open) in Y.*

PROOF: Let $\{C_\alpha\}_{\alpha<\varkappa}$ be a covering of Y with $\lambda(C_\alpha) = 0$. Using the property of regularity, for each $\alpha < \varkappa$ and each $n \in \mathbb{N}$ we can find cozero (resp. open) sets $V_{\alpha,n}$ such that $C_\alpha \subset V_{\alpha,n+1} \subset V_{\alpha,n}$, $\lambda(V_{\alpha,n}) \leq \dfrac{1}{n}$ and $V_{\alpha,1} = Y$. We set $D_\alpha = \bigcap_n V_{\alpha,n}$. Then $C_\alpha \subset D_\alpha$ and $\lambda(D_\alpha) = 0$.

We now define a function $\varphi: Y \to \mathbb{N}^\varkappa \times Y$ by setting $\varphi(y) = (f(y), y)$, where $f(y)$ is given by

$$f(y)(\alpha) = \begin{cases} 1, & \text{if } y \in D_\alpha \\ \min\{n : y \notin V_{\alpha,n}\}, & \text{if } y \notin D_\alpha. \end{cases}$$

For every $\alpha < \varkappa$ we have $\{y \in Y : f(y)(\alpha) = 1\} = D_\alpha$ and $\{y \in Y : f(y)(\alpha) = n > 1\} = V_{\alpha,n-1} \setminus V_{\alpha,n}$, so f is $\mathscr{B}(Y)$ (resp. $\mathscr{B}_0(Y)$) to $\mathscr{B}(\mathbb{N}^\varkappa)$ measurable. It follows that φ is also measurable and so we can define a measure μ on $\mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}(Y)$ (resp. on $\mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}_0(Y)$) by setting $\mu(A) = \lambda(\varphi^{-1}(A))$.

Next we define

$$G_{\alpha,n} = \{x \in \mathbb{N}^\varkappa : x(\alpha) = n\} \times V_{\alpha,n}$$

for each $\alpha < \varkappa$ and each $n \in \mathbb{N}$. The family $\{G_{\alpha,n}\}$ is a covering of $\mathbb{N}^\varkappa \times Y$. Indeed, if $(x, y) \in \mathbb{N}^\varkappa \times Y$ then $y \in C_\alpha \subset D_\alpha$ for some $\alpha < \varkappa$. So $(x, y) \in G_{\alpha,n}$, where $n = x(\alpha)$.

Finally, we check that each of the elements of the above covering has μ-measure zero. To do this, it is enough to observe that

$$\varphi^{-1}(G_{\alpha,n}) = \begin{cases} D_\alpha, & \text{if } n = 1 \\ \emptyset, & \text{if } n > 1. \end{cases}$$

The proof is complete.

If $\mathscr{B}(\mathbb{N}^\varkappa \times Y) = \mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}(Y)$, then under the assumptions of the above theorem (for a Baire measure λ) we have that $\mathbb{N}^\varkappa \times Y$ is not almost Lindelöf. For instance, this is true when Y is a product of separable metric spaces. This fact will be used several times in the sequel. We note here that if $Y = \mathbb{N}^{\aleph_0}$ we can reduce Theorem 2.1 to the case of [6] and [7] using a closed covering of Y.

Corollary 2.2: *Let Y be a noncompact separable metric space and $\varkappa$ an infinite cardinal. The space $Y^\varkappa$ is almost Lindelöf if and only if there is no probability Baire measure on $Y^\varkappa$ such that $Y^\varkappa$ is the union of $\leq \varkappa$ sets of measure zero.*

Proof: Assume that there is a probability Baire measure on $Y^\varkappa$ such that $Y^\varkappa$ is the union of $\leq \varkappa$ sets of measure zero. By Theorem 2.1, $\mathbb{N}^\varkappa \times Y^\varkappa$ is not almost Lindelöf. Since $\mathbb{N}^\varkappa \times Y^\varkappa$ is homeomorphic to a closed subspace of $Y^\varkappa \times Y^\varkappa \cong Y^\varkappa$, it follows that $Y^\varkappa$ is not almost Lindelöf.

Now assume that $Y^\varkappa$ is not almost Lindelöf. Then there is a probability Baire measure μ on $Y^\varkappa$ and a cozero covering $\mathscr{U}$ of $Y^\varkappa$ of sets of μ-measure zero. Since the weight of $Y^\varkappa$ is equal to $\varkappa$, there is a subcovering of $\mathscr{U}$ of cardinality $\leq \varkappa$. This completes the proof.

By the above corollary, the almost Lindelöf property in a fixed power of separable metric spaces is a Baire measurable property. If we restrict ourselves to powers of $\mathbb{R}$ we have:

COROLLARY 2.3: *For every infinite cardinal $\varkappa$ the following are equivalent:*

(i) *$\mathbb{R}^\varkappa$ is almost Lindelöf;*

(ii) *there is a family $\{X_\alpha\}_{\alpha<\varkappa}$ of noncompact completely regular spaces such that $\prod_{\alpha<\varkappa} X_\alpha$ is almost Lindelöf;*

(iii) *for every family $\{X_\alpha\}_{\alpha<\varkappa}$ of Polish spaces, $\prod_{\alpha<\varkappa} X_\alpha$ is almost Lindelöf.*

PROOF: (iii) $\rightarrow$ (i) $\rightarrow$ (ii) are trivial.

(ii) $\rightarrow$ (i). Since $\prod_{\alpha<\varkappa} X_\alpha$ is almost Lindelöf, it is realcompact. Therefore each X_α is a realcompact noncompact space and so it contains a copy of $\mathbb{N}$ as a closed subspace. Now $\mathbb{N}^\varkappa$, being homeomorphic to a closed subspace of $\prod_{\alpha<\varkappa} X_\alpha$, is almost Lindelöf. Since $\mathbb{R}^\varkappa$ and $\mathbb{N}^\varkappa$ are Baire isomorphic, it follows from Corollary 2.2 that $\mathbb{R}^\varkappa$ is almost Lindelöf.

(i) $\rightarrow$ (iii) We observe that $\prod_{\alpha<\varkappa} X_\alpha$ is almost Lindelöf if (and only if) $\prod_{\alpha\in A} X_\alpha$ is almost Lindelöf, where $A = \{\alpha < \varkappa : X_\alpha \text{ is infinite}\}$ (cf. [12, Theorem 5.3]). Thus we can assume that each X_α is infinite. Since every infinite Polish space is Baire isomorphic to $\mathbb{N}^{\aleph_0}$ (if uncountable) or to $\mathbb{N}$ (if countable), $\prod_{\alpha<\varkappa} X_\alpha$ is Baire isomorphic to $\mathbb{N}^\varkappa$ which is almost Lindelöf. If $\prod_{\alpha<\varkappa} X_\alpha$ were not almost Lindelöf then $\prod_{\alpha<\varkappa} X_\alpha$, hence also $\mathbb{N}^\varkappa$, would be the union of some family of $\leq \varkappa$ sets of measure zero (for some probability Baire measure). This contradicts Corollary 2.2 and completes the proof.

The next result due to Fremlin [3] contains some other useful characterizations of the almost Lindelöf property in $\mathbb{R}^\varkappa$. We include a proof which uses Theorem 2.1 and is simpler than that of Fremlin.

THEOREM 2.4:([3, Theorem 7]): *For every infinite cardinal $\varkappa$ the following are equivalent:*

(i) *$\mathbb{R}^\varkappa$ is almost Lindelöf;*

(ii) *in any Radon measure space the union of $\leq \varkappa$ sets of measure zero has inner measure zero;*

(iii) *in any Radon measure space the union of $\leq \varkappa$ closed sets of measure zero has inner measure zero.*

PROOF: $\sim$(ii) $\rightarrow$ $\sim$(i). Assume that (ii) fails. Then, using the property of regularity, there is a compact space Y and a Radon measure $\lambda \neq 0$ on

Y such that Y is the union of $\leq \varkappa$ sets of λ-measure zero. By Theorem 2.1, there exists a probability measure μ defined on $\mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}_0(Y)$ and a covering $\{U_\alpha \times V_\alpha\}_{\alpha<\varkappa}$ of $\mathbb{N}^\varkappa \times Y$ such that U_α is cozero in $\mathbb{N}^\varkappa$, V_α is open in Y and $\mu(U_\alpha \times V_\alpha) = 0$. As in the proof of [12, Theorem 5.3], using the compactness of Y we find a family $\mathscr{C}$ of some finite intersections of the sets $U_\alpha, \alpha < \varkappa$, such that $\cup \mathscr{C} = \mathbb{N}^\varkappa$ and $\mathrm{pr}_{\mathbb{N}^\varkappa}(\mu)(C) = 0$ for all $C \in \mathscr{C}$. It follows that $\mathbb{N}^\varkappa$, hence also $\mathbb{R}^\varkappa$, is not almost Lindelöf.

(ii) $\rightarrow$ (iii) is trivial.

$\sim$(i) $\rightarrow$ $\sim$(iii). If (i) fails, there exists a probability Baire measure μ on $\mathbb{R}^\varkappa$ and a cozero covering $\{U_\alpha\}_{\alpha<\varkappa}$ of $\mathbb{R}^\varkappa$ such that $\mu(U_\alpha) = 0$. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ the the one-point compactification of $\mathbb{R}$ and define a Radon measure ν on $\bar{\mathbb{R}}^\varkappa$ by setting $\nu(B) = \mu(B \cap \mathbb{R}^\varkappa)$ for all Baire sets B in $\bar{\mathbb{R}}^\varkappa$. Let V_α be a cozero set in $\bar{\mathbb{R}}^\varkappa$ with $U_\alpha = V_\alpha \cap \mathbb{R}^\varkappa$ and set $Z_\alpha = \{x \in \bar{\mathbb{R}}^\varkappa : x(\alpha) = \infty\}$. Then $\bar{\mathbb{R}}^\varkappa = \bigcup_{\alpha<\varkappa} Z_\alpha \cup \bigcup_{\alpha<\varkappa} V_\alpha$ and $\nu(Z_\alpha) = \nu(V_\alpha) = 0$. Since each V_α is a countable union of zero sets in $\bar{\mathbb{R}}^\varkappa$, we conclude that (iii) fails.

We note that in (iii) of Corollary 2.3 we can replace the completeness of the separable metric spaces X_α by the weaker assumption that every measure on X_α is Radon. Indeed, let us assume that $\mathbb{R}^\varkappa$ is almost Lindelöf and that $\prod_{\alpha<\varkappa} X_\alpha$ is not. We proceed as in the last part of the proof of the above theorem considering a metrizable compactification $\bar{X}_\alpha$ of each X_α. By the assumption for X_α, there is a σ-compact subset C_α of X_α with $\mathrm{pr}_{X_\alpha}(\mu)(C_\alpha) = 1$. Then we define $Z_\alpha = \{x \in \prod_{\alpha<\varkappa} \bar{X}_\alpha : x(\alpha) \in \bar{X}_\alpha - C_\alpha\}$, V_α, $\alpha < \varkappa$, and a Radon measure ν on $\prod_{\alpha<\varkappa} \bar{X}_\alpha$ as above and we conclude that $\prod_{\alpha<\varkappa} \bar{X}_\alpha$ is covered by $\leq \varkappa$ sets of ν-measure zero, contradicting Theorem 2.4. However, as we will see later, the result is not valid for arbitrary separable metric spaces.

§3. Products of $\mathbb{R}^\varkappa$ with metric spaces

This section is concerned with the almost Lindelöf property in spaces of the form $\mathbb{R}^\varkappa \times Y$, where Y is a separable metric space and $\varkappa$ is an infinite cardinal. First we note that the separability of Y is not essential. Indeed, if Y is an arbitrary metric space then $\mathbb{R}^\varkappa \times Y$ is almost Lindelöf if and only if Y and $\mathbb{R}^\varkappa \times Z$, for every closed separable subset Z of Y, are almost Lindelöf. This follows from the well-known result that a metric space Y is almost Lindelöf if and only if every Baire measure on Y is supported by a closed separable subset. Thus, in the sequel the separability of Y can be replaced by the almost Lindelöf property.

The next proposition together with Theorem 2.1 will be used for a characterization of the almost Lindelöf property in $\mathbb{R}^\varkappa \times Y$.

PROPOSITION 3.1: *Let X be a compact space, $(Y, \mathscr{S})$ a measurable space and μ a probability measure defined on $\mathscr{B}(X) \otimes \mathscr{S}$ (resp. on $\mathscr{B}_0(X) \otimes \mathscr{S}$ with $\mathrm{pr}_X(\mu)$ regular) such that $X \times Y$ is the union of $\leq \varkappa$ sets of μ-measure zero for some cardinal $\varkappa$. Let λ denote the measure $\mathrm{pr}_Y(\mu)$ on Y. Then, either Y is the union of $\leq \varkappa$ sets of λ-measure zero, or X is the union of $\leq \varkappa$ sets of ν-measure zero for some probability Baire (resp. regular Borel) measure ν on X.*

PROOF: Let $\{R_\alpha\}_{\alpha<\varkappa}$ be a covering of $X \times Y$ such that $\mu(R_\alpha) = 0$. By [2] and [16], there is a strict λ-disintegration of μ, that is, a family $(u(y))_{y \in Y}$ of probability Radon measures on X such that for every $R \in \mathscr{B}(X) \otimes \mathscr{S}$ (resp. $R \in \mathscr{B}_0(X) \otimes \mathscr{S}$)

$$\mu(R) = \int_Y u(y)(R^y)\, d\lambda,$$

where $R^y = \{x \in X : (x, y) \in R\}$. In particular, we have

$$\int_Y u(y)(R_\alpha^y)\, d\lambda = 0$$

for every $\alpha < \varkappa$. Therefore each of the sets

$$\{y \in Y : u(y)(R_\alpha^y) > 0\}, \ \alpha < \varkappa,$$

has λ-measure zero. If these sets cover Y we have finished. Otherwise, there is $y_0 \in Y$ such that $u(y_0)(R_\alpha^{y_0}) = 0$ for all $\alpha < \varkappa$. Since X is the union of the family $\{R_\alpha^{y_0}\}_{\alpha<\varkappa}$, the measure $\nu = u(y_0)$ has the desired properties.

The following corollary summarizes Theorem 2.1 and Proposition 3.1.

COROLLARY 3.2: *Let Y be any (nonempty) completely regular space and $\varkappa$ an infinite cardinal. Then the following are equivalent:*

(i) *there exists a probability measure μ defined on $\mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}(Y)$ (resp. on $\mathscr{B}(\mathbb{N}^\varkappa) \otimes \mathscr{B}_0(Y)$ with $\mathrm{pr}_Y(\mu)$ regular) such that $\mathbb{N}^\varkappa \times Y$ is the union of $\leq \varkappa$ sets of μ-measure zero;*

(ii) *the same as* (i) *with each of the sets of μ-measure zero of the form $U \times V$ where U is cozero in $\mathbb{N}^\varkappa$ and V is cozero (resp. open) in Y;*

(iii) *either there exists a probability Baire (resp. regular Borel) measure λ on Y such that Y is the union of $\leq \varkappa$ sets of λ-measure zero, or $\mathbb{N}^\varkappa$ is not almost Lindelöf.*

PROOF: (ii) $\to$ (i) is rivial and (i) $\to$ (iii) follows from Proposition 3.1 (since $\mathbb{N}^\varkappa$ is Baire isomorphic to the compact space $\{0,1\}^\varkappa$) and Corollary 2.2.

(iii) $\to$ (ii). If $\mathbb{N}^\varkappa$ is not almost Lindelöf then there exists a probability Baire measure ν on $\mathbb{N}^\varkappa$ such that $\mathbb{N}^\varkappa$ is the union of $\leq \varkappa$ cozero sets of ν-measure zero. It is easy to see that, for any fixed $y_0 \in Y$, the measure μ on $\mathbb{N}^\varkappa \times Y$ defined by $\mu(A) = \nu(A^{y_0})$ has the desired properties. On the other hand, if there exists a measure λ on Y as stated in (iii), then (ii) follows from Theorem 2.1. We note that if λ is regular Borel, then by the construction of μ we have $\mathrm{pr}_Y(\mu) = \lambda$, so $\mathrm{pr}_Y(\mu)$ is regular.

THEOREM 3.3: *Let Y be a separable metric space and $\varkappa$ an infinite cardinal. Then the following are equivalent:*

(i) *$\mathbb{R}^\varkappa \times Y$ is almost Lindelöf;*

(ii) *$\mathbb{R}^\varkappa$ is almost Lindelöf and there is no probability Baire measure on Y such that Y is the union of $\leq \varkappa$ sets of measure zero.*

PROOF: $\sim$(i) $\to$ $\sim$(ii). Assume that $\mathbb{R}^\varkappa \times Y$ is not almost Lindelöf. Since the weight of $\mathbb{R}^\varkappa \times Y$ is equal to $\varkappa$, there is a probability Baire measure μ on $\mathbb{R}^\varkappa \times Y$ such that $\mathbb{R}^\varkappa \times Y$ is the union of $\leq \varkappa$ (cozero) sets of μ-measure zero. Since $\mathbb{R}^\varkappa$ is Baire isomorphic to the compact space $\{0,1\}^\varkappa$, we can apply Proposition 3.1. Thus either Y is the union of $\leq \varkappa$ sets of $\mathrm{pr}_Y(\mu)$-measure zero, or $\mathbb{R}^\varkappa$ is the union of $\leq \varkappa$ sets of ν-measure zero for some probability Baire measure ν on $\mathbb{R}^\varkappa$. In the latter case $\mathbb{R}^\varkappa$ is not almost Lindelöf (by Corollary 2.2). Thus in either case (ii) fails.

$\sim$(ii) $\to$ $\sim$(i). Of course, if $\mathbb{R}^\varkappa$ is not Lindelöf (i) fails. Now assume that Y is the union of $\leq \varkappa$ sets of measure zero for some probability Baire measure on Y. By Theorem 2.1, $\mathbb{N}^\varkappa \times Y$ is not almost Lindelöf. Since $\mathbb{N}^\varkappa \times Y$ is homeomorphic to a closed subspace of $\mathbb{R}^\varkappa \times Y$ it follows that $\mathbb{R}^\varkappa \times Y$ is not almost Lindelöf.

If we require $\mathbb{R}^\varkappa \times Y$ to be almost Lindelöf for every separable metric space Y, then we get a stronger condition. As the next theorem shows, this condition can be characterized using Lindelöf M-spaces instead of compact spaces used in Theorem 2.4.

We recall that a completely regular space X is a Lindelöf M-space if and only if it admits a perfect function onto a separable metric space or, equivalently, if it is homeomorphic to a closed subspace of a product of a compact and a separable metric space (see [13] for additional information and the definition of the M-property). We also note that every Baire measure on a Lindelöf M-space (in fact, on any Lindelöf space) has a unique extension to a regular Borel measure.

THEOREM 3.4: *Let $\varkappa$ be an infinite cardinal. Then the following are equivalent:*

(i) *$\mathbb{R}^{\varkappa} \times Y$ is almost Lindelöf for every separable metric space Y;*

(ii) *for every regular Borel measure on a Lindelöf M-space, the union of $\leq \varkappa$ sets of measure zero has inner measure zero;*

(iii) *for every regular Borel measure on a Lindelöf M-space, the union of $\leq \varkappa$ closed sets of measure zero has inner measure zero.*

PROOF: $\sim$(ii)$\rightarrow$$\sim$(i). Assume that (ii) fails. Using the property of regularity and the fact that the Lindelöf M-property is hereditary on closed subsets, there exists a Lindelöf M-space X and a probability regular Borel measure μ on X such that X is the union of $\leq \varkappa$ sets of μ-measure zero. We consider X as a subspace of $Z \times Y$ for some compact space Z and some separable metric space Y and we extend μ to a measure $\tilde{\mu}$ on $\mathscr{B}_0(Z \times Y)$ by setting $\tilde{\mu}(A) = \mu(A \cap X)$.

In order to apply Proposition 3.1 for $\tilde{\mu}$, we first check that $\mathscr{B}_0(Z \times Y) = \mathscr{B}_0(Z) \otimes \mathscr{B}(Y)$. To do this, it is enough to show that $G \in \mathscr{B}_0(Z) \otimes \mathscr{B}(Y)$ for every open $G \subset Z \times Y$. Let $\{V_n\}_{n \in \mathbb{N}}$ be a base for the topology of Y. Then $G = \bigcup_{i \in I} (U_i \times V_{n_i})$ where U_i is open in Z and $n_i \in \mathbb{N}$ for all $i \in I$. If we set $I_n = \{i \in I : n_i = n\}$, then $G = \bigcup_{n \in \mathbb{N}} (\bigcup_{i \in I_n} U_i \times V_n)$ and so $G \in \mathscr{B}_0(Z) \otimes \mathscr{B}(Y)$.

Now, by Proposition 3.1, either Y is the union of $\leq \varkappa$ sets of $\mathrm{pr}_Y(\tilde{\mu})$-measure zero, or Z is the union of $\leq \varkappa$ sets of ν-measure zero for some Radon probability measure ν on Z. In the latter case $\mathbb{R}^{\varkappa}$ is not almost Lindelöf (by Theorem 2.4), while in the former $\mathbb{R}^{\varkappa} \times Y$ is not almost Lindelöf (by Theorem 3.3). Thus in either case (i) fails.

(ii)$\rightarrow$(iii) is trivial.

$\sim$(i)$\rightarrow$$\sim$(iii). We have that $\mathbb{R}^{\varkappa} \times Y$ is not almost Lindelöf for some separable metric space Y. If $\mathbb{R}^{\varkappa}$ is not almost Lindelöf then (iii) fails (Theorem 2.4). So we can assume that $\mathbb{R}^{\varkappa}$ is almost Lindelöf. Then, by Theorem 3.3, Y is the union of $\leq \varkappa$ sets of λ-measure zero for some probability measure λ. Using Theorem 2.1, we can find a probability Baire measure μ on $\mathbb{N}^{\varkappa} \times Y$ and a cozero covering $\{U_\alpha\}_{\alpha < \varkappa}$ of $\mathbb{N}^{\varkappa} \times Y$ such that $\mu(U_\alpha) = 0$ for all $\alpha < \varkappa$.

We now show that there is a Baire isomorphism $\varphi : \mathbb{N}^{\varkappa} \times Y \rightarrow \{0,1\}^{\varkappa} \times Y$ such that the image of every cozero set in $\mathbb{N}^{\varkappa} \times Y$ is a countable union of zero sets in $\{0,1\}^{\varkappa} \times Y$. To do this, we note that there is a function $f : \mathbb{N}^{\aleph_0} \rightarrow \{0,1\}^{\aleph_0}$ which is continuous one-to-one and onto such that the image of every open set in $\mathbb{N}^{\aleph_0}$ is an F_σ in $\{0,1\}^{\aleph_0}$ (see [9, p. 442]). Then we define $\varphi : (\mathbb{N}^{\aleph_0})^{\varkappa} \times Y \rightarrow (\{0,1\}^{\aleph_0})^{\varkappa} \times Y$

by setting $\varphi((x_\alpha)_{\alpha<\varkappa}, y) = ((f(x_\alpha))_{\alpha<\varkappa}, y)$. Identifying $(\mathbb{N}^{\aleph_0})^\varkappa$ and $(\{0,1\}^{\aleph_0})^\varkappa$ with $\mathbb{N}^\varkappa$ and $\{0,1\}^\varkappa$, we see that φ has the desired properties.

Finally, using φ we transfer μ and the above cozero covering of $\mathbb{N}^\varkappa \times Y$ to a measure ν and a zero covering of $\{0,1\}^\varkappa \times Y$. Since $\{0,1\}^\varkappa \times Y$ is a Lindelöf M-space, it follows that (iii) fails.

§4. Applications

We now come to some applications of axioms of set theory. As Fremlin has shown, using Theorem 2.4, if Martin's Axiom is true then $\mathbb{R}^\varkappa$ is almost Lindelöf for every $\varkappa < c$ ([3, Proposition 8]). Using this result and [10, §4, Theorem 1], we deduce immediately from Theorem 3.3 the following.

COROLLARY 4.1: *If Martin's Axiom is true, then $\mathbb{R}^\varkappa \times Y$ is almost Lindelöf for every separable metric space Y and every $\varkappa < c$.*

If we assume that c is a real-valued measurable cardinal the situation is different (Corollary 4.3). The next proposition is needed for this purpose. We recall that a measure is said to be m-additive, for some cardinal m, if the union of less than m sets of measure zero has measure zero.

PROPOSITION 4.2: *Assume that there exists a real-valued measurable cardinal $m \leq c$ and an m-additive probability measure μ defined on all subsets of m such that the measure algebra of μ is not generated by less than m of its elements. Then $\mathbb{R}^\varkappa$ is almost Lindelöf for every $\varkappa < m$.*

PROOF: Suppose that $\mathbb{R}^\varkappa$ is not almost Lindelöf for some $\varkappa < m$. Then there is a probability Baire measure on $\mathbb{R}^\varkappa$ such that $\mathbb{R}^\varkappa$ is the union of $\leq \varkappa$ sets of measure zero. Since $\mathbb{R}^\varkappa$ is Baire isomorphic to $\{0,1\}^\varkappa$, this is true for $\{0,1\}^\varkappa$ and let ν be the unique extension of this measure to a Radon measure on $\{0,1\}^\varkappa$. Also, let λ be the usual product measure on $\{0,1\}^\varkappa$. By [4, Proposition 1D], there is a function $f: \{0,1\}^\varkappa \to \{0,1\}^\varkappa$ such that $f(\lambda) = \nu$, i.e. $\nu(A) = \lambda(f^{-1}(A))$ for every Borel set $A \subset \{0,1\}^\varkappa$.

Since m is a regular cardinal, we can assume that μ is homogeneous and, using Maharam's representation theorem of measure algebras, we can find a family $\{B_\alpha\}_{\alpha<\varkappa}$ of stochastically independent subsets of m of μ-measure 1/2. Then we define $g: m \to \{0,1\}^\varkappa$ by

$$g(\xi)(\alpha) = \begin{cases} 1, & \text{if } \xi \in B_\alpha \\ 0, & \text{if } \xi \notin B_\alpha. \end{cases}$$

It is easy to see that $g(\mu) = \lambda$.

Finally, we set $h = \text{fog}$ and we have $h(\mu) = \nu$. Since μ is m-additive and ν is not, this leads to a contradiction.

COROLLARY 4.3: *If c is real-valued measurable, then*

(i) *$\mathbb{R}^{\varkappa}$ is almost Lindelöf for every $\varkappa < c$; and*

(ii) *there exists a separable metric space Y such that $\mathbb{R}^{\varkappa} \times Y$ is not almost Lindelöf for any uncountable cardinal $\varkappa$.*

PROOF: (i) Assume that c is real-valued measurable and let μ be any c-additive measure defined on all subsets of c and vanishing on singletons. By an unpublished result of Fremlin and Kunen, μ satisfies the hypothesis of Proposition 4.2 for $m = c$ and so (i) follows.

(ii) According to a result of Solovay (see [14]), if c is real-valued measurable there exists a non-Lebesgue measurable set of reals Y of cardinality $\aleph_1$. Then the Lebesgue outer measure induces a nonzero measure on Y and using Theorem 3.3 we deduce that $\mathbb{R}^{\aleph_1} \times Y$ is not almost Lindelöf. Therefore $\mathbb{R}^{\varkappa} \times Y$ is not almost Lindelöf for any $\varkappa \geq \aleph_1$.

Another weaker axiom that can be used instead of the assumption that c is real-valued measurable is the Measure Extension Axiom (MEA) which is stated as follows: there exists a non-separable probability measure space $(X, \mathscr{S}, \mu)$ such that for every countable family $\mathscr{C}$ of subsets of X, μ can be extended to a measure on the σ-algebra generated by $\mathscr{S}$ and $\mathscr{C}$. This axiom was formulated by Prikry and its consistency with ZFC was proved by Carlson [1].

Prikry [14] has proved that MEA implies the existence of a non-Lebesque measurable set of cardinality $\aleph_1$ and $\aleph_1 < c$. As we see from the proof of Corollary 4.3, part (ii) remains valid if we assume MEA. Moreover, MEA is sufficient for part (i) of Corollary 4.3 at least for $\varkappa = \aleph_1$.

Corollary 4.3 shows that, under a set theoretical assumption, a product of an almost Lindelöf and a separable metric space need not be almost Lindelöf and so answers in the negative a question of Gardner [5, p. 108].

REMARKS: As in [3] we define $\varkappa_0$ as the least cardinal $\varkappa$ such that $\mathbb{R}^{\varkappa}$ is not almost Lindelöf. It is noted there that $\aleph_1 \leq \varkappa_0 \leq c$ and that Martin's Axiom implies $\varkappa_0 = c$. By Corollary 4.3-(i), if c is real-valued measurable then $\varkappa_0 = c$ (and MEA implies $\varkappa_0 > \aleph_1$). Further, if ρ is the least cardinal $\varkappa$ such that $2^{\varkappa} > c$, then $\rho \leq \varkappa_0$ whenever ρ is real-valued measurable. This follows from Proposition 4.2 since, by a result of

Prikry, the hypothesis of 4.2 is satisfied if $m = \rho$ is real-valued measurable.

Using powers of arbitrary separable metric spaces, one might similarly define a cardinal $\varkappa_1$, namely the least cardinal $\varkappa$ such that $Y^\varkappa$ is not almost Lindelöf for some separable metric space Y. However, $\varkappa_1 = \aleph_1$. This follows from the next example, due to Fremlin, of a separable metric space Y such that $Y^{\aleph_1}$ is not almost Lindelöf.

EXAMPLE 4.4: Consider the unit circle $T \subset \mathbb{R}^2$ as a group and let λ be the normalized Lebesgue measure on it. List the uncountable compact subsets on T as $\{K_\theta\}_{\theta < c}$. If $c = \aleph_1$ then $\mathbb{R}^{\aleph_1}$ is not almost Lindelöf and so we can assume $c > \aleph_1$. (In any case the construction below works but this assumption makes it clearer what order we have to do things in.) Choose $\{\alpha_\xi\}_{\xi < \aleph_1}$ in T such that

$$\alpha_\eta - \alpha_\xi = \alpha_{\eta'} - \alpha_{\xi'} \Rightarrow \eta = \eta',\ \xi = \xi' \text{ or } \eta = \xi,\ \eta' = \xi'.$$

Choose $\{\beta_{\zeta\theta}\}_{\zeta < \aleph_1, \theta < c}$ in lexicographic order such that $\beta_{\zeta\theta} \in K_\theta$ and

$$\beta_{\zeta\theta} - \alpha_\eta + \alpha_\xi = \beta_{\zeta'\theta'} - \alpha_{\eta'} + \alpha_{\xi'} \Rightarrow \zeta = \zeta',\ \theta = \theta'$$

[There will be less than c points to dodge each time.] Set

$$Y = \{\beta_{\zeta\theta} - \alpha_\eta : \eta \leq \zeta,\ \theta < c\}.$$

Now define $f: T \to T^{\aleph_1}$ by $f(\alpha) = (\alpha - \alpha_\xi)_{\xi < \aleph_1}$ and let μ be the Baire measure $f(\lambda)$ on $T^{\aleph_1}$. We claim that $\mu^*(Y^{\aleph_1}) = 1$. Indeed, if B is a Baire set containing $Y^{\aleph_1}$, then $f^{-1}(B) \supset \bigcap_{\xi \leq \zeta} (Y + \alpha_\xi)$ for some $\zeta < \aleph_1$. Since $\beta_{\zeta\theta} \in \bigcap_{\xi \leq \zeta} (Y + \alpha_\xi) \cap K_\theta$ for every ζ, θ, the outer λ-measure of $\bigcap_{\xi \leq \zeta} (Y + \alpha_\xi)$ is one, so $\lambda(f^{-1}(B)) = \mu(B) = 1$.

Let ν be the induced Baire measure on $Y^{\aleph_1}$. We show that there is a covering of $Y^{\aleph_1}$ by cozero sets of ν-measure zero. To do this, it is enough to show that $f(T) \cap Y^{\aleph_1} = \emptyset$ or, equivalently, $\bigcap_{\gamma < \aleph_1} (Y + \alpha_\gamma) = \emptyset$. Suppose, if possible, that $\alpha \in \bigcap_{\gamma < \aleph_1} (Y + \alpha_\gamma)$. Then for each $\gamma < \aleph_1$ there are $\theta(\gamma) < c$, $\zeta(\gamma) < \aleph_1$ and $\xi(\gamma) \leq \zeta(\gamma)$ such that $\alpha = \beta_{\zeta(\gamma)\theta(\gamma)} - \alpha_{\xi(\gamma)} + \alpha_\gamma$. By the choice of $\beta_{\zeta\theta}$, all pairs $\zeta(\gamma), \theta(\gamma)$ must be the same; say $\zeta(\gamma) = \zeta$, $\theta(\gamma) = \theta$ for every γ. Then $\alpha = \beta_{\zeta\theta} - \alpha_{\xi(\gamma)} + \alpha_\gamma$ for every γ, and $\alpha_{\xi(\gamma)} - \alpha_\gamma = \beta_{\zeta\theta} - \alpha$ for every γ. By the choice of α_γ, this can happen only if $\xi(\gamma) = \gamma$ for every γ. But now $\gamma = \xi(\gamma) \leq \zeta(\gamma) = \zeta$ for every γ which is absurd. Therefore $Y^{\aleph_1}$ is not almost Lindelöf.

Acknowledgement

I wish to express my thanks to Professors D.H. Fremlin for communicating to me Example 4.4 and K. Prikry for useful discussions.

The research of this paper was partially supported by NSF grant MCS 78-01525.

REFERENCES

[1] T. Carlson: Extending Lebesgue measure by countably many sets, manuscript, 1979.

[2] G.A. Edgar: Disintegration of measures and the vector-valued Radon–Nikodym theorem. *Duke Math. J. 42* (1975) 447–450.

[3] D.H. Fremlin: Uncountable powers of $\mathbb{R}$ can be almost Lindelöf. *Manuscripta Math. 22* (1977) 77–85.

[4] D.H. Fremlin: Measurable and almost continuous functions. *Manuscripta Math. 33* (1981) 387–405.

[5] R.J. Gardner: The regularity of Borel measures and Borel measurecompactness. *Proc. London Math. Soc. 30* (1975) 95–113.

[6] S.H. Hechler: On $\mathbb{N}^{\aleph_1}$ and the almost Lindelöf property. *Proc. Amer. Math. Soc. 52* (1975) 353–355.

[7] J.H.B. Kemperman and D.Maharam: $\mathbb{R}^c$ is not almost Lindelöf. *Proc. Amer. Math. Soc. 24* (1970) 772–773.

[8] R.B. Kirk: Measures in topological spaces and B-compactness. *Indag. Math. 31* (1969) 172–183.

[9] K. Kuratowski: *Topology*, Vol. I. Academic Press, 1966.

[10] D.A. Martin and R.M. Solovay: Internal Cohen extensions. *Ann. Math. Logic 2* (1970) 143–178.

[11] W. Moran: The additivity of measures on completely regular spaces. *J. London Math. Soc. 43* (1968) 633–639.

[12] W. Moran: Measures and mappings on topological spaces. *Proc. London Math. Soc. 19* (1969) 493–508.

[13] K. Morita: A survey of the theory of M-spaces. *Gen. Topology Appl. 1* (1971) 49–55.

[14] K. Prikry: On an application of a theorem of Maharam. Measure Theory and Appl., Proc. 1980 Conf., Northern Illinois University, DeKalb, Illinois, 1981, p. 243.

[15] K.A. Ross and A.H. Stone: Products of separable spaces. *Amer. Math. Monthly 71* (1964) 398–403.

[16] M. Valadier: Désintégration d'une mesure sur un produit *C.R. Acad. Sci. Paris Sér. A-B 276* (1973), *A33–A35.*

(Oblatum 29-I-1982)

School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455
U.S.A.

Chair I of Mathematical Analysis
Athens University
Panepistemiopolis
Athens 621
Greece