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THE NUMBER OF POINTS ON CERTAIN FAMILIES OF HYPERSURFACES OVER FINITE FIELDS

Neal Koblitz[1]

Formulas are given for the number of points on monomial deformations of a diagonal hypersurface and on hypersurfaces in families connected to generalized hypergeometric functions. By expanding this number as a Fourier series in multiplicative characters of the parameter, one discovers a simple but striking fact: the Fourier coefficients are Jacobi sums. The resulting formulas are analogous to classical Barnes integrals for hypergeometric functions.

§1. Introduction

Let $\mathbf{F}_q$ be the finite field of $q = p^f$ elements. Let $N(\alpha)$ denote the number of $\mathbf{F}_q$-points on the hypersurface in $P^{n-1}_{\mathbf{F}_q}$ defined by the homogeneous equation

$$\alpha_1 X_1^{h_1^{(1)}} \dots X_n^{h_n^{(1)}} + \dots + \alpha_r X_1^{h_1^{(r)}} \dots X_n^{h_n^{(r)}} = 0 \tag{1.1}$$

(which we shall abbreviate $\sum \alpha_i X^{h^{(i)}} = 0$). Here α is an r-tuple of nonzero elements of $\mathbf{F}_q$. Our purpose is to study formulas expressing $N(\alpha)$ in terms of Jacobi and Gauss sums, and to call attention to analogies with the classical integral formulas which express hypergeometric functions in terms of the gamma function.

If $\chi: \mathbf{F}_q^* \to K^*$ is a multiplicative character with values in a field of characteristic zero and $\psi: \mathbf{F}_q \to K^*$ is an additive character, then the Gauss sum

$$g(\chi) = \sum_{x \in \mathbf{F}_q^*} \chi(x)\psi(x)$$

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is the finite-field analog of the gamma function

$$\Gamma(s) = \int_{x\in\mathbf{R}^+} x^s e^{-x}\frac{dx}{x},$$

where $x \mapsto x^s$ is a multiplicative character on R^+, and e^{-x} corresponds to $\psi(x)$. The Jacobi sum $J(\chi_1, \chi_2) = \sum \chi_1(x)\chi_2(1-x)$ is the analog of the beta function

$$B(s_1, s_2) = \int_0^1 x^{s_1}(1-x)^{s_2}\frac{dx}{x(1-x)}.$$

We shall see that the number $N(\lambda)$ of points on a certain family of hypersurfaces (monomial deformations of diagonal hypersurfaces; λ parametrizes the deformations) can be expressed as a sum of terms each of which is the finite-field analog of the integral ($\lambda \in \mathbf{C}$; $h_i, w_i \in \mathbf{Z}$)

$$\Gamma\left(\frac{1}{d}\right)\cdots\Gamma\left(\frac{d-1}{d}\right)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma\left(h_1 s+\frac{w_1}{d}\right)\cdots\Gamma\left(h_n s+\frac{w_n}{d}\right)}{\Gamma\left(s+\frac{1}{d}\right)\cdots\Gamma\left(s+\frac{d-1}{d}\right)}\Gamma(-s)(-\lambda^d)^s\,ds$$

where the path of integration from $-i\infty$ to $i\infty$ in the complex s-plane curves so as to keep all poles of $\Gamma\left(h_i s + \frac{w_i}{d}\right)$ to its left and all poles of $\Gamma(-s)$ to its right. This expression is the Barnes integral for a hypergeometric function associated to the family of hypersurfaces over $\mathbf{C}$.

In some cases, formal analogies of this type have been found to be surface manifestations of an underlying p-adic theory. For instance, Jacobi sums are used to count points on the Fermat curve $x^d + y^d = 1$ (see, e.g., [11]), just as the beta function is used to evaluate the periods of the same curve considered complex-analytically; and the Jacobi sums are equal to an expression in terms of the p-adic gamma-function which is essentially the same as the expression for the beta function in terms of the complex-analytic gamma function (see [9], Ch. III). However, no such underlying p-adic theory has yet been developed in full generality to "explain" the analogy between $N(\lambda)$ and the classical Barnes integral.

We ultimately want a formula for $N(\alpha)$, the number of points on (1.1). However, it turns out to be easier to compute the number $N^*(\alpha)$ of $\mathbf{F}_q$-points on (1.1) all of whose coordinates are nonzero. Delsarte [3] and Furtado Gomida [6] found an expression for $N^*(\alpha)$ in terms of Jacobi sums. Roughly speaking, if $N^*(\alpha)$, considered as a function of $\alpha \in (\mathbf{F}_q^*)^r$, is expanded as a Fourier sum over the characters of $(\mathbf{F}_q^*)^r$, then the Fourier coefficients are Jacobi sums. In §2 we give a new, more geometric proof of this fact.

For the general hypersurface (1.1) it is more cumbersome to express $N(\alpha)$ itself in terms of Jacobi sums. Namely, one has to write $N(\alpha) = \sum N_S^*(\alpha)$, where the sum is over all subsets S of $\{1,\ldots,n\}$ and $N_S^*(\alpha)$ denotes the number of points with nonzero coordinates on the $(n-1-\#S)$-dimensional hypersurface obtained by intersecting (1.1) with $X_{s_1} = \ldots = X_{s_{\#S}} = 0$, $s_j \in S$.

However, in the case of a monomial deformation of a diagonal hypersurface, i.e.,

$$X_1^d + \ldots + X_n^d = d\lambda X_1^{h_1} X_2^{h_2} \ldots X_n^{h_n}, \tag{1.2}$$

in §3 we shall prove a formula for $N(\lambda) = N(1, 1, \ldots, 1, -d\lambda)$ which is just as simple as the formula for $N^*(\lambda)$.

The formula for $N(\lambda)$ in §3 is a finite-field analog of the Pochhammer–Barnes integral for certain hypergeometric functions which in the classical case ($\lambda \in \mathbf{C}$) are associated to the hypersurface (1.2). Classical analogs are discussed in §4.

Finally, in §§5–6 we give formulas for $N(\lambda)$ for certain other families of hypersurfaces connected to generalized hypergeometric functions.

§2. Jacobi sums and $N^*(\alpha)$ for general hypersurfaces

We first recall the definition and elementary properties of Jacobi sums.

Let $\chi_{1/(q-1)}: \mathbf{F}_q^* \to K^*$ be a fixed generator of the character group of $\mathbf{F}_q^*$ with values in an algebraically closed field K of characteristic zero (such as the complex numbers $\mathbf{C}$ or an algebraically closed p-adically complete field Ω). If, for example, $K = \mathbf{C}$, we can arbitrarily fix a primitive root of $\mathbf{F}_q^*$ and determine $\chi_{1/(q-1)}$ by letting it take that root to $e^{2\pi i/(q-1)}$. If $K = \Omega$, it is natural to take $\chi_{1/(q-1)}$ to be the Teichmüller character.

For $s \in \dfrac{1}{q-1}\mathbf{Z}/\mathbf{Z}$, we let $\chi_s = (\chi_{1/(q-1)})^{s(q-1)}$. For any s (including 0) we agree to let $\chi_s(0) = 0$.

If $s_1, \ldots, s_r \in \dfrac{1}{q-1}\mathbf{Z}/\mathbf{Z}$ and $\sum s_i \not\equiv 0 \bmod \mathbf{Z}$, we define the Jacobi sum

$$J(s_1, \ldots, s_r) = \sum_{\substack{x_1, \ldots, x_r \in \mathbf{F}_q \\ x_1 + \ldots + x_r = 1}} \chi_{s_1}(x_1) \ldots \chi_{s_r}(x_r), \quad r > 1; \; J(s_1) = 1. \tag{2.1}$$

This will be our definition even if some of the characters are trivial, i.e., $s_i = 0$. If $\sum s_i \equiv 0 \bmod \mathbf{Z}$, then we modify the definition (2.1) by inserting the factor q on the right. An easy computation shows that if $\sum s_i \equiv 0 \bmod \mathbf{Z}$ (but not all $s_i \equiv 0 \bmod \mathbf{Z}$), then

$$J(s_1,\ldots,s_r) = -\frac{q}{q-1} \sum_{\substack{x_1,\ldots,x_r \in \mathbf{F}_q \\ x_1+\ldots+x_r=0}} \chi_{s_1}(x_1)\ldots\chi_{s_r}(x_r), \tag{2.2}$$

and also

$$J(s_1,\ldots,s_r) = \begin{cases} -q\chi_{s_r}(-1)J(s_1,\ldots,s_{r-1}), & \text{if } s_r \not\equiv 0 \bmod \mathbf{Z}; \\ -J(s_1,\ldots,s_{r-1}), & \text{if } s_r \equiv 0 \bmod \mathbf{Z}. \end{cases} \tag{2.3}$$

Jacobi sums can be expressed in terms of Gauss sums. Let $\psi: \mathbf{F}_q \to K^*$ be an additive character, fixed once and for all. For $s \in \frac{1}{q-1}\mathbf{Z}/\mathbf{Z}$, we let $g(s)$ denote the Gauss sum

$$g(s) = \sum_{x \in \mathbf{F}_q} \chi_s(x)\psi(x).$$

Gauss sums satisfy:

$$g(s)g(-s) = q\chi_s(-1) \text{ if } s \neq 0; \qquad g(0) = -1. \tag{2.4}$$

Jacobi sums satisfy the following relations if all of the s_i are nonzero and their sum is also nonzero ([7], p. 100):

$$J(s_1,\ldots,s_r) = \frac{g(s_1)\ldots g(s_r)}{g(s_1+\ldots+s_r)}; \tag{2.5}$$

$$J(-s_1,\ldots,-s_r) = \frac{q^{r-1}}{J(s_1,\ldots,s_r)}. \tag{2.6}$$

If, say, $s_r = 0$, $r > 1$, then $J(s_1,\ldots,s_r) = -J(s_1,\ldots,s_{r-1})$, so that (2.5) and (2.6) still hold if some of the s_i vanish, provided that $\sum s_i \not\equiv 0 \bmod \mathbf{Z}$ and the r in q^{r-1} in (2.6) is replaced by the number r' of nonzero s_i. Next, if $\sum s_i \equiv 0 \bmod \mathbf{Z}$ (but not all $s_i \equiv 0 \bmod \mathbf{Z}$), then the q inserted on the right in (2.1) ensures that (2.5) holds, and (2.6) holds with $q^{r'}$ in the numerator on the right. Finally, if $s_1 \equiv \ldots \equiv s_r \equiv 0 \bmod \mathbf{Z}$, then by (2.1) we have

$$J(0,\ldots,0) = q \cdot \#\{x \in \mathbf{F}_q^{*r} \mid \sum x_i = 1\} = (q-1)^r - (-1)^r. \tag{2.7}$$

Now, suppose we have an algebraic variety V defined over a finite field $\mathbf{F}_q$ and we want to determine the number $N(V) = N_{\mathbf{F}_q}(V)$ of $\mathbf{F}_q$-points on it. Since these points are the $\bar{\mathbf{F}}_q$-points of V fixed by the q-th power Frobenius map $F:(\ldots, X_i, \ldots) \mapsto (\ldots, X_i^q, \ldots)$, it follows that

$$N_{\mathbf{F}_q}(V) = \#\{X \in V | F(X) = X\}. \tag{2.8}$$

If we have a group G acting on V, then it is convenient to split up $N(V)$ into pieces $N(V, \chi) = N_{\mathbf{F}_q}(V, \chi)$, where $\chi: G \to K^*$ is a character with values in an algebraically closed field K of characteristic zero. $N(V, \chi)$ is defined as follows:

$$N_{\mathbf{F}_q} V, \chi) = \frac{1}{\#G} \sum_{\xi \in G} \chi^{-1}(\xi) \#\{X \in V | F \circ \xi(X) = X\}. \tag{2.9}$$

REMARK: The zeta-function of V is defined as $\exp(\sum_n N_{\mathbf{F}_{q^n}}(V) T^n/n)$, and the L-function of V corresponding to χ is defined as $\exp(\sum_n N_{\mathbf{F}_{q^n}}(V, \chi) T^n/n)$. More generally, $N_{\mathbf{F}_q}(V, \rho)$ is defined for any finite dimensional representation ρ of G by replacing χ^{-1} by Trace ρ^{-1} on the right in (2.9). If ρ is the regular representation of G, then this definition reduces to (2.8).

In all of our examples, G will be abelian, so the only irreducible representations will be one-dimensional characters χ. In that case (G abelian) we have

LEMMA 1: $N_{\mathbf{F}_q}(V) = \sum_{\text{characters } \chi \text{ of } G} N_{\mathbf{F}_q}(V, \chi).$

PROOF: This is immediate from (2.9) and (2.8), by orthogonality of characters. Alternately, it follows from the decomposition of the regular representation as a direct sum of characters and the additivity of $N_{\mathbf{F}_q}(V, \rho)$ with respect to direct sum of ρ's.

The same definitions apply to the number of points with nonzero coordinates, and we have

LEMMA 2: $N^*_{\mathbf{F}_q}(V) = \sum_{\text{characters } \chi \text{ of } G} N^*_{\mathbf{F}_q}(V, \chi).$

The simplest example of a variety V with a large group action is the diagonal hypersurface of degree d in $P^{n-1}_{\mathbf{F}_q}$, where we assume $d | q - 1$:

$$X_1^d + \ldots + X_n^d = 0. \tag{2.10}$$

Let $D_{d,n}$ denote this hypersurface. The group μ_d^n of n-tuples of d-th roots of 1 in $\mathbf{F}_q^*$ acts on $D_{d,n}$: $\xi = (\xi_1, \ldots, \xi_n)$ takes the point $(X_1, \ldots, X_n)$ to $(\xi X_1, \ldots, \xi_n X_n)$. The diagonal $\Delta = \{(\xi, \xi, \ldots, \xi)\} \subset \mu_d^n$ acts trivially, and μ_d^n/Δ acts faithfully. The character group of μ_d^n/Δ is in one-to-one correspondence with the n-tuples

$$w = (w_1, \ldots, w_n),\ 0 \leq w_i < d, \text{ for which } \sum w_i \equiv 0 \pmod{d}, \tag{2.11}$$

where

$$\chi_w(\xi) \overset{\text{def}}{=} \chi(\xi^w), \qquad \xi^w = \xi_1^{w_1} \ldots \xi_n^{w_n}$$

(χ is a fixed primitive character of μ_d, such as the restriction to μ_d of $\chi_{1/(q-1)}$ in our earlier notation). In [11] it is shown that for $G = \mu_d^n/\Delta$ we have

$$\mathrm{N}_{\mathbf{F}_q}(D_{d,n}, \chi_w) = \begin{cases} 0 & \text{if some but not all } w_i = 0, \\ \dfrac{q^{n-1} - 1}{q - 1} & \text{if all } w_i = 0; \\ -\dfrac{1}{q} J\left(\dfrac{w_1}{d}, \ldots, \dfrac{w_n}{d}\right) & \text{if all } w_i \neq 0. \end{cases} \tag{2.12}$$

(In Weil's notation,

$$-\frac{1}{q} J\left(\frac{w_1}{d}, \ldots, \frac{w_n}{d}\right) = \chi_{w_n/d}(-1)\, J\left(\frac{w_1}{d}, \ldots, \frac{w_{n-1}}{d}\right)$$
$$= \sum_{x_1 + \ldots + x_{n-1} + 1 = 0} \chi_{w_1/d}(x_1) \ldots \chi_{w_{n-1}/d}(x_{n-1}) \tag{2.13}$$

is denoted $j\left(\dfrac{w}{d}\right)$. Also, Weil does not use the definition (2.9), but (2.12) is essentially what he proves.)

The purpose of this section is to use information on the number of points on diagonal hypersurfaces to derive a formula for the number $N^*(\alpha)$ of points with all coordinates nonzero on the hypersurface (1.1).

THEOREM 1 ([3, 6]):

$$N^*(\alpha) = \sum_w \chi_w^{-1}(\alpha) c_{\chi_w}, \tag{2.14}$$

where the summation is over all $w \in (\mathbf{Z}/(q-1)\mathbf{Z})^r$, $\sum w_i \equiv 0 \mod q-1$, *which index the characters of* μ_{q-1}^r/Δ, *for which*

$$\sum_i h_j^{(i)} w_i \equiv 0 \mod q - 1 \textit{ for all } j = 1, \ldots, n; \tag{2.15}$$

and for such w,

$$c_{\chi_w} = -\frac{1}{q}(q-1)^{n-r} J\left(\frac{w_1}{q-1}, \dots, \frac{w_r}{q-1}\right), \tag{2.16}$$

unless $w = (0, 0, \dots, 0)$, *in which case*

$$c_{\chi_0} = (q-1)^{n-r} \frac{(q-1)^{r-1} - (-1)^{r-1}}{q}.$$

PROOF: We have

$$N^*(\alpha) = \frac{1}{q-1} \#\{X \in \mathbf{F}_q^{*n} | \sum \alpha_i X^{h(i)} = 0\}, \tag{2.17}$$

and

$$c_\chi = \frac{1}{(q-1)^r} \sum_{\alpha \in \mathbf{F}_q^{*r}} N^*(\alpha) \chi(\alpha). \tag{2.18}$$

Now define

V: $\sum_{i=1}^{r} Z_i^{q-1} X^{h(i)} = 0$ in $P_{\bar{\mathbf{F}}_q}^{r-1} \times P_{\bar{\mathbf{F}}_q}^{n-1}$ with variables (Z, X);

$\tilde{V}$: $\sum_{i=1}^{r} Z_i^{q-1} Y^{(q-1)h(i)} = 0$ in $P_{\bar{\mathbf{F}}_q}^{r-1} \times P_{\bar{\mathbf{F}}_q}^{n-1}$ with variables (Z, Y).

Let $\mu^r = (\mu_{q-1})^r$. Then $\mu^r \times \mu^n$ acts on $\tilde{V}$ by $(\xi, \zeta)(Z, Y) = (\xi Z, \zeta Y)$; $\mu^r \times \{1\}$ acts on V by $\xi(Z, X) = (\xi Z, X)$; and V is the quotient of $\tilde{V}$ by the action of $\{1\} \times \mu^n$.

If w is an r-tuple of integers, $0 \le w < q-1$, $\sum w_i \equiv 0 \bmod q-1$, let $\chi = \chi_w = (\chi_{w_1}, \dots, \chi_{w_r})$: $\mu^r \to \mathbf{C}^*$, and $\tilde{\chi} = \tilde{\chi}_w = (\chi_{w_1}, \dots, \chi_{w_r}, 1, \dots, 1)$: $\mu^r \times \mu^n \to \mathbf{C}^*$. Then the following two expressions are clearly equal:

$$N^*(V, \chi) = \frac{1}{(q-1)^r} \sum_{\xi \in \mu^r} \chi(\xi) \frac{1}{(q-1)^2} \#\{(Z, X) \in \bar{\mathbf{F}}_q^{*r+n} | \; Z_i^{q-1} = \xi_i,\ X_j \in \mathbf{F}_q,\ \sum Z_i^{q-1} X^{h(i)} = 0\}; \tag{2.19}$$

$$N^*(\tilde{V}, \tilde{\chi}) = \frac{1}{(q-1)^{r+n}} \sum_{\xi, \zeta \in \mu^r \times \mu^n} \tilde{\chi}(\xi, \zeta) \frac{1}{(q-1)^2} \#\{(Z, Y) \in \bar{\mathbf{F}}_q^{*r+n} | \; Z_i^{q-1} = \xi_i,\ Y_j^{q-1} = \zeta_j,\ \sum (Z_i Y^{h(i)})^{q-1} = 0\}. \tag{2.20}$$

(Use $X_j = Y_j^{q-1} = \zeta_j$ to see that they are equal.) In addition, by (2.17) and (2.18),

$$c_\chi = \frac{1}{(q-1)^{r-1}} N^*(V, \chi)$$

(take $\alpha_i = Z_i^{q-1} = \xi_i$ in (2.19)).

The diagonal hypersurface $D = D_{q-1,r}$ in $P_{\mathbf{F}_q}^{r-1}$ with variables W is given by

$$W_1^{q-1} + \ldots + W_r^{q-1} = 0.$$

Now notice that $\tilde{V}$ and $D \times P^{n-1}$ are birationally isomorphic; namely, on the complement of the Y-coordinate hyperplanes they are isomorphic via the map

$$W_i = Z_i Y^{h^{(i)}}, \quad Z_i = W_i Y^{-h^{(i)}}, \qquad 1 \le i \le r.$$

Then we have

$$\begin{aligned}
(q-1)^{r-1} c_\chi &= N^*(V, \chi) = N^*(\tilde{V}, \tilde{\chi}) = \\
&= \frac{1}{(q-1)^{r+n+2}} \sum_{\xi,\zeta} \chi(\xi) \# \{(W, Y) \in \bar{\mathbf{F}}_q^{*r+n} \mid \\
&\qquad W_i^{q-1} = \xi_i \zeta^{h^{(i)}},\ Y_j^{q-1} = \zeta_j,\ \textstyle\sum W_i^{q-1} = 0\} \\
&= \frac{1}{(q-1)^{r+n+2}} \sum_{\xi,\zeta} \chi(\ldots, \xi_i \zeta^{-h^{(i)}}, \ldots) \# \{(W, Y) \in \bar{\mathbf{F}}_q^{*r+n} \mid \\
&\qquad W_i^{q-1} = \xi_i,\ Y_j^{q-1} = \zeta_j,\ \textstyle\sum W_i^{q-1} = 0\}
\end{aligned}$$

(after replacing ξ_i by $\xi_i \zeta^{-h^{(i)}}$, $i = 1, \ldots, r$)

$$\begin{aligned}
&= \frac{1}{(q-1)^{r+2}} \sum_{\zeta} \chi_{w_1}(\zeta^{-h^{(1)}}) \ldots \chi_{w_r}(\zeta^{-h^{(r)}}) \sum_{\xi} \chi(\xi) \# \{W \in \bar{\mathbf{F}}_q^{*r} \mid \\
&\qquad W_i^{q-1} = \xi_i,\ \textstyle\sum W_i^{q-1} = 0\} \\
&= \begin{cases} 0 \text{ unless } \sum_i h_j^{(i)} w_i \equiv 0 \bmod q-1 \text{ for all } j = 1, \ldots, n; \\ (q-1)^{n-1} \left(\frac{1}{(q-1)^{r+1}} \sum_{\xi} \chi(\xi) \# \{W \in \bar{\mathbf{F}}_q^{*r} \mid W_i^{q-1} = \xi_i, \sum W_i^{q-1} \right. \\ \left. \qquad = 0\} \right) \text{ if this condition holds.} \end{cases}
\end{aligned}$$

If $w \neq (0, 0, \ldots, 0)$, then the expression in the large parentheses is equal to

$$\frac{1}{q-1} \sum_{\xi, \Sigma\xi_i = 0} \chi(\xi) = -\frac{1}{q} J\left(\frac{w}{q-1}\right)$$

by (2.2) (here we use $w/(q-1)$ to abbreviate the r-tuple of $w_i/(q-1)$). Thus, in this case

$$c_\chi = -\frac{1}{q}(q-1)^{n-r} J\left(\frac{w}{q-1}\right),$$

as claimed. If $w = (0, \ldots, 0)$, then an easy computation gives

$$c_\chi = (q-1)^{n-r}\frac{1}{q}((q-1)^{r-1} - (-1)^{r-1}). \qquad \text{Q.E.D.}$$

§3. Monomial deformations of diagonal hypersurfaces

We now consider the hypersurface (1.2), where we assume that $d | q-1$, $h_1 + \ldots + h_n = d$, and g.c.d. $(d, h_1, \ldots, h_n) = 1$. Let $h = (h_1, \ldots, h_n)$.

Let W denote the set of n-tuples $w = (w_1, \ldots, w_n)$, $0 \le w_i < q-1$, $\sum w_i \equiv 0 \bmod q-1$.

Theorem 1 can be applied to this hypersurface. Let $N^*(\lambda)$ denote the number of $\mathbf{F}_q$-points with nonzero coordinates on the hypersurface (this was $N^*(1, 1, \ldots, 1, -d\lambda)$ in the earlier notation).

Corollary 1: *For* $\lambda \neq 0$,

$$N^*(\lambda) = \frac{1}{q-1} \sum_{s \in \frac{d}{q-1}\mathbf{Z}/\mathbf{Z},\, w \in W} q^{\varepsilon(s)} J\left(\frac{w+sh}{d}\right) \chi_s(d\lambda), \tag{3.1}$$

where $\varepsilon(s) = -1$ *if* $s = 0$ *and* 0 *otherwise, and* $(w+sh)/d$ *denotes the* n-*tuple of* $(w_i + sh_i)/d$.

Proof: In Theorem 1 set $r = n+1$, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1$, $\alpha_r = -d\lambda$, and replace w_j by u_j, $j = 1, \ldots, n, n+1$. Then the condition (2.15) becomes

$$du_j \equiv -h_j u_{n+1} \pmod{q-1} \text{ for } j = 1, \ldots, n.$$

Since g.c.d. $(d, h_1, \ldots, h_n) = 1$, it follows that $d | u_{n+1}$.

Now $\chi_w^{-1}(\alpha)$ in Theorem 1 is $\chi_{u_{n+1}/(q-1)}^{-1}(-d\lambda)$ in our context, using the notation for characters of $\mathbf{F}_q^*$ defined at the beginning of §2. So (2.14) and (2.16) give us

$$N^*(\lambda)=\frac{1}{q-1}\sum_{u_j\equiv -h_j\frac{u_{n+1}}{d}\left(\bmod\frac{q-1}{d}\right)}-\frac{1}{q}J\left(\frac{u_1}{q-1},\dots,\frac{u_{n+1}}{q-1}\right)\chi_{u_{n+1}/(q-1)}^{-1}(-d\lambda)$$

$\left(\text{with } \frac{1}{q}((q-1)^n-(-1)^n) \text{ in place of } -\frac{1}{q}J \text{ when } u_1=\dots=u_{n+1}=0\right)$.

Let $s=-u_{n+1}/(q-1)\in\frac{d}{q-1}\mathbf{Z}/\mathbf{Z}$, so that $u_j=\frac{q-1}{d}(sh_j+w_j)$, $0\le w_j<d, j=1,\dots,n$. Note that $u_1+\dots+u_{n+1}\equiv 0 \pmod{q-1}$ implies that

$$\sum_{j=1}^n w_j=\frac{d}{q-1}\sum_{j=1}^n u_j-s\sum_{j=1}^n h_j=\frac{d}{q-1}\sum_{j=1}^{n+1}u_j\equiv 0 \pmod d.$$

Thus,

$$N^*(\lambda)=\frac{1}{q-1}\sum_{s,w}-\frac{1}{q}J\left(\frac{w_1+sh_1}{d},\dots,\frac{w_n+sh_n}{d},-s\right)\chi_s(-d\lambda)$$

(with $(q-1)^n-(-1)^n$ in place of $-J$ when $s=0$, $w=0$). Then (2.3) and (2.7) give us the corollary. Q.E.D.

REMARK: Corollary 1 can also be proved directly, without Theorem 1, as follows. The hypersurface (1.2) has an action of the subgroup $G\subset\mu_d^n/\Delta$ consisting of elements which preserve the monomial $X^h = X_1^{h_1}\dots X^{h_n}$, i.e.,

$$G=\{\xi\in\mu_d^n \mid \xi^h=1\}/\Delta.$$

The characters χ_w of μ_d^n/Δ which act trivially on the subgroup G are precisely the powers of χ_h. Thus, the character group $\hat{G}$ of G corresponds to equivalence classes of the w in (2.11), where $w\sim w'$ if $w-w'$ is a multiple (mod d) of the n-tuple h. Each equivalence class contains d n-tuples w', because g.c.d.$(h_1,\dots,h_n,d)=1$.

Then one can look at the number $N^*(\lambda,\chi_w)$ of χ_w-points (see (2.9)). After some computation, one finds that

$$N^*(\lambda,\chi_w)=\frac{1}{q-1}\sum_{s\in\frac{d}{q-1}\mathbf{Z}/\mathbf{Z},\,w'\sim w} q^{\varepsilon(s)}J\left(\frac{w'+sh}{d}\right)\chi_s(d\lambda), \tag{3.2}$$

where $\varepsilon(s) = -1$ if $s = 0$ and 0 otherwise. Finally, Corollary 1 is obtained by summing over $\chi_w \in \hat{G}$, by Lemma 2.

We now show that in the case of monomial deformations of diagonal hypersurfaces, an equally simple result applies to the total number of points, including those where a coordinate vanishes. As far as we know, this is not true for the general hypersurfaces in Theorem 1.

THEOREM 2:

$$N_{\mathbf{F}_q}(\lambda) = N_{\mathbf{F}_q}(0) + \sum \frac{g\left(\dfrac{w+sh}{d}\right)}{g(s)} \chi_s(d\lambda),$$

in which $g((w+sh)/d) = \prod g((w_i + sh_i)/d)$, *the summation is over n-tuples w satisfying* (2.11) *and over* $s \in \dfrac{d}{q-1}\mathbf{Z}/\mathbf{Z}$, *and* $N_{\mathbf{F}_q}(0) = N_{\mathbf{F}_q}(D_{d,n})$ *is equal to the sum of the terms* (2.12) *over all w in* (2.11).

PROOF: The theorem is trivial for $\lambda = 0$, since $\chi_s(0) = 0$. So suppose that $\lambda \neq 0$.

It is obvious from (1.2) that the contribution to $N(\lambda)$ from points where a coordinate vanishes is independent of λ, i.e.,

$$N(\lambda) - N^*(\lambda) = N(0) - N^*(0). \tag{3.3}$$

Now $N^*(0) = \sum N^*(0, \chi_w)$, by Lemma 2, where the sum is over n-tuples w as in (2.11). Thus, by Corollary 1 and (3.3),

$$N(\lambda) - N(0) = \frac{1}{q-1}\left(\sum_{0 \neq s \in \frac{d}{q-1}\mathbf{Z}/\mathbf{Z},\, w} J\left(\frac{w+sh}{d}\right)\chi_s(d\lambda) + \right.$$
$$\left. + \sum_w \left(\frac{1}{q} J\left(\frac{w}{d}\right) - (q-1)N^*(0,\chi_w)\right)\right). \tag{3.4}$$

First suppose $w \neq (0, 0, \ldots, 0)$. A computation similar to Weil's proof of (2.12) gives

$$N^*(0, \chi_w) = -\frac{1}{q} J\left(\frac{w}{d}\right),$$

and hence

$$\frac{1}{q} J\left(\frac{w}{d}\right) - (q-1)N^*(0,\chi_w) = J\left(\frac{w}{d}\right). \tag{3.5}$$

Next, if $w = (0, 0, \ldots, 0)$, then it follows from the definition of $N^*(0, \chi_w)$ that

$$(q-1)N^*(0, \chi_0) = \#\{U \in \mathbf{F}_q^{*n} \mid \sum U_i = 0\},$$

whereas by (2.7)

$$\frac{1}{q} J(0, \ldots, 0) = \#\{U \in \mathbf{F}_q^{*n} \mid \sum U_i = 1\}.$$

Let A_n be the first and B_n the second of these numbers. I claim that $B_n - A_n = (-1)^{n-1}$. This clearly holds for $n = 2$. Suppose it holds for n. A simple counting argument gives

$$A_n + A_{n+1} = \#\{(U_0, U_1, \ldots, U_n) \in \mathbf{F}_q^{*n+1} \mid \sum U_i = 0\} + \\ + \#\{(0, U_1, \ldots, U_n) \in \{0\} \times \mathbf{F}_q^{*n} \mid \sum U_i = 0\} = (q-1)^n,$$

and similarly,

$$B_n + B_{n+1} = (q-1)^n.$$

Subtracting these two equalities gives $B_{n+1} - A_{n+1} = -(B_n - A_n)$, and hence the induction step. Thus,

$$\frac{1}{q} J(0, \ldots, 0) - (q-1)N^*(0, \chi_{(0, \ldots, 0)}) = (-1)^{n-1} = \frac{g(0) \ldots g(0)}{g(0)}. \tag{3.6}$$

Finally, using (2.5) (which holds for all Jacobi sums except $J(0, \ldots, 0)$), we have from (3.4), (3.5) and (3.6)

$$N(\lambda) - N(0) = \frac{1}{q-1} \sum_{s, w} g\left(\frac{w_1 + sh_1}{d}\right) \ldots g\left(\frac{w_n + sh_n}{d}\right) \frac{1}{g(s)} \chi_s(d\lambda),$$

as desired. Q.E.D.

REMARK: Theorem 2 suggests the question: Are there broader classes of hypersurfaces for which the points with a coordinate vanishing can be included in such a natural way?

§4. Classical analog

The Gauss sum

$$g(s) = \sum_{x \in \mathbf{F}_q^*} \chi_s(x)\psi(x)$$

is the finite-field analog of the gamma function

$$\Gamma(s) = \int_{x \in \mathbf{R}^+} x^s e^{-x} \frac{dx}{x},$$

where $x \mapsto x^s$ is a multiplicative character on $\mathbf{R}^+$, and e^{-x} corresponds to $\psi(x)$. The Jacobi sum $J(s_1, s_2)$ is the analog of the beta function

$$B(s_1, s_2) = \int_0^1 x^{s_1}(1-x)^{s_2} \frac{dx}{x(1-x)}.$$

We shall see that $N^*(\lambda, \chi_w)$ (see (3.2)), or, more precisely, the corresponding part of the sum in Theorem 2 for $N(\lambda) - N(0)$, is a finite-field analog of a hypergeometric function.

Thus, we consider for some fixed w

$$\sum_{s \in \frac{d}{q-1}\mathbf{Z}/\mathbf{Z},\, w' \sim w} \frac{g\left(\dfrac{w+sh}{d}\right)}{g(s)} \chi_s(d\lambda). \tag{4.1}$$

Replacing s by ds and summing over $s \in \dfrac{1}{q-1}\mathbf{Z}/\mathbf{Z}$, we see that the resulting sum over s is the same as the sum (4.1) over s and w':

$$\sum_{s \in \frac{1}{q-1}\mathbf{Z}/\mathbf{Z}} \frac{g\left(hs + \dfrac{w}{d}\right)}{g(ds)} \chi_{ds}(d\lambda). \tag{4.2}$$

We now rewrite (4.2) using the following identity for Gauss sums [2]:

$$\prod_{j=0}^{d-1} g\left(s + \frac{j}{d}\right) = \chi_{-ds}(d) g(ds) \prod_{j=1}^{d-1} g\left(\frac{j}{d}\right).$$

Then (4.2) becomes

$$\prod g\left(\frac{j}{d}\right)\sum_s \frac{g\left(h_1 s + \frac{w_1}{d}\right)\dots g\left(h_n s + \frac{w_n}{d}\right)}{g(s)g\left(s+\frac{1}{d}\right)\dots g\left(s+\frac{d-1}{d}\right)}\chi_{ds}(\lambda)$$

$$= \frac{1}{q}\prod g\left(\frac{j}{d}\right)\sum_s \frac{g\left(h_1 s + \frac{w_1}{d}\right)\dots g\left(h_n s + \frac{w_n}{d}\right)}{g\left(s+\frac{1}{d}\right)\dots g\left(s+\frac{d-1}{d}\right)} g(-s)\chi_s(-\lambda^d).$$

This expression is analogous to the Barnes type integral ($\lambda \in \mathbf{C}$):

$$\Gamma\left(\frac{1}{d}\right)\dots\Gamma\left(\frac{d-1}{d}\right)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma\left(h_1 s + \frac{w_1}{d}\right)\dots\Gamma\left(h_n s + \frac{w_n}{d}\right)}{\Gamma\left(s+\frac{1}{d}\right)\dots\Gamma\left(s+\frac{d-1}{d}\right)}$$

$$\cdot\,\Gamma(-s)(-\lambda^d)^s\,ds, \tag{4.3}$$

where the path of integration from $-i\infty$ to $i\infty$ in the complex s-plane curves so as to keep all poles of $\Gamma\left(h_i s + \frac{w_i}{d}\right)$ to its left and all poles of $\Gamma(-s)$ to its right. As in the more familiar case of the Barnes integral of $\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\Gamma(-s)(-z)^s$, which gives $\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)$ (see [12], pp 286–288), our integral (4.3) can be evaluated as the sum of residues at $s = k$. That is, the integral (4.3) is equal to

$$\Gamma\left(\frac{1}{d}\right)\dots\Gamma\left(\frac{d-1}{d}\right)\sum_{k=0}^{\infty}\frac{\Gamma\left(h_1 k + \frac{w_1}{d}\right)\dots\Gamma\left(h_n k + \frac{w_n}{d}\right)}{\Gamma\left(k+\frac{1}{d}\right)\dots\Gamma\left(k+\frac{d-1}{d}\right)k!}\lambda^{dk}$$

$$= \prod_{1\le j\le n}\Gamma\left(\frac{w_j}{d}\right)\sum_{k=0}^{\infty}\frac{\prod_{\substack{1\le j\le n\\ 1\le i<h_j}}\left(\frac{w_j}{dh_j}+\frac{i}{h_j}\right)_k}{\left(\frac{1}{d}\right)_k\dots\left(\frac{d-1}{d}\right)_k k!}\left(\left(\prod_j h_j^{h_j}\right)\lambda^d\right)^k$$

(where $(x)_k \underset{\text{def}}{=} x(x+1)\dots(x+k-1)$)

$$= \prod_j \Gamma\left(\frac{w_j}{d}\right) {}_dF_{d-1}\left(\begin{matrix} \dots \frac{w_j}{dh_j} + \frac{i}{h_j} \dots \\ \frac{1}{d} \dots \frac{d-1}{d} \end{matrix} ; (\prod_j h_j^{h_j})\lambda^d\right).$$

These are the hypergeometric functions associated to the hypersurface (1.2) over **C** (see, e.g., [4, 8]). (Here the upper exponents of ${}_dF_{d-1}$ run through the h_j values $(w_j + id)/dh_j$, $i = 1, \dots, h_j - 1$, for each j; no exponent appears when $h_j = 0$.)

REMARK: This close analog between classical and finite-field formulas suggests the question of whether there is a similar classical analog to Theorem 1, i.e., a multiple Barnes integral which is associated to the general hypersurface (1.1) ($\alpha_i \in \mathbf{C}$).

In the next two sections we shall see some other families of hypersurfaces over finite fields which have formulas involving Jacobi sums for the number of points, and which in the complex analytic case correspond to integral analogs of these formulas.

§5. Hypersurfaces connected to hypergeometric functions

Let

$$y^d = x^a(1 - x)^{b-a}(1 - \lambda x_1 \dots x_n)^{-a_{n+1}}, \tag{5.1}$$

$\lambda \in \mathbf{F}_q$, be the equation of an n-dimensional hypersurface, depending on a parameter λ, in affine space over $\mathbf{F}_q$. Here $d > 1$ is an integer dividing $q - 1$; $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, $a_i, b_i \in \mathbf{Z}$, $0 < a_i, b_j < d$ for $i = 1, \dots, n + 1$, $j = 1, \dots, n$; $\text{g.c.d.}(a_i, b_i, d) = 1$, $a_i \neq b_i$ for $i = 1, \dots, n$; $\text{g.c.d.}(a_{n+1}, d) = 1$; $x^a \underset{\text{def}}{=} x_1^{a_1} \dots x_n^{a_n}$, $(1 - x)^{b-a} \underset{\text{def}}{=} (1 - x_1)^{b_1 - a_1} \dots (1 - x_n)^{b_n - a_n}$.

If equation (5.1) is considered over the complex numbers for $\lambda \in \mathbf{C}$, then this hypersurface has periods which are essentially the hypergeometric function ${}_{n+1}F_n$. More precisely, for $|\lambda| < 1$, $\alpha = (a_1/d, \dots, a_{n+1}/d)$, $\beta = (b_1/d, \dots, b_n/d)$, we have (see, for example, [5]):

$${}_{n+1}F_n\left(\begin{matrix}\alpha \\ \beta\end{matrix}; \lambda\right) = \frac{\Gamma(\beta_1) \dots \Gamma(\beta_n)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)\Gamma(\beta_1 - \alpha_1) \dots \Gamma(\beta_n - \alpha_n)} \cdot$$

$$\cdot \int_{(0,1)^n} x_1^{\alpha_1} \dots x_n^{\alpha_n}(1 - x_1)^{\beta_1 - \alpha_1} \dots (1 - x_n)^{\beta_n - \alpha_n}(1 - \lambda x_1 \dots x_n)^{-\alpha_{n+1}} d^*x, \tag{5.2}$$

where $d^*x = \prod dx_i/x_i(1-x_i)$ is the measure on $(0,1)^n$ obtained by pulling back $\prod dt_i$ on $\mathbf{R}^n$ by the map $t_i = \log(x_i/(1-x_i))$ from $(0,1)^n$ to $\mathbf{R}^n$.

But we wish to consider the hypersurface (5.1) over $\mathbf{F}_q$, and compute the number $N(\lambda) = N_{\mathbf{F}_q}(\lambda)$ of points $(x_1, \ldots, x_n, y) \in \mathbf{F}_q^{n+1}$ satisfying (5.1). It is again convenient to split up $N(\lambda)$ into pieces $N(\lambda, \chi)$, where $\chi: G \to K^*$ is a character of the group $G = \mu_d$ of d-th roots of 1 in $\mathbf{F}_q^*$ with values in an algebraically closed field K of characteristic zero. G acts on points on the hypersurface in the obvious way: $\xi \in G$ leaves the x_i fixed and takes y to ξy.

THEOREM 3: *Let $N(\lambda, \chi)$ be defined by* (2.9) *for the hypersurface* (5.1) *and for $G = \mu_d$. Let χ be the restriction to μ_d of $\chi_{w/(q-1)}: \mathbf{F}_q^* \to K^*$ (w is an integer uniquely determined modulo d). Then for χ nontrivial (i.e., $w \not\equiv 0$* mod *d)*

$$N(\lambda, \chi) = \prod_{i=1}^{n} \frac{g\left(\frac{w}{d} a_i\right) g\left(\frac{w}{d}(b_i - a_i)\right)}{g\left(\frac{w}{d} b_i\right)} \left(1 + \frac{1}{q-1} \sum_{s \in \frac{1}{q-1}\mathbf{Z}/\mathbf{Z}} c_s \chi_s(\lambda)\right), \tag{5.3}$$

where

$$c_s = c_{s,w} = \frac{q}{g(s)} \prod_{i=1}^{n+1} \frac{g\left(s + \frac{w}{d} a_i\right)}{g\left(\frac{w}{d} a_i\right)} \prod_{i=1}^{n} \frac{g\left(\frac{w}{d} b_i\right)}{g\left(s + \frac{w}{d} b_i\right)}, \tag{5.4}$$

with $g\left(s + \frac{w}{d} b_i\right)$ replaced by $-q$ if $s + \frac{w}{d} b_i \equiv 0$ mod $\mathbf{Z}$.

PROOF: For $(x, y) = (x_1, \ldots, x_n, y) \in \bar{\mathbf{F}}_q^{n+1}$, the condition $F \circ \xi(x, y) = (x, y)$ means that $x \in \mathbf{F}_q^n$ and $\xi y^q = y$, i.e., $y^{q-1} = \xi^{-1}$ or $y = 0$. Since χ is nontrivial, it is easy to see that the terms in (2.9) with $y = 0$ drop out. Hence the sum (2.9) is equal to

$$\frac{1}{d} \sum \chi(y^{q-1}),$$

where the summation is over $x \in \mathbf{F}_q^n$ and $y \in \bar{\mathbf{F}}_q$ satisfying (5.1) for which $y^{q-1} \in \mu_d$. Let $u = y^d$. Then $y^{q-1} \in \mu_d$ if and only if $u \in \mathbf{F}_q^*$. For each $u \in \mathbf{F}_q^*$ there are d different y with $u = y^d$. Hence, this sum is (if we take $\chi(0)$

$= 0)$: $\sum \chi(u^{(q-1)/d})$, where the summation is over $(x, u) \in \mathbf{F}_q^{n+1}$ for which (5.1) holds with u in place of y^d. By the definition of w, the map $u \mapsto \chi(u^{(q-1)/d})$ is simply $\chi_{w/d}$. Thus, replacing u by the right side of (5.1), we have

$$N(\lambda, \chi) = \sum_{x \in \mathbf{F}_q^n} \chi_{w/d}(x^a(1-x)^{b-a}(1-\lambda x_1 \ldots x_n)^{-a_{n+1}}). \tag{5.5}$$

We now multiply (5.5) by $\chi_{-s}(\lambda)$ for arbitrary fixed $s \in \frac{1}{q-1}\mathbf{Z}/\mathbf{Z}$, and we sum over $\lambda \in \mathbf{F}_q^*$. Making the change of variables $t = \lambda x_1 \ldots x_n$ and noting that the sum vanishes unless all $x_1, \ldots, x_n$ are nonzero, we have

$$\begin{aligned}
\sum_{\lambda \in \mathbf{F}_q^*} N(\lambda, \chi)\chi_{-s}(\lambda) &= \sum_{(x,t) \in \mathbf{F}_q^{n+1}} \chi_{w/d}(x^a(1-x)^{b-a}(1-t)^{-a_{n+1}})\chi_{-s}(t)\chi_s(x_1 \ldots x_n) \\
&= \left(\sum_t \chi_{-s}(t)\chi_{-\frac{w}{d}a_{n+1}}(1-t)\right)\left(\prod_{i=1}^n \sum_{x_i} \chi_{s+\frac{w}{d}a_i}(x_i)\chi_{\frac{w}{d_i}(b_i-a_i)}(1-x_i)\right) \\
&= J\left(-\frac{w}{d}a_{n+1}, -s\right)\prod_{i=1}^n J\left(s+\frac{w}{d}a_i, \frac{w}{d}(b_i-a_i)\right) \\
&= \chi_s(-1)g(-s)\frac{g\left(s+\frac{w}{d}a_{n+1}\right)}{g\left(\frac{w}{d}a_{n+1}\right)}\prod_{i=1}^n \frac{g\left(s+\frac{w}{d}a_i\right)g\left(\frac{w}{d}(b_i-a_i)\right)}{g\left(s+\frac{w}{d}b_i\right)},
\end{aligned} \tag{5.6}$$

with $g\left(s+\frac{w}{d}b_i\right)$ replaced by $-q$ if $s+\frac{w}{d}b_i \in \mathbf{Z}$. Note that

$$\chi_s(-1)g(-s) = \begin{cases} q/g(s) & \text{if } s \neq 0; \\ q-1+\dfrac{q}{g(s)} & \text{if } s = 0. \end{cases}$$

If $\lambda \neq 0$, then $N(\lambda, \chi)$ as a function of $\lambda \in \mathbf{F}_q^*$ can be expanded in characters χ_s of $\mathbf{F}_q^*$, and (5.3) for $\lambda \neq 0$, $w \neq 0$ asserts that the coefficient of χ_s is equal to

$$\frac{1}{q-1}\prod_{i=1}^n \frac{g\left(\frac{w}{d}a_i\right)g\left(\frac{w}{d}(b_i-a_i)\right)}{g\left(\frac{w}{d}b_i\right)} \cdot \begin{cases} c_s & \text{if } s \neq 0; \\ q-1+c_0 & \text{if } s = 0, \end{cases}$$

where c_s is given by (5.4). But this is what we computed in (5.6).

If $\lambda = 0$, then by (5.5)

$$N(0,\chi) = \sum_{x\in\mathbf{F}_q^n} \chi_{w/d}(x^a(1-x)^{b-a}) = \prod_{i=1}^{n} \frac{g\left(\frac{w}{d}a_i\right) g\left(\frac{w}{d}(b_i - a_i)\right)}{g\left(\frac{w}{d}b_i\right)},$$

so that (5.3) also holds for $\lambda = 0$. Q.E.D.

REMARKS: 1. By Lemma 1, if we want a formula for $N(\lambda)$, we need only add the expressions (5.3) for $w = 1, \ldots, d-1$, and then add $N(\lambda, \chi_{\text{trivial}})$. It remains to compute $N(\lambda, \chi_{\text{trivial}})$, i.e., the case $w = 0$. By (2.9), $N(\lambda, \chi_{\text{trivial}}) = \frac{1}{d} \#\{x, y, \xi \mid x \in \mathbf{F}_q^n,\ y \in \bar{\mathbf{F}}_q,\ \xi \in \mu_d,\ y^{q-1} = \xi^{-1}$ or $y = 0$, and (5.1) holds$\}$

$= \#\{(x,u) \in \mathbf{F}_q^{n+1} \mid (5.1)$ holds with u in place of $y^d\}$
$= q^n$,

since for any $x \in \mathbf{F}_q^n$ the equation (5.1) uniquely determines u.

2. $N(\lambda, \chi)$ is essentially a finite-field analog of the hypergeometric function (5.2). More precisely, we introduce Gauss sum terms to correspond to the gamma terms in (5.2), i.e., we define

$$\begin{aligned} {}_{n+1}F_{n,\mathbf{F}_q}\left(\begin{matrix}\alpha\\ \beta\end{matrix}; \lambda\right) &= \prod_{i=1}^{n} \frac{g(\beta_i)}{g(\alpha_i) g(\beta_i - \alpha_i)} \cdot \\ &\cdot \sum_{x\in\mathbf{F}_q^n} \chi_{\alpha_1}(x_1)\chi_{\beta_1-\alpha_1}(1-x_1)\ldots\chi_{\alpha_n}(x_n)\chi_{\beta_n-\alpha_n}(1-x_n)\chi_{-\alpha_{n+1}}(1-\lambda x_1\ldots x_n) \\ &= \prod_{i=1}^{n} \frac{g(\beta_i)}{g(\alpha_i) g(\beta_i - \alpha_i)} \cdot \\ &\cdot \sum_{x\in\mathbf{F}_q^n} \chi_{1/d}(x^a(1-x)^{b-a}(1-\lambda x_1\ldots x_n)^{-a_{n+1}}), \end{aligned}$$

where $\alpha_i = a_i/d$, $\beta_j = b_j/d \in \frac{1}{q-1}\mathbf{Z}/\mathbf{Z} - \{0\}$, $\lambda \in \mathbf{F}_q$. (Compare with (5.2) and (5.5).) Theorem 3 (with $w = 1$) then says that this function equals

$$1 + \frac{1}{q-1} \sum_s c_s \chi_s(\lambda), \tag{5.7}$$

where c_s is given by (5.4) with $w = 1$. For $s \neq 0$, we have $q/g(s) = \chi_s(-1)g(-s)$ by (2.4). Hence, the part of the sum in (5.7) over $s \neq 0$ can be rewritten

$$\frac{g(\beta_1)\dots g(\beta_n)}{g(\alpha_1)\dots g(\alpha_{n+1})}\sum_{s\neq 0}\frac{g(\alpha_1+s)\dots g(\alpha_{n+1}+s)}{g(\beta_1+s)\dots g(\beta_n+s)}g(-s)\chi_s(-\lambda).$$

Thus, Theorem 3 can be thought of as a finite field analog of the Pochhammer–Barnes integral formula ([10], p. 102) for the classical hypergeometric function (5.2), according to which

$${}_mF_n\begin{pmatrix}\alpha_1\dots\alpha_m\\ \beta_1\dots\beta_n\end{pmatrix};\lambda\Bigg)=\frac{\Gamma(\beta_1)\dots\Gamma(\beta_n)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_m)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(\alpha_1+s)\dots\Gamma(\alpha_m+s)}{\Gamma(\beta_1+s)\dots\Gamma(\beta_n+s)}\cdot \Gamma(-s)(-\lambda)^s ds$$

(here no α_i or β_j can be zero or a negative integer, and the path of integration from $-i\infty$ to $i\infty$ in the complex s-plane curves so as to keep all poles of $\Gamma(\alpha_i+s)$ to its left and all poles of $\Gamma(-s)$ to its right).

3. It is not hard to generalize Theorem 3 to the case where $\lambda x_1\dots x_n$ in (5.1) is replaced by another monomial $\lambda x_1^{h_1}\dots x_n^{h_n}$.

§6. Hypersurfaces connected to hypergeometric functions of several variables

A hypergeometric function of n variables $\lambda_1,\dots,\lambda_n\in\mathbf{C}$, $|\lambda_i|<1$, and $2n+1$ "exponents" $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,\gamma$ can be defined as follows ([1], p. 115), where we use the abbreviations $\alpha=(\alpha_1,\dots,\alpha_n)$, $x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}$, $(1-\lambda x)^{-\beta}=(1-\lambda_1x_1)^{-\beta_1}\dots(1-\lambda_nx_n)^{-\beta_n}$, etc.:

$$F(\alpha,\beta,\gamma;\lambda)=\frac{\Gamma(\gamma)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)\Gamma(\gamma-\alpha_1-\dots-\alpha_n)}\int\dots\int x^\alpha(1-\lambda x)^{-\beta}\cdot$$
$$\cdot(1-x_1-\dots-x_n)^{\gamma-\alpha_1-\dots-\alpha_n}\frac{dx_1\dots dx_n}{x_1\dots x_n(1-x_1-\dots-x_n)},$$

where the integration is over nonnegative x_i with $\sum x_i\le 1$.

Thus, over a finite field $\mathbf{F}_q$ it is natural to consider the n-dimensional affine hypersurface

$$y^d=x^a(1-\lambda x)^{-b}(1-x_1-\dots-x_n)^{c-a_1-\dots-a_n},\qquad (6.1)$$

where $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbf{F}_q^n$ are parameters, $d|q-1$, $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$, $a_i,b_j,c\in\mathbf{Z}$, $0<a_i,b_j,c<d$, g.c.d.$(b_i,d)=1$, g.c.d.$(c-a_1-\dots-a_n,d)=1$, $x^a=x_1^{a_1}\dots x_n^{a_n}$, $(1-\lambda x)^{-b}=(1-\lambda_1x_1)^{-b_1}\dots(1-\lambda_nx_n)^{-b_n}$.

Let $G = \mu_d$ act by $\xi(x, y) = (x, \xi y)$ for $(x, y) \in \bar{\mathbf{F}}_q^{n+1}$, and let $\chi: G \to K^*$ be the restriction to μ_d of $\chi_{w/(q-1)}: \mathbf{F}_q^* \to K^*$, as in §5. Define $N(\lambda)$ and $N(\lambda, \chi)$ by (2.8)–(2.9) with V given by equation (6.1).

THEOREM 4: *Let $\lambda \in \mathbf{F}_q^{*n}$. Then for χ nontrivial*

$$N(\lambda, \chi) = \frac{1}{q-1} \sum_{s \in \left(\frac{1}{q-1}\mathbf{Z}/\mathbf{Z}\right)^n} c_s \chi_{s_1}(\lambda_1) \ldots \chi_{s_n}(\lambda_n),$$

where

$$c_s = J\left(\frac{w}{d}a_1 + s_1, \ldots, \frac{w}{d}a_n + s_n, \frac{w}{d}(c - a_1 - \ldots - a_n)\right) \cdot \\ \cdot J\left(-\frac{w}{d}b_1, -s_1\right) \ldots J\left(-\frac{w}{d}b_n, -s_n\right).$$

For χ trivial

$$N(\lambda, \chi_{\text{trivial}}) = q^n.$$

The proof is completely analogous to the proof of Theorem 3.

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