G. ANGERMÜLLER
W. BARTH

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ELLIPTIC FIBRES ON ENRIQUES SURFACES

G. Angermüller and W. Barth

1. Introduction

It is known [5] that an Enriques surface X admits holomorphic fibrations over $\mathbb{P}_1$, where the general fibre is a smooth elliptic curve. The aim of this note is to classify the singular fibres. Since X is algebraic and $b_2(X) = 10$, such a fibre can have at most 9 irreducible components. In Kodaira's notation [6] (see also section 2 below) the following types of (non-multiple) singular elliptic fibres have 9 components or less:

I_b, $b \leq 9$,
I_b^*, $b \leq 4$,
II, II*, III, III*, IV, IV*.

For each of these types we give an example of an Enriques surface and an elliptic fibration, in which it appears as singular fibre. So we prove the

THEOREM: *For each of the types in the list above, there is an Enriques surface admitting an elliptic fibration with a singular fibre of this type.*

The Enriques surfaces are found as follows [5]: If $B \subset \mathbb{P}_1 \times \mathbb{P}_1$ is a (reduced) curve of bidegree (4,4), there is a double cover $Y \to \mathbb{P}_1 \times \mathbb{P}_1$ ramified over B. If B has at worst simple singularities, the minimal desingularisation $\bar{Y}$ of Y is a $K3$-surface. If the polynomial defining B is invariant under an involution τ of $\mathbb{P}_1 \times \mathbb{P}_1$ with 4 fixed points (not on B), this τ lifts to an involution $\bar{\sigma}$ of $\bar{Y}$ without fixed points, and $X = \bar{Y}/\bar{\sigma}$ is an Enriques surface. Elliptic fibrations of X are induced by the projections of $\mathbb{P}_1 \times \mathbb{P}_1$ on its factors and singular fibres on X are created by singularities of the branch curve B or special positions of this curve with respect to the projections.

It should be mentioned that an example of a fibre of type II* was already given by Horikawa [5]. But a catalog of all possible fibres seems not to exist so far.

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Our method very easily produces examples for all types of fibres, except for the types I_5 and I_7, where some computations are necessary, and I_9, where the computations are a little cumbersome. The basefield always is $\mathbb{C}$, the field of complex numbers.

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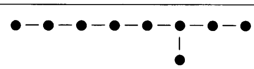
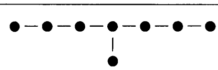
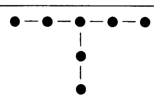
2. Kodaira's table of singular fibres

First we recall Kodaira's classification [6] of singular fibres in elliptic fibrations, which are not multiple fibres. Since we have to compare these fibres with simple singularities of the branch curve we prefer to use the A–D–E notation instead of Kodaira's one.

Table 1.

Kodaira's notation	Description, resp. dual graph		Notation here
I_0	nonsingular elliptic		–
I_1	irreducible rational with node		$\tilde{A}_0$
I_b, $b \geq 2$	cycle of b nonsingular rational curves		$\tilde{A}_{b-1}$
I_b^*, $b \geq 0$		($b + 5$ curves)	$\tilde{D}_{b+4}$
II	irreducible rational with cusp		A_0'
II*		(9 curves)	$\tilde{E}_8$
III	two smooth rational curves touching		A_1'
III*		(8 curves)	$\tilde{E}_7$
IV	three smooth rational curves meeting in one point		A_2'
IV*		(7 curves)	$\tilde{E}_6$

An Enriques surface X is algebraic and $b_2(X) = 10$. By Zariski's lemma each curve C properly contained in a fibre X_s has selfintersection $C^2 < 0$. So the components of X_s span in $H^2(X, \mathbb{R})$ a subspace of dimension $b_2(X_s)$. Since there also is the class of a hyperplane section, for any fibre X_s on an Enriques surface, $b_2(X_s) \leq 9$. If X_s is of type $\tilde{A}_n, \ldots, \tilde{E}_8$, the subscript is one less than $b_2(X_s)$. So only the fibres $\tilde{A}_n (n \le 8)$, A'_0, A'_1, A'_2, $\tilde{D}_n (n \le 8)$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ can appear as such fibres.

3. Enriques surfaces and double coverings of $\mathbb{P}_1 \times \mathbb{P}_1$

Here we recall the representation of (so-called) non-special Enriques surfaces in terms of double coverings of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ [5].

Let $((u_0 : u_1), (v_0 : v_1))$ be bi-homogeneous coordinates on $\mathbb{P}_1 \times \mathbb{P}_1$. A curve $B \subset \mathbb{P}_1 \times \mathbb{P}_1$ of bidegree $(4, 4)$ is given by an equation

$$f(u, v) = \sum_{\substack{i_0 + i_1 = 4 \\ j_0 + j_1 = 4}} c_{i_0 i_1 j_0 j_1} u_0^{i_0} u_1^{i_1} v_0^{j_0} v_1^{j_1} = 0.$$

There is a unique normal surface Y admitting a double covering $\pi : Y \to \mathbb{P}_1 \times \mathbb{P}_1$ ramified precisely over B. If B is nonsingular, we have

$$K_Y = 0$$

by the adjunction formula, and for $e(Y) = \sum_0^4 (-1)^i b_i(Y)$ we find

$$\begin{aligned} e(Y) &= 2e(\mathbb{P}_1 \times \mathbb{P}_1) - e(B) = \\ &= 2 \cdot 4 - (2 - 2g(B)) = 24, \end{aligned}$$

because $g(B) = 9$. So Y is a $K3$-surface by classification of surfaces. If B has at most simple singularities, Brieskorn's simultaneous resolution [2] shows that the minimal desingularisation $\bar{Y}$ of Y is a deformation of a $K3$-surface, hence a $K3$-surface itself. Let $\bar{\pi} : \bar{Y} \to \mathbb{P}_1 \times \mathbb{P}_1$ be the map induced by π.

Now denote by τ the involution

$$((u_0 : u_1), (v_0 : v_1)) \mapsto ((u_1 : u_0), (v_1 : v_0))$$

of $\mathbb{P}_1 \times \mathbb{P}_1$ and assume that $f(u, v)$ is invariant under τ. Since Y can be described in terms of f, the involution τ lifts to Y, such that the points $y \in Y$ lying over the four fixed points

$$((1 : \pm 1), (1 : \pm 1))$$

of τ are invariant. If B does not pass through any of these fixed points, we may compose the lifted involution on Y with the involution interchanging the two sheets to obtain an involution $\sigma : Y \to Y$ without fixed points. σ lifts to an involution $\bar{\sigma}: \bar{Y} \to \bar{Y}$, again without fixed points. So $\bar{Y}/\bar{\sigma} = X$ is an Enriques surface.

Fix one of the projections $\mathbb{P}_1 \times \mathbb{P}_1$, say

$$p_1: ((u_0:u_1), (v_0:v_1)) \mapsto (u_0:u_1).$$

Then $p = p_1 \circ \bar{\pi}: \bar{Y} \to \mathbb{P}_1$ is a fibration of $\bar{Y}$ over $\mathbb{P}_1$. If B is reduced, the general fibre $\bar{Y}_{(u_0:u_1)}$ is a double covering of the line

$$L_{(u_0:u_1)} = p_1^{-1}(u_0:u_1)$$

ramified in the four points of $L_{(u_0:u_1)} \cap B$, hence an elliptic curve. The fibration p commutes with $\bar{\sigma}$, hence induces an elliptic fibration $X \to \mathbb{P}_1$. For $(u_0:u_1) \neq (1:\pm 1)$, the fibre $\bar{Y}_{(u_0:u_1)}$ is isomorphic with its image in X. The two fibres $\bar{Y}_{(1:\pm 1)}$ however are invariant under $\bar{\sigma}$, so they are mapped 2:1 on two 2-fold fibres of X, which are called the *halfpencils*.

To classify the singular fibres of X, it suffices therefore to classify the singular fibres of $\bar{Y}$.

4. Singular fibres and simple singularities of the branch curve B

Here we consider the situation locally above a line on the quadric $\mathbb{P}_1 \times \mathbb{P}_1$. So denote by $L \subset \mathbb{P}_1 \times \mathbb{C}$ the line $\mathbb{P}_1 \times \{0\}$ and let $B \subset \mathbb{P}_1 \times \mathbb{C}$ be a curve satisfying:

a) B has only A–D–E singularities along L,

b) the intersection number $(B.L)$ is 4.

Let $\bar{Y}$ be the minimal desingularization of Y, the double cover of $\mathbb{P}_1 \times \mathbb{C}$ ramified over B. Denote by $\bar{L} \subset \bar{Y}$ the curve mapped flatly onto L. The total inverse image $F \subset \bar{Y}$ of L is a fibre in an elliptic fibration containing $\bar{L}$. We need the relation between the behaviour of B near L and the type of F. This relation should be well-known, but since we do not have a reference, we give it here explicitly.

The result is contained in Table 2. The first column shows the intersection multiplicities in the (at most four) points of $B \cap L$ (resp. $\overline{B \backslash L} \cap L$). The second column gives the singularities of B (resp. $B' = \overline{B \backslash L}$) in these points. Except for one case (marked by an *) the type of F is determined uniquely by these data. For simplicity, a smooth point of B is several times considered as singularity of "type A_0".

PROOF: We check all cases and subcases distinguished by the pattern of intersection numbers.

Table 2.

Case 1: *B does not split off L*

intersection numbers	singularities of B	type of F
1 1 1 1	none	nonsingular
1 1 2	$A_k,\ k \geq 0$	$\tilde{A}_k$
2 2	$A_k, A_l, k, l \geq 0$	$\tilde{A}_{k+l+1}$
1 3	$A_k, k = 0, 1, 2$ D_k E_k	A'_k $\tilde{D}_k$ $\tilde{E}_k$
4	$A_k, k = 0, 1$ $A_k, k \geq 3$ * D_k * D_5 E_6	A'_{k+1} $\tilde{D}_{k+1}$ $\tilde{D}_{k+1}$ $\tilde{E}_6$ $\tilde{E}_7$

* If $k = 5$ and L touches the nonsingular (resp. singular) branch, we have type $\tilde{D}_6$ (resp. $\tilde{E}_6$).

Case 2: *B splits off L, say* $B = B' \cup L$

interaction numbers	singularities of B'	type of F
1 1 1 1	none	$\tilde{D}_4$
1 1 2	$A_k, k \geq 0$	$\tilde{D}_{k+5}$
2 2	$A_k, A_l, k, l \geq 0$	$\tilde{D}_{k+l+6}$
1 3	$A_k, k = 0, 1, 2$	$\tilde{E}_{k+6}$
4	$A_k, k = 0, 1$	$\tilde{E}_{k+7}$

CASE 1:

1 1 2: The curve $\bar{L}$ is irreducible rational. If B is smooth ($k = 0$), $\bar{L}$ has a node over the point where B touches L (type $\tilde{A}_0$). If $k \geq 1$, the fibre is reducible, hence contains smooth curves only, and $\bar{L}$ must meet the A_k-string in $\bar{Y}$ over the singularity of B in two distinct points. This creates a loop and F must be of type $\tilde{A}_k$.

2 2: The curve $\bar{L}$ decomposes in two copies of $\mathbb{P}_1$. Both the trees A_k and A_l over the intersections meet these two curves in distinct points. This creates a loop and F must be of type $\tilde{A}_{k+l+1}$.

1 3: The curve $\bar{L}$ is irreducible and meets the exceptional configuration (over the point where B touches L) in one point. If B is nonsingular, $\bar{L}$ has a cusp. If B has singularities A_1 or A_2, the fibre must be of type A'_1 or A'_2, because it contains only 2, resp. 3 components, and no cycle.

Singularities A_k with $k \geq 3$ cannot occur by Lemma 1 below. If B has a singularity of type E_k, there is only one way to add one vertex for $\bar{L}$ and to arrive at a diagram from Table 1, namely at $\tilde{E}_k$. The situation is analogous for a D_k-singularity except in the case $k = 8$, where the D_k-diagram could be completed to $\tilde{D}_8$ or $\tilde{E}_8$. But blowing up the singularity once, one finds that $\tilde{E}_8$ is impossible.

4: $\bar{L}$ consists of two copies of $\mathbb{P}_1$. If B is smooth, they touch (and F is of type A'_1), if B has a singularity A_1, the three curves cannot form a cycle, hence meet in one point (F is of type A'_2). By Lemma 1 below, B cannot have an A_2 singularity in L. If B has an A_k-singularity, $k \geq 3$, then the two components of $\bar{L}$ intersect the A_k-tree over this singularity in two distinct points. Except for $k = 6, 7$, only a $\tilde{D}_{k+1}$ can be formed to give a configuration from Table 1. But adding two components to a tree A_6, resp. A_7, obtaining an $\tilde{E}_7$, resp. $\tilde{E}_8$, would not be symmetric under the involution interchanging the sheets of $\bar{Y}$ over $\mathbb{P}_1 \times \mathbb{C}$. If the singularity of B has type D_k, the two components of $\bar{L}$ can complete the D_k-configuration to give a fibre of type $\tilde{D}_{k+1}$. This is the only possibility, symmetric under sheet-exchange, unless $k = 5$. But if B has a D_5-singularity, there are indeed two possibilities (the asterisk in Table 2): L touches either the nonsingular or the singular branch. After blowing up once, one finds the former to lead to an $\tilde{D}_6$ fibre, and the latter to type $\tilde{E}_6$. If B has an E_k-singularity, the fibre must be of type $\tilde{E}_{k+1}$, so $k = 6$ or 7. But two vertices cannot be added to E_7 in a symmetric way. The only possibility is E_6 leading to $\tilde{E}_7$.

CASE 2: Here $\bar{L}$ is always irreducible, lying bijectively over L.

1 1 1 1: The branch curve B has four A_1-singularities, each introducing one component. The fibre is of type $\tilde{D}_4$, because it is the only one with 5 components without a cycle.

1 1 2: The curve B has two A_1-singularities and one A_3 (if B' is smooth), resp. D_{k+3} (if B' has an A_k double point). The only way to combine them to a diagram from Table 1 leads to a $\tilde{D}_{k+5}$-fibre.

2 2: Let the singularities of B' on L have type A_k and A_l, $k, l \geq 0$. Then B has singularities of type A_3 (if $k = 0$), resp. D_{k+3}, and A_3 (if $l = 0$), resp. D_{l+3}. Since the two trees resolving them cannot meet in F, there is only one type for F, namely $\tilde{D}_{k+l+6}$, except if $k = 0$, $l = 2$ which case might lead to $\tilde{E}_8$. But this possibility is excluded, because after blowing up once, one finds that $\bar{L}$ must meet the middle curve in any A_3-tree.

1 3: By Lemma 1 below, B' is either smooth near L or has a double point A_1 or A_2. So B has one A_1-singularity and another one of type A_5, D_6, or E_7. The only way to combine their trees, is to form a fibre of type $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$.

4: If B' is smooth, B has on L an A_7-singularity, if B' has an ordinary double point, then B has a D_8. There cannot be other singularities on B', because then B would not have a simple singularity. Blowing up twice in the first case, and once in the second, one checks that F cannot be of type $\tilde{D}_7$, resp. $\tilde{D}_8$. Hence the fibre is of type $\tilde{E}_7$ or $\tilde{E}_8$.

LEMMA 1: *Let $P \in B$ be a singularity of type A_k and C a smooth curve through P.*

i) *if $k \geq 3$, then $i_P(B, C) \neq 3$*

ii) *if $k = 2$, then $i_P(B, C) \neq 4$.*

PROOF: Let σ be the blowing up of P and $E = \sigma^{-1}P$ the exceptional curve. The proper transform $\bar{B}$ of B has a double point A_{k-2} in $\bar{P} \in E$. Let $\bar{C}$ be the proper transform of C. Then

$$i_P(B, C) = 2(E, \bar{C}) + i_{\bar{P}}(\bar{B}, \bar{C}).$$

If $k \geq 3$, then $i_{\bar{P}}(\bar{B}, \bar{C}) = 0$ or ≥ 2. This proves i). If $k = 2$, then E and $\bar{B}$ touch at $\bar{P}$. So, if $i_{\bar{P}}(\bar{B}, \bar{C}) = 2$, then $(E, \bar{C}) \geq 2$ too, a contradiction. This proves ii).

5. The SQH-criterion for A_n-singularities, $n \leq 5$

To detect A_n-singularities we apply a computational technique developed in [4] and [3].

A convergent power series $f(x, y)$ is called *semi-quasihomogeneous* with respect to weights $a, b \in \mathbb{Q}$ if all terms in $f(x^a, y^b)$ have degree ≥ 1. The terms in $f(x^a, y^b)$ of degree 1 determine the part of $f(x, y)$ of weight 1. If this part of weight 1 describes an isolated singularity at the origin, the singularity at 0 with equation $f(x, y) = 0$ is called SQH of weights a, b.

Recognition principle for A_n's [1, 4]: *If a curve singularity is* SQH *with weights $a = \frac{1}{2}$ and $b = \frac{1}{n+1}$, then it is biholomorphically equivalent with the singularity A_n of equation*

$$x^2 + y^{n+1} = 0.$$

We shall apply this to a polynomial

$$f(x, y) = Q(x, y) + C(x, y) + D(x, y) + xy\bar{C}(x, y) + x^2y^2\bar{Q}(x, y),$$

where Q, C, D are homogeneous of degree 2, 3, 4 respectively, and $\bar{P}(x, y) = P(y, x)$ for any polynomial P.

Criterion: *If Q is not a square, then the singularity with equation $f = 0$ is an A_1. If there is a linear form $L(x, y) \neq 0$ with $Q = L^2$, then this singularity is of type*
A_2 if $L \nmid C$,
A_3 if $C = LQ_1$ and $L \nmid D - \frac{1}{4}Q_1^2$,
A_4 if $D - \frac{1}{4}Q_1^2 = LC_1$ and $L \nmid xy\overline{LQ}_1 - \frac{1}{2}Q_1C_1$,
A_5 if $xy\overline{LQ}_1 - \frac{1}{2}Q_1C_1 = LD_1$ and $L \nmid (xy\bar{L})^2 - \frac{1}{4}C_1^2 - \frac{1}{2}Q_1D_1$.

Proof: Assume $Q = L^2$. After interchanging x and y, if necessary, we may assume $\partial L/\partial x \neq 0$. So any coordinate change $x_1 = L +$ (higher order terms), $y_1 = y$ will be biholomorphic.

First, let $x_1 = L$, $y_1 = y$. Then

$$f = L^2 + C + D + xy\bar{C} + x^2y^2\bar{L}^2 = \\ = x_1^2 + c_1y_1^3 + r_1(x_1, y_1)$$

where, for the weights $\frac{1}{2}, \frac{1}{3}$ all terms in r_1 have degree > 1. So f is SQH of weights $\frac{1}{2}, \frac{1}{3}$ if $c_1 \neq 0$, which is the case if $L = x_1$ does not divide C.

If next $C = LQ_1$, then

$$f = L^2 + LQ_1 + D + xy\overline{LQ}_1 + x^2y^2\bar{L}^2 \\ = (L + \tfrac{1}{2}Q_1)^2 + (D - \tfrac{1}{4}Q_1^2) + xy\overline{LQ}_1 + x^2y^3\bar{L}^2.$$

Putting $x_2 = L + \frac{1}{2}Q_1$ and $y_2 = y$ we obtain

$$f = x_2^2 + c_2y_2^4 + r_2(x_2, y_2),$$

where all monomials in r_2 have degree > 1 for the weights $\frac{1}{2}, \frac{1}{4}$. So f is SQH of weights $\frac{1}{2}, \frac{1}{4}$ if $c_2 \neq \theta$. This c_2, the coefficient of y_2^4 in $D - \frac{1}{4}Q_1^2$, as polynomial in x_2 and y_2, is also the coefficient of y^4 in $D - \frac{1}{4}Q_1^2$, as polynomial in L and y, because L and x_2 differ by a quadratic term. So f is SQH of weights $\frac{1}{2}, \frac{1}{4}$ if L does not divide $D - \frac{1}{4}Q_1^2$.

If next $C = LQ_1$ and $D - \frac{1}{4}Q_1^2 = LC_1$, then

$$f = (L + \tfrac{1}{2}Q_1)^2 + LC_1 + xy\overline{LQ}_1 + x^2y^2\overline{L}^2$$
$$= (L + \tfrac{1}{2}Q_1 + \tfrac{1}{2}C_1)^2 + (xy\overline{LQ}_1 - \tfrac{1}{2}Q_1C_1) + (x^2y^2\overline{L}^2 - \tfrac{1}{4}C_1^2).$$

Putting $x_3 = L + \frac{1}{2}Q_1 + \frac{1}{2}C_1$, $y_3 = y$ we obtain

$$f = x_3^2 + c_3y_3^5 + r_3(x_3, y_3).$$

So f is SQH of weights $\frac{1}{2}, \frac{1}{5}$ if $c_3 \neq 0$. Here c_3 is the coefficient of y^5 in $xy\overline{LQ}_1 - \frac{1}{2}Q_1C_1$ as polynomial of L and y. Then $c_3 \neq 0$ if and only if L does not divide this polynomial.

If finally $C = LQ_1$, $D - \frac{1}{4}Q_1^2 = LC_1$, and $xy\overline{LQ}_1 - \frac{1}{2}Q_1C_1 = LD_1$, then

$$f = (L + \tfrac{1}{2}Q_1 + \tfrac{1}{2}C_1)^2 + LD_1 + (x^2y^2\overline{L}^2 - \tfrac{1}{4}C_1^2) =$$
$$= (L + \tfrac{1}{2}Q_1 + \tfrac{1}{2}C_1 + \tfrac{1}{2}D_1)^2 + (x^2y^2\overline{L}^2 - \tfrac{1}{4}C_1^2 - \tfrac{1}{2}Q_1D_1) -$$
$$- \tfrac{1}{2}C_1D_1 - \tfrac{1}{4}D_1^2.$$

Putting $x_4 = L + \frac{1}{2}Q_1 + \frac{1}{2}C_1 + \frac{1}{2}D_1$, $y_4 = y$ we obtain

$$f = x_4^2 + c_4y_4^6 + r_4(x_4, y_4).$$

Arguing as above finds that f is SQH of weights $\frac{1}{2}, \frac{1}{6}$ if L does not divide $(xy\overline{L})^2 - \frac{1}{4}C_1^2 - \frac{1}{2}Q_1D_1$.

6. Examples

On $\mathbb{P}_1 \times \mathbb{P}_1$ we use bihomogeneous coordinates $(u_0{:}u_1)$, $(v_0{:}v_1)$. Since τ interchanges u_0 and u_1 as well as v_0 and v_1, a polynomial $f(u_0, u_1, v_0, v_1)$, bihomogeneous of degree 4, is invariant under τ if it is of the form

$$\begin{aligned} f = {} & a_0(u_0^4v_0^4 + u_1^4v_1^4) + a_1(u_0^3u_1v_0^4 + u_0u_1^3v_1^4) + \\ & + a_2(u_0^2u_1^2v_0^4 + u_0^2u_1^2v_1^4) + a_3(u_0u_1^3v_0^4 + u_0^3u_1v_1^4) + \\ & + a_4(u_1^4v_0^4 + u_0^4v_1^4) + a_5(u_0^4v_0^3v_1 + u_1^4v_0v_1^3) + \\ & + a_6(u_0^3u_1v_0^3v_1 + u_0u_1^3v_0v_1^3) + a_7(u_0^2u_1^2v_0^3v_1 + u_0^2u_1^2v_0v_1^3) + \\ & + a_8(u_0u_1^3v_0^3v_1 + u_0^3u_1v_0v_1^3) + a_9(u_1^4v_0^3v_1 + u_0^4v_0v_1^3) + \\ & + a_{10}(u_0^4v_0^2v_1^2 + u_1^4v_0^2v_1^2) + a_{11}(u_0^3u_1v_0^2v_1^2 + u_0u_1^3v_0^2v_1^2) + \\ & + a_{12}u_0^2u_1^2v_0^2v_1^2. \end{aligned}$$

As line L we choose the line $u_1 = 0$, on which we shall use the points

$$P_1:(1:0)(1:0) \quad P_2:(1:0)(0:1).$$

The curve B with equation $f = 0$ passes through P_1 iff

$$a_0 = 0,$$

which we shall assume from now on.

Since we usually work in P_1, we also need the inhomogeneous expansion of f in coordinates $u = \frac{u_1}{u_0}$ and $v = \frac{v_1}{v_0}$:

$$\begin{aligned} f(u,v) = {} & a_1(u + u^3v^4) + a_2(u^2 + u^2v^4) + a_3(u^3 + uv^4) + \\ & + a_4(u^4 + v^4) + a_5(v + u^4v^3) + a_6(uv + u^3v^3) + \\ & + a_7(u^2v + u^2v^3) + a_8(u^3v + uv^3) + a_9(u^4v + v^3) + \\ & + a_{10}(v^2 + u^4v^2) + a_{11}(uv^2 + u^3v^2) + a_{12}u^2v^2. \end{aligned}$$

Examples of $\tilde{A}_k$-fibres, $k = 0, 1, 2, 3, 5, 7$.

Using the SQH-criterion, one easily writes down conditions on the coefficients of f, guaranteeing that B, the curve with equation $f = 0$, has at P_1 double points of certain types:

conditions		singularity at P_1
$a_5 = 0$	$a_1 \neq 0$	A_0
$a_1 = a_5 = 0$	$4a_2a_{10} \neq a_6^2$	A_1
$a_1 = a_2 = a_5 = a_6 = 0$	$a_3 \neq 0$	A_2
$a_1 = a_2 = a_3 = a_5 = a_6 = 0$	$4a_4a_{10} \neq a_7^2$	A_3
$a_i = 0$ for $i \leq 7$	$a_8 \neq 0$	A_5
$a_i = 0$ for $i \leq 8$	$a_9 \neq 0$	A_7

If $a_{10} \neq 0$ the curve B intersects the line L with equation $u_1 = 0$ at P_1 with multiplicity 2. If a_4 and a_{12} vary independently (the first four cases), the linear system $|B|$ determined by the vanishing conditions on the a_i has no fixed point other than P_1 and τP_1. By Bertini's theorem, the general curve in $|B|$ is nonsingular away from P_1 and τP_1. It intersects L with multiplicities 211, and at P_1 it has the double point A_k, $k = 0, 1, 2, 3$ from the table above. It therefore defines a double covering $Y \to \mathbb{P}_1 \times \mathbb{P}_1$ ramified over B, such that the fibre $F \subset \bar{Y}$ over L has type $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{A}_2$, resp. $\tilde{A}_3$ (Table 2).

If $a_i = 0$ for $i \leq 7$, the linear system $|B|$ has the lines $v_0 \cdot v_1 = 0$ as fixed components. Since a_9 and a_{12} vary independently, the system $|B - \{v_0 v_1 = 0\}|$ has only P_1 and τP_1 as fixed points. So the general curve in $|B|$ is nonsingular, except for P_1 and τP_1 and two ordinary double points on each of the lines $v_0 = 0$, $v_1 = 0$. It determines a surface carrying over L a fibre F of type $\tilde{A}_5$, resp. $\tilde{A}_7$ (Table 2).

Examples of $\tilde{A}_k$-fibres, $k = 4, 6$.

Interchanging v_0 and v_1 leaves in the polynomial f the coefficients a_2, a_7, a_{10}, a_{11}, a_{12} invariant, whereas it permutes a_0 with a_4, a_1 with a_3, a_5 with a_9, and a_6 with a_8. The linear system of curves B with equation $f = 0$ having singularities at P_1 and P_2 therefore is specified by the conditions

$$a_0 = a_1 = a_3 = a_4 = a_5 = a_9 = 0.$$

The inhomogeneous expansion of such an f at P_1 is

$$\begin{aligned} & a_6(uv + u^3v^3) + a_8(u^3v + uv^3) + \\ & + a_2(u^2 + u^2v^4) + a_7(u^2v + u^2v^3) + \\ & + a_{10}(v^2 + u^4v^2) + a_{11}(uv^2 + u^3v^2) + a_{12}u^2v^2. \end{aligned}$$

Its expansion at P_2 is identical, except for the permutation of a_6 and a_8. To apply section 5, we give Q, C, D at P_1 explicitly (to obtain them at P_2, interchange a_6 and a_8):

$$\begin{aligned} Q(u, v) &= a_2u^2 + a_6uv + a_{10}v^2 \\ C(u, v) &= uv(a_7u + a_{11}v) \\ D(u, v) &= uv(a_8u^2 + a_{12}uv + a_8v^2). \end{aligned}$$

Let us put

$$L(u, v) = a_7u + a_{11}v,$$

then P_1 is a singularity of type

A_1 if $4a_2a_{10} \neq a_6^2$,
A_2 if $4a_2a_{10} = a_6^2$, $a_6 \neq 0$, $Q \neq \text{const} \cdot L^2$,
A_3 if $Q = L^2$, $L \nmid D - \frac{1}{4}u^2v^2$.

Fix $a_7 \neq 0$ and $a_{11} \neq 0$ and put

$$a_2 = a_7^2,\ a_8 = -2a_7a_{11},\ a_{10} = a_{11}^2.$$

Any curve with these coefficients has at P_2 a singularity of type A_2. Denote by f_1 a polynomial formed with these coefficients and

$$a_6 \neq \pm a_8,\ a_{12} = 0.$$

Then $f_1 = 0$ has an A_1 at P_1. The curve intersects each of the four lines $u_0u_1v_0v_1 = 0$ in two of the points P_1, P_2, τP_1, τP_2 only. So for general λ, a_{12} the curve B with equation $\lambda f_1 + a_{12}u_0^2u_1^2v_0^2v_1^2 = 0$ is nonsingular, except for the points P_1, P_2, τP_1, and τP_2, where it has A_1, resp. A_2-singularities. The corresponding Enriques surface carries a fibre of type $\tilde{A}_4$ over L (Table 2).

Next denote by f_2 a polynomial with the coefficients a_2, a_7, a_8, a_{10}, a_{11} as above and

$$a_6 = -a_8,\ a_{12} = 0.$$

Again, for general a_{12} the curve B with equation $f_2 + a_{12}u_0^2u_1^2v_0^2v_1^2 = 0$ is nonsingular except for P_1, P_2, τP_1, τP_2. At P_2 it has an A_2, and at P_1 an A_3-singularity, provided that L does not divide

$$D - \tfrac{1}{4}u^2v^2 = uv(a_8u^2 + (a_{12} - \tfrac{1}{4})uv + a_8v^2),$$

which is the case for general a_{12}. So for general a_{12} the curve defines an Enriques surface with a fibre of type $\tilde{A}_6$ over L (Table 2).

Example of an $\tilde{A}_8$-fibre

We put $a_1 = a_3 = a_4 = a_5 = a_9 = 0$ as in the preceding example, and for some $a \in \mathbb{C}^*$ we put

$$a_6 = 2a^2,\ a_8 = -2a^2$$
$$a_2 = a_{10} = a^2,\ a_7 = a_{11} = a$$
$$a_{12} = \tfrac{1}{4} - 4a^2.$$

Then f has at P_1 the inhomogeneous expansion

$$\begin{aligned} f(u,v) = {} & 2a^2(uv + u^3v^3) - 2a^2(u^3v + uv^3) \\ & + a^2(u^2 + u^2v^4) + a(u^2v + u^2v^3) \\ & + a^2(v^2 + u^4v^2) + a(uv^2 + u^3v^2) + (\tfrac{1}{4} - 4a^2)u^2v^2. \end{aligned}$$

At P_2, just as above, the curve B with equation $f = 0$ has a singularity A_2. The SQH-criterion shows that at P_1, this curve has a singularity A_5.

The curve B cannot split off any of the four lines $u_0u_1v_0v_1 = 0$. So B intersects these lines only in the four points P_1, P_2, τP_1, τP_2. So any irreducible component of B passes through one of these points, which is a double point A_n on B. This shows that B is reduced. In any of the four fixed points $(1:\pm 1)$, $(1:\pm 1)$ of τ, the polynomial $f(\pm 1, \pm 1)$ in a has constant term $\frac{1}{4} \neq 0$. So for general a, the curve B does not pass through any of these four points.

Lemma 2 below shows that B has no other singularities than double points. So it defines a double covering leading to an Enriques surface, which has a fibre of type $\tilde{A}_8$ (Table 2).

LEMMA 2: *For any $a \neq 0$, the curve*

$$\{(u,v) \in \mathbb{C}^2 : f(u,v) = 0\}$$

has no singularities of multiplicity ≥ 3.

PROOF: Assume that $(p, q) \neq (0,0)$ is an m-fold point. As $f(u,v) = f(v,u)$, the point (q,p) has multiplicity m too. We distinguish two cases:

1. $p \neq q$: The line $u + v = r$ with $r = p + q$ passes through both the singular points. Then there must be $\alpha \neq \beta \in \mathbb{C}$ such that $(x-\alpha)^m(x-\beta)^m$ divides $f(x, r-x)$, which by inspection is a polynomial of degree 5. For $m \geq 3$ the only possibility is that $f(x, r-x)$ vanishes identically. Its constant term being a^2r^2, this is excluded if $r \neq 0$. But if $r = 0$, we have $f(x, r-x) = \frac{1}{4}x^2$, and arrive at a contradiction again.

2. $p = q \neq 0$. Now $x^2(x-p)^m$ divides

$$f(x,x) = 4a^2x^2g(x),$$

where

$$g(x) = x^4 + \frac{1}{2a}x^3 + \left(\frac{1}{16a^2} - 2\right)x^2 + \frac{1}{2a} + 1.$$

This polynomial g satisfies the identity

$$g(x) = x^4 g\left(\frac{1}{x}\right).$$

So if $m \geq 3$, necessarily $p = \pm 1$, and

$$g(x) = (x \pm 1)^4 = x^4 \pm 4x + 6x^2 \pm 4x^3 + 1.$$

Comparing coefficients we find the contradiction

$$\frac{1}{2a} = \pm 4, \frac{1}{16a^2} - 2 = 6.$$

Examples of A_k'-fibres, $k = 0, 1, 2$

Consider the linear system of curves B defined by equations $f = 0$, where $a_1 = a_5 = a_6 = a_{10} = 0$. Since a_4 and a_{12} vary independently, P_1 and τP_1 are its only base points. By Bertini's theorem the general curve in this system will be nonsingular away from these points. The same holds for any linear system containing the one above. We consider three systems contained in the one defined by $a_5 = a_{10} = 0$. For the general member B always a_9 will be nonzero, i.e., B will intersect L in P_1 with multiplicity 3. The following table gives the definition of the linear systems, the singularity at P_1 of its general members (SQH-test from section 5), and the type of the fibre F lying over L (Table 2):

linear system	singularity	fibre
$a_5 = a_{10} = 0$	A_0	A_0'
$a_1 = a_5 = a_{10} = 0$	A_1	A_1'
$a_1 = a_5 = a_6 = a_{10} = 0$	A_2	A_2'

Example of $\tilde{D}_4$-fibres

The linear system of curves B splitting off L and τL is given by $a_4 = a_5 = a_9 = a_{10} = 0$. As a_3 and a_{12} vary independently, there are no fixed points away from L and τL. The restriction of $B' = B - L$ to L has equation

$$a_1 v_0^4 + a_6 v_0^3 v_1 + a_{11} v_0^2 v_1^2 + a_8 v_0 v_1^3 + a_3 v_1^4 = 0.$$

So the curves B' cut out on L the complete linear system of degree 4. The general curve B in the linear system will be nonsingular away from

L and will have four distinct double points on L. By Table 2, over L will lie a fibre of type $\tilde{D}_4$.

Examples of $\tilde{D}_k$-fibres $k = 5, 6, 7$

Consider the linear system of curves B defined by equations $f = 0$ where

$$a_1 = a_2 = a_4 = a_5 = a_6 = a_7 = a_9 = a_{10} = a_{11} = 0.$$

It splits off L and τL, and the only fixed points of $|B'| = |B - L - \tau L|$ are P_1 and τP_1. As long as $4a_3a_{11} \neq a_8^2$, i.e., $a_8 \neq 0$, the curve B' will intersect L in two distinct points different from P_1. The following table defines three systems containing the one above, shows the singularity of B' at P_1 for the general member B, and gives the type of the fibre over L (Table 2)

linear system	singularity	fibre
$a_1 = a_4 = a_5 = a_6 = a_9 = a_{10} = 0$	A_0	$\tilde{D}_5$
in addition $a_2 = 0$	A_1	$\tilde{D}_6$
in addition $a_7 = a_{11} = 0$	A_2	$\tilde{D}_7$

Examples of $\tilde{D}_8$-fibres

Consider the system $|B| = |B' + L + \tau L|$ defined by $a_1 = a_3 = a_4 = a_5 = a_6 = a_8 = a_9 = a_{10} = 0$. It splits off L and τL, and the system $|B'|$ has the fixed points P_1, P_2, τP_1 and τP_2. As long as a_{11} and a_{12} vary independently, there are no other base points. All curves B' touch L in both points P_1 and P_2, as long as $a_{11} \neq 0$. The inhomogeneous expansion of f/u at P_1 is

$$a_2(u + uv^4) + a_7(uv + uv^3) + a_{11}(v^2 + u^2v^2) + a_{12}uv^2.$$

Since f is invariant under interchanging v_0 and v_1, the expansion at P_2 is the same. So B' is smooth at both points, if $a_2 \neq 0$, but if $a_2 = 0$ and $a_7 = 0$, it carries an A_1 at both points. From Table 2 we find that the fibre over L is of type $\tilde{D}_8$.

Examples of fibres $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$

Consider the linear system of curves $B = L + \tau L + B'$ with equations $f = 0$, where $a_1 = a_4 = a_5 = a_6 = a_9 = a_{10} = a_{11} = 0$. It contains L and τL as fixed components and the curves B' intersect L at P_1 with multiplicity 3 as long as $a_8 \neq 0$. The same holds for the subsystem of curves where additionally $a_2 = a_7 = 0$. Since a_3 and a_{12} vary independently, the system of curves B' has P_1 and τP_1 as its only fixed points. By Bertini's theorem the general curve B is nonsingular away from L.

The following table defines three linear systems, shows the singularity at P_1 of $B' = B - L$ for its general member B (SQH-Test from section 5), and the type of fibre F lying over L (Table 2).

linear system	singularity of B'	fibre
$a_1 = a_4 = a_5 = a_6 = a_9 = a_{10} = a_{11} = 0$	A_0	$\tilde{E}_6$
in addition $a_2 = 0$	A_1	$\tilde{E}_7$
in addition $a_7 = 0$	A_2	$\tilde{E}_8$

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Gerhard Angermüller
Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr.1 1/2
D-8520 Erlangen
Federal Republic of Germany

Wolf Barth
Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr. 1 1/2
D-8520 Erlangen
Federal Republic of Germany