J. S. MILNE

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ZERO CYCLES ON ALGEBRAIC VARIETIES IN NONZERO CHARACTERISTIC: ROJTMAN'S THEOREM

J. S. Milne*

Let X be a smooth projective variety over an algebraically closed field k, and let $CH_0(X)$ be the Chow group of zero cycles on X modulo rational equivalence; further, let Alb_X be the Albanese variety of X and $\mathrm{Alb}(X)$ the group of k-points of Alb_X. A map $\phi: X \to Alb_X$ induces a map $S: CH_0(X) \to \mathrm{Alb}(X)$ whose restriction to the cycle classes of degree zero is independent of ϕ. If we let $M \mapsto M(l)$ denote the functor taking an abelian group to its l-primary component, l a prime, then an important theorem of Rojtman [16] states that

$$S(l): CH_0(X)(l) \to \mathrm{Alb}(X)(l)$$

is an isomorphism for all l not equal to the characteristic of k. Bloch [1], [2] has given two further proofs of the same result. The purpose of this paper is to remove the restriction on l.

THEOREM 0.1: *Let k have characteristic $p \neq 0$; then*

$$S(p): CH_0(X)(p) \to \mathrm{Alb}(X)(p)$$

is an isomorphism.

We can therefore conclude (without restriction on the characteristic of k) that $CH_0(X)$ has a filtration

$$CH_0(X) \supset CH_0(X)_{\deg 0} \supset CH_0(X)_{\mathrm{tors}} \supset 0$$

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with quotients

$$CH_0(X)/CH_0(X)_{\deg 0} \xrightarrow{\approx} \mathbb{Z}$$
$$CH_0(X)_{\deg 0}/CH_0(X)_{\text{tors}} \text{ uniquely divisible}$$
$$CH_0(X)_{\text{tors}} \xrightarrow{\approx} \mathrm{Alb}(X)_{\text{tors}}.$$

In particular, if k is the algebraic closure of a finite field then $CH_0(X)_{\deg 0} \xrightarrow{\approx} \mathrm{Alb}(X)$.

The result provides the following interpretation of the abelian fundamental group $\pi_1(X)^{ab}$ of X: there is an exact sequence

$$0 \to \mathrm{Hom}(\mu_M, NS_X)^* \to \pi_1(X)^{ab} \to T(CH_0(X)) \to 0$$

in which,

μ_M = group scheme of Mth roots of 1, M sufficiently large,

$NS_X = Pic_X/Pic_X^0$, Pic_X the Picard scheme of X and Pic_X^0 the Picard variety,

* denotes the dual of a finite group,

$$T(CH_0(X)) = \varprojlim {}_mCH_0(X),\ {}_mCH_0(X) = \mathrm{Ker}(CH_0(X) \xrightarrow{m} CH_0(X)).$$

The first six sections of the paper are devoted to the proof of (0.1). The proof uses the sheaves K_rO_X of Quillen [15], and in particular the formula $H^n(X_{\text{Zar}}, K_nO_X) = CH_0(X)$, and the sheaves $\nu(r)$ introduced in [11]. The result of K. Kato [8] on the surjectivity of the $d\log$ symbol plays a crucial role in the proof (see 4.1). As a by-product of the proof we obtain the p-part of the following statement: when X is a surface $H^1(X_{\text{Zar}}, K_2O_X)$ modulo its divisible subgroup is isomorphic to $\mathrm{Hom}(\mu_M, NS_X)$ for all $M \gg 0$. (The l-part, $l \neq \mathrm{char}(k)$, is due to Bloch; see [2, Lecture 5].) In the seventh section, we re-interpret the proof so as to point out the similarities with the proof of Bloch (for $l \neq p$) in [2]. The final section is devoted to showing that similar methods can be used to prove that for X a projective smooth surface over a finite field, $CH_0(X)(p)$ is finite.

§1 The Albanese

Let X be a complete variety over an algebraically closed field k.

PROPOSITION 1.1: *For any integer m, there is a canonical isomorphism*

$$H^1(X_{\text{et}}, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\approx} \operatorname{Hom}(\mu_m, Pic_X)$$

PROOF: See, for example, [12, III 4.16].

Assume further that X is smooth. As $\operatorname{Ext}^1_k(\mu_m, A) = 0$ for any abelian variety A [13, p II. 14-2], the sequence

$$0 \to \operatorname{Hom}(\mu_m, Pic^0_X) \to \operatorname{Hom}(\mu_m, Pic_X) \to \operatorname{Hom}(\mu_m, NS_X) \to 0$$

is exact. The e_m-pairing identifies ${}_mAlb_X$ with the Cartier dual of ${}_mPic^0_X$, and therefore

$$\operatorname{Hom}(\mu_m, Pic^0_X) = \operatorname{Hom}(\mu_m, {}_mPic^0_X) = \operatorname{Hom}({}_mAlb_X, \mathbb{Z}/m\mathbb{Z}).$$

Consequently, on taking Hom of the above exact sequence into $\mathbb{Z}/m\mathbb{Z}$ and using (1.1), we obtain an exact sequence

$$0 \to \operatorname{Hom}(\mu_m, NS_X)^* \to H^1(X, \mathbb{Z}/m\mathbb{Z})^* \to {}_m\mathrm{Alb}(X) \to 0 \tag{1.2}$$

Since $(NS_X)_{\text{tors}}$ is a finite group scheme, $\varprojlim_m \operatorname{Hom}(\mu_m, NS_X) = 0$ and $\varinjlim_m \operatorname{Hom}(\mu_m, NS_X) = \operatorname{Hom}(\mu_M, NS_X)$ and $M \gg 0$. Therefore, on passing to the direct and inverse limits in (1.2), we obtain exact sequences

$$0 \to H^1(X_{\text{et}}, \hat{\mathbb{Z}})^* \xrightarrow{\approx} \mathrm{Alb}(X)_{\text{tors}} \to 0 \tag{1.3}$$

$$0 \to \operatorname{Hom}(\mu_M, NS_X)^* \to H^1(X_{\text{et}}, \mathbb{Q}/\mathbb{Z})^* \to T(\mathrm{Alb}(X)) \to 0 \tag{1.4}$$

We remark that the second map in (1.2) can be identified with the map

$$H^1(X, \mathbb{Z}/m\mathbb{Z})^* = \pi_1(X)^{ab}/m\pi_1(X)^{ab} \to \pi_1(Alb_X)/m\pi_1(Alb_X) = {}_m\mathrm{Alb}(X)$$

induced by any map $X \to Alb_X$. From this point of view, it is clear that for any smooth subvariety $i\colon Y \hookrightarrow X$, the diagram

$$\begin{array}{ccc} H^1(Y_{\mathrm{et}}, \mathbb{Z}/m\mathbb{Z})^* & \to & H^1(X_{\mathrm{et}}, \mathbb{Z}/m\mathbb{Z})^* \\ \downarrow & & \downarrow \\ \mathrm{Alb}(Y) & \to & \mathrm{Alb}(X) \end{array} \tag{1.5}$$

commutes (with the obvious horizontal maps).

§2 First reductions

LEMMA 2.1: *For any smooth projective variety X over an algebraically closed field and any integer m, $S: {}_mCH_0(X) \to {}_m\mathrm{Alb}(X)$ is surjective.*

PROOF: If X has dimension one, S is an isomorphism $\mathrm{Pic}(X)_{\mathrm{tors}} \to \mathrm{Jac}(X)_{\mathrm{tors}}$. Otherwise [5, III 7.9] shows that if Y is a hypersurface section of X, then any connected finite covering of X pulls back to a connected covering of Y; in particular $H^1(X_{\mathrm{et}}, \mathbb{Z}/m\mathbb{Z}) \to H^1(Y_{\mathrm{et}}, \mathbb{Z}/m\mathbb{Z})$ is injective. Choose Y to be smooth; then the lemma follows by induction from

$$\begin{array}{ccccc} {}_mCH_0(Y) & \xrightarrow{S} \!\!\!\!\to & {}_m\mathrm{Alb}(Y) & \twoheadleftarrow & H^1(Y, \mathbb{Z}/m\mathbb{Z})^* \\ \downarrow & & \downarrow & & \downarrow\!\!\!\downarrow \\ {}_mCH_0(X) & \xrightarrow{S} & {}_m\mathrm{Alb}(X) & \twoheadleftarrow & H^1(X, \mathbb{Z}/m\mathbb{Z})^* \end{array}$$

LEMMA 2.2: *Suppose that $S: CH_0(X)_{\mathrm{tors}} \to \mathrm{Alb}(X)_{\mathrm{tors}}$ is injective for all smooth projective surfaces X over a given algebraically closed field k; then S is injective for all smooth projective varieties over k.*

PROOF: See [2, 5.2].

For the remainder of the paper, X will be a smooth projective surface over a field k; in §3–§7, k will be algebraically closed, and in §8, k will be finite. The field of rational functions $k(X)$ on X will be denoted by F.

§3 The K-groups

We write K_2F for the second Milnor (equivalently, Quillen) K-group of F. The symbol $(f, g) \mapsto \dfrac{df}{f} \wedge \dfrac{dg}{g} : F^\times \times F^\times \to \Omega^2_{F/k}$ defines a homomorphism $d\log^2 : K_2F \to \Omega^2_{F/k}$.

LEMMA 3.1: *The kernel of* $d\log^2$ *contains* $K_2F(p)$.

PROOF: For any n, let $f_n = (f, 0, \ldots, 0)$ be the multiplicative representative of $f \in F^\times$ in the Witt ring W_nF. If we let $W_n\Omega^2_{F/k}$ denote the de Rham–Witt group [6], then $(f,g) \mapsto \frac{df_n}{g_n} \wedge \frac{dg_n}{g_n} : F^\times \times F^\times \to W_n\Omega^2_{F/k}$ defines a homomorphism $d\log^2_n : K_nF \to W_n\Omega^2_{F/k}$. Thus $z \in K_2F$ gives rise to an element $(d\log^2_n(z)) \in W.\Omega^2_{F/k}$, which is zero if z is killed by a power of p because $W.\Omega^2_{F/k}$ has no p-torsion; in particular, we then have $d\log^2(z) = d\log^2_1(z) = 0$.

Recall [15, §5] that there is an exact sequence of sheaves on X_{Zar}

$$0 \to K_2\mathcal{O}_X \to i_*K_2F \xrightarrow{\alpha} \bigoplus_Z i_*k(Z)^\times \xrightarrow{\beta} \bigoplus_P i_*\mathbb{Z} \to 0 \tag{3.2}$$

in which the sums are over the irreducible curves Z and closed points P on X, and i_*K_2F, $i_*k(Z)^\times$, and $i_*\mathbb{Z}$ denote respectively constant sheaves on X, Z, and P extended by zero to X. The maps α and β are known to have the following descriptions ([7, II 2.4]):

$K_2F \to k(Z)^\times$, $\{f, g\} \mapsto (-1)^{mn}\dfrac{f^n}{g^m}$, $m = \mathrm{ord}_Z(f)$, $n = \mathrm{ord}_Z(g)$;

$k(Z)^\times \to \mathbb{Z}$, $f \mapsto \sum_{\pi(Q)=P} \mathrm{ord}_Q(f \circ \pi)$, $\pi : \tilde{Z} \to Z$ the normalization of Z. As $k(Z)^\times$ and $\mathbb{Z}$ have no p-torsion, $K_2\mathcal{O}_X(p) = K_2F(p)$ and

$$0 \to K_2\mathcal{O}_X/K_2\mathcal{O}_X(p) \to K_2F/K_2F(p) \to \oplus\, i_*k(Z)^\times \to \oplus\, i_*\mathbb{Z} \to 0$$

is exact. Moreover (3.1) shows that $d\log^2$ factors through $K_2F/K_2F(p)$. These two remarks allow us to replace $K_2\mathcal{O}_X$ and K_2F by $K_2\mathcal{O}_X/K_2\mathcal{O}_X(p)$ and $K_2F/K_2F(p)$ in the rest of the paper: henceforth we may assume $K_2\mathcal{O}_X$ and K_2F have no p-torsion.

§4 The sheaves $\nu(r)$

As in [11], we define for a scheme (or ring) S of characteristic p sheaves $\nu(r)_S = \mathrm{Ker}(C^{-1} - 1 : \Omega^r_S \to \Omega^r_S/d\Omega^{r-1}_S)$, where C^{-1} is the inverse Cartier operator. For example, if $\{x, y\}$ is a p-basis for F over F^p, so that $\Omega^2_{F/k} = F\,dx \wedge dy$, then

$$\nu(2)_F = \left\{ f\frac{dx}{x} \wedge \frac{dy}{y} \,\middle|\, (f^p - f)\frac{dx}{x} \wedge \frac{dy}{y} \in d\Omega^1_{F/k} \right\}.$$

Clearly $d\log^2$ factors through $\nu(2)_F$. The following is a special case of a result of Kato.

PROPOSITION 4.1: *The map* $d\log^2 : K_2F \to \nu(2)$ *is surjective* (*as a map of groups*).

PROOF: Since the proof in [8, §1] simplifies substantially in our special case, we include it. For any finite separable extension F' of F there is a commutative diagram

$$\begin{tikzcd} K_2F' \arrow[r, "d\log^2"] \arrow[d, "\mathrm{norm}"'] & \nu(2)_{F'} \arrow[d, "\mathrm{trace}"] \\ K_2F \arrow[r, "d\log^2"] & \nu(2)_F, \end{tikzcd}$$

from which it follows that we can replace F in the proof by an extension field having no finite extension of degree prime to p.

Let $\omega = f\dfrac{dx}{x} \wedge \dfrac{dy}{y}$, $f \in F$, be such that $(f^p - f)\dfrac{dx}{x} \wedge \dfrac{dy}{y} \in d\Omega^1_{F/k}$; we have to show that ω is logarithmic.

LEMMA 4.2: *There exists* $g \in F^p[x]^\times$ *such that* $g^i\omega \in d\Omega^1_{F/k}$ *for* $i = 1, \ldots, p-1$.

PROOF: Let $E = F^p[x]$. The space $\Omega^2_{F/k}/d\Omega^1_{F/k}$ has dimension 1 over F^p $\left(\text{with basis } \dfrac{dx}{x} \wedge \dfrac{dy}{y}\right)$, and so the F^p-linear map $\phi : E \to \Omega^2/d\Omega^1$, $z \mapsto z\omega \pmod{d\Omega^1}$ has a kernel of dimension $p-1$. The F^p-bilinear pairing

$$(g, \eta) \mapsto g\eta : E \times \Omega^1_{E/F^p} \to \Omega^1_{E/F^p}/dE \quad (\approx F^p)$$

is non-degenerate, and so there exists an $\eta \in \Omega^1_{E/F^p}$ such that $\mathrm{Ker}(\phi) = \{z \mid z\eta \in dE\}$. Write $\eta = a\dfrac{dx}{x}$, $a = \sum_{0 \le i \le p-1} a_i^p x^i$, $a_i \in F$. If $a_0 = 0$, then $\eta \in dE$ and $\omega \in d\Omega^1_{F/k}$, which is impossible because then $f^p\dfrac{dx}{x} \wedge \dfrac{dy}{y}$ would be in $d\Omega^1_{F/k}$. Thus $a_0 \neq 0$, and we can choose a $u \in F^\times$ such that $u^p a = ua_0$. Then $C(u^p\eta) = uC(\eta) = ua_0\dfrac{dx}{x} = u^p\eta$, and so, by [3, 2.6, Proposition 7], $u^p\eta$ is logarithmic, say $u^p\eta = \dfrac{dg}{g}$ with $g \in E^\times$. This g has the property that $g^i \in \mathrm{Ker}(\phi)$, $i = 1, \ldots, p-1$.

Clearly, $g \notin F^p$, and so $\{g, y\}$ forms a p-basis for F over F^p. Therefore we can write $\omega = f'\dfrac{dg}{g} \wedge \dfrac{dy}{y}$, $f' \in F$. Let $f' = \sum_{0 \le i,j \le p-1} a_{ij}^p g^i y^j$, $a_{ij} \in F$; then $a_{ij} = 0$ when $j = 0$, $i \neq 0$, for otherwise we could not have $g^i\omega = g^i f'\dfrac{dg}{g} \wedge \dfrac{dy}{y}$ in $d\Omega^1_{F/k}$ for $i = 1, \ldots, p-1$. Moreover, $\omega = f'\dfrac{dg}{g} \wedge \dfrac{dy}{y} \in \operatorname{Ker}(C^{-1} - 1)$ implies $f'^p = a_{00}^p$. Consequently if we regard $f'\dfrac{dg}{g}$ as an element of $\Omega^1_{F/E}$, then

$$C\left(f'\frac{dy}{y}\right) = C\left(a_{00}^p\frac{dy}{y} + \sum_{j \neq 0} a_{ij}^p g^i y^j \frac{dy}{y}\right) = a_{00}\frac{dy}{y} = f'\frac{dy}{y}.$$

Thus [3, 2.6, Proposition 7] shows there exists $h \in F^\times$ such that $f'\dfrac{dy}{y} = \dfrac{dh}{h}$ in $\Omega^1_{F/E}$, i.e., such that $f'\dfrac{dy}{y} = \dfrac{dh}{h} + \eta'$ with $\eta' \in \Omega^1_{E/F^p}$. Hence $\omega = \dfrac{dg}{g} \wedge \dfrac{dh}{h} + \dfrac{dg}{g} \wedge \eta' = \dfrac{dg}{g} \wedge \dfrac{dh}{h}$.

PROPOSITION 4.3: *There is an exact commutative diagram of sheaves on* X_{Zar},

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bigoplus_Z i_* k(Z)^\times & \xrightarrow{\beta} & \bigoplus_P i_*\mathbb{Z} & \to & 0 \\
 & & & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\
0 \to K_2\mathcal{O}_X & \to & i_*K_2F & \xrightarrow{\alpha} & \bigoplus_Z i_* k(Z)^\times & \xrightarrow{\beta} & \bigoplus_P i_*\mathbb{Z} & \to & 0 \\
\downarrow & & \downarrow{\scriptstyle d\log^2} & & \downarrow{\scriptstyle d\log^1} & & \downarrow{\scriptstyle d\log^0} & & \\
0 \to \nu(2)_X & \to & i_*\nu(2)_F & \xrightarrow{\alpha_1} & \bigoplus_Z i_*\nu(1)_{k(Z)} & \xrightarrow{\beta_1} & \bigoplus_P i_*\mathbb{Z}/p\mathbb{Z} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & &
\end{array}$$

PROOF: Define α_1 and β_1 by the following maps:

$$\nu(2)_F \to \nu(1)_{k(Z)}, \frac{df}{f} \wedge \frac{dg}{g} \mapsto \operatorname{ord}_Z(g)\frac{df}{f} - \operatorname{ord}_Z(f)\frac{dg}{g};$$

$$\nu(1)_{k(Z)} \to \mathbb{Z}/p\mathbb{Z}, \frac{df}{f} \mapsto \sum_Q \operatorname{ord}_Q(f \circ \pi) \pmod p.$$

It is clear that the lower two squares commute (and hence that α_1 is well-defined). The map $K_2O_X \to \nu(2)_X$ is defined so as to make the left hand square commute. Clearly the columns are exact and we have already noted that the first two rows are exact. A diagram chase now shows that the bottom row is exact.

Let $\bar{F}$ be the separable algebraic closure of F; then $\nu(2)_{\bar{F}}$ is a module over the Galois group $\mathrm{Gal}(\bar{F}/F)$.

LEMMA 4.4: $H^i(F, \nu(2)_{\bar{F}}) = 0$, $i > 0$.

PROOF: As the p-cohomological dimension of F is 1, we only have to show that $H^1(F, \nu(2)_{\bar{F}}) = 0$. From the exact sequence

$$0 \to \nu(2)_{\bar{F}} \to \Omega^2_{\bar{F}/k} \xrightarrow{C-1} \Omega^2_{\bar{F}/k} \to 0$$

we see that $H^1(F, \nu(2)_{\bar{F}}) = \mathrm{coker}(C-1)$. Let $\omega = f\dfrac{dx}{x} \wedge \dfrac{dy}{y}$ and let $g \in \bar{F}$ be such that $g - g^p = f$; then $(C-1)g^p\dfrac{dx}{x} \wedge \dfrac{dy}{y} = \omega$, and so the class of ω dies in $H^1(F', \nu(2)_{\bar{F}})$, $F' = F[g]$. It therefore suffices to show that $H^1(G, \nu(2)_{F'}) = 0$, F'/F Galois of degree p with Galois group G. For this we can show instead [17, IX.4] that $0 = H^2(G, \nu(2)_{F'}) = \nu(2)_F/\mathrm{trace}(\nu(2)_{F'})$. The diagram at the start of the proof of (4.1) shows that this would follow from knowing that norm: $K_2F' \to K_2F$ is surjective. But F is a C_2-field [9], and therefore [7, I.4, Prop. 1] proves the surjectivity.

PROPOSITION 4.5: *The canonical map* $H^i(X_{\mathrm{Zar}}, \nu(2)) \to H^i(X_{\mathrm{et}}, \nu(2))$ *is an isomorphism for all* i.

PROOF: The bottom row of the diagram in (4.3) can be regarded as a resolution of $\nu(2)$ on X_{et}, and it will suffice to show that each of its terms is acyclic for the étale topology. Let i denote $\mathrm{spec}\, F \hookrightarrow X$; then (4.4) (applied to an X' étale over X) shows that $R^j i_* \nu(2) = 0$ for all $j > 0$, and so the Leray spectral sequence shows that $H^i(X_{\mathrm{et}}, i_*\nu(2)_F) \simeq H^i(F, \nu(2)_{\bar{F}}) = 0$ for $i > 0$. The remaining two terms can be treated similarly (but more easily).

§5 Duality

Set $\nu_n(r) = K_rO_X/p^nK_rO_X$, regarded as a sheaf on X_{Zar} (or X_{et}); then $d\log^2$ defines maps $\nu_n(2) \twoheadrightarrow \nu(2)$.

LEMMA 5.1: *The map* $H^i(X_{\mathrm{Zar}}, \nu_1(2)) \to H^i(X_{\mathrm{Zar}}, \nu(2))$ *is surjective if* $i = 0$ *and is bijective if* $i > 0$.

PROOF: It follows from (4.3) that there is an exact sequence

$$0 \to C \to \nu_1(2) \to \nu(2) \to 0$$

with $C = \mathrm{Ker}(i_* K_2 F \to i_* \nu(2)_F)/p\, i_* K_2 F$ a constant sheaf.

Let $\eta: H^2(X_{\mathrm{et}}, \nu(2)) \overset{\approx}{\to} \mathbb{Z}/p\mathbb{Z}$ be the map induced by the trace map $H^2(X_{\mathrm{et}}, \Omega^2_{X/k}) = H^2(X_{\mathrm{Zar}}, \Omega^2_{X/k}) \overset{\approx}{\to} k$. Define η_n to be the composite of the maps

$$H^2(X_{\mathrm{et}}, \nu_n(2)) \to H^2(X_{\mathrm{et}}, \nu_{(n)}(2)) \xrightarrow{\eta(n)} \mathbb{Z}/p^n\mathbb{Z}$$

(see [11, p. 199]); then

$$\begin{array}{ccc} H^2(X_{\mathrm{et}}, \nu_n(2)) & \xrightarrow{\eta_n} & \mathbb{Z}/p^n\mathbb{Z} \\ \downarrow & & \downarrow \\ H^2(X_{\mathrm{et}}, \nu(2)) & \xrightarrow{\eta} & \mathbb{Z}/p\mathbb{Z} \end{array}$$

commutes. An easy case of [11, Thm 2.4] states that

$$H^{2-i}(X_{\mathrm{et}}, \mathbb{Z}/p\mathbb{Z}) \times H^i(X_{\mathrm{et}}, \nu(2)) \to H^2(X_{\mathrm{et}}, \nu(2) \overset{\approx}{\to} \mathbb{Z}/p\mathbb{Z}$$

is a non-degenerate pairing of finite groups.

PROPOSITION 5.2: *Consider*

$$\begin{array}{ccc} & H^i(X_{\mathrm{Zar}}, \nu_n(2)) & \\ & \downarrow & \\ H^{2-i}(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}) \times & H^i(X_{\mathrm{et}}, \nu_n(2)) & \to H^2(X_{\mathrm{et}}, \nu_n(2) \xrightarrow{\eta_n} \mathbb{Z}/p^n\mathbb{Z}. \end{array}$$

The induced map of finite groups $\phi^i_n: H^i(X_{\mathrm{Zar}}, \nu_n(2)) \to H^{2-i}(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})^*$ *is an isomorphism for* $i \geq 1$ *and is surjective for* $i = 0$.

PROOF: For $n = 1$ the proposition follows from (5.1), (4.5), and the above-mentioned duality. The general case is proved by induction on n, using the exact sequence $0 \to \nu_{n-1}(2) \to \nu_n(2) \to \nu_1(2) \to 0$ and the five-lemma.

§6 Completion of the proof

The sequence $0 \to K_2O_X \xrightarrow{p^n} K_2O_X \to \nu_n(2) \to 0$ gives rise to a boundary map $H^1(X_{\mathrm{Zar}}, \nu_n(2)) \to H^2(X_{\mathrm{Zar}}, K_2O_X)_{\mathrm{tors}} = CH_0(X)_{\mathrm{tors}}$.

LEMMA 6.1: *The diagram*

$$\begin{array}{ccc} H^1(X_{\mathrm{Zar}}, \nu_n(2)) & \to & CH_0(X)_{\mathrm{tors}} \\ \downarrow \phi_n^1 & & \downarrow s \\ H^1(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})^* & \to & \mathrm{Alb}(X) \end{array}$$

commutes.

PROOF: Let $i: Y \hookrightarrow X$ be a smooth curve on X. From the exact sequence

$$0 \to \nu_n(2)_X \to i_*\nu_n(2)_F \to \bigoplus_Z i_*\nu_n(1)_{k(Z)} \to \bigoplus_P i_*\nu_n(0) \to 0$$

we obtain an isomorphism $\nu_n(1)_Y \xrightarrow{\approx} \underline{H}^1_Y(X_{\mathrm{Zar}}, \nu(2)_X)$, and hence a map i_*

$$H^0(Y, \nu_n(1)) \xrightarrow{\approx} H^0(Y, \underline{H}^1_Y(\nu_n(2)) \xrightarrow{\approx} H^1_Y(X, \nu_n(2)) \to H^1(X, \nu_n(2)).$$

Consider the diagrams

$$\begin{array}{ccccc} H^0(Y_{\mathrm{Zar}}, \nu_n(1)) & \xrightarrow{\approx} & H^1(Y_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})^* & \to & \mathrm{Alb}(Y) \\ \downarrow i_* & \mathrm{a} & \downarrow & \mathrm{b} & \downarrow \\ H^1(X_{\mathrm{Zar}}, \nu_n(2)) & \xrightarrow{\approx} & H^1(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})^* & \to & \mathrm{Alb}(X) \end{array}$$

$$\begin{array}{ccccc} H^0(Y, \nu_n(1)) & \to & H^1(Y, O_Y^\times)_{\mathrm{tors}} & \xrightarrow{s} & \mathrm{Alb}(Y) \\ \downarrow i_* & \mathrm{c} & \downarrow & \mathrm{d} & \downarrow \\ H^1(X, \nu_n(2)) & \to & H^2(X, K_2O_X)_{\mathrm{tors}} & \xrightarrow{s} & \mathrm{Alb}(X) \end{array}$$

We observed in §1 that the square (b) commutes, and it is straightforward to verify that the remaining squares commute. (For (a) it is useful to note that $\nu(1)_Y \to \underline{H}^1_Y(\nu(2))$ is induced by the map $\Omega^1_Y \to \underline{H}^1_Y(\Omega_X)$ [1] of [4, III 8.1]; for (c) and (d) one notes the middle vertical map can be defined by the same process as i_*, and is equal to the obvious map $CH_0(Y)_{\mathrm{tors}} \to CH_0(X)_{\mathrm{tors}}$.) To complete the proof we use that the composed maps in the top rows of the two diagrams are equal [10, Appendix].

We may now complete the proof of Theorem (0.1). On passing to the direct limit in the diagram

$$\begin{array}{ccccccccc} 0 \to & H^1(X_{\mathrm{Zar}}, K_2O_X)/p^nH^1(X, K_2O_X) & \to & H^1(X, \nu_n(2)) & \to & {}_{p^n}CH_0(X) & \to 0 \\ & \downarrow & & \approx\downarrow \phi_n^1 & & \downarrow S & \\ 0 \to & \mathrm{Hom}(\mu_{p^n}, NS_X)^* & \to & H^1(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})^* & \to & {}_{p^n}\mathrm{Alb}(X) & \to 0 \end{array}$$

we obtain a diagram

$$\begin{array}{ccccccc} 0 \to \varinjlim H^1(K_2O_X)/p^nH^1(K_2O_X) & \to & \varinjlim H^1(X, \nu_n(2)) & \to & CH_0(X)(p) & \to 0 \\ & & \approx\downarrow & & \downarrow S & \\ & 0 \to & H^1(X_{\mathrm{et}}, \mathbb{Z}_p)^* & \overset{\approx}{\to} & \mathrm{Alb}(X)(p) & \to 0 \end{array}$$

Thus S is injective, and is therefore an isomorphism. On returning to the first diagram, we now see that

$$H^1(X_{\mathrm{Zar}}, K_2O_X)/p^nH^1(X, K_2O_X) \to \mathrm{Hom}(\mu_{p^n}, NS_X)^*$$

is an isomorphism. Thus $H^1(X, K_2O_X)$ modulo its p-divisible subgroup is finite, and is isomorphic to $\mathrm{Hom}(\mu_{p^M}, NS_X)^*$, any $M \gg 0$.

§7 Comments on the proof

We show that, when appropriately interpreted, the above proof is quite similar to the proof (for $l \neq \mathrm{char}(k)$) in [2]. (In fact, except for the proof of (4.1), the above proof pre-dates and helped suggest that in [2]).

Write $H^i(Y_{\mathrm{fl}}, \mu_p^{\otimes r}) = H^{i-r}(Y_{\mathrm{et}}, \nu(r))$. This notation is suggested by the facts that these groups behave similarly to the groups $H^i(Y_{\mathrm{et}}, \mu_l^{\otimes r})$, $l \neq \mathrm{char}(k)$, and that for $r = 0,1$ they are in fact the cohomology groups of $\mu_p^{\otimes 0} = \mathbb{Z}/p\mathbb{Z}$ and $\mu_p^{\otimes 1} = \mu_p$. The essential ingredients of the two proofs are the following.

(7.1) The canonical maps $K_2F \to H^2(F, \mu_l^{\otimes 2})$ are surjective (Galois cohomology $l \neq \mathrm{char}(k)$, "flat" cohomology $l = \mathrm{char}(k)$); see [2, 5.7] and (4.1).

(7.2) There is an exact sequence of sheaves on X_{Zar},

$$0 \to \mathscr{H}^2(\mu_l^{\otimes 2}) \to i_*H^2(F, \mu_l^{\otimes 2}) \to \bigoplus_Z i_*H^1(k(Z), \mu_l) \to \bigoplus_P i_*H^0(k(P), \mathbb{Z}/l\mathbb{Z}) \to 0$$

where $\mathscr{H}^2(\mu_l^{\otimes 2})$ is the sheaf associated with $U \mapsto H^2(U, \mu_l^{\otimes 2})$ (étale cohomology $l \neq \mathrm{char}(k)$, "flat" cohomology $l = \mathrm{char}(k)$); the maps are compatible with those on the Gersten–Quillen sequence (3.2); cf. [2, 4.17, p 5.8]; see (4.3)

(7.3) There is a duality theorem

$$H^r(X, \mathbb{Z}/l\mathbb{Z}) \times H^{4-r}(X, \mu_l^{\otimes 2}) \to H^4(X, \mu_l^{\otimes 2}) \approx \mathbb{Z}/l\mathbb{Z},$$

and the diagram

$$\begin{array}{ccccc} H^1(\mathscr{H}^2(\mu_l^{\otimes 2})) & \leftarrow & H^1(K_2O_X/lK_2O_X) & \to & {}_lH^2(K_2O_X) \\ \downarrow & & & & s\downarrow \\ H^3(X, \mu_l^{\otimes 2}) & \to & H^1(X, \mathbb{Z}/l\mathbb{Z})^* & \to & {}_l\mathrm{Alb}(X) \end{array}$$

commutes; cf. [2, 5.6]; see (6.1).

§8 Case of a surface over a finite field

In this section we let X be a smooth projective surface over a finite field k; $\bar{k}$ denotes the algebraic closure of k and $\bar{X} = X \otimes_k \bar{k}$.

THEOREM 8.1: *The group $CH_0(X)(p)$ is finite.*

LEMMA 8.2: *There is a canonical injective map $H^1(X_{\mathrm{Zar}}, \nu_n(2)) \to H^2(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})^*$.*

PROOF: The argument in (5.1) shows that

$$H^i(X_{\mathrm{Zar}}, \nu_1(2)) \to H^i(X_{\mathrm{Zar}}, \nu(2))$$

is surjective for $i = 0$ and bijective for $i > 0$.

It is obvious that

$$H^i(X_{\mathrm{Zar}}, \nu(2)) \to H^i(X_{\mathrm{et}}, \nu(2))$$

is an isomorphism for $i = 0$ and is injective for $i = 1$.

The duality theorem [10, 1.9] shows that there is an isomorphism

$$H^i(X_{\mathrm{et}}, \nu(2)) \to H^{3-i}(X_{\mathrm{et}}, \mathbb{Z}/p\mathbb{Z})^*$$

for all i.

On combining these statements, we find that

$$H^i(X_{\mathrm{Zar}}, \nu_1(2)) \to H^{3-i}(X_{\mathrm{et}}, \mathbb{Z}/p\mathbb{Z})^*$$

is surjective for $i = 0$ and injective for $i = 1$.

A five-lemma argument, using induction on n, now completes the proof (cf. (5.2)).

LEMMA 8.3: *The order of $H^2(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})$ is bounded, independently of n.*

PROOF: Since $H^2(X_{\mathrm{et}}, \mathbb{Z}/p\mathbb{Z})$ is finite (finiteness theorem in étale cohomology) it suffices to show that $H^2(X_{\mathrm{et}}, \mathbb{Z}_p) \stackrel{\mathrm{def}}{=} \varprojlim H^2(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})$ is finite. Let $\Gamma \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$ have canonical generator σ and, for M a Γ-module, write M_Γ and M^Γ respectively for the kernel and cokernel of $\sigma - 1 : M \to M$. The Hochschild–Serre spectral sequence for $\bar{X}/X$ gives exact sequences

$$0 \to H^1(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma \to H^2(X, \mathbb{Z}/p^n\mathbb{Z}) \to H^2(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})^\Gamma \to 0,$$

which in the limit become

$$0 \to H^1(\bar{X}, \mathbb{Z}_p)_\Gamma \to H^2(X, \mathbb{Z}_p) \to H^2(\bar{X}, \mathbb{Z}_p)^\Gamma \to 0$$

The eigenvalues of σ acting on $H^r(\bar{X}, \mathbb{Z}_p)$ are the eigenvalues of the Frobenius automorphism on $H^r(\bar{X}, \mathbb{Q}_l)$ that are p-adic units [6]. Consequently, the Riemann hypothesis shows that $H^1(\bar{X}, \mathbb{Z}_p)_\Gamma$ and $H^2(\bar{X}, \mathbb{Z}_p)^\Gamma$ are finite.

The theorem now follows from considering the diagram

$$\begin{array}{c} H^1(X_{\mathrm{Zar}}, K_2O_X) \to H^1(X_{\mathrm{Zar}}, \nu_n(2)) \to {}_{p^n}H^2(X, K_2O_X) \to 0 \\ \hookdownarrow \\ H^2(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})^*. \end{array}$$

COROLLARY 8.4: *Assume X has no global differential 2-form fixed by the Cartier operator; then the canonical map*

$$CH_0(X) \to \pi_1(X)^{ab}$$

has a dense image and a kernel that is uniquely divisible by p.

PROOF: The canonical map sends the class of a point to the Frobenius automorphism corresponding to the point. Since these Frobenius elements are well-known to be dense, the map has dense image. In particular, the map

$$CH_0(X)/p^nCH_0(X) \to \pi_1(X)^{ab}/p^n\pi_1(X)^{ab}$$

is surjective, and to complete the proof of the corollary it suffices to show that the orders of these groups satisfy

$$[CH_0(X)/p^nCH_0(X)] \leq [\pi_1(X)^{ab}/p^n\pi_1(X)^{ab}].$$

From the last diagram in the proof of (8.1) we see that

$$[{}_{p^n}CH_0(X)] \leq [H^2(X_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z})]$$

The sequences

$$0 \to H^r(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma \to H^{r+1}(X, \mathbb{Z}/p^n\mathbb{Z}) \to H^{r+1}(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})^\Gamma \to 0$$

show that

$$\begin{aligned}[H^2(X, \mathbb{Z}/p^n\mathbb{Z})] &= [H^1(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma][H^2(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})^\Gamma] \\ &= [H^1(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})^\Gamma][H^2(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma] \\ &= ([H^1(X, \mathbb{Z}/p^n\mathbb{Z})]/[H^0(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma])[H^2(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma].\end{aligned}$$

Thus

$$p^n[H^2(X, \mathbb{Z}/p^n\mathbb{Z})] = [H^1(X, \mathbb{Z}/p^n\mathbb{Z})][H^2(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma],$$

and we can conclude that

$$\begin{aligned}[CH_0(X)/p^nCH_0(X)] &= p^n[{}_{p^n}CH_0(X)] \\ &\leq [\pi_1(X)^{ab}/p^n\pi_1(X)^{ab}][H^2(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma].\end{aligned}$$

We now use the hypothesis on X to show that $H^2(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})_\Gamma = 0$. By induction, it suffices to prove this for $n = 1$, but $H^2(\bar{X}, \mathbb{Z}/p\mathbb{Z})_\Gamma$ is dual to $H^0(\bar{X}, \nu(2))^\Gamma = H^0(X, \nu(2)) = \operatorname{Ker}(C - 1 : H^0(X, \Omega^2) \to H^0(X, \Omega^2))$.

REMARK 8.5: Of course it is to be expected that, even without the hypothesis on X, $CH_0(X) \to \pi_1(X)^{ab}$ is injective with dense image. (We note that Paršin has withdrawn his claim, made in the announcement [14, Cor. to Thm. 6], to be able to prove this; in fact, as Kato has pointed out, even the statement of Theorem 6 is false.)

Note (added 2-2-82)

(a) K. Kato and S. Saito have announced a proof that, for any smooth projective surface over a finite field, the canonical map $CH_0(X) \to \pi_1(X)^{ab}$ is injective (cf. (8.4) and (8.5)).

(b) S. Bloch and O. Gabber have announced proofs that

$$d\log^n : K_nF/pK_nF \to \nu(n)_F$$

is injective, where K_nF is the Milnor K-group, (at least) in the case that F is a field of characteristic p. Thus the sheaf C occurring in the proof of (5.1) is zero.

(c) S. Bloch has pointed out to the author that some of the above arguments can be extended to prove the following result: Let X be a smooth projective variety over an algebraically closed field k of characteristic p, and assume that $\Gamma(X, d\Omega^2_{X/k}) = 0$; then ${}_pCH^2(X)$ and $H^1(X_{\mathrm{Zar}}, K_2\mathcal{O}_X)/pH^1(X_{\mathrm{Zar}}, K_2\mathcal{O}_X)$ are both finite.

To see this, consider the exact sequence

$$0 \longrightarrow \Omega^2_{\mathrm{cl}} \longrightarrow \Omega^2 \xrightarrow{\ d\ } d\Omega^2 \longrightarrow 0.$$

From its cohomology sequence and the assumption, we deduce that

$$H^0(X, \Omega^2_{\mathrm{cl}} \xrightarrow{\ i\ } H^0(X, \Omega^2)$$

is an isomorphism and $H^1(X, \Omega^2_{\mathrm{cl}}) \to H^1(X, \Omega^2)$ is injective (Zariski cohomology). Standard arguments now show that

$$\mathrm{Ker}(C - i : H^1(X, \Omega^2_{\mathrm{cl}}) \to H^1(X, \Omega^2)$$

is finite and

$$\mathrm{Coker}(C - i : H^0(X, \Omega^2_{\mathrm{cl}}) \to H^0(X, \Omega^2))$$

is zero. Recall [11, 1.5] that $\nu(2)$ can also be defined as

$$\mathrm{Ker}(C - 1 : \Omega^2_{\mathrm{cl}} \to \Omega^2).$$

Let $Q = \mathrm{Coker}(C - 1)$ (for the Zariski topology) and consider

$$\begin{tikzcd} \phi \arrow[r] & \nu(2) \arrow[r] & \Omega^2_{\mathrm{cl}} \arrow[r, "C-1"] \arrow[dr, "j"'] & \Omega^2 \arrow[d, phantom, "\circlearrowleft"] \\ & & & Q \end{tikzcd}$$

It follows from the above that $\operatorname{Coker} H^0(j)$ and $\operatorname{Ker} H^1(j)$ are finite, and this shows that $H^1(X_{\mathrm{Zar}}, \nu(2))$ is finite. As in (5.1) one shows that

$$H^1(X_{\mathrm{Zar}}, \nu_1(2)) \xrightarrow{\approx} H^1(X_{\mathrm{Zar}}, \nu(2))$$

(for many results in the paper, it is not necessary to assume that X is a surface) and now the exact sequence

$$H^1(X_{\mathrm{Zar}}, K_2O_X) \xrightarrow{p} H^1(X_{\mathrm{Zar}}, K_2O_X) \longrightarrow H^1(X_{\mathrm{Zar}}, \nu(2)) \longrightarrow {}_pCH^2(X) \longrightarrow 0$$

gives the result.

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Department of Mathematics
The University of Michigan
Ann Arbor, MI 48109
U.S.A.