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VECTOR BUNDLES ON THE CONE OVER A CURVE

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Let $X \subset \mathbb{P}^n_k$ be a projectively normal curve over an algebraically closed field k, and let $C(X) \subset \mathbb{A}^{n+1}$ be the affine cone over X. The problem studied in this paper is to determine whether $K_0(C(X)) = \mathbb{Z}$, where K_0 denotes the Grothendieck group of vector bundles on $C(X)$ (see [2] for definitions). This is an important special case of a question raised by Murthy, as to whether $K_0(A) = \mathbb{Z}$ for any normal graded ring $A = \bigoplus_{n \geq 0} A_n$, finitely generated over $A_0 = k$, where k is a field (see Bass [1]). Spencer Bloch recently showed that $K_0(A) \neq \mathbb{Z}$ for $A = \mathbb{C}[X, Y, Z]/(Z^2 - X^3 - Y^7)$ giving a counterexample to Murthy's question. However, one still suspected that the result would be true for cones. Partial positive results were known (see Varley [3]).

It turns out that the problem has a very different flavour in characteristic 0 than in positive characteristics. First consider the case of characteristic $p > 0$. We have

THEOREM 1: *Let* $A = \bigoplus_{n \geq 0} A_n$ *be a normal graded ring, finitely generated over* $A_0 = k$, *where* k *is algebraically closed of characteristic* $p > 0$. *Suppose that* A *is Cohen–Macaulay, and that the vertex* (*corresponding to the ideal* $\bigoplus_{n > 0} A_n$) *is the only singularity of* Spec A. *Then* $A_0(\text{Spec}\, A) = 0$.

Here $A_0(\text{Spec}\, A)$ denotes the subgroup of $K_0(A)$ generated by the classes of smooth points of Spec A. Now $K_0(A)$ is generated by the class of the trivial line bundle, and classes of sub-varieties not meeting the singular locus (see [3]). Since Pic $A = (0)$ for a normal graded ring A, we deduce (see §1)

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COROLLARY (1.3): *Let* A *be as in Theorem* 1. *Suppose that* $\dim A = 2$. *Then* $K_0(A) = \mathbb{Z}$.

This answers Murthy's question affirmatively in the two dimensional case, and thus includes the result on cones over curves.

Next, we have a partial positive result in characteristic 0.

THEOREM 2: *Let* $X \subset \mathbb{P}^n_k$ *be a projectively normal curve, where* k *is an algebraically closed field of characteristic* 0. *Assume that* X *is not contained in a hyperplane, and that* $\deg X \leq 2n - 1$. *Then* $A_0(C(X)) = 0$, *where* $C(X) \subseteq \mathbb{A}^{n+1}$ *is the affine cone over* X.

As a consequence, we obtain

THEOREM 2′: *Let* X/k *be a curve of genus* g, *and* D *a divisor on* X *such that* $deg\, D \geq 2g + 1$. *Then* $A_0(C(X)) = 0$, *where* $C(X)$ *is the cone over* X *in the embedding* $X \hookrightarrow \mathbb{P}^{|D|}$ *given by the complete linear system* $|D|$.

Using the cancellation theorem of Murthy and Swan, we can formulate the above theorems as follows.

THEOREM: *Let* k *be an algebraically closed field, and let* $A = \bigoplus_{n \geqq 0} A_n$ *be a finitely generated graded* k*-algebra with* $A_0 = k$. *Then every projective module over* A *is free, in each of the following cases:*

i) $\operatorname{char} k = p > 0$, *and* A *is normal of dimension* 2.

ii) $\operatorname{char} k = 0$, *and* $\operatorname{Spec} A$ *is the cone over a projectively normal curve* X *properly contained in* $\mathbb{P}^n$, *and satisfying* $\deg X \leq 2n - 1$.

iii) $\operatorname{char} k = 0$, *and* $A = \bigoplus_{n \geqq 0} H^0(X, \mathcal{O}_X(nD))$ *where* X/k *is a smooth curve of genus* g, *and* D *a divisor on* X *satisfying* $\deg D \geq 2g + 1$.

Finally, we construct an infinite family of examples of cones over $\mathbb{C}$ which admit non-trivial vector bundles. Let L denote the field of algebraic numbers.

THEOREM 3: *Let* $X \subset \mathbb{P}^n_L$ *be a projectively normal curve such that* $H^1(X, \mathcal{O}_X(1)) \neq 0$. *Then if* $C(X_{\mathbb{C}})$ *denotes the cone over the corresponding complex curve, we have* $K_0(C(X_{\mathbb{C}})) \neq \mathbb{Z}$. (In fact, a slight modification of our argument will show that $K_0(C(X_{\mathbb{C}}))$ is uncountable).

One remarkable fact about theorem 3 is the following. For a curve $X \subset \mathbb{P}^n_{\mathbb{C}}$, let $Y \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ denote the projective cone over X. Let $Z \xrightarrow{\pi} Y$ be the blow up of Y at the vertex. Then $Y \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(1))$. The Leray spectral sequence applied to the map π yields an exact sequence

$$\begin{aligned} 0 \to H^1(Y, \mathcal{O}_Y) \to H^1(Z, \mathcal{O}_Z) \to \Gamma(Y, R^1\pi_*\mathcal{O}_Z) \to \\ \to H^2(Y, \mathcal{O}_Y) \to H^2(Z, \mathcal{O}_Z) \to 0. \end{aligned}$$

Since Z is a ruled surface, $p_g(Z) = 0$, and $q = g(X)$, the genus of X. (In fact, all elements of $H^1(Z, \mathcal{O}_Z)$ are pulled back from $H^1(X, \mathcal{O}_X)$). Now $R^1\pi_*\mathcal{O}_Z$ is a torsion sheaf on Y supported only at the vertex of the cone. From the formal function theorem (see [7]), $\Gamma(R^1\pi_*\mathcal{O}_Z)$ has a filtration whose associated graded module is $\bigoplus_{m \geqq 0} H^1(E, I^m/I^{m+1})$ where E is the exceptional set, and I is its sheaf of ideals on Z. Now E is a section of the fibration $Z \to X$, hence $E \cong X$. One easily checks that $I/I^2 \cong \mathcal{O}_X(1)$, and thus $I^m/I^{m+1} \cong \mathcal{O}_X(m)$. Also, the map $H^1(Z, \mathcal{O}_Z) \to \Gamma(R^1\pi_*\mathcal{O}_Z)$ maps the former isomorphically onto $H^1(E, \mathcal{O}_E)$. Hence $h^2(Y, \mathcal{O}_Y)$ vanishes precisely when $H^1(X, \mathcal{O}_X(1)) = 0$. Thus, the curves $X \subset \mathbb{P}^n_{\mathbb{C}}$ with $H^1(X, \mathcal{O}_X(1)) \neq 0$ correspond precisely to the cones Y with "*geometric genus*" (i.e. $h^2(\mathcal{O})) > 0$. Hence, Theorem 3 may be regarded as an analogue for cones of a famous result of Mumford on the infinite dimensionality of the Chow group of zero cycles on a surface with $p_g > 0$ (see [5]). In fact, one might conjecture that at least for cones, $A_0(C(X)) = 0 \Leftrightarrow p_g(Y) = 0$ (where p_g stands for $h^2(\mathcal{O})$); this is the analogue of a conjecture of Bloch for smooth surfaces with $p_g = 0$ (see [13], ch. 1 for motivation and further references for that conjecture).

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§1) Results in characteristic $p > 0$

In this section we prove theorem 1. In this paper, the Chow group of zero cycles will always be the subgroup of the Grothendieck group K_0 generated by the classes of smooth points. In particular, it is *not* the same (in general) as the Chow group of Fulton [6] when the variety we are dealing with is singular. The proof of the theorem is based on two lemmas:

LEMMA (1.1): *Let Y be an affine normal variety with isolated singularities over an algebraically closed field (of arbitrary characteristic). If $U \subset Y$ is an open (dense) set, then $A_0(Y)$ is generated by the classes of smooth points of U. $A_0(Y)$ is a divisible group.*

Proof: If Y is a curve, the result holds from the theory of Jacobians. In general, if $P \in Y$ is a smooth point, we can find a curve $C \subset Y$ such that $P \in C$, $C \cap U \neq \phi$, and C misses the singular locus of Y; we may take C to be smooth. Then there is a natural map $\operatorname{Pic} C \to A_0(Y)$, and the class of P is in the image of this map. Hence the result follows from the previous case.

Lemma (1.2): *Let* $A = \bigoplus_{n \geqq 0} A_n$ *be a graded normal ring of dimension* 1, *where* $A_0 = k$, *and* A *is finitely generated over* A_0. *Then* $A \cong k[t]$, *where* t *is homogeneous* (*perhaps of degree* $d > 1$).

Proof: It is amusing to give two proofs. First, an algebraic one. Let $M = \bigoplus_{n > 0} A_n$. Then A_M is a P.I.D. as A is normal. Since MA_M is generated by one element, but also has a set of homogeneous generators, it is generated by one homogeneous element (Nakayama's Lemma). Let $MA_M = fA_M$, with $f \in A$ homogeneous, and let $g \in A$ be any homogeneous element of positive degree. Since $g \in f \cdot A_M$, $g = u \cdot f^n = \dfrac{u_1}{u_2} \cdot f^n$, where $u_1, u_2 \in A - M$. Comparing homogeneous terms of lowest degree on both sides of $u_2 g = u_1 f^n$, we see that we may assume u_1, u_2 to be homogeneous. Since $u_i \notin M$, $u_i \in A_0 = k$. Thus $A = k[f]$ (since every element of A is a finite sum of homogeneous elements).

The second proof is geometric – since $\operatorname{Spec} A$ is an affine curve over k with a non-trivial $\mathbb{G}_m$-action, it is a rational curve. Since it is normal, and has no units (because A is graded) apart from k^*, it must be $\mathbb{A}_k^1$. The group of automorphisms of $\mathbb{A}_k^1$ fixing a point is $\mathbb{G}_m$; hence the grading on the coordinate ring of $\mathbb{A}_k^1$ induced from A must be the usual one.

Proof of Theorem 1: We first give a simple proof in the case when A is the homogeneous coordinate ring of a plane curve.

Let $X = \operatorname{Proj} A \subset \mathbb{P}_k^2$ be a smooth plane curve, and let $C(X) \subset \mathbb{A}^3$ be the cone over X (so that $C(X) = \operatorname{Spec} A$). Let $0 \in C(X)$ be the vertex, and let $\pi : C(X) - \{0\} \to X$ be the projection. Let $P \in C(X)$ be a smooth point and $\pi(P) = \bar{P}$. Choose a line $l \subset \mathbb{P}^2$ such that $l \cap X = \{P_1, \ldots, P_n\}$, where $P_1 = \bar{P}$, and $n = \deg X$, and the P_i are distinct. Then $\pi^{-1}(l) \cup \{0\} = S_1 \cap C(X)$, where $S_1 \subset \mathbb{A}^3$ is a plane (the cone over the line l). Thus $S_1 \cap C(X) = l_1 \cup \ldots \cup l_n$, where $\pi(l_i - \{0\}) = P_i$, and the l_i are lines on S_1 which concur at 0. We can choose coordinates x and y on S_1 so that l_1 is the y-axis, $P \in l_1$ is the point $(0, 1)$, and the lines $l_2, \ldots, l_n$ respectively have slopes $\lambda_2, \ldots, \lambda_n$. Thus

$$S_1 \cap C(X) = \operatorname{Spec} k[x, y]/(x \cdot \prod_{i=2}^{n} (y - \lambda_i x))$$

Consider the function $f = 1 - (y - \lambda_2 x)^r \prod_{i=3}^n (y - \lambda_i x)$ (where $r > 0$ will be chosen in a moment). Then f is identically equal to 1 on $l_2 \cup \ldots \cup l_n$. On $l_1, x = 0$, so that $f|_{l_1} = 1 - y^{r+n-2}$. Choose $r > 0$ to be the smallest integer such that $r + n - 2 = p^\nu$, where $p = \operatorname{char} k$. Then $f|_{l_1} = (1 - y)^{p^\nu}$. Thus, the zero cycle (P) represents a p^ν-torsion element of $\operatorname{Pic}(S_1 \cap C(X))$, and hence a p^ν-torsion element of $A_0(C(X))$. Since $A_0(C(X))$ is generated by the classes of smooth points $P \in C(X)$, we have $p^\nu \cdot A_0(C(X)) = 0$. Now the divisibility of $A_0(C(X))$ forces $A_0(C(X)) = 0$.

Now we give the proof in the general case, based on the same idea. Let $\mathcal{O}(1)$ denote the sheaf on $\operatorname{Proj} A$ associated to the graded module $\bigoplus_{n>0} A_n$ (see [7], ch. II). Fix a large integer $m > 0$ such that $\mathcal{O}(m)$ embeds $\operatorname{Proj} A$ is some projective space, and let $\dim \operatorname{Proj} A = r$ (so that $\dim A = r + 1$). Let $0 \in \operatorname{Spec} A$ be the vertex, and let $\pi : \operatorname{Spec} A - \{0\} \to \operatorname{Proj} A$ be the projection. Let $U \subset \operatorname{Spec} A - \{0\}$ be the inverse image of the locus of smooth points of $\operatorname{Proj} A$, and let $P \in U$. Let $\pi(P) = \bar{P}$. Choose r general hyperplane sections of $\operatorname{Proj} A$ through $\bar{P}$, so that the intersection of all of them with $\operatorname{Proj} A$ consists of $d = \deg(\operatorname{Proj} A)$ points $P_1 = \bar{P}, P_2, \ldots, P_d$, where all the P_i are smooth. Let $Y = \pi^{-1}(\{P_1, \ldots, P_d\}) \cup \{0\}$, so that $Y = \operatorname{Spec}(A/(f_1, \ldots, f_r))$ where f_i is homogeneous of degree m for each i. Since A is Cohen–Macaulay, and height $(f_1, \ldots, f_r) = r$, the ring $B = A/(f_1, \ldots, f_r)$ is a reduced graded ring of dimension 1. The minimal primes $\mathcal{P}_1, \ldots, \mathcal{P}_d$ of B satisfy $\operatorname{Spec}(B/\mathcal{P}_i) = \pi^{-1}(P_i) - \{0\}$. Let $\tilde{B}$ denote the normalisation of B (i.e. its integral closure in its total quotient ring), and let $I \subset B$ be the conductor of $B \to \tilde{B}$ (see [12] for the definition). We have the well known exact sequence (where * denotes the unit group)

$$0 \to B^* \to \tilde{B}^* \to (\tilde{B}/I\tilde{B})^*/(B/IB)^* \to \operatorname{Pic} B \to \operatorname{Pic} \tilde{B} \to 0$$

From lemma (1.2), $\tilde{B} \cong \bigoplus_{i=1}^{d} k[t]$, so that $\operatorname{Pic} \tilde{B} = 0$. Thus we have a surjection $(\tilde{B}/I\tilde{B})^*/(\operatorname{Image} \tilde{B}^*) \twoheadrightarrow \operatorname{Pic} B$. Since $\tilde{B} \cong \bigoplus_{i=1}^{d} k[t]$, $\tilde{B}^* \cong \bigoplus_{i=1}^{d} k^*$; also $\tilde{B}/I\tilde{B} \cong \bigoplus_{i=1}^{d} k[t]/(t^{n_i})$ for some exponents $n_i > 0$. Now $\left(\frac{k[t]}{(t^n)}\right)^* = k^* R_n$, where R_n is p^ν-torsion for any ν such that $p^\nu \geq n$ (this boils down to the identity

$$(1 + a_1 t + a_2 t^2 + \ldots + a_k t^k)^{p^\nu} = 1 + a_1^{p^\nu} t^{p^\nu} + \ldots + a_k^{p^\nu} t^{k \cdot p^\nu}).$$

Thus if $p^\nu \geq \sup n_i$, then $p^\nu \cdot (\operatorname{Pic} B) = (0)$. If we can find a value of $\nu > 0$

such that v is independent of the choice of the initial point $P \in U$, and the hyperplane sections $f_1, \ldots, f_r$, then as in the case of plane curves we can conclude that $p^v \cdot A_0(\operatorname{Spec} A) = 0$; hence $A_0(\operatorname{Spec} A) = 0$ by divisibility. The rest of the proof will consist in showing that by shrinking U to some non-empty open subset V, and for any choices of $f_1, \ldots, f_r$ corresponding to any given point $P \in V$, the resulting exponent v is bounded by some preassigned number depending only on A.

Factor the inclusion $B \hookrightarrow \tilde{B}$ as $B \hookrightarrow \bigoplus_{i=1}^{d} B/\mathscr{P}_i$ and

$$\bigoplus_{i=1}^{d} B/\mathscr{P}_i \hookrightarrow \bigoplus_{i=1}^{d} (B/\mathscr{P}_i)^{\sim} \cong \bigoplus_{i=1}^{d} k[t].$$

If J_1 and J_2 are the respective conductors, then $J_1 J_2 \tilde{B} \subset I\tilde{B} = I$. Thus if $J_1\tilde{B} = \oplus t^{n_{i1}} k[t]$, and $J_2\tilde{B} = \oplus t^{n_{i2}} k[t]$, it is enough to separately bound the exponents n_{i1} and n_{i2} for all i.

Let $A = k[\phi_1, \ldots, \phi_s]$ where $\phi_i \in A$ is homogeneous of degree m_i for each i. For any homogeneous prime $\mathscr{P}_0$ of A of height r (corresponding to a point $\mathscr{P}_0 \in \operatorname{Proj} A$), $A/\mathscr{P}_0 \cong k[t]$, where t has degree e (say) in the grading induced from A. Thus $A/\mathscr{P}_0 \hookrightarrow k[u]$, where $u^e = t$, and u has degree 1. Clearly ϕ_i is mapped to an element of degree m_i in $k[u]$, which is homogeneous i.e. $\phi_i \mapsto \alpha_i \cdot u^{m_i}$ for some $\alpha_i \in k$. So $A/\mathscr{P}_0 \cong$ $\cong k[\alpha_1 u^{m_1}, \ldots, \alpha_s u^{m_s}] \subset k[u]$. Now $\alpha_i = 0 \Leftrightarrow \phi_i$ (considered as a section of $\mathscr{O}(m_i)$) vanishes at $\mathscr{P}_0 \in \operatorname{Proj} A$. Hence deleting the finite set of zeroes of the sections $\phi_1, \ldots, \phi_s \in \Gamma(\operatorname{Proj} A, \oplus\, \mathscr{O}(m))$, and correspondingly shrinking our open set $U \subset \operatorname{Spec} A - \{0\}$ to a smaller open set V, we may assume that none of the α_i vanish when $\mathscr{P}_0$ is one of the primes we construct by taking hyperplane sections of $\operatorname{Proj} A$. But if $\alpha_1, \ldots, \alpha_s$ are non-zero, then $k[\alpha_1 u^{m_1}, \ldots, \alpha_s u^{m_s}] = k[u^{m_1}, u^{m_2}, \ldots, u^{m_s}]$ i.e. all the $B/\mathscr{P}_i$ are isomorphic. Since $m_1, \ldots, m_s$ depends only on A, this bounds the exponents n_{i2} for J_2.

We claim that if $J = \bigcap_i (\mathscr{P}_i + \bigcap_{j \neq i} \mathscr{P}_j)$, then $J \subset J_1$. By definition of the conductor, J_1 is the largest ideal in B which is a $\bigoplus_{i=1}^{d} B/\mathscr{P}_i$-module. Hence, to verify the claim, it is enough to prove the following –

given $a_1, \ldots, a_d \in J$, there exists $a \in B$ such that $a - a_i \in \mathscr{P}_i$ (i.e. $a \mapsto (\bar{a}_1, \ldots, \bar{a}_d)$ under $B \hookrightarrow \oplus\, B/\mathscr{P}_i$). But if $a_i = b_i + c_i$, with $b_i \in \mathscr{P}_i$, $c_i \in \bigcap_{j \neq i} \mathscr{P}_j$, then $a = \sum_{i=1}^{d} c_i$ works, since $a - a_i = \sum_{j \neq i} c_j - b_i \in \mathscr{P}_i$ as desired. Now suppose $f \in J$ satisfies $f = (\beta_1 t^{v_1}, \ldots, \beta_d t^{v_d})$ where $\beta_1 \ldots \beta_d \neq 0$. Then clearly $v_i \geq n_{i2}$ for all i. So if we can suitably bound v_i, we will be done. In fact, by symmetry it suffices to find f with suitably bounded v_1, and $\beta_1 \neq 0$.

Consider the homomorphisms $A \to B \to B/\mathscr{P}_i$, where we identify $B/\mathscr{P}_i$ with $k[u^{m_1}, \ldots, u^{m_s}] \subset k[u]$. Then for each $i \neq 1$, we can find μ, ν such that $1 \leq \mu$, $\nu \leq s$, and $\alpha_{1\mu}^{m_\nu} \cdot \alpha_{i\nu}^{m_\mu} - \alpha_{1\nu}^{m_\mu} \cdot \alpha_{i\mu}^{m_\nu} \neq 0$ (i.e. if two points of Proj A, namely P_1 and P_i, are distinct, then they have distinct "weighted homogeneous" coordinates – recall that $\phi_\mu \mapsto \alpha_{i\mu} \cdot u^{m_\mu}$ under $A \to B \to B/\mathscr{P}_i$). Let $\gamma \in A$ be the element defined by $\gamma = \gamma_{\mu,\nu} = \alpha_{i\nu}^{m_\mu} \cdot \phi_\mu^{m_\nu} - \alpha_{i\mu}^{m_\nu} \cdot \phi_\nu^{m_\mu}$. Then the image of γ in $B/\mathscr{P}_i$ is

$$\alpha_{i\nu}^{m_\mu}(\alpha_{i\mu}u^{m_\mu})^{m_\nu} - \alpha_{i\mu}^{m_\nu}(\alpha_{i\nu}u^{m_\nu})^{m_\mu} = (\alpha_{i\nu}^{m_\mu} \cdot \alpha_{i\mu}^{m_\nu} - \alpha_{i\mu}^{m_\nu} \cdot \alpha_{i\nu}^{m_\mu}) \cdot u^{m_\mu m_\nu} = 0$$

i.e. the image $\bar{\gamma}$ of γ in B actually lies in $\mathscr{P}_i$. The image of γ in $B/\mathscr{P}_1$ is $[\alpha_{i\nu}^{m_\mu} \cdot \alpha_{1\mu}^{m_\nu} - \alpha_{i\mu}^{m_\nu} \cdot \alpha_{1\nu}^{m_\mu}] \cdot u^{m_\nu \cdot m_\mu} = \delta_i \cdot u^{t_i}$, where $\delta_i \neq 0$, and t_i is bounded by a number depending only on A. Since $\mathscr{P}_1 + J = \mathscr{P}_1 + \bigcap_{j \neq 1} \mathscr{P}_j$, the element $\gamma_0 = \prod_{i=2}^d \bar{\gamma}_{\mu,\nu}$ is such that the image of γ_0 in $B/\mathscr{P}_1$ is actually in $J \cdot B/\mathscr{P}_1$, and hence in $J_1 \cdot B/\mathscr{P}_1$. But $\gamma_0 \cdot k[u] = u^{t_2 + \cdots + t_d} k[u]$, and $t_2 + \ldots + t_d$ is bounded by a number depending only on A.

This completes the proof of Theorem 1.

Corollary (1.3): *Let A be as in Theorem 1. Then if* $\dim A = 2$, *we have $K_0(A) = \mathbb{Z}$. Hence all vector bundles on* Spec A *are trivial.*

Proof: By a remark of Murthy (see [1]) we know that Pic $A = (0)$. By the standard argument using the cancellation theorem of Bass, it suffices to prove that vector bundles of rank 2 represent trivial elements of $K_0(A)$ to prove that $K_0(A) = \mathbb{Z}$. Now we can find a section of a given vector bundle of rank 2 which has isolated zeroes at smooth points of Spec A. If P is the projective A-module of global sections of the bundle, we have an exact sequence $0 \to L \to P^* \to I \to 0$, where $I \subset A$ is the ideal of zeroes of the chosen section, P^* is the dual projective module. An argument using the determinant shows that $L \cong \Lambda^2 P^* \in \text{Pic}\, A$ i.e. $L \cong A$. Next, $[A/I] \in K_0(A)$ gives an element of $A_0(\text{Spec}\, A)$ which is trivial by Theorem 1. Hence $[A] = [I]$ in $K_0(A)$. Putting these facts together gives $[P^*] \cong [A^{\oplus 2}]$ i.e. all vector bundles of rank 2 are stably trivial. This proves $K_0(A) = \mathbb{Z}$. Now the cancellation theorem of Murthy and Swan [4] proves that all vector bundles are trivial.

The argument needed to deduce the triviality of vector bundles from the vanishing of the Chow group works in all characteristics (see section 2 of this paper).

§2) Some positive results in characteristic 0

In this section we obtain partial positive results for cones over smooth projective curves in characteristic zero. Our result is a slight improvement on results of Varley (see [3]). The proof is based on an idea of Ojanguren [8], who used it to prove the result for plane cubics.

THEOREM 2: *Let* $X \subset \mathbb{P}^n_k$ *be a projectively normal curve over the algebraically closed field* k *of characteristic* 0. *Assume that* X *is not contained in a hyperplane, and has degree atmost* $2n - 1$. *Then* $A_0(C(X)) = 0$, *where* $C(X) \subset \mathbb{A}^{n+1}$ *is the affine cone over* X. *Hence vector bundles on* $C(X)$ *are trivial.* (See the proof of Corollary (1.3).)

PROOF: The triviality of vector bundles follows from the vanishing of the Chow group, using the triviality of line bundles (a remark of Murthy – see [1]) and the cancellation theorem of Murthy and Swan [4]. The proof of the vanishing of $A_0(C(X))$ is based on two lemmas.

Let $\deg X = d \leq 2n - 1$, and set $r = d - n$.

LEMMA (2.1): *Assume that* $P \in X$ *is not a Weierstrass point. Let* $H \subset \mathbb{P}^n$ *be the osculating hyperplane to* X *at* P, *so that the zero cycle* $(H \cdot X) = n(P) + \sum_{i=1}^{r} (P_i)$ (*where the* $P_i \in X$ *may not be distinct from each other or from* P, *in general*). *Then* $\{P, P_1, \ldots, P_r\}$ *span a* $\mathbb{P}^r \subset \mathbb{P}^n$.

PROOF: Suppose that $\{P, P_1, \ldots, P_r\} \subset L \subset \mathbb{P}^n$, where L is a linear space of dimension $r - 1$. Since the space $\hat{L}$ of hyperplanes (in the dual projective space) which contain L is a $\mathbb{P}^{n-r}$, we have

$$h^0(\mathcal{O}_X(D - P - \sum_{i=1}^{r} P_i)) \geq n - r + 1 \ \ (\text{where } \mathcal{O}_X(D) = \mathcal{O}_X(1))$$

Choosing the representative $n(P) + \sum_{i=1}^{r} (P_i) \in |D|$, we have

$$h^0(\mathcal{O}_X(n - 1)P)) \geq n - r + 1.$$

Now $\deg X = d$, and $\dim |D| = n$. Since $n > d/2$, the divisor D is non-special by Clifford's Theorem (see [7], ch. IV). Hence by the Riemann–Roch theorem, the genus g of X satisfies

$$g = \deg D - \dim |D| = d - n = r.$$

Since $n - 1 = g + (n - r - 1)$,

$$h^0(\mathcal{O}_X(n - 1)P)) \geq n - r + 1 \Rightarrow h^0(\mathcal{O}_X(gP)) \geq (n - r + 1) - (n - r - 1) = 2$$

i.e. $P \in X$ is a Weierstrass point.

Lemma (2.2): *There is a non-empty open set $U \subset X$ with the following property – if $P \in U$, then there exist points $P_0, P_1, \dots, P_{r-1}$ of X such that (i) $P, P_0, \dots, P_{r-1}$ span a $\mathbb{P}^r \subset \mathbb{P}^n$, and (ii) if H is the osculating hyperplane to P_0, then $(H \cdot X) = n(P_0) + (P) + \sum_{i=1}^{r-1} P_i$.*

Proof: The set S in the dual projective space of hyperplanes which parametrizes osculating hyperplanes is birational to X. As s ranges over S, the hyperplane sections $(H(s) \cdot X)$ have the form $(H(s) \cdot X) = nP(s) + \sum_{i=1}^{r} P_i(s)$ (through the individual $P_i(s)$ don't make sense, the zero cycle $\sum_{i=1}^{r} P_i(s)$ does). Then the lemma amounts to the claim that $\sum_{i=1}^{r} P_i(s)$ is not independent of s. Suppose that the lemma is false. Let $L \subset \mathbb{P}^n$ be the $\mathbb{P}^{r-1}$ spanned by $P_1, \dots, P_r$ (for general $s \in S$, $P_1(s), \dots, P_r(s)$ span a $\mathbb{P}^{r-1}$, by Lemma (2.1)). Projection from L to a suitable $\mathbb{P}^{n-r}$ yields a curve $\bar{X} \subset \mathbb{P}^{n-r}$ with the following property – if $P \in \bar{X}$ is general, then there exists a hyperplane $H_P \subset \mathbb{P}^{n-r}$ such that the local intersection multiplicity $(H_P \cdot \bar{X})_P \geq n$ (choose H_P to be the image of a suitable osculating hyperplane to X). But this is impossible – at a general point of a curve in $\mathbb{P}^{n-r}$, the maximum local intersection multiplicity with a hyperplane is $n - r$. This contradiction finishes the proof of the lemma.

We now prove theorem 2. Let $0 \in C(X)$ be the vertex, and $\phi : C(X) - (0) \to X$ the projection. Let $P \in \phi^{-1}(U)$, where $U \subset X$ is the open set of lemma (2.2). Then if $\bar{P} = \phi(P)$, we can find $P_0, \dots, P_{r-1} \in X$ and a hyperplane H such that $(H \cdot X) = n(P_0) + (\bar{P}) + \sum_{i=1}^{r-1} (P_{r-1})$. Then $\phi^{-1}(H) \cup \{0\} \cong \mathbb{A}^n \subset \mathbb{A}^{n+1}$ (by abuse of notation, let ϕ also denote the projection $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$). Also, $(\phi^{-1}(H) \cup \{0\}) \cap C(X) = l_0 \cup l_1 \dots \cup l_{r-1} \cup \bar{l}$, where l_i and $\bar{l}$ are lines through 0 in $\phi^{-1}(H) \cup \{0\}$. Since $P, P_0, \dots, P_{r-1}$ can be chosen to span a $\mathbb{P}^r \subset \mathbb{P}^n$ by lemma (2.2), the lines $\bar{l}, l_0, \dots, l_{r-1}$ span an $\mathbb{A}^{r+1} \subset \phi^{-1}(H) - \{0\}$, and satisfy $\phi(l_i - \{0\}) = P_i$, $\phi(\bar{l} - \{0\}) = \bar{P}$, and $P \in \bar{l} - \{0\}$. The lines $\bar{l}, l_1, \dots, l_{r-1}$ occur with multiplicity 1 in the intersection $(\phi^{-1}(H) \cup \{0\}) \cap C(X)$, while l_0 occurs with multiplicity n.

There exists a unique linear subspace $L \cong \mathbb{A}^r$, with $L \subset \operatorname{span}\{\bar{l}, l_0, \dots, l_{r-1}\}$, such that $P \in L$, and $L \cap \operatorname{span}\{l_0, \dots, l_{r-1}\} = \phi$. (This is just the unique $\mathbb{A}^r$ through P which is parallel to $\operatorname{span}\{l_0, \dots, l_{r-1}\} \cong \mathbb{A}^r$). If $f = 0$ is the equation of L in the affine space $\mathbb{A}^{r+1} = \operatorname{span}\{\bar{l}, l_0, \dots, l_{r-1}\}$, then the restriction of f to the curve $Y = (\phi^{-1}(H) \cup \{0\}) \cap C(X)$ is a regular function on Y whose divisor of zeroes is (P). Thus $(P) = 0$ in $\operatorname{Pic}^0(Y)$, and hence in $A_0(C(X))$. By lemma (1.1), this proves the result.

We easily deduce theorem 2′ from theorem 2.

THEOREM 2′: *Let X be a smooth curve of genus g over an algebraically closed field of characteristic* 0. *Let D be a divisor on X such that* $\deg D \geq 2g+1$. (Thus D is very ample – see [7], ch. IV). *Assume that X is projectively normal in this embedding. Then $A_0(A)=0$, where $A = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$.*

PROOF: Since $\deg D \geq 2g+1$, D is non-special. Hence by the Riemann–Roch theorem, $n = \dim |D| = \deg D - g$. We claim that $\deg D \leq 2n-1$ (so that theorem 2 applies). For

$$\begin{aligned} 2n - 1 - \deg D &= 2(\deg D - g) - 1 - \deg D = \\ &= \deg D - (2g+1) \geq 0. \end{aligned}$$

REMARK: In fact, a result of Castelnuovo implies that for the range of degrees in theorem 2′, X will always be projectively normal. See [12], p. 52.

§3) A counterexample in characteristic 0

In this section we construct examples of cones over projectively normal complex curves which admit non-trivial vector bundles. Let L denote the field of algebraic numbers.

THEOREM 3: *Let $X \subset \mathbb{P}^n_L$ be a projectively normal curve such that $H^1(X, \mathcal{O}_X(1)) \neq 0$. Then $K_0(C(X_{\mathbb{C}})) \neq \mathbb{Z}$.*

COROLLARY (3.1): *Let X be a non-hyperelliptic curve, defined over L. Then $K_0(A) \neq \mathbb{Z}$, where $A = \bigoplus_{n \geq 0} H^0(X_{\mathbb{C}}, \omega_{X_{\mathbb{C}}}^{\otimes n})$. (The cone over the canonical embedding.)*

COROLLARY (3.2): *Let $X \subset \mathbb{P}^2_L$ be a smooth curve of degree at least* 4. *Then $C(X_{\mathbb{C}})$ admits non-trivial vector bundles.*

This is in contrast to the situation in characteristic $p > 0$, and to the situation for analytic vector bundles (since any analytic vector bundle on a contractible Stein space is trivial). The method of proof is based on an idea of Spencer Bloch. He showed that $\mathbb{C}[x,y,z]/(z^7 - x^2 - y^3)$ provides a counterexample to the statement of theorem 1 in characteristic 0. Let me sketch his idea.

Let $X = \operatorname{spec} \mathbb{C}[x,y,z]/(z^7 - x^2 - y^3)$. Then the origin is the only singular point of X. Let $\bar{X}$ be a projective surface containing X as an open subset, such that $\bar{X}_{\mathrm{sing}} = X_{\mathrm{sing}} = \{0\}$, the origin. Let $\pi : \tilde{X} \to \bar{X}$ be a resolution of the singularity. Then $\tilde{X}$ can be chosen so that $\pi^{-1}(\{0\})$ is a cuspidal rational curve E. Now $K_0(\tilde{X}) = \mathbb{Z} \oplus \mathrm{Pic}(\tilde{X})$, and $SK_1(\tilde{X}) \cong \mathrm{Pic}(\tilde{X}) \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^{\oplus n}$ for some n, since $\tilde{X}$ is a rational surface. Also $SK_1(E) \cong \Omega^1_{\mathbb{C}/\mathbb{Z}}$, the module of Kahler differentials of $\mathbb{C}$ (see [7] for the definition, some properties and references). Since $\mathbb{C}$ has uncountable transcendence degree over $\mathbb{Q}$, and $\mathrm{Hom}_{\mathbb{C}}(\Omega^1_{\mathbb{C}/\mathbb{Z}}, \mathbb{C}) =$ (vector space of all derivations $\mathbb{C} \to \mathbb{C}$), $\Omega^1_{\mathbb{C}/\mathbb{Z}}$ is a $\mathbb{C}$-vector space of uncountable dimension.

Now one considers the diagram

$$\begin{tikzcd}
K_1(\tilde{X}) \arrow[r, "\alpha"] & K_1(E) \arrow[r, "\beta"] & K_0(\tilde{X}, E) \arrow[r] & K_0(\tilde{X}) \arrow[r] & K_0(E) \\
K_1(\bar{X}) \arrow[r, "\alpha'"] \arrow[u] & K_1(\{0\}) \arrow[r, "\beta'"] \arrow[u] & K_0(\bar{X}, \{0\}) \arrow[r] \arrow[u, "\pi^*"] & K_0(\bar{X}) \arrow[r] \arrow[u] & K_0(\{0\}) \arrow[u]
\end{tikzcd}$$

Here $K_0(\tilde{X}, E)$ and $K_0(\bar{X}, P)$ are relative K-groups (we give the definitions below). Clearly α' is onto, as $K_1(\{0\}) = \mathbb{C}^*$; hence $\beta' = 0$. It turns out that points of $\bar{X} - \{0\}$ admit cycle classes in $K_0(\bar{X}, \{0\})$, and similarly for $K_0(\tilde{X}, E)$. Define $F_0K_0(\tilde{X}, E)$ to be the subgroup of $K_0(\tilde{X}, E)$ generated by classes of points of $\tilde{X} - E$, and similarly define $F_0K_0(\bar{X}, \{0\})$. Evidently $\pi^* : F_0K_0(\bar{X}, \{0\}) \twoheadrightarrow F_0K_0(\tilde{X}, E)$ as $\pi : \tilde{X} - E \xrightarrow{\sim} \bar{X} - \{0\}$. One main ingredient of the proof is a geometric description of $\beta|_{SK_1(E)}$. A class in $SK_1(E)$ is represented by finite sets of points of $E - E_{\mathrm{sing}}$, together with non-zero elements of the residue fields of each of the points. If $P_1, \ldots, P_r \in E - E_{\mathrm{sing}}$, and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ (where we think of $\alpha_i \in \mathbb{C}(P_i)^*$), we choose a curve $C \subset \tilde{X}$ which meets E transversally at $P_1, \ldots, P_r$. If C meets E at additional points $P_{r+1}, \ldots, P_s$, assume that these intersections are also transverse and the points $P_i \in E$ are all smooth. Let $\alpha_i \in \mathbb{C}(P_i)^*$ be set equal to 1 for $r + 1 \leq i \leq s$. Choose a rational function $f \in \mathbb{C}(C)^*$, such that $f(P_i) = \alpha_i$ $(1 \leq i \leq s)$. Then the element $\beta((\{P_1, \alpha_1\}, \ldots, \{P_s, \alpha_s\})$ $=$ (cycle class of the divisor of $f) \in F_0K_0(\tilde{X}, E)$. Once one has this, one can show that $F_0K_0(\tilde{X}, E) \neq 0$, and hence $F_0K_0(\bar{X}, \{0\}) \neq 0$, as desired.

In our case, we have to work harder, because SK_1 of the ambient surface maps onto SK_1 of the exceptional set when we resolve the singularity of the cone. However, if we work with a *multiple* of the exceptional set, then obstructions to the triviality of vector bundles appear.

Let $X_L \subset \mathbb{P}^n_L$ be our given curve, and let Y_L be the affine cone over X_L. We will make use of the following convention – unless "L" appears as a subscript on the symbol for a variety, *we will be working over* $\mathbb{C}$. Let $\tilde{Y}$ be the blow up of Y at P. Then $\tilde{Y} \cong \mathbb{V}(\mathcal{O}_X(-1))$, a ruled surface over X, and the exceptional set $\pi^{-1}(P) = E_0$ (where $P \in Y$ is the vertex) is a section of $\tilde{Y} \to X$ with normal bundle $\cong \mathcal{O}_X(-1)$. Let E be the subscheme

$2E_0$: thus if I is the sheaf of ideals of E_0 on $\tilde{Y}$, then E is defined by the sheaf I^2. We write "$2P$" for the scheme $\operatorname{spec} A/M^2$, where $Y = \operatorname{spec} A$, and M is the maximal ideal of P.

For any scheme T, let $\mathscr{P}(T)$ denote the category of locally free sheaves of finite rank; $\mathscr{H}(T)$ will denote the category of coherent sheaves of finite homological dimension on T. If $S \subset T$ is a subscheme, let $\mathscr{H}(S, T)$ denote the category of $\mathcal{O}_T$-modules which are coherent, of finite homological dimension, and vanish on $T - S$. If S is a single point $x \in T$, we may write $\mathscr{H}_x$ for $\mathscr{H}(x, T)$.

Now we define the relative K-groups and cycle classes in them. Let $i: Y \hookrightarrow X$ be a closed immersion. We have a natural map $i^*: \mathscr{P}(X) \to \mathscr{P}(Y)$. For any exact category $\mathscr{C}$, let $BQ\mathscr{C}$ be the topological space (together with its natural base point) as defined by Quillen [9]. Then we have a natural map (of based spaces) $BQ\mathscr{P}(X) \xrightarrow{i^*} BQ\mathscr{P}(Y)$.

Let $F(i^*)$ denote the homotopy fiber of i^* (for a map $(X, P) \xrightarrow{f} (Y, P')$ of based spaces, the *homotopy fiber* is the set of pairs (ω, x) where $x \in X$, $\omega: [0, 1] \to Y$ is a path, with $\omega(0) = P'$, $\omega(1) = f(x)$. The base point is (ω_0, P) where $\omega_0: [0, 1] \to P'$). One of the basic properties of the homotopy fiber is that its homotopy groups fit into a long exact sequence with those of the domain and range. So if we set $K_n(X, Y) = \pi_{n+1}(F(i^*), *)$, where $* \in F(i^*)$ is the base point, then we have a long exact sequence

$$\ldots \to K_n(X, Y) \to K_n(X) \to K_n(Y) \to K_{n-1}(X, Y) \to \ldots$$

A general reference for the definitions and basic properties of higher K-groups is the fundamental paper [9] of Quillen. A summary of Quillen's results, and some applications to questions in the theory of algebraic cycles, can be found in Bloch's lecture notes [13].

Let $Z \subset X - Y$ be a subscheme, closed in X, and of finite homological dimension. Then we claim that there is a natural cycle class $[Z] \in K_0(X, Y)$. To construct it, we use the category $\mathscr{H}_0(X) \subset \mathscr{H}(X)$, defined to be the full subcategory consisting of all coherent $\mathcal{O}_X$-modules $\mathscr{F}$ satisfying $\operatorname{Tor}_i^{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_Y) = 0$ for $i > 0$. Then the map $i: Y \hookrightarrow X$ induces a functor $\bar{i}^*: \mathscr{H}_0(X) \to \mathscr{H}(Y)$. By Quillen's resolution theorem [9], the maps $BQ\mathscr{P}(X) \to BQ\mathscr{H}_0(X)$ and $BQ\mathscr{P}(Y) \to BQ\mathscr{H}(Y)$ are homotopy equivalences. Hence the natural induced map between the homotopy fibers $F(i^*)$ and $F(\bar{i}^*)$ is also a homotopy equivalence. The inclusion $j: Z \hookrightarrow X$ induces a functor $j_*: \mathscr{P}(Z) \to \mathscr{H}_0(X)$, since $Z \cap Y = \phi$; and the composite functor $\bar{i}^* \circ j_*: \mathscr{P}(Z) \to \mathscr{H}(Y)$ is the 0-functor. Hence the induced map $BQ\mathscr{P}(Z) \to BQ\mathscr{H}(Y)$ maps everything to the base point.

Hence we have an induced natural map $BQ\mathscr{P}(Z) \to F(\bar{i}^*)$, and thus a map $K_0(Z) \to K_0(X, Y)$. The image of $[\mathcal{O}_Z] \in K_0(Z)$ under this map is the required cycle class; by construction, it maps to the usual cycle class in $K_0(X)$ under the natural map $K_0(X, Y) \to K_0(X)$.

The relative K-groups, and the cycle classes, enjoy the following naturality properties. If $i: Y \hookrightarrow X$ and $i': Y' \hookrightarrow X'$, and $\pi: X \to X'$ is a morphism such that $\pi^{-1}(Y') = Y$, then we have a diagram

$$\begin{tikzcd}
\ldots \arrow[r] & K_n(X, Y) \arrow[r] & K_n(X) \arrow[r] & K_n(Y) \arrow[r] & K_{n-1}(X, Y) \arrow[r] & \ldots \\
\ldots \arrow[r] & K_n(X', Y') \arrow[r] \arrow[u] & K_n(X') \arrow[r] \arrow[u] & K_n(Y') \arrow[r] \arrow[u] & K_{n-1}(X', Y') \arrow[r] \arrow[u] & \ldots
\end{tikzcd}$$

Further, let $Z' \subset X'$ be a subscheme of finite homological dimension, satisfying $Z' \cap Y' = \phi$, and $\mathrm{Tor}_i^{\mathcal{O}_{X'}}(\mathcal{O}_{Z'}, \mathcal{O}_X) = 0$ for $i > 0$. Let $\mathscr{H}_1(X')$ denote the category of coherent $\mathcal{O}_{X'}$-modules $\mathscr{F}$ of finite homological dimension satisfying $\mathrm{Tor}_i^{\mathcal{O}_{X'}}(\mathscr{F}, \mathcal{O}_{Y'}) = \mathrm{Tor}_i^{\mathcal{O}_{X'}}(\mathscr{F}, \mathcal{O}_X) = \mathrm{Tor}_i^{\mathcal{O}_{X'}}(\mathscr{F}, \mathcal{O}_Y) = 0$ for $i > 0$. Then consider the commutative square of categories (where $Z = \pi^{-1}(Z')$)

$$\begin{tikzcd}
\mathscr{P}(Z) \arrow[r] & \mathscr{H}_0(X) \\
\mathscr{P}(Z') \arrow[r] \arrow[u, "\pi^*"] & \mathscr{H}_1(X') \arrow[u, "\pi^*"']
\end{tikzcd}$$

This gives the equation $\pi^*([Z']) = [Z]$ in $K_0(X, Y)$. Two cases where the hypothesis are satisfied are when $X - Y \to X' - Y'$ is an open immersion, and when the map π is flat. In our applications, we only work with $K_0(X, Y)$ where $X - Y$ is smooth.

There is one technical point that we systematically ignore. When we say that a diagram of categories

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "H"'] & \mathscr{B} \arrow[d, "G"] \\
\mathscr{C} \arrow[r, "K"] & \mathscr{D}
\end{tikzcd}$$

commutes, what we often mean is that the functors $G \circ F$ and $K \circ H$ are naturally equivalent. Thus, the corresponding diagram of classifying spaces

$$\begin{tikzcd}
B\mathscr{A} \arrow[r, "B(F)"] \arrow[d, "B(H)"'] & B\mathscr{B} \arrow[d, "B(G)"] \\
B\mathscr{C} \arrow[r, "B(K)"] & B\mathscr{D}
\end{tikzcd}$$

only commutes upto homotopy. The induced map on homotopy fibers $F(B(F)) \to F(B(K))$ depends on the choice of this homotopy i.e. on the choice of the equivalence of functors $G \circ F \cong K \circ H$. However, in all our situations, there is always one "natural" choice of the equivalence – for example, there is an obvious choice of a natural isomorphism $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$; this is the kind of choice which has to be made consistently. More details will appear in my thesis. I wish to thank Professor Swan for pointing out that some care is needed here.

We need to make use of certain results from K-theory. We give them in a sequence of lemmas. Recall that L denotes the field of algebraic numbers.

Lemma (3.2) (Van der Kallen [14]): *Let $\mathcal{O}$ be a regular local ring containing L, and let $\mathcal{O}[t]/(t^2)$ be the ring of dual numbers over $\mathcal{O}$. Then $K_2(\mathcal{O}[t]/(t^2))$ fits into the exact sequence*

$$0 \to \Omega^1_{\mathcal{O}/L} \to K_2(\mathcal{O}[t]/(t^2)) \to K_2(\mathcal{O}) \to 0.$$

The isomorphism $\ker(K_2(\mathcal{O}[t]/t^2) \to K_2(\mathcal{O})) \to \Omega^1_{\mathcal{O}/L}$ is given follows: the kernel is generated by symbols of the form $\{u, 1 + vt\}$ where $u \in \mathcal{O}^*$, $v \in \mathcal{O}$, and

$$\{u, 1 + vt\} \to v \cdot \frac{du}{u} \in \Omega^1_{\mathcal{O}/L}.$$

(Note that $\Omega^1_{\mathcal{O}/L} = \Omega^1_{\mathcal{O}/\mathbb{Z}}$, since $L/\mathbb{Q}$ is separable algebraic.)

From now on, all differentials will be relative to L unless indicated otherwise.

Lemma (3.3) (Localisation sequence [11]): *Let $U \hookrightarrow X$ be an open immersion, where U is affine, and $X - U$ is defined by an ideal sheaf which is locally principal and generated by a non zero-divisor. Let H be the category of coherent $\mathcal{O}_X$-modules which are 0 on U and have homological dimension 1 on X. Then we have a localisation sequence*

$$\ldots \to K_{q+1}(U) \to K_q(H) \to K_q(X) \to K_q(U) \to \ldots .$$

Now we come back to cones. Recall that $E \subset \tilde{Y}$ is the non-reduced scheme "$2E_0$" where E_0 is the exceptional set. For any finite subscheme $S \subset E$, the localisation sequence gives

$$\ldots \to K_2(E) \to K_2(E - S) \to K_1(\mathscr{H}(S, E)) \to K_1(E) \to K_1(E - S) \to \ldots$$

Taking limits over all such S (see Quillen [9], p. 96) we get

$$\ldots \to K_2(E) \to K_2(F) \to \coprod_{\substack{x \in E \\ x \text{ closed}}} K_1(\mathscr{H}_x) \to K_1(E) \to K_1(F) \to \ldots$$

where F is the local ring at the generic point of E. Define $SK_1(E) = \operatorname{Ker}(K_1(E) \to K_1(F))$. Then we have a presentation

$$K_2(F) \to \coprod_{x \in E} K_1(\mathscr{H}_x) \to SK_1(E) \to 0.$$

Since it is difficult to work with $\mathscr{H}_x$, we wish to obtain another viewpoint on $SK_1(E)$. To do this, replace E by $\mathcal{O}_{x,E}$, for any closed point $x \in E$, in the above argument. We obtain an exact sequence

$$K_2(\mathcal{O}_{x,E}) \to K_2(F) \to K_1(\mathscr{H}_x) \to 0,$$

because $K_1(\mathcal{O}_{x,E}) \hookrightarrow K_1(F)$ (since K_1(local ring) = units). Now let $x \in E_0$ be a smooth closed point; since any infinitesimal deformation of the regular local ring $\mathcal{O}_{x,E_0}$ is trivial (as $\mathcal{O}_{x,E_0}$ is essentially of finite type over $\mathbb{C}$), we see that $\mathcal{O}_{x,E} \cong \mathcal{O}_{x,E_0}[t]/(t^2)$. Hence lemma (3.2) applies to give a diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{\mathcal{O}_{x,E_0}} \arrow[r] \arrow[d] & K_2(\mathcal{O}_{x,E}) \arrow[r] \arrow[d] & K_2(\mathcal{O}_{x,E_0}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{\mathbb{C}(E_0)} \arrow[r] & K_2(F) \arrow[r] & K_2(\mathbb{C}(E_0)) \arrow[r] & 0
\end{tikzcd}$$

By a result of Dennis and Stein [10], $K_2(\mathcal{O}_{x,E_0}) \hookrightarrow K_2(\mathbb{C}(E_0))$. Also, $\Omega^1_{\mathcal{O}_{x,E_0}} \hookrightarrow \Omega^1_{\mathbb{C}(E_0)}$, since $\Omega^1_{\mathcal{O}_{x,E_0}}$ is a free $\mathcal{O}_{x,E_0}$-module, and the inclusion is just the localisation at the generic point. Hence $K_2(\mathcal{O}_{x,E}) \hookrightarrow K_2(F)$.

Let $\eta \in E$ be the generic point; for any point $x \in E$ let $i_x : \overline{\{x\}} \hookrightarrow E$. Then we have constructed an exact sequence of sheaves (for the Zariski topology)

$$0 \to \mathscr{K}_{2,E} \to (i_\eta)_* K_2(F) \to \coprod_{\substack{x \in E \\ x \text{ closed}}} (i_x)_* K_1(\mathscr{H}_x) \to 0,$$

where $\mathscr{K}_{2,E}$ is the sheaf associated to the presheaf $U \mapsto K_2(\Gamma(U, \mathcal{O}_E))$. Here $K_2(F), K_1(\mathscr{H}_x)$ are regarded as constant sheaves supported on a subvariety.

Since $(i_\eta)_* K_2(F)$, $(i_x)_* K_1(\mathscr{H}_x)$ are flasque, they have no higher cohomology, and so we can use the above resolution of $\mathscr{K}_{2,E}$ to compute its cohomology. Hence, we obtain an isomorphism $SK_1(E) \cong H^1(E, \mathscr{K}_{2,E})$, since both are presented as $\operatorname{coker}(K_2(F) \to \coprod_{x \in E} K_1(\mathscr{H}_x))$.

Next, we go back to the identification of $\mathcal{O}_{x,E}$ with $\mathcal{O}_{x,E_0}[t]/(t^2)$. The map $\operatorname{Ker}(K_2(\mathcal{O}_{x,E}) \to K_2(\mathcal{O}_{x,E_0})) \to \Omega^1_{\mathcal{O}_{x,E_0}}$ was given by

$$\{u, 1 + vt\} \mapsto v\frac{du}{u}.$$

This is not quite canonical, as it involves the choice of t generating $\operatorname{Ker}(\mathcal{O}_{x,E} \to \mathcal{O}_{x,E_0})$. However, $v\frac{du}{u} \otimes t \in \Omega^1_{\mathcal{O}_{x,E_0}}, \otimes_{\mathcal{O}_{E_0}} I/I^2$ is clearly canonical. Thus we obtain an exact sequence of sheaves

$$0 \to I/I^2 \otimes_{\mathcal{O}_{E_0}} \Omega^1_{E_0}, \to \mathscr{K}_{2,E} \to \mathscr{K}_{2,E_0} \to 0.$$

In fact, this exact sequence splits naturally, using the fibration $\tilde{Y} \to X$ together with the isomorphism $X \cong E_0$ to split the inclusion $E_0 \hookrightarrow E$. Hence, we have a naturally split exact sequence

$$0 \to H^1(E_0, \Omega^1_{E_0} \otimes_{\mathcal{O}_{E_0}} I/I^2) \to SK_1(E) \to SK_1(E_0) \to 0.$$

Now $E_0 \cong X_L \times_L \mathbb{C}$. Hence $\Omega^1_{E_0} \cong (\Omega^1_{X_L} \otimes_L \mathbb{C}) \oplus (\mathcal{O}_{X_L} \otimes_L \Omega^1_{\mathbb{C}})$. This gives a corresponding splitting of $\Omega^1_{E_0} \otimes_{\mathcal{O}_{E_0}} I/I^2$. Since $\Omega^1_{X_L} \otimes_L \mathbb{C} = \Omega^1_{X/\mathbb{C}}$, and $I/I^2 \cong \mathcal{O}_X(1)$, we have

$$H^1(E_0, (\Omega^1_{X_L} \otimes_L \mathbb{C}) \otimes_{\mathcal{O}_{E_0}} I/I^2) = H^1(X, \mathcal{O}_X(1) \otimes \Omega^1_{X/\mathbb{C}}) = 0$$

by Serre duality.

Again using $E_0 \cong X_L \times \mathbb{C}$, and the Künneth formula, the other direct summand reduces to $H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \Omega^1_{\mathbb{C}}$.

LEMMA (3.4): *$K_i(\tilde{Y}) \cong K_i(X)$, and the natural maps $K_i(\tilde{Y}) \to K_i(E)$ are injective. Further,* $\operatorname{coker}(K_i(\tilde{Y}) \to K_i(E)) \cong \operatorname{Ker}(K_i(E) \to K_i(E_0))$.

PROOF: Since $\tilde{Y} \to X$ is an $\mathbb{A}^1$-bundle, the first claim follows from [9], sec. 7, prop. (4.1). The remaining claims just exploit the fact that in $E_0 \hookrightarrow E \hookrightarrow \tilde{Y} \to X$, the composite $E_0 \to X$ is an isomorphism.

In particular, $H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \Omega^1_{\mathbb{C}} \hookrightarrow K_0(\tilde{Y}, E)$. The next task is to imitate the geometric construction of the boundary map used by Spencer Bloch to show that at least some of these elements land in $F_0K_0(\tilde{Y}, E)$.

From the diagram

$$\begin{tikzcd}
K_1(\tilde{Y}) \arrow[r] & K_1(E) \arrow[r, "\delta"] & K_0(\tilde{Y}, E) \arrow[r] & K_0(\tilde{Y}) \\
K_1(Y) \arrow[r] \arrow[u] & K_1(2P) \arrow[r, "\psi"] \arrow[u, "\pi^*"] & K_0(Y, 2P) \arrow[r, "\phi"] \arrow[u, "\pi^*"] & K_0(Y) \arrow[u]
\end{tikzcd}$$

and the fact that $SK_1(2P) = 0$, we claim that if $\alpha \in H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \Omega^1_{\mathbb{C}}$ is non-zero, and $\partial\alpha = \pi^*\delta$, then $\phi(\delta) \in K_0(Y)$ is also non-zero. For suppose $\delta = \psi(\gamma)$. Then $\pi^*(\gamma) - \alpha \in \mathrm{Image}(K_1(\tilde{Y}) \to K_1(E))$, which maps isomorphically to $K_1(E_0)$. But, by changing γ by an element of $K_1(Y)$ (in fact, an element of $\mathbb{C}^*$) we may assume $\gamma \mapsto 0$ in $K_1(P)$. Clearly $\pi^*(\gamma) \to 0$ in $K_1(E_0)$, from

$$\begin{tikzcd}
K_1(E) \arrow[r] & K_1(E_0) \\
K_1(2P) \arrow[r] \arrow[u] & K_1(P) \arrow[u]
\end{tikzcd}$$

Since $\alpha \mapsto 0$ in $K_1(E_0)$, $\pi^*(\gamma) - \alpha \mapsto 0$ in $K_1(E_0)$. But this forces $\pi^*\gamma - \alpha = 0$ i.e. $\pi^*(\gamma) = \alpha$. Since $K_1(2P) \xrightarrow{\pi^*} K_1(E) \to K_1(F)$ is injective (use the grading) while $\alpha \in SK_1(E)$, this forces $\alpha = 0$.

Now let $C \subset \tilde{Y}$ be a smooth (possibly disconnected) affine closed curve. Then we claim there is a map between the sequence of $(C, C \cap E)$ and $(\tilde{Y}, E)$. Let $j: C \hookrightarrow \tilde{Y}, j': C \cap E \to E$. Then we have a diagram

$$\begin{tikzcd}
\mathscr{H}_0(\tilde{Y}) \arrow[r] & \mathscr{H}(E) \\
\mathscr{P}(C) \arrow[r] \arrow[u, "j_*"] & \mathscr{P}(C \cap E) \arrow[u, "j'_*"']
\end{tikzcd}$$

($\mathscr{H}_0$ was defined when we introduced cycle classes).

This induces the maps between the sequences.

Lemma (3.5): *Let C be a smooth affine curve, $S \subset C$ a finite subscheme, $\mathcal{O}_{S,C}$ the semilocal ring of S on C. Then there is a commutative diagram (upto sign)*

$$\begin{tikzcd}
 & K_0(C, S) \\
\mathcal{O}^*_{S,C} \arrow[ur, "\eta"] \arrow[dr, "\varepsilon"'] & \\
 & K_1(S) \arrow[uu, "\partial"']
\end{tikzcd}$$

where $\partial : K_1(S) \to K_0(C, S)$ *is the boundary map of the pair* (C, S), $\varepsilon : \mathcal{O}^*_{S,C} \to K_1(S)$ *is the natural map on units, and* η *sends* $f \in \mathcal{O}^*_{S,C}$ *to the cycle class of the divisor* (f) *of* f *on* C.

PROOF: It clearly suffices to check that $\partial \circ \varepsilon(f) = \eta(f)$ for all $f \in \mathrm{Image}(\mathcal{O}_C \to \mathcal{O}_{S,C}) \cap \mathcal{O}^*_{S,C}$. Such an f can be regarded as a morphism $C \to \mathbb{A}^1$, and $[(f)] \in K_0(C, S)$ is just $f^*([0])$, where $[0] \in K_0(\mathbb{A}^1, f(S))$; and $f(S) \subset \mathbb{A}^1 - \{0\}$. So we are reduced to checking the claim in the case when $C \cong \mathbb{A}^1$, $S \subset \mathbb{A}^1 - \{0\}$, and $f = t$, the standard function on $\mathbb{A}^1$. The image of t in $K_1(S)$ is a unit. If $\mathbb{A}^1 = \operatorname{Spec} k[t]$, $S = \operatorname{Spec} k[t]/I$, then we have a diagram of rings

$$\begin{tikzcd}
k[t] \arrow[r] \arrow[d] & k[t, t^{-1}] \arrow[d] \\
k[t] \arrow[r] & k[t]/I
\end{tikzcd}$$

This induces a map between the localisation sequence for $(k[t], k[t, t^{-1}])$ and the exact sequence of the pair $(\mathbb{A}^1, S)$. In terms of categories, we have a diagram

$$\begin{tikzcd}
\mathscr{H}_0(\mathbb{A}^1) \arrow[r] \arrow[d] & \mathscr{H}_0(\mathbb{G}_m) \arrow[d] \\
\mathscr{H}_0(\mathbb{A}^1) \arrow[r] & \mathscr{H}(S)
\end{tikzcd}$$

Hence we have a diagram of spaces

$$\begin{tikzcd}
BQ\mathscr{P}(\{0\}) \arrow[r] \arrow[d] & BQ\mathscr{H}_0(\mathbb{A}^1) \arrow[r] \arrow[d] & BQ\mathscr{H}_0(\mathbb{G}_m) \arrow[d] \\
F(i^*) \arrow[r] & BQ\mathscr{H}_0(\mathbb{A}^1) \arrow[r] & BQ\mathscr{H}(S)
\end{tikzcd}$$

since the homotopy fiber in the localisation sequence is known to be homotopy equivalent to $BQ\mathscr{P}(\{0\})$. The induced map $BQ\mathscr{P}(\{0\}) \to BQ\mathscr{H}_0(\mathbb{A}^1)$ is the one which was used to define the cycle class of $[0]$ in $K_0(\mathbb{A}^1, S)$. So the lemma will follow if we can show that for $t \in K_1(k[t, t^{-1}])$, $\partial(t) = \pm [0] \in K_0(\{0\})$ in the localisation sequence. This is proved in Quillen [9].

We need one more lemma. Let $C \subset \tilde{Y}$ be as before, and let $\Pi \in \mathcal{O}_{C \cap E, Y}$ be a local generator for the ideal sheaf of C on $\tilde{Y}$. Then we have a diagram of localisation sequences

$$\begin{tikzcd}
K_2(\mathcal{O}_{C \cap E, E}) \arrow[r] & K_2(F) \arrow[r] & K_1(\mathscr{H}(C \cap E, E)) \arrow[r] & 0 \\
K_2(\mathcal{O}_{C \cap E, Y}) \arrow[r] \arrow[u] & K_2(\mathcal{O}_{C \cap E, Y}[\Pi^{-1}]) \arrow[r] \arrow[u] & K_1(\mathcal{O}_{C \cap E, C}) \arrow[r] \arrow[u, "\alpha"'] & 0
\end{tikzcd}$$

(induced from the diagram

$$\begin{tikzcd}
BQ\mathscr{H}(C \cap E, E) \arrow[r] & BQ\mathscr{H}(\mathcal{O}_{C\cap E,E}) \arrow[r] & BQ\mathscr{H}(F) \\
BQ\mathscr{P}(\mathcal{O}_{C\cap E,E}) \arrow[r] \arrow[u] & BQ\mathscr{H}_0(\mathcal{O}_{C\cap E,Y}) \arrow[r] \arrow[u] & BQ\mathscr{H}_0(\mathcal{O}_{C\cap E,Y}[\Pi^{-1}]) \arrow[u]
\end{tikzcd}$$

Here $\mathscr{H}_0(\mathcal{O}_{C\cap E,Y})$ is the category of coherent $\mathcal{O}_{C\cap E,Y}$ modules M satisfying $\mathrm{Tor}_i(M, \mathcal{O}_{C\cap E,E}) = 0$ for $i > 0$, and similarly for $\mathcal{O}_{C\cap E,Y}[\Pi^{-1}]$. Note that both rings are regular). There is another map

$$\begin{tikzcd}
\beta: K_1(\mathcal{O}_{C\cap E,C}) = \mathcal{O}^*_{C\cap E} \arrow[r, "\varepsilon"] \arrow[rr, bend right, "\beta"'] & K_1(C \cap E) \arrow[r, "j'_*"] & K_1(\mathscr{H})
\end{tikzcd}$$

(where $\mathscr{H} = \mathscr{H}(C \cap E, E)$).

LEMMA (3.6): $\alpha = \beta$.

PROOF: $\mathscr{P}(\mathcal{O}_{C\cap E,C}) \rightarrow \mathscr{H}(C \cap E, E)$ factors through the full subcategory $\mathscr{P}(C \cap E) \hookrightarrow \mathscr{H}(C \cap E, E)$.

The point of Lemma (3.6) is to use symbols for calculations, to avoid dealing with $\mathscr{H}(C \cap E, E)$. Lemmas (3.5) and (3.6) give the geometric description of the boundary map $K_1(E) \rightarrow K_0(\tilde{Y}, E)$, since we know that $K_2(\mathcal{O}_{C\cap E,Y}[\Pi^{-1}]) \rightarrow K_1(\mathcal{O}_{C\cap E,C})$ is the tame symbol (see Quillen [9]).

Finally we are ready to prove the theorem. Let $x_0, \ldots, x_n$ be homogeneous coordinates on $\mathbb{P}^n_L$. Let $a_0, \ldots, a_n \in \mathbb{C}$ be algebraically independent over L, and let $\Pi = a_0 + a_1 x_1/x_0 + \ldots + a_n x_n/x_0$ be a rational function on $\tilde{Y}$. The divisor of zeroes of Π consists of a union of fibres of the map $p: \tilde{Y} \rightarrow X$; indeed, if $H \subset \mathbb{P}^n_{\mathbb{C}}$ is the hyperplane $a_0 x_0 + \ldots + a_n x_n = 0$, the divisor is just $p^{-1}(H \cap X)$. Let D be a derivation of $\mathbb{C}$ extending $\partial/\partial a_0$ of $L[a_0, \ldots, a_n]$. Then $(d\Pi, D) = 1$, where we regard D as a derivation on functions on $\tilde{Y}$ which is 0 on functions defined over L. Now the homogeneous coord-x_0 may be regarded as a regular function on $\tilde{Y}$ in the ideal of E_0, which generates that ideal at points of $X - \{x_0 = 0\}$ (under the identification of X with E_0). Since $(\Pi = 0) \cap E_0$ consists of points not defined over L, x_0 generates I (and hence I/I^2) at these points.

Let ϕ be a rational function on $\tilde{Y}$ which is regular at the points $\{t_1, \ldots, t_p\} = (x_0 = 0) \cap E_{\mathbb{C}}$. Then $S = \phi(d\Pi/\Pi) \otimes x_0$ (where ϕ, Π are restricted to E_0) represents an element of $H^1(E_0, \Omega^1_{E_0} \otimes I/I^2) \subset SK_1(E)$, whose boundary is a relative 0-cycle. Hence Theorem 3 is proved if this class is nonzero. Clearly it is enough to show that $(S, D) \in H^1(E_0, I/I^2)$ is nonzero (where $(\ , D)$ denotes contraction with the derivation D). We will do this using Serre Duality, in its formulation in terms of residues (see [7]). Let $\omega \in H^0(E_0, \mathcal{O}_{E_0}(-1) \otimes \omega_{E_0})$, the dual vector space to $H^1(E_0, I/I^2)$; assume ω is defined over L. Then ω is nonzero at $t_1, \ldots, t_p$, and $x_0\omega/\Pi$ has a simple pole with nonzero residue at each t_i. On the affine curve $\Pi = 0$, we can find a regular function ϕ with prescribed values at each t_i. Thus, by properly choosing ϕ, we can arrange that $\sum_{i=1}^{p} \mathrm{res}_{t_i}[(\phi/\Pi) \otimes x_0\omega]$ is nonzero. This finishes the proof.

REMARKS: 1) The proof in fact shows that $K_0(Y)$ is uncountably generated, since there are uncountably many mutually algebraically independent choices of numbers $a_0, \ldots, a_n$.

2) Since derivations of the form $\partial/\partial a$ (with a running through a transcendence base for $\mathbb{C}$) span the dual of $\Omega^1_{\mathbb{C}}$, at least if we allow infinite linear combinations, one can show that $\mathrm{image}(SK_1(E) - K_0(\tilde{Y}, E)) = F_0K_0(\tilde{Y}, E)$. Hence $K_0(\tilde{Y}, E)$ is generated by algebraic cycles. This is no longer clear if the curve X is not defined over a number field, since the vector space Ω^1_k may play some role, where k is a field of definition of X. However, theorem 3 is still valid; in the final step of the proof, choose $a_0, \ldots, a_n$ to be algebraically independent over k.

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