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AN L_2-ISOLATION THEOREM FOR YANG-MILLS FIELDS OVER COMPLETE MANIFOLDS

J. Dodziuk and Min-Oo

1. Introduction

In this note we extend the results of the preceding paper [4] to the case of non compact complete manifolds. Beyond the method of [4] we only make use of appropriate cut-off functions as in [2]. This cut-off trick is due to Andreotti and Vesentini (see [5, Th. 26]). It is the point of view of [2] that every vanishing theorem based on a Weitzenböck identity generalizes from the compact to the complete case for L_2-forms. On the other hand the results in [4] are proved by applying a Sobolev inequality to a Weitzenböck formula for certain bundle valued harmonic forms. Thus it is not surprising that the L_2-isolation theorem of the preceding paper extends to complete manifolds.

We shall use freely the notation and formulae of [4]. However the isoperimetric constant c_1 will have to be replaced by another isoperimetric constant $c_0 = c_0(M)$ defined as follows:

$$c_0 = \inf_D \frac{(\mathrm{vol}(\partial D))^4}{(\mathrm{vol}(D))^3},$$

where D ranges over all open, relatively compact subsets of M with smooth boundary. M is assumed from now on to be a noncompact, complete, oriented, 4-dimensional, Riemannian manifold.

We begin by stating the results. First, our method yields a simple proof of the following result of C.-L. Shen [6].

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THEOREM 1: *Assume that*

(i) $$k_- - \frac{1}{\sqrt{3}}|\Omega_-| \geq 0$$

with strict inequality holding at some point of M.

Suppose further that β is a harmonic section of the bundle $\Lambda^2_- \otimes E$ satisfying the decay condition:

(ii) $$\lim_{R\to\infty} \frac{1}{R^2} \int_{B_{2R}(x_0)\setminus B_R(x_0)} |\beta|^2 = 0$$

for some point $x_0 \in M$. Then $\beta \equiv 0$.

In particular, if ω is a sourceless Yang-Mills field such that Ω_- satisfies (i) and (ii), then $\Omega_- \equiv 0$.

As a generalization of Theorem 2 of [4] we obtain the following

THEOREM 2: *Assume that the curvature of M satisfies $k_- \geq 0$. If $\|\Omega_-\|_2^2 < c_0/108$, then every square integrable harmonic section of $\Lambda^2_- \otimes E$ vanishes identically. In particular, if ω is a sourceless Yang-Mills field with $\|\Omega_-\|_2^2 < c_0/108$ then $\Omega_- \equiv 0$.*

Of course, this theorem is of interest only if $c_0(M) > 0$. This is the case for $\mathbb{R}^4$ with the flat metric and we obtain the following:

COROLLARY: *Let ω be a sourceless Yang-Mills field over $\mathbb{R}^4$ equipped with a complete conformally flat metric. If $\int_{\mathbb{R}^4}|\Omega_-|^2 < \pi^2 2^5/3^3$, then $\Omega_- \equiv 0$.*

This corollary yields an improvement of the constant in Theorem 3 of [4].

THEOREM 3: *Let ω be a sourceless Yang-Mills field over S^4 with a conformally flat metric. If*

$$\frac{1}{2}\int_{S^4} |\Omega|^2 < 2\pi^2\left(|p_1(E)| + \frac{16}{27}\right)$$

then ω is either self-dual or anti-self-dual.

Finally the following result gives a lower bound of the spectrum of the Laplacian Δ^{ω}_-.

THEOREM 4: *Suppose* $2k_- \geq \mu > 0$. *If*

$$\|\Omega_-\|_2^2 \leq \frac{c_0}{108}, \textit{ then } \operatorname{Spec}(\Delta_-^\omega) \subset [\mu, \infty).$$

We now prove the theorems stated above. For a given $x_0 \in M$ we can construct (cf. [5]) a family $\{\lambda_R\}_{R>0}$ of Lipschitz continuous function $\lambda_R : M \to \mathbb{R}$ with the following properties

$$\text{(2)}\qquad \begin{aligned} &\text{(i) } \operatorname{supp} \lambda_R \subset B_{2R}(x_0)\\ &\text{(ii) } 0 \leq \lambda_R \leq 1\\ &\text{(iii) } \lambda_R|_{B_R(x_0)} \equiv 1\\ &\text{(iv) } \lim_{R\to\infty} \lambda_R = 1\\ &\text{(v) } |d\lambda_R| < \frac{C}{R} \text{ a.e.,}\end{aligned}$$

where $d\lambda_R$ exists almost everywhere since λ_R is Lipschitz and the constant C is independent of R. In what follows we shall write λ for λ_R. Set $\beta_- = \beta$ in the Weitzenböck identity (3.3) of [4] and take the inner product with $\lambda^2\beta$. Integration by parts, which is permitted since $\operatorname{supp} \lambda^2\beta \subset B_{2R}(x_0)$ is compact, now yields

$$\text{(3)}\qquad \begin{aligned}(\Delta_-^\omega\beta, \lambda^2\beta) = (\nabla\beta, \nabla(\lambda^2\beta)) + \left(\frac{\kappa}{6}\beta, \lambda^2\beta\right)\\ - (\beta \circ W_-, \lambda^2\beta) - (\Omega_-, [\beta, \lambda^2\beta]).\end{aligned}$$

Leibnitz rule shows that

$$\text{(4)}\qquad (\nabla\beta, \nabla(\lambda^2\beta)) = \|\nabla(\lambda\beta)\|_2^2 - \|d\lambda \otimes \beta\|_2^2.$$

Hence if $\Delta_-^\omega\beta = 0$, estimating the last three terms on the right hand side of (3) as in [4], we obtain

$$\text{(5)}\qquad \begin{aligned}\|d\lambda \otimes \beta\|_2^2 &\geq \int_M \left(\frac{\kappa}{6} - \mu_-\right)|\lambda\beta|^2 - \frac{2}{\sqrt{3}}\int_M |\Omega_-||\lambda\beta|^2\\ &\geq 2\int_M \left(k_- - \frac{1}{\sqrt{3}}|\Omega_-|\right)|\lambda\beta|^2.\end{aligned}$$

Observe that for $\lambda = \lambda_R$, (2) implies

$$\text{(6)}\qquad \|d\lambda \otimes \beta\|_2^2 \leq \frac{c^2}{R^2}\int_{B_{2R}(x_0)/B_R(x_0)} |\beta|^2.$$

Passing to the limit as $R \to \infty$ in (5) we see that under the assumptions of Theorem 1

$$\int_M \left(k_- - \frac{1}{\sqrt{3}}|\Omega_-|\right)|\beta|^2 = 0.$$

Hence $\beta = 0$ on an open set. By the unique continuation theorem of Aronszajn, Krzywicki and Szarski (cf. [5]) $\beta \equiv 0$ and Theorem 1 is proved.

To prove Theorem 2 we use the Sobolev inequality of P. Li [3, Lemma 6]

$$\|\nabla f\|_2^2 \geq \frac{1}{9}\sqrt{c_0}\,\|f\|_4^2$$

for compactly supported functions, which implies in our case that

$$\|\nabla(\lambda\beta)\|_2^2 \leq \frac{1}{9}\sqrt{c_0}\,\|\lambda\beta\|_4^2. \tag{7}$$

Now assuming $\Delta_-^\omega\beta = 0$, substituting (4) into (3), using (7) together with the definition of k_- and the pointwise estimate (3.8) of [4], we obtain

$$\|d\lambda \otimes \beta\|_2^2 \geq \left(\frac{1}{9}\sqrt{c_0} - \frac{2}{\sqrt{3}}\|\Omega_-\|_2\right)\|\lambda\beta\|_4^2$$
$$+ 2\int k_-|\lambda\beta|^2.$$

Theorem 2 now follows by passing to the limit as $R \to \infty$, since by (6) $\lim_{R\to\infty}\|d\lambda \otimes \beta\|_2^2 = 0$ if β is square integrable.

The corollary follows from Theorem 2 by substituting the value $c_0(\mathbb{R}^4) = 2^7\pi^2$. Theorem 3 follows from the corollary since $\mathbb{R}^4$ with the flat metric is conformally equivalent to $S^4\setminus\{\text{pt.}\}$ with the standard metric and because the Yang-Mills functional is conformally invariant.

We now turn to the proof of Theorem 4. The Laplacian Δ_-^ω is essentially self-adjoint on $C_0^\infty(\Lambda_-^2 \otimes E)$. This is a consequence of completeness (cf. [1]). Thus it suffices to estimate $(\Delta_-^\omega\beta, \beta)$ for compactly supported β. From the Weitzenböck identity (3.3) of [4], the Sobolev inequality (7), the definition of k_- and the estimate (3.8) of [4] we obtain through integration by parts the following estimate:

$$(\Delta_-^\omega\beta, \beta) \geq \left(\frac{1}{9}\sqrt{c_0} - \frac{2}{\sqrt{3}}\|\Omega_-\|_2\right)\|\beta\|_4^2 + 2\int k_-|\beta|^2$$
$$\geq \mu\|\beta\|_2^2$$

provided the assumptions of Theorem 4 are satisfied. This proves the theorem.

REFERENCES

[1] P.R. CHERNOFF: Essential self-adjointness of powers of generators of hyperbolic equations. *Journal of Functional Analysis 12* (1973) 401–414.

[2] J. DODZIUK: Vanishing theorems for square integrable harmonic forms, *Geometry and analysis*, papers dedicated to the memory of V.K. Patodi, Indian Academy of Sciences and Tata Institute of Fundamental Research, Bombay, 1981, pp. 21–27.

[3] P. LI: *On the Sobolev constant and the p-Spectrum of a compact Riemannian manifold*, Ann. Scient. Éc. Norm. Sup. 4^e serie, t. 13 (1980) 451–469.

[4] MIN-OO: An L_2-isolation theorem for Yang-Mills fields, this journal.

[5] G. DERHAM: *Variétés différentiables*, Hermann, Paris 1973.

[6] C.-L. SHEN: *Gap-phenomena of Yang-Mills fields over complete manifolds*, preprint.

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Added in proof

In "Best Constant in Sobolev Inequality", *Ann. Mat. Pure Appl. 110* (1976) 353–372, G. Talenti shows that the best constant in the Sobolev inequality $\|\nabla f\|^2 \geq c\|f\|_4^2$ for functions on R^4 is $c = (8\pi/\sqrt{6})$. Using this we can improve the Corollary of Theorem 2 and Theorem 3. In the corollary the constant $\pi^2 2^5/3^3$ can be replaced by $8\pi^2$ and in Theorem 2 16/27 may be replaced by 2. The statements obtained this way are *optimal*. In fact, Bourguignon and Lawson, in "Stability and Isolation Phenomena for Yang-Mills Fields", *Commun. Math. Phys. 79* (1981) 189–230, exhibit Yang-Mills fields on S^4 with its canonical metric for which the pointwise norm $|\Omega_-| \equiv \sqrt{3}$, and hence $\|\Omega_-\|_2^2 = 8\pi^2$. Analyzing the case of equality carefully, we can show that if $\|\Omega_-\|_2^2 = 8\pi^2$, then $|\Omega_-| \equiv \sqrt{3}$. Such fields have been classified by Bourguignon and Lawson.