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Finite determinacy and topological triviality. II : sufficient conditions and topological stability

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FINITE DETERMINACY AND TOPOLOGICAL TRIVIALITY II: SUFFICIENT CONDITIONS AND TOPOLOGICAL STABILITY

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In this paper we continue the investigation of the topological triviality of unfoldings begun in part I of this paper [2]. There it was proven that for weighted homogeneous polynomial germs $f_0: k^s, 0 \to k^t, 0$ ($k = \mathbb{R}$ or $\mathbb{C}$) which are finitely $\mathscr{A}$-determined (or infinitesimally stable off the subspace of non-positive weight) any unfolding which does not decrease weights (an unfolding of non-negative weight) is topological trivial. From this there followed, for example, consequences about the topological versality of unfoldings which were not versal.

The usefulness of these results depends upon our ability to establish that germs are, for example, finitely $\mathscr{A}$-determined. Mather's algebraic characterization of finite $\mathscr{A}$-determinacy [8 III] still leaves a generally difficult algebraic problem to be solved. Most work on $\mathscr{A}$-determinacy has concentrated on improving the order of determinacy (see e.g. Duplessis [4] or Gaffney [5]). The first main result we obtain is a sufficient condition that a linear unfolding of a germ be finitely $\mathscr{A}$-determined. Neither the germ nor the unfolding need be weighted homogeneous; but in the applications this has been the case. Also, in the weighted homogeneous case, the method also establishes in an analogous manner a sufficient condition for the unfolding to be infinitesimally stable off the subspace of non-positive weight.

For a linear unfolding f of a germ f_0, we determine which relations between the extended $\mathscr{A}$-tangent space of f_0 and the deformations of f_0 obtained from f do not lift to relations for f itself. The failure of the relations to lift is measured by a series of inductively defined linear maps

τ_l defined from $\ker(\tau_{l-1})$ to a quotient of $\mathrm{coker}(\tau_{l-1})$. Here τ_l measures the failure of the relations to lift up to the l-th order. The sufficient condition for finite $\mathscr{A}$-determinacy is that one of these τ_l be surjective.

The description of the condition in this general form still sounds formidable. However, for an important class of examples it can be further simplified (and almost always reduces to the surjectivity of τ_1). This class, which consists of the uni-maximal germs (see §1), includes the uni-modal germs. For this class, an important role is played by the Euler relations. Our second main result concerns uni-maximal germs which have a special type of non-singular pairing on the space of deformations. This establishes that the Euler relations are already sufficient to guarantee that τ_1 is surjective (in general, we show that a weaker form of the pairing is still useful in proving τ_1 is surjective). Such pairings are extensions of the natural pairings on the Jacobian algebra used by Looijenga in his work on the simple elliptic singularities [7].

When we specialize these results to germs defining isolated hypersurface singularities, we recover results of Looijenga [7] and Wirthmüller [13] in terms of certain unfolding being of finite $\mathscr{A}$-codimension (or infinitesimally stable off a subspace). We also apply the criteria to two examples of finite map germs which have uni-modal singularities (§8). We do not require that the finite map germs be complete intersections; and in the equi-dimensional case we obtain the example computed by Ronga [11]. Examples of uni-modal complete intersections (of positive dimension) together with special properties which simplify the sufficient condition will appear in the final part of this paper.

Lastly, we consider the question of topological stability of unfoldings of the uni-maximal germs which satisfy the sufficient conditions. In his paper [7], Looijenga referred to the fact that the unfoldings of the simple elliptic singularities which he constructs are topologically stable germs. To clarify the role that such germs play in topological stability, we prove our final result (§9). If we form a stratum from the union of $\mathscr{K}$-orbits associated to the deformations of maximal weight for a uni-maximal germ, then any germ transverse to this stratum is topologically stable.

§1 Preliminaries

We will use the same notation as in part I [2] (in particular, we refer the reader to §§0–2). We will consider a polynomial germ $f_0 : k^s, 0 \to k^t, 0$ ($k = \mathbb{R}$ or $\mathbb{C}$) which is weighted homogeneous and finitely $\mathscr{K}$-determined. If f_0 has rank 0, then we recall that the deformation space is given by

$$N(f_0) = m_x \theta(f_0)/T\mathscr{K}_e f_0.$$

Then, $N(f_0)$ also has a weighting. We will say that f_0 is a *unimodal germ* if $\dim_k N(f_0)_{\geqq} = 1$. In general, we let $N(f_0)_{\max} = N(f_0)_m$ where $m = \max\{k : N(f_0)_k \neq 0\}$ and denote m by "max wt". We are interested in the case when max $wt \geq 0$. We will say that f_0 is a *uni-maximal germ* if $\dim_k N(f_0)_{\max} = 1$.

Furthermore, if $N(f_0)_{\max} = N(f_0)_m$ with $m \geq 0$, then we will refer to an unfolding versal in weight $< m$, as an *unfolding versal in non-maximal weight*.

To describe the general form of the sufficient condition, we indicate the form of the linear maps which measure the failure of the lifting of relations between $T\mathscr{A}_e f_0$ and the deformations of weight $< m$.

To describe this, we more generally consider any finitely $\mathscr{K}$-determined germ $f_0 : k^s, 0 \to k^t, 0$ of rank 0 and let $N \subset m_x \theta(f_0)$ be a subspace which maps injectively into $N(f_0)$ under the canonical projection π. Also, we let $\{\phi_i\}_{i=1}^q \subset m_x \theta(f_0)$ be a set which maps to a basis for the complement of $\pi(N)$.

We define a map

$$\tau_0 : \theta_s \oplus \theta_t \oplus \mathscr{C}_y\{\delta_i\}_{i=1}^q \to \theta(f_0)$$

Recall that

$$\theta_s = \mathscr{C}_x\{\varepsilon_i'\}, \ \varepsilon_i' = \frac{\partial}{\partial x_i}; \ \theta_t = \mathscr{C}_y\{\varepsilon_i\}, \ \varepsilon_i = \frac{\partial}{\partial y_i};$$

and $\theta(f_0) = \mathscr{C}_x\{\varepsilon_i\}$. Then τ_0 is defined by

$$\tau_0\left(\sum g_i(x)\varepsilon_i', \sum h_i(y)\varepsilon_i, \sum k_i(y)\delta_i\right) = \sum g_i \frac{\partial f_0}{\partial x_i} + \sum h_i \varepsilon_i + \sum k_i \phi_i.$$

In particular, $\ker(\tau_0)$ contains the relations involving $T\mathscr{A}_e f_0$ and $\mathscr{C}_y\{\phi_i\}$.

We also observe that

$$\text{Im}(\tau_0) = \mathscr{C}_x \left\{ \frac{\partial f_0}{\partial x_i} \right\} + \mathscr{C}_y \{\varepsilon_i, \phi_j\}.$$

In §3 we give a simple condition which guarantees $\text{coker}(\tau_0) \cong N$.

Given $W \subset \mathscr{C}_{x,u}\{\varepsilon_i\}$ a linear subspace, we denote $m_u^k W / m_u^{k+1} W$ by $(m_u^k / m_u^{k+1})W$. If $V \subset (m_u^k / m_u^{k+1})W$, then we let $m_u V$ denote the image in $(m_u^{k+1} / m_u^{k+2})W$. Hence, for a quotient $W' = (m_u^k / m_u^{k+1})W/V$, $m_u W' = (m_u^{k+1} / m_u^{k+2})W/m_u V$. The only possible point of confusion is in realizing that $m_u V$ is meant to be taken in $(m_u^{k+1} / m_u^{k+2})W$ (see below).

For the data $\{\phi_j\}_{j=1}^q$ and N, we will define a series of linear mappings extending τ_0

$$\tau_l : \ker(\tau_{l-1}) \to m_u \, \text{coker}(\tau_{l-1})/\mathscr{E}_{l-1} \qquad l \geq 1.$$

Here $\mathscr{E}_{l-1}$ measures an indeterminacy; and we view $\theta(f_0) = \mathscr{C}_{x,u}\{\varepsilon_i\}/m_u \mathscr{C}_{x,u}\{\varepsilon_i\}$ with u-local coordinates for $(k^q, 0)$, so that $m_u \, \text{coker}(\tau_0) \overset{\text{def}}{=} (m_u/m_u^2)\,(\theta(f_0)/\text{Im}(\tau_0))$. For these mappings, $\mathscr{E}_0$ and $\mathscr{E}_1 = 0$, so if $\text{coker}(\tau_0) \cong N$ we obtain

$$\tau_1 : \ker(\tau_0) \to (m_u/m_u^2)N.$$

For the case of weighted homogeneous germs f_0, we will be interested in the case where $\pi(N) = N(f_0)_{\geq m}$ for some $m \geq 0$ with N generated by weighted homogeneous germs of exact weight $\{\phi_j'\}$, and $\{\phi_j\}$ will be a basis of weighted homogeneous germs of exact weight for $N(f_0)_{<m}$. Then

$$\tau_1 : \ker(\tau_0) \to (m_u/m_u^2)N(f_0)_{\geq m}.$$

Also, for studying infinitesimal stability off the subspace of non-positive weight (for weighted homogeneous germs) we also define a series of linear maps τ_l^+, $l \geq 0$ ($\tau_0^+ = \tau_0$), where

$$\tau_l^+ : \ker(\tau_{l-1}^+) \to m_{u_+} \, \text{coker}(\tau_{l-1}^+)/\mathscr{E}_{l-1}^+$$

and u_+ denotes the u-coordinates of positive weight.

Again $\mathscr{E}_l^+ = 0$ for $l = 0$ or 1, and

$$\tau_1^+ : \ker(\tau_0) \to (m_{u_+}/m_{u_+}^2)N(f_0)_{\geq m}.$$

For the uni-maximal germs, the study of τ_l (or τ_l^+) is simplified when

a certain pairing involving $N(f_0)$ is non-singular. We first recall the local algebra of f_0, $Q(f_0) = \mathscr{C}_x/m_y\mathscr{C}_x$. Then, $N(f_0)$ is a $Q(f_0)$-module. This module structure together with the natural projection $p = N(f_0) \to N(f_0)_{max}$ induces a pairing

$$m_xQ(f_0) \times N(f_0)_{<max} \to N(f_0)_{max} \qquad (1.1)$$

sending $(g, \psi) \to p(g \cdot \psi)$. This, in turn, induces a dual map

$$\chi : m_xQ(f_0) \to \mathrm{Hom}_k(N(f_0)_{<max}, N(f_0)_{max}).$$

We can define a weighting on $\mathrm{Hom}_k(N(f_0)_{<max}, N(f_0)_{max})$ so that a non-zero h which vanishes except on $N(f_0)_j$ has weight $= \max wt - j$. Then, χ preserves weights. We can analogously define

$$\chi_+ : m_xQ(f_0) \to \mathrm{Hom}_k(N(f_0)_-, N(f_0)_{max}).$$

It is this $\mathrm{Im}(\chi)$ which is related to $\mathrm{Im}(\tau_1)$ on the Euler relations. Next, we define

DEFINITION 1.2: $N(f_0)$ has a *strong non-singular pairing* if
i) $N(f_0)_{<max}$ is 0 in weights $= -d_i$, all i, and
ii) χ is surjective.

For such a situation, (1.1) becomes a non-singular pairing on a quotient of $m_xQ(f_0)$. Alternately we can view $\mathrm{Hom}(N(f_0), N(f_0)_{max})$ as the dual $Q(f_0)$-module to $N(f_0)$ by viewing $N(f_0)_{max}$ as a 1-dimensional k-vector space. Then, χ being surjective is equivalent to the projection $p : N(f_0) \to N(f_0)_{max}$ being the generator of $\mathrm{Hom}_k(N(f_0), N(f_0)_{max})$ as a $Q(f_0)$-module. In general, for any weighted subspace N of $N(f_0)_{<max}$, we define

DEFINITION 1.3: $N(f_0)$ *has a non-singular pairing on* N if (i) $N_l = 0$ when $l = -d_i$, any i, and (ii) under the restriction map, χ maps onto $\mathrm{Hom}_k(N, N(f_0)_{max})$.

§2 Statement of the main results on sufficient conditions

With the preliminary definitions we have given, we can state the sufficient conditions for an unfolding to be finitely $\mathscr{A}$-determined or infinitesimally stable off the subspace of non-positive weight. In fact, for the statement involving infinitesimal stability, we are referring to the

complexification of the germ if $k = \mathbb{R}$. From now on, whenever we refer to an analytic germ f being infinitesimally stable off the subspace of non-positive weight it will be understood that we are referring to the complexification of f when $k = \mathbb{R}$.

Let $f_0 : k^s, 0 \to k^t, 0$ be a weighted homogeneous polynomial germ of rank 0 and let $\{\phi_i\}_{i=1}^q$ be a set of weighted homogeneous germs of exact weight which project to a basis for $N(f_0)_{<m}$ for some $m \geq 0$. We consider the unfolding $f(x, u) = (f_0(x) + \sum u_i \phi_i, u)$ versal in weight $< m$. We then have the maps τ_l (and τ_l^+) $l \geq 0$. If $\{\phi'_j\}$ is a set of weighted homogeneous germs of exact weight which project to a basis of $N(f_0)_{\geq m}$, then we abbreviate the condition $\mathscr{m}_y \phi'_j \subset \mathrm{Im}(\tau_0)$ by writing $\mathscr{m}_y \cdot N(f_0)_{\geq m} \subset \mathrm{Im}(\tau_0)$. Since $\mathrm{Im}(\tau_0)$ is a $\mathscr{C}_y$-module, to verify this conclusion, it is enough to verify $y_i \cdot \phi'_j \in \mathrm{Im}(\tau_0)$. Also, we shall see that the inclusion is independent of the choice of basis $\{\phi'_j\}$ for $N(f_0)_{\geq m}$.

The general form of the sufficient condition is given by the following

.THEOREM 1: *Given f_0 a weighted homogeneous polynomial germ and f an unfolding of f_0 versal in weight $< m$ (as above), suppose that $\mathscr{m}_y N(f_0)_{\geq m} \subset \mathrm{Im}(\tau_0)$.*

(i) *if some τ_l is surjective then f is finitely $\mathscr{A}$-determined,*

(ii) *if some τ_l^+ is surjective then f is infinitesimally stable off the subspace of non-positive weight.*

As a corollary of the method of proof, we can also give a sufficient condition for the finite $\mathscr{A}$-determinacy of a linear unfolding f of a finitely $\mathscr{K}$-determined germ $f_0 : k^s, 0 \to k^t, 0$. Given $N \subset \mathscr{m}_x \theta(f_0)$ which injects into $N(f_0)$ and $\{\phi_j\}_{j=1}^q$ projecting to a basis for the complement of the image of N then we have

THEOREM 2: *If for the above germ f_0, $\mathscr{m}_y N \subset \mathrm{Im}(\tau_0)$, then in order that the linear unfolding $f(x, u) = (f_0 + \sum u_i \phi_i, u)$ be finitely $\mathscr{A}$-determined it is sufficient that τ_l be surjective for some $l \geq 0$.*

REMARK 1: The conditions $\mathscr{m}_y N \subset \mathrm{Im}(\tau_0)$ could be replaced by $\mathscr{m}_y^l N \subset \mathrm{Im}(\tau_0)$ and there would be corresponding τ_k. However, for our present purposes such a generalization is not needed. In fact, only the version for weighted homogeneous germs has so far been used in the analysis of uni-modal germs.

Next, we can say precisely how in the uni-maximal case, a strong non-singular pairing gives information about τ_1 (or τ_1^+).

THEOREM 3: *Let f_0 be a weighted homogeneous polynomial germ which is uni-maximal (of rank 0). Suppose that $N(f_0)$ has a strong non-singular*

pairing, and let f be an unfolding of f_0 versal in non-maximal weight. Then,

(i) τ_1^+ *is surjective so f is infinitesimally stable off the subspace of non-positive weight*

(ii) *if $N(f_0)_0$ or $N(f_0)_+ = 0$, then τ_1 is surjective and f is finitely $\mathcal{A}$-determined.*

REMARK 2: In §6 we shall show even more; namely, that the conditions imply that τ_1^+ (or τ_1 as appropriate) is surjective when restricted to the "Euler relations". Even when $N(f_0)$ only has a non-singular pairing on a weighted subspace N we can still obtain useful information about the image of τ_1 (or τ_1^+) restricted to the 'Euler relations'. This can then be used in establishing the surjectivity of τ_1 (or τ_1^+) as in §8.

REMARK 3: For a finite germ $f_0 : k^s, 0 \to k^t, 0$, we can construct other germs $f_1 : k^{s'}, 0 \to k^{t'}, 0$ with the same local algebra as f_0 but with $t' - s' \neq t - s$ (so in particular f_1 cannot be obtained as an unfolding). In §7, we shall see that applicability of theorem 1 to f_0 relates to its applicability to f_1.

§3 Construction of τ_l and τ_l^+

We consider a weighted homogeneous polynomial germ $f_0 : k^s, 0 \to k^t, 0$, and choose a fixed set $\{\phi_j\}_{j=1}^q$ of weighted homogeneous germs of exact weight whose projections form a basis for $N(f_0)_{<m}$ where $m \geq 0$ is a fixed integer. Then, the mapping τ_0, described in §1, is defined. Here, we will describe the construction of the τ_l (and τ_l^+) and indicate how the definition of the τ_l does not depend on f_0 being weighted homogeneous. Although we define all τ_l, for most applications, it is sufficient to understand the definition of τ_1.

First, we remark that if we assign weights

$$wt(\varepsilon_i') = -wt(x_i), \; wt(\varepsilon_i) = -wt(y_i), \text{ and } wt(\delta_i) = wt(\phi_i)$$

then τ_0 is a k-linear mapping preserving weights. We begin with a lemma.

LEMMA 3.1: *In the preceding situation, if*

$$\mathscr{m}_y N(f_0)_{\geq m} \subset \mathrm{Im}(\tau_0)$$

then, there is a weight preserving isomorphism

$$\mathrm{coker}(\tau_0) \xrightarrow{\sim} N(f_0)_{\geq m}.$$

This, in turn, follows from

$$N(f_0)_{\geq m} \oplus \text{Im}(\tau_0) = \theta(f_0) \tag{3.2}$$

or equivalently

$$\text{Im}(\tau_0) = \langle \varepsilon_i, \phi_j \rangle + T\mathcal{K}_e \cdot f_0 \tag{3.3}$$

($\langle v_i \rangle$ denotes the vector space spanned by the $v_1, \ldots,$ etc.).

PROOF: First, we recall

$$\text{Im}(\tau_0) = \mathcal{C}_x \left\{ \frac{\partial f_0}{\partial x_i} \right\} + \mathcal{C}_y \{\varepsilon_i, \phi_j\}.$$

Let $\{\phi'_j\}$ project to a basis for $N(f_0)_{\geq m}$. Then

$$T\mathcal{K}_e \cdot f_0 + \langle \varepsilon_i, \phi_j, \phi'_k \rangle = \mathcal{C}_x \{\varepsilon_i\}. \tag{3.4}$$

By the preparation theorem,

$$\mathcal{C}_x \left\{ \frac{\partial f_0}{\partial x_i} \right\} + \mathcal{C}_y \{\varepsilon_i, \phi_j, \phi'_k\} = \mathcal{C}_x \{\varepsilon_i\}.$$

The hypothesis $m_y N(f_0)_{\geq m} \subset \text{Im}(\tau_0)$ then implies

$$\mathcal{C}_x \left\{ \frac{\partial f_0}{\partial x_i} \right\} + \mathcal{C}_y \{\varepsilon_i, \phi_j\} + \langle \phi'_j \rangle = \mathcal{C}_x \{\varepsilon_i\}.$$

Thus,

$$\text{Im}(\tau_0) + \langle \phi'_j \rangle = \theta(f_0). \tag{3.5}$$

However,

$$\text{Im}(\tau_0) \subseteq T\mathcal{K}_e \cdot f_0 + \langle \varepsilon_i, \phi_j \rangle \tag{3.6}$$

and by assumption

$$T\mathcal{K}_e \cdot f_0 \oplus \langle \varepsilon_i, \phi_j \rangle \oplus \langle \phi'_j \rangle = \mathcal{C}_x \{\varepsilon_i\}.$$

Thus, (3.5) is a direct sum and (3.6) must be an equality. The result follows. $\square$

For the construction we assume $m_y N(f_0)_{\geq m} \subset \mathrm{Im}(\tau_0)$. For *any* unfolding $f(x, u)$ of f_0, we will define an auxiliary map

$$\tilde{\tau}(f): \theta_s \oplus \theta_t \oplus \mathscr{C}_y\{\delta_j\} \to \mathscr{C}_{x,u}\{\varepsilon_i\}$$

and a sequence of maps $\tau_i(f)$. Then, we obtain our desired τ_i and τ_i^+ as: $\tau_i = \tau_i(f)$ (and $\tilde{\tau} = \tilde{\tau}(f)$) for f the unfolding versal in weight $< m$ given by $f(x, u) = (f_0(x) + \sum u_i \phi_i, u)$ and $\tau_i^+ = \tau_i(f_+)$ (and $\tilde{\tau}^+ = \tilde{\tau}(f_+)$) for f_+ the negative versal unfolding $f_+(x, u_+) = (f_0(x) + \sum u_{i+} \phi_i, u_+)$ (where the sum is over ϕ_i of $wt < 0$). To define $\tilde{\tau}(f)$, we represent f as usual as $f(x, u) = (\bar{f}(x, u), u)$. We first define for f a map analogous to τ_0 for f_0; namely,

$$\tau(f): \mathscr{C}_{x,u}\{\varepsilon_i'\}_{i=1}^s \oplus \mathscr{C}_{y,u}\{\varepsilon_i\}_{i=1}^t \oplus \mathscr{C}_{y,u}\{\delta_i\} \to \mathscr{C}_{x,u}\{\varepsilon_i\}_{i=1}^t \simeq \theta(\bar{f})$$

defined by

$$\tau(f)(\sum g_i \varepsilon_i', \sum h_i \varepsilon_i, \sum k_i \delta_i) = \sum g_i \cdot \frac{\partial \bar{f}}{\partial x_i} + \sum h_i \varepsilon_i + \sum k_i \phi_i.$$

We note

$$\mathrm{Im}(\tau(f)) = \mathscr{C}_{x,u}\left\{\frac{\partial \bar{f}}{\partial x_i}\right\} + \mathscr{C}_{y,u}\{\varepsilon_i, \phi_j\}.$$

We let $\tilde{\tau}(f) = \tau(f) | \theta_s \oplus \theta_t \oplus \mathscr{C}_y\{\delta_i\}$. It is important to note that under this restriction, $\tilde{\tau}(f) | \theta_t \oplus \mathscr{C}_y\{\delta_i\}$ is a homomorphism over f^* (and not f_0^*) so, in particular,

$$\tilde{\tau}(f)(0, \sum h_i(y)\varepsilon_i, \sum k_i(y)\delta_i) = \sum h_i(y \circ \bar{f})\varepsilon_i + \sum k_i(y \circ \bar{f})\phi_i.$$

The key fact about $\tilde{\tau}(f)$ for the definition of the $\tau_i(f)$ is that for $\psi \in \theta_s \oplus \theta_t \oplus \mathscr{C}_y\{\delta_i\}$

$$\tilde{\tau}(f)(\psi) \equiv \tau_0(\psi) \bmod m_u \mathscr{C}_{x,u}\{\varepsilon_i\}$$

so that if $\psi \in \ker(\tau_0)$ then   (3.7)

$$\tilde{\tau}(f)(\psi) \in m_u \mathscr{C}_{x,u}\{\varepsilon_i\}.$$

Now let pr_1 denote the composition

$$m_u \mathscr{C}_{x,u}\{\varepsilon_i\} \to m_u \mathscr{C}_{x,u}\{\varepsilon_i\}/m_u \mathrm{Im}(\tau_0) + m_u^2 \mathscr{C}_{x,u}\{\varepsilon_i\} \stackrel{\mathrm{def}}{=} m_u \cdot \mathrm{coker}(\tau_0).$$

Then, we define $\tau_1(f) = pr_1 \circ \tilde{\tau}(f)$

$$\tau_1(f) : \ker(\tau_0) \to m_u \cdot \operatorname{coker}(\tau_0) \simeq (m_u/m_u^2)N(f_0)_{\geq m}.$$

Inductively, we define

$$\tau_l(f) : \ker(\tau_{l-1}(f)) \to m_u \cdot \operatorname{coker}(\tau_{l-1}(f))/\mathscr{E}_{l-1}$$

where

$$\mathscr{E}_{l-1} = pr_l(m_u \cdot \operatorname{Im}(\tilde{\tau}(f)) \cap m_u^l \mathscr{C}_{x,u}\{\varepsilon_i\})$$

and pr_l denotes projection $m_u^l \mathscr{C}_{x,u}\{\varepsilon_i\} \to m_u \operatorname{coker}(\tau_{l-1}(f))$.

The inductive definition of $\tau_l(f)$ will be such that:

(i) $\ker(\tau_{l-1}(f))$ consists of those $\psi \in \ker(\tau_0)$ such that there is a $\phi \in m_u \cdot \operatorname{Im}(\tilde{\tau}(f))$ so that $\tilde{\tau}(f)(\psi) - \phi \in m_u^l \theta(f)$.

(ii) $m_u \cdot \operatorname{coker}(\tau_{l-1})/\mathscr{E}_{l-1} \simeq m_u^l \theta(f)/(m_u \quad \operatorname{Im}(\tilde{\tau}(f)) \cap m_u^l \theta(f)) + m_u^{l+1}\theta(f)$.

Note by (3.7) that $m_u^l \cdot \operatorname{Im}(\tilde{\tau}(f)) \equiv m_u^l \cdot \operatorname{Im}(\tau_0) \bmod m_u^{l+1}\theta(f)$ so

$$m_u^l \cdot \operatorname{Im}(\tau_0) \subset (m_u \operatorname{Im}(\tilde{\tau}(f)) \cap m_u^l \theta(f)) + m_u^{l+1}\theta(f).$$

Then we inductively define $\tau_l(f)$ by

$$\tau_l(f)(\psi) = pr_l'(\tilde{\tau}(f)(\psi) - \phi)$$

where pr_l' denotes the projection

$$m_u^l \mathscr{C}_{x,u}\{\varepsilon_i\} \to m_u \cdot \operatorname{coker}(\tau_{l-1}(f))/\mathscr{E}_{l-1}.$$

It follows that $\tau_l(f)$ is well-defined up to the choice of ϕ, i.e., up to $pr_l'(\phi - \phi')$, where $\phi - \phi' \in m_u \operatorname{Im}(\tilde{\tau}(f)) \cap m_u^l \theta(f)$. Thus, the indeterminacy is in $\mathscr{E}_{l-1}$.

Lastly, we verify the inductive assumptions on $\ker(\tau_l(f))$ and $m_u \cdot \operatorname{coker}(\tau_l)/\mathscr{E}_l$. By the inductive assumption, $\psi \in \ker(\tau_l(f))$ if there is a $\phi \in m_u \operatorname{Im}(\tilde{\tau}(f))$ such that $\tilde{\tau}(f)(\psi) - \phi \in m_u^{l+1}\theta(f)$.
Also,

$$\operatorname{coker}(\tau_l(f)) \simeq m_u^l \theta(f)/M + m_u^{l+1}\theta(f) \tag{3.8}$$

where $M \subset \mathscr{C}_u\{\operatorname{Im}(\tilde{\tau}(f))\} \cap m_u^l \theta(f)$.
Thus, $m_u \cdot M \subset m_u \operatorname{Im}(\tilde{\tau}(f)) \cap m_u^{l+1}\theta(f)$.

Hence, the inductive verification of (ii) also follows.

REMARK 1: From the definition, it follows that $\mathscr{E}_1 = 0$; thus

$$\tau_2 : \ker(\tau_1) \to \mathscr{m}_u \operatorname{coker}(\tau_1).$$

Beyond this the τ_l are increasingly difficult to work with. Fortunately, all of the calculations involving unimodal singularities, with but one exception to be described in the last part of this paper, only involve τ_1.

REMARK 2: Note that if we: 1) replace $N(f_0)_{\geq m}$ by an $N \subset \mathscr{m}_x \theta(f_0)$ which injects into $N(f_0)$, 2) let $\{\phi_j\} \subset \mathscr{m}_x \theta(f_0)$ be a set which projects onto a basis for the complement of N, and 3) drop any references to weights then the construction of the $\tilde{\tau}(f)$ and $\tilde{\tau}_l(f)$ goes through without change (even including lemma 3.1).

§4 Infinitesimal stability off of a subspace

Before proceeding with the proof of theorem 1, we first find it necessary to establish the converse of proposition 5.1 in Part I. We refer the reader to the notation used for that proposition. There we considered an unfolding f of $f_0 : k^s, 0 \to k^t, 0$ with both germs analytic and f_0 finitely $\mathscr{K}$-determined. We also let f_0 and f denote representatives of the germs or their complexifications in the case $k = \mathbb{R}$. Then, with the notation of that section we defined for $S = f^{-1}(y, u) \cap \sum(f) \cap U$

$$V = \{(y, u) \in W : \bar{f}(\cdot, u) : \mathbb{C}^s, S \to \mathbb{C}^t, (y, u)$$
$$\text{is not infinitesimally stable}\}$$

we let I be the ideal of holomorphic germs vanishing on V.

Proposition 5.1 established a torsion condition involving I; its converse is given by the following

PROPOSITION 4.1: *Let f_0 and f be holomorphic germs represented by mappings defined in neighbourhoods of 0 as above. We also let $I' \subset \mathscr{C}_{y,u}$ be an ideal such that there is a $k > 0$ so that*

$$I'^k \theta(\bar{f}) \subset \operatorname{Im}(\tau(f)).$$

Then, on some possibly smaller neighbourhood W', $V \subset V(I')$; so, in particular, $\bar{f}(\cdot, u)$ is infinitesimally stable off of $V(I') \cap W'$.

PROOF: In the proof of proposition 5.1 of part I, we constructed a coherent sheaf $\mathcal{N}(\bar{f})$ with $\mathrm{supp}(\mathcal{N}(\bar{f})) = V$. Let $\{\phi_i\}_{i=1}^r$ denote a set of generators for $\mathcal{N}(\bar{f})$ as a $\mathcal{O}_W$ module. By shrinking W if necessary to a W', and restricting all sheaves, we may choose generators for I', $\{g_i\} \subset \mathcal{O}_{W'}$ and holomorphic mappings $h_{ijl} \in \mathcal{O}_U$, $k_{ijl} \in \mathcal{O}_{W'}$ so that

$$g_i^k \cdot \phi_j = \sum h_{ijl}\frac{\partial \bar{f}}{\partial x} + \sum k_{ijl}\varepsilon_l. \tag{4.2}$$

Now, at a point $(y_1, u_1) \in W' - V(I')$, some $g_i(y_1, u_1) \neq 0$. Thus, g_i is a unit at y_1 and (4.2) implies that $g_i^k\phi_j = 0$ in $\mathcal{N}(\bar{f})_{(y_1, u_1)}$. As the $\{g_i^k \cdot \phi_j\}_{j=1}^r$ generate $\mathcal{N}(\bar{f})_{(y_1, u_1)}$ we conclude $\mathcal{N}(\bar{f})_{(y_1, u_1)} = 0$. Thus, $\mathrm{Supp}(\mathcal{N}(\bar{f})) \cap W' \subset V(I')$.

§5 Establishing the general sufficient conditions

Our central goal in proving theorem 1 is to reduce it to condition that for appropriate unfoldings f of $f_0 : k^s, 0 \to k^t, 0$, $\mathrm{Im}(\tau(f))$ has finite codim in $\mathscr{C}_{x,u}\{\varepsilon_i\}_{i=1}^t = \theta(f)$. Thus, in what follows we fix a basis $\{\phi_j\}_{j=1}^q$ for $N(f_0)_{<m}$ for a fixed $m \geq 0$. Thus, τ_0 is defined and we suppose $m_y \cdot N(f_0)_{\geq m} \subset \mathrm{Im}(\tau_0)$. We consider any unfolding of f_0, $f(x, u) = (\bar{f}(x, u), u)$. The maps $\tau(f)$ and $\tau_t(f)$ are defined. We recall from §3 that

$$\mathrm{Im}(\tau(f)) = \mathscr{C}_{x,u}\left\{\frac{\partial \bar{f}}{\partial x_i}\right\} + \mathscr{C}_{y,u}\{\varepsilon_i, \phi_j\}$$

LEMMA 5.1: *Consider f_0 as above such that $m_y N(f_0)_{\geq m} \subset \mathrm{Im}(\tau_0)$. Let f be any unfolding of f_0. In order that*

$$\dim_k \theta(\bar{f})/\mathrm{Im}(\tau(f)) < \infty$$

it is sufficient that there is a $k > 0$ so that

$$m_u^k \cdot N(f_0)_{\geq m} \subset \mathrm{Im}(\tau(f)) + m_u^k \mathrm{Im}(\tau_0) + m_u^{k+1}\theta(\bar{f}). \tag{5.2}$$

PROOF: We know by (3.2) that if $\{\phi_j'\}$ projects to a basis of $N(f_0)_{\geq m}$ then

$$\mathrm{Im}(\tau_0) + \langle \phi_j' \rangle = \theta(f_0).$$

Thus, by the preparation theorem,

$$\mathrm{Im}(\tau(f)) + \mathscr{C}_u\{\phi_j'\} = \theta(\bar{f}) \tag{5.3}$$

We let

$$E_k = \mathrm{Im}(\tau(f)) + m_u^k\{\phi_j'\}.$$

By (5.3), it is sufficient to show $E_k \subseteq \mathrm{Im}(\tau(f))$ for some $k \geq 0$. In fact, by Nakayama's lemma applied to the finitely generated $\mathscr{C}_u$-module $(E_k/\mathrm{Im}(\tau(f)))$ it is sufficient to show $E_k \subseteq E_{k+1}$ (then

$$m_u(E_k/\mathrm{Im}(\tau(f))) = E_{k+1}/\mathrm{Im}(\tau(f)) = E_k/\mathrm{Im}(\tau(f))).$$

Next, by (5.3) we have

$$m_u^k \cdot \theta(\bar{f}) \subset \mathrm{Im}(\tau(f)) + m_u^k\{\phi_j'\}$$

hence

$$E_k = \mathrm{Im}(\tau(f)) + m_u^k\theta(\bar{f}). \tag{5.4}$$

Also, by (3.7)

$$\mathrm{Im}(\tau(f)) \equiv \mathrm{Im}(\tau_0) \quad (\mathrm{mod}\ m_u\theta(\bar{f})).$$

Thus,

$$m_u^k\,\mathrm{Im}(\tau(f)) \equiv m_u^k\,\mathrm{Im}(\tau_0)\,\mathrm{mod}\,m_u^{k+1}\theta(\bar{f}) \qquad k \geq 0.$$

Hence

$$E_{k+1} = \mathrm{Im}(\tau(f)) + m_u^k\,\mathrm{Im}(\tau_0) + m_u^{k+1}\theta(\bar{f}).$$

Then, the condition that $E_k \subseteq E_{k+1}$ is exactly (5.2).   □

We specialize to two unfoldings: $f(x, u) = (f_0 + \sum u_i\phi_i, u)$, the unfolding versal in weight $< m$, and $f_+(x, u_+) = (f_0 + \sum u_{i+}\phi_i, u)$ (where the sum is over ϕ_i of $wt(\phi_i) < 0$), the negative versal unfolding. Then, we relate our conditions by

LEMMA 5.5: *For f_0, f, f_+ as above we have*
(i) *f is a finitely $\mathscr{A}$-determined germ if and only if*

$$\dim_k \theta(\bar{f})/\mathrm{Im}(\tau(f)) < \infty \tag{5.6}$$

(ii) f is infinitesimally stable off the subspace of non-positive weight if and only if

$$\dim_k \theta(\bar{f}_+)/\mathrm{Im}(\tau(f_+)) < \infty. \tag{5.7}$$

PROOF: We have for (i)

$$T\mathscr{A}_e \cdot f = \mathscr{C}_{x,u}\left\{\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial u_j}\right\} + \mathscr{C}_{y,u}\{\varepsilon_i\}.$$

Since $\dfrac{\partial f}{\partial u_j} = \varepsilon_{t+j} + \phi_j = \bar{\varepsilon}_{t+j}$ and $\dfrac{\partial f}{\partial x_i} = \dfrac{\partial \bar{f}}{\partial x_i}$, we have

$$T\mathscr{A}_e \cdot f = \mathscr{C}_{x,u}\left\{\frac{\partial \bar{f}}{\partial x_i}\right\} + \mathscr{C}_{x,u}\{\bar{\varepsilon}_{t+j}\} + \mathscr{C}_{y,u}\{\varepsilon_i\}_{i=1}^t + \mathscr{C}_{y,u}\{\phi_j\}$$

$$= \mathrm{Im}(\tau(f)) + \mathscr{C}_{x,u}\{\bar{\varepsilon}_{t+j}\}.$$

As $\theta(f) = \theta(\bar{f}) \oplus \mathscr{C}_{x,u}\{\bar{\varepsilon}_{t+j}\}$, and $\mathrm{Im}(\tau(f)) \subset \theta(\bar{f})$ we conclude that $T\mathscr{A}_e \cdot f$ has finite codimension in $\theta(f)$ if and only if (5.6) holds.

For (ii), let the set of polynomial germs of exact weight $\{\psi_j\} \in \theta(\bar{f}_+)$ project to a basis for $\theta(\bar{f}_+)/\mathrm{Im}(\tau(f_+))$. If (5.7) holds we may assume $\{\psi_j\}$ is finite. Thus,

$$\mathrm{Im}(\tau(f_+)) + \langle \psi_j \rangle = \theta(\bar{f}_+).$$

By the preparation theorem applied to f viewed as an unfolding of f_+ we have

$$\mathrm{Im}(\tau(f)) + \mathscr{C}_{u'}\{\psi_j\} = \theta(\bar{f}) \tag{5.8}$$

(where u' denotes the u-coordinates of non-positive weight). Then, as $\{\psi_j\}$ is finite, we let

$$M = \max\{wt(\psi_j)\} \quad \text{and} \quad N = \min\{wt(\psi_j)\}.$$

Since $wt(u_i') \leq 0$, we have

$$\theta(\bar{f})_{>M} \subset \mathrm{Im}(\tau(f)).$$

Then, if $l > M - N$,

$$\mathscr{m}_{y,u_+}^l \cdot \psi_j \subset \mathrm{Im}(\tau(f)). \tag{5.9}$$

Now multiplying (5.8) by m_{y,u_+}^l we obtain

$$m_{y,u_+}^l \cdot \theta(\bar{f}) \subset \text{Im}(\tau(f)) + m_{y,u_+}^l \cdot \mathscr{C}_{u'}\{\psi_j\}.$$

Since the right hand side of (5.9) is a $\mathscr{C}_{u'}$-module we have

$$m_{y,u_+}^l \cdot \mathscr{C}_{u'}\{\psi_j\} \subset \text{Im}(\tau(f)),$$

hence

$$m_{y,u_+}^l \cdot \theta(\bar{f}) \subset \text{Im}(\tau(f)). \tag{5.10}$$

Thus, by proposition 4.1, f is infinitesimally stable off the subspace of non-positive weight. Conversely, by proposition 5.1 of part I we obtain (5.9). Since f_0 is finitely $\mathscr{K}$-determined so is f; thus, at least $\theta(\bar{f})/\text{Im}(\tau(f))$ ·is a finitely generated $\mathscr{C}_{y,u}$-module. Then, we repeat the type of argument used in proposition 5.4 of part I to conclude first that $\theta(\bar{f})/\text{Im}(\tau(f))$ is a finitely generated $\mathscr{C}_{u'}$-module and then reducing mod $m_{u'}$ we conclude $\theta(\bar{f}_+)/\text{Im}(\tau(f_+))$ has finite k-dimension. □

We can now easily complete the proofs of the theorems. By the preceding lemma, it is sufficient to show:

$$\dim_k \theta(\bar{f})/\text{Im}(\tau(f)) < \infty \text{ if some } \tau_k(f) \text{ is surjective.}$$

In turn, by lemma (5.1), it is sufficient to show that for some $k \geq 0$ if $\tau_k(f)$ is surjective,

$$m_u^k \cdot N(f_0)_{\geq m} \subset \text{Im}(\tau(f)) + m_u^k \text{Im}(\tau_0) + m_u^{k+1} \theta(\bar{f}).$$

Now, $\text{Im}(\tilde{\tau}(f)) \subset \text{Im}(\tau(f))$; thus, so is $\mathscr{C}_u\{\text{Im}(\tilde{\tau})\}$, which contains $\mathscr{E}_k$. Also, by (3.8)

$$m_u \text{coker}(\tau_{k-1}(f))/\mathscr{E}_{k-1} \xrightarrow{\sim} m_u^k \theta(\bar{f})/M + m_u^{k+1} \theta(\bar{f})$$

where $M \subset \mathscr{C}_u(\text{Im}(\tilde{\tau}) \cap m_u^k \theta(\bar{f}))$, which contains $m_u^k \text{Im}(\tau_0)$ (mod $m_u^{k+1} \theta(\bar{f})$). As $\tau_k(f)$ is surjective

$$m_u^k \cdot N(f_0)_{\geq m} \subset \text{Im}(\tau(f)) + m_u^k \text{Im}(\tau_0) + m_u^{k+1} \theta(\bar{f}). \qquad □$$

REMARK: Again, nowhere in the proof for the finite $\mathscr{A}$-determinacy is there dependence on the weighted homogeneity and the proof works as well for the non-weighted homogeneous case.

As a final corollary of the proof we have

COROLLARY 5.11: *If f_0 is a weighted homogeneous germ such that $m_y \cdot N(f_0)_{\geq m} \subset \mathrm{Im}(\tau_0)$ and τ_1 is surjective then*

$$\mathscr{A}\text{-codim } f = \dim_k N(f_0)_{\geq m}$$

for f the unfolding versal in weight $< m$.

PROOF: We have $m_u \theta(\bar{f}) \subset \mathrm{Im}(\tau(f))$ so

$$\mathscr{A}\text{-codim } f \leq \dim_k \theta(\bar{f})/(\mathrm{Im}(\tau(f)) + m_u \theta(\bar{f})) \leq \dim_k N(f_0)_{\geq m}.$$

On the other hand,

$$T\mathscr{A}_e \cdot f \subseteq T\mathscr{K}_e \cdot f + \langle \varepsilon_i \rangle_{i=1}^{t+q}$$

which has codimension $= \dim_k N(f_0)_{\geq m}$.

§6 Sufficient conditions via non-singular pairings

In this section we investigate the role of non-singular pairings on $N(f_0)$ in determining the image of τ_1 for an unfolding of f_0 versal in non-maximal weight. We shall see that such pairings lead to considerable information about τ_1 restricted to the Euler relations.

We consider a (finitely $\mathscr{K}$-determined) weighted homogeneous germ $f_0 : k^s, 0 \rightarrow k^t, 0$ of rank 0. We let $\{\phi_j\} \subset m_x \theta(f_0)$ be weighted homogeneous germs of exact weight which project to a basis for $N(f_0)_{<\max}$. For now, we do not assume f_0 is unimaximal, but we do assume $m_y N(f_0)_{\max} \subset \mathrm{Im}(\tau_0)$. We also let $f(x, u) = (f_0 + \sum u_i \phi_i, u)$ be the unfolding versal in non-maximal weight (and we assume the maximal weight ≥ 0).

If we choose a basis $\{\phi_j'\}$ of germs of exact weight for $N(f_0)_{\max}$, then $(m_u/m_u^2)N(f_0)_{\max}$ has a basis $\{u_i \phi_j'\}$. By assigning to $u_i \phi_j'$ its weight as an element of $\mathscr{C}_{x,u}\{\varepsilon_i\}$, we obtain a weighting for $(m_u/m_u^2)N(f_0)_{\max}$. Then, it readily follows that τ_1 preserves weights.

Now, by the weighted homogeneity of f_0 there is the Euler relation

$$\sum a_i x_i \frac{\partial f_0}{\partial x_i} - \sum d_i f_{0i} \varepsilon_i = 0$$

where $a_i = wt(x_i)$ and $d_i = wt(f_{0i})$ for $f_{0i} = y_i \circ f_0$. If $e = \sum a_i x_i \varepsilon_i' - \sum d_i y_i \varepsilon_i$, then the Euler relation states $\tau_0(e) = 0$. If g is any other weighted homogeneous germ in $\mathscr{C}_x$, then we also obtain another Euler relation

$$g \cdot \tau_0(e) = \sum a_i g x_i \frac{\partial f_0}{\partial x_i} - \sum d_i g y_i \varepsilon_i = 0.$$

To use this relation in the computation of τ_1 we must express it as an element of $\ker(\tau_0)$. However, $g \cdot y_i \varepsilon_i \in m_y \mathscr{C}_x\{\varepsilon_i\}$ and by (3.3), $m_y \mathscr{C}_x\{\varepsilon_i\} \subset \operatorname{Im}(\tau_0)$. Thus, $g y_i \varepsilon_i = \tau_0(\lambda_i)$; and hence

$$\tau_0(\sum a_i g x_i \varepsilon_i' - \sum d_i \lambda_i) = \sum a_i g x_i \frac{\partial f_0}{\partial x_i} - \sum d_i g y_i \varepsilon_i = 0.$$

We can be even more precise than this about the λ_i. By lemma 3.1, τ_0 is surjective in all weights $\neq \max wt$; hence, if $wt(g\varepsilon_i) \neq \max wt$, then $g\varepsilon_i = \tau_0(\lambda_i')$ and we can choose $\lambda_i = y_i \lambda_i'$ with $wt(\lambda_i') = wt(g\varepsilon_i)$. For such a weighted homogeneous g so that $wt(g\varepsilon_i) \neq \max wt$ for any i, we define

$$\psi_g = \sum a_i x_i g \varepsilon_i' - \sum d_i y_i \lambda_i' \in \ker(\tau_0)$$

We let the vector space spanned by such ψ_g be denoted by E. We think of E as representing a subset of the Euler relations. It is on this subset that we obtain information about τ_1. Note that there is some indeterminacy in the definition of the ψ_g and hence E. Nonetheless, we claim that $\tau_1(E)$ is well defined.

If we had represented $g \cdot \varepsilon_i = \tau_0(\lambda_i'')$ and obtained instead ψ_g', then $\psi_g - \psi_g' = \sum d_i y_i(\lambda_i' - \lambda_i'')$ with $\lambda_i' - \lambda_i'' \in \ker(\tau_0)$. That $\tau_1(E)$ is well-defined is a consequence of the following lemma

LEMMA 6.1: *Suppose that for f_0, $m_y \cdot N(f_0)_{\max} \subset \operatorname{Im}(\tau_0)$. If $\psi \in \theta_s \oplus \theta_t \oplus \mathscr{C}_y\{\delta_i\}$ and f is the unfolding versal in non-maximal weight defined via the $\{\phi_j\}$, then*

$$\tilde{\tau}(y_i \psi) \equiv (y_i \circ \tilde{f}) \tau_0(\psi) \bmod(m_u \operatorname{Im}(\tau_0) + m_u^2 \theta(\tilde{f})).$$

(Recall $\tilde{\tau} = \tilde{\tau}(f)$ for the above unfolding f.)

PROOF: Since both sides are linear in ψ it is sufficient to verify this on each component θ_s and $\theta_t \oplus \mathscr{C}_y\{\delta_i\}$.

If $\psi \in \theta_s$.

$$\tilde{\tau}(y_i \cdot \psi) = (y_i \circ f_0)\tilde{\tau}(\psi).$$

Then,

$$\tilde{\tau}(y_i\psi) - (y_i \circ \bar{f})\tau_0(\psi) = (y_i \circ f_0)(\tau(\psi) - \tau_0(\psi)) - (y_i \circ \bar{f} - y_i \circ f_0)\tau_0(\psi)$$

By (3.7) $\tilde{\tau}(\psi) - \tau_0(\psi) \in m_u\theta(\bar{f})$ so

$$(y_i \circ f_0)(\tilde{\tau}(\psi) - \tau_0(\psi)) \in m_u(m_y \cdot \mathscr{C}_x\{\varepsilon_i\}) + m_u^2\theta(\bar{f})$$

However, by (3.3) $m_y\mathscr{C}_x\{\varepsilon_i\} \subset \operatorname{Im}(\tau_0)$. Also, $y_i \circ \bar{f} - y_i \circ f_0 \in m_u\mathscr{C}_{x,u}$ and $\tau_0(\psi) \subset \mathscr{C}_x\left\{\dfrac{\partial f_0}{\partial x_i}\right\}$; thus,

$$(y_i \circ \bar{f} - y_i \circ f_0)\tau_0(\psi) \in m_u\mathscr{C}_x\left\{\frac{\partial f_0}{\partial x_i}\right\} + m_u^2\theta(\bar{f}) \subset m_u\operatorname{Im}(\tau_0) + m_u^2\theta(\bar{f}).$$

Next, for $\psi \in \theta_t \oplus \mathscr{C}_y\{\delta_i\}$, $\tilde{\tau}(y_i\psi) = (y_i \circ \bar{f})\tilde{\tau}(\psi)$. Thus,

$$\tilde{\tau}(y_i\psi) - (y_i \circ \bar{f})\tau_0(\psi) = (y_i \circ \bar{f})(\tilde{\tau}(\psi) - \tau_0(\psi)).$$

As before,

$$(y_i \circ \bar{f})(\tilde{\tau}(\psi) - \tau_0(\psi)) \in m_u\operatorname{Im}(\tau_0) + m_u^2\theta(\bar{f}). \qquad \square$$

In the special case when $\psi \in \ker(\tau_0)$, we obtain

$$\tilde{\tau}(y_i \cdot \psi) \equiv 0 \bmod(m_u\operatorname{Im}(\tau_0) + m_u^2\theta(\bar{f})).$$

Thus, $\tau_1(E)$ is well-defined. In fact, this lemma provides us with a simplified way to compute $\tau_1(E)$.

LEMMA 6.2: *With the preceding notation*

$$\tau_1(\psi_g) = pr_1(g \cdot \tilde{\tau}(e)).$$

PROOF:

$$\tau_1(\psi_g) = pr_1(\tilde{\tau}(\sum a_i g x_i \varepsilon_i' - \sum d_i y_i \lambda_i'))$$

where $\tau_0(\lambda_i') = g\varepsilon_i$. Then

$$\tilde{\tau}(\sum a_i g x_i \varepsilon_i') = g \cdot \tilde{\tau}(\sum a_i x_i \varepsilon_i').$$

Also, by the preceding lemma

$$\tilde{\tau}(\sum d_i y_i \lambda_i') \equiv \sum d_i (y_i \circ f) \tau_0(\lambda_i') \bmod(m_u \operatorname{Im}(\tau_0) + m_u^2 \theta(f))$$

$$\equiv \sum d_i g \tilde{\tau}(y_i \varepsilon_i)$$

$$\equiv g \tilde{\tau}(\sum d_i y_i \varepsilon_i).$$

Thus,

$$\tau_1(\psi_g) = pr_1(g \cdot \tilde{\tau}(\sum a_i x_i \varepsilon_i' - \sum d_i y_i \varepsilon_i)) = pr_1(g \cdot \tilde{\tau}(e)). \qquad \square$$

REMARK: A similar argument works for τ_1^+.

We can now state a result which explains the role of a non-singular pairing in computing $\tau_1(E)$ or $\tau_1^+(E)$.

We begin by observing that given the choice $\{\phi_j\}$ of basis for $N(f_0)_{<\max}$, then for $l \neq \max wt$ there is a natural isomorphism

$$\operatorname{Hom}_k(N(f_0)_{<\max}, N(f_0)_{\max})_l \xrightarrow{\sim} ((m_u/m_u^2)N(f_0)_{\max})_l \qquad (6.3)$$

sending $h \to \sum_i - wt(u_i)u_i h(\phi_i)$.

We have an analogous isomorphism for all l

$$\operatorname{Hom}_k(N(f_0)_-, N(f_0)_{\max})_l \xrightarrow{\cong} ((m_{u_+}/m_{u_+}^2)N(f_0)_{\max})_l \qquad (6.4)$$

Then, via these isomorphisms, we relate the images of τ_1 and the dual map χ (recall §1) by

THEOREM 6.5: *If f_0 is a uni-maximal germ with $m_y N(f_0)_{\max} \subset \operatorname{Im}(\tau_0)$ then in weights $l \neq \max wt + d_i$, any i ($d_i = wt\, y_i$):*
  (i) *via the isomorphism (6.3), $\operatorname{Im}(\chi)_l$ is isomorphic to $\tau_1(E)_l$*
  (ii) *via the isomorphism (6.4), $\operatorname{Im}(\chi_+)_l$ is isomorphic to $\tau_1^+(E)_l$.*

PROOF: By lemma 6.2, if $wt\, g \neq \max wt + d_i$, any i,

$$\tau_1(\psi_g) = pr_1(g \cdot \tilde{\tau}(e)).$$

Also, by the Euler relation

$$\tilde{\tau}(e) = - \sum wt(u_j)u_j \phi_j.$$

Thus,

$$\tau_1(\psi_g) = -\sum wt(u_j)pr_1(u_j g \cdot \phi_j)$$

$$= \sum -wt(u_j)u_j\chi(g)(\phi_j)$$

$$= \chi(g) \text{ via the isomorphism (6.3).} \qquad \square$$

An identical argument works for τ_1^+ and χ_+.

As a corollary we have the result which implies theorem 3.

COROLLARY 6.6: *If f_0 is a uni-maximal germ so that $N(f_0)$ has a strong non-singular pairing and $m_y N(f_0)_{max} \subset \text{Im}(\tau_0)$, then*

(i) $\tau_1^+(E) = (m_{u_+}/m_{u_+}^2)N(f_0)_{max}$
(ii) *if $N(f_0)_0$ or $N(f_0)_+ = 0$, then*

$$\tau_1(E) = (m_u/m_u^2)N(f_0)_{max}.$$

PROOF: By the definition of strong non-singular pairing, χ is surjective (hence χ_+ is also surjective) and $(m_u/m_u^2)N(f_0)_{max}$ is 0 in weights $= \max wt + d_i$, any i. Thus, by the preceding theorem τ_1 or τ_1^+ has the desired properties. $\qquad \square$

As the next corollary we have the consequences of the calculations of Looijenga and Wirthmüller.

COROLLARY 6.7: *Let $f_0: k^s, 0 \to k, 0$ be a non-simple weighted homogeneous germ defining an isolated hypersurface singularity. Then there is an unfolding versal in non-maximal weight f such that:*

(i) *f is infinitesimally stable off the space of non-positive weight*
(ii) *if f_0 is simple elliptic or $N(f_0)_0 = 0$, then f is finitely $\mathcal{A}$-determined.*

PROOF: In this case, $\theta(f_0)/T\mathcal{K}_e \cdot f_0 \simeq \mathscr{C}_x/\Delta(f_0)$ where $\Delta(f_0)$ is the ideal generated by $\dfrac{\partial f_0}{\partial x_i}$. Thus, $\theta(f_0)/T\mathcal{K}_e \cdot f_0$ has a natural structure of an algebra. Furthermore, the pairing reduces to the multiplication $\mathscr{C}_x/\Delta(f_0) \times \mathscr{C}_x/\Delta(f_0) \to \mathscr{C}_x/\Delta(f_0) \to N(f_0)_{max}$ composed with projection. By Grothendieck local duality theory (see e.g. [1] or [6]) this is a non-singular pairing. The remaining conditions for a strong non-singular pairing are trivially satisfied. $\qquad \square$

Finally, as a last corollary we observe

COROLLARY 6.8: *Let f_0 be a uni-maximal germ with*

$m_y \cdot N(f_0)_{\max} \subset \mathrm{Im}(\tau_0)$ *and a non-singular pairing on* N. *Then,*

$$\dim_k \tau_1(E) \geq \dim_k N.$$

PROOF: By the non-singular pairing property,

$$\dim_k \mathrm{Im}(\chi) \geq \dim_k N \text{ and } N_l = 0 \text{ for } l = -d_i, \text{ any } i.$$

Thus, by Theorem 6.5 the result follows.

§7 Finite map germs

A finite map germ $f_0 : k^s, 0 \to k^t, 0$ is a germ with $s \leq t$ which is finitely $\mathcal{K}$-determined (this is equivalent to $\dim_k Q(f_0) < \infty$ [8, IV]). For any embedding $i : k^t, 0 \to k^{t+r}, 0$, $f_1 = i \circ f_0$ is another finite map germ with the same local algebra. If $f : k^{s+q}, 0 \to k^{t+q}$ is an infinitesimally stable unfolding of f_0, then there is a simple way to construct from f the infinitesimally stable unfolding of f_1. Suppose i is the identity $i : k^t$, $0 \to k^t \times \{0\}, 0$, and $\{\tilde{\phi}_i\}$ projects to a basis for $m_x Q(f_0)$. Then, defining $g(x, v) = (\sum v_{1i} \tilde{\phi}_i, \ldots, \sum v_{ti} \tilde{\phi}_i)$ we obtain the infinitesimally stable unfolding F of f_1 given by

$$F(x, u, v) = (\bar{f}(x, u), g(x, v), u, v).$$

If f is only finitely $\mathcal{A}$-determined, it is natural to ask whether for such a construction, F is also finitely $\mathcal{A}$-determined. Unfortunately, the answer is no; however, the construction is close to being correct. In this section we give the correct version.

To see why it fails consider $f_0(x, y) = (x^2 + \varepsilon y^4, xy^3 + ty^5)$ $\varepsilon = \pm 1$. We will see in §8 that for $\varepsilon + t^2 \neq 0$, there is a negative versal unfolding which is finitely $\mathcal{A}$-determined. However, if we apply the preceding procedure to $(x^2 + \varepsilon y^4, xy^3 + ty^5, 0)$, we do not obtain a finitely $\mathcal{A}$-determined germ. If we look at the $\mathcal{K}$-orbit structure in the two dimensional section $(x^2 + \varepsilon y^4, xy^3 + ty^5, uy^6)$ in the (t, u) parameters, then the unfolding fails to be transverse to the $\mathcal{K}$-orbits of $(x^2 + \varepsilon y^4, xy^3 + ty^5, y^6)$.

section by F t $\mathcal{K}$ orbits of $(x^2 + \varepsilon y^4, xy^3 + ty^5, y^6)$
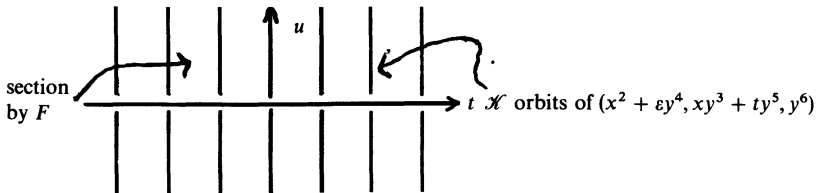
This example, in fact, suggests the correct way to obtain a finitely $\mathscr{A}$-determined germ.

We consider a uni-maximal f_0 of rank 0, and let

$$b = \max wt + \max\{wt(y_i)\}.$$

The condition which we need involves an extension of χ to $\tilde{\chi}: m_x Q(f_0) \to \mathrm{Hom}_k(\tilde{N}(f_0)_{<\max}, \ N(f_0)_{\max})$ where $\tilde{N}(f_0) = \theta(f_0)/T\mathscr{K}_e \cdot f_0$. By standard elementary linear algebra, if $\tilde{\chi}$ is injective in weight $\leq b$, then the other dual linear map

$$\tilde{N}(f_0) \to \mathrm{Hom}_k((m_x Q(f_0)_{\leq b}, N(f_0)_{\max}) \tag{7.1}$$

is surjective. Note there is no way the original χ could be injective in weight $= b$ by weight considerations.

Next, we let $\{g_i\}_{i=1}^r \subset \mathscr{C}_x$ be a set of weighted homogeneous germs which *span* $Q(f_0)_{>b}$, and $\{\psi_j\}$ a set of weighted homogeneous germs which *form a basis* for $(m_x Q(f_0))_{\leq}{}^b$. We define $g(x) = (g_1(x), \ldots, g_r(x))$,

$$f_1(x) = (f_0(x), g(x)), \ \ g(x, v) = g(x) + \sum_{i,j} v_{ij} \psi_j \varepsilon_i$$

(summed over $i = 1, \ldots, q$ and $j = t + 1, \ldots, t + r$). Also, we let $f(x, u) = (f_0(x) + \sum u_i \phi_i, u)$, and

$$F(x, u, v) = (\bar{f}(x, u), \ g(x, v), u, v).$$

Then, we have

PROPOSITION 7.2: *For the above weighted homogeneous germ f_0 of rank 0 with $\tilde{\chi}$ injective in weights $\leq b$ and $m_y N(f_0)_{\max} \subset \mathrm{Im}(\tau_0)$, if some $\tau_l(f)$ (or $\tau_l^+(f)$) is surjective, then $m_{y_1} N(f_1)_{\max} \subset \mathrm{Im}(\tau_0(f_1))$, and for the same l, $\tau_l(F)$ (or $\tau_l^+(F)$) is surjective. Hence, F is either finitely $\mathscr{A}$-determined or infinitesimally stable off the subspace of non-positive weight.*

Note: Here y_1 represent the coordinates for $(k^{t+r}, 0)$.

PROOF: We have

$$\mathrm{Im}(\tau_0(f_1)) = \mathscr{C}_x\left\{\frac{\partial f_1}{\partial x_i}\right\} + \mathscr{C}_{y_1}\{\varepsilon_i, \phi_j, \psi_j \varepsilon_{t+i}, \varepsilon_{t+i}\}.$$

However, $\{\varepsilon, \psi_j\}$ projects to a basis for $Q(f_1) \backsimeq Q(f_0)/m_{y_1} \cdot Q(f_0)$. Thus, by the preparation theorem $\mathscr{C}_x\{\varepsilon_{t+i}\} = \mathscr{C}_{y_1}\{\varepsilon_{t+i}, \psi_j \varepsilon_{t+i}\}$. Hence

$\mathscr{C}_x\{\varepsilon_{t+i}\} \subset \mathrm{Im}(\tau_0(f_1))$. Thus, $N(f_1)_{\max} = N(f_0)_{\max}$ in the sense that we may choose $\phi \in \theta(f_0)$ which projects to a generator of $N(f_0)_{\max}$, and $(\phi, 0)$ projects to a generator of $N(f_1)_{\max}$. It is sufficient to project to $\mathscr{C}_x\{\varepsilon_i\}_{i=1}^t$ and show

$$\mathscr{m}_{y_1} \cdot N(f_0)_{\max} \subset \mathscr{C}_x\left\{\frac{\partial f_0}{\partial x_i}\right\} + \mathscr{C}_{y_1}\{\varepsilon_i, \phi_j\}. \tag{7.3}$$

However, the right hand side of (7.3) contains $\mathrm{Im}(\tau_0(f_0))$. By our assumptions and lemma 3.1,

$$\mathscr{m}_{y_1} \cdot N(f_0)_{\max} \subset \mathscr{C}_x\{\varepsilon_i\}_{>\max} \subset \mathrm{Im}(\tau_0(f_0)).$$

Thus, (7.3) is established.

Secondly, suppose $\tau_l(f)$ is surjective. Since $\mathscr{C}_x\{\varepsilon_{t+i}\} \subset \mathrm{Im}(\tau_0(f_1))$, we have that for any element $\psi \in \ker(\tau_0(f_0))$ there is a $\psi_1 \in \mathscr{C}_{y_1}\{\varepsilon_{t+i}, \psi_j \varepsilon_{t+i}\}$ so that $\psi + \psi_1 \in \ker(\tau_0(f_1))$ and

$$\tilde{\tau}(f)(\psi) \equiv \tilde{\tau}(F)(\psi + \psi_1) \bmod(\mathscr{m}_{u,v} \mathrm{Im}(\tau_0(f_1)))$$

Thus, $\mathrm{Im}(\tau_l(f)) \subset \mathrm{Im}(\tau_l(F))$. If $\pi : \mathscr{m}_{u,v}^l N(f_1)_{\max} \to \mathscr{m}_{u,v} \mathrm{coker}(\tau_{l-1}(F))/\mathscr{E}_{l-1}$ denotes the projection, then $\pi(\mathscr{m}_u^l N(f_1)_{\max}) \subset \mathrm{Im}(\tau_l(F))$.

Lastly, for each i, j with $j > t$, $y_{1j} \cdot \phi_i \in \mathrm{Im}(\tau_0(f_0))$; thus, there are elements of $\ker(\tau_0(f_1))$ of the form

$$\zeta = \zeta_1 + y_{1j}\phi_k + \sum \mathscr{m}_{ij}(y_1) \cdot \psi_i \varepsilon_{t+j}$$

with $\zeta_1 \in \theta_s \oplus \theta_t \oplus \mathscr{C}_y\{\delta_i\}$. Then,

$$\tau_1(F)(\zeta) = \sum v_{ji} \pi_{\max}(\psi_i \phi_k) \bmod(\mathscr{m}_u N(f_1)_{\max})$$

where $\pi_{\max}$ denotes projection onto $N(f_1)_{\max}$.

Then, by the surjectivity of the map (7.1), it follows that in $(\mathscr{m}_{u,v}/\mathscr{m}_{u,v}^2)N(f_1)_{\max}$ we have

$$\mathscr{m}_v \cdot N(f_1)_{\max} \subset \mathrm{Im}(\tau_1(F)) + \mathscr{m}_u \cdot N(f_1)_{\max}. \tag{7.4}$$

Hence, in $(\mathscr{m}_{u,v}^{k+n}/\mathscr{m}_{u,v}^{k+n+1})N(f_1)_{\max}$, (7.4) implies

$$\mathscr{m}_u^k \cdot \mathscr{m}_v^n N(f_1)_{\max} \subset \mathscr{m}_u^k \mathscr{m}_v^{n-1} \mathrm{Im}(\tau_1(F)) + \mathscr{m}_u^{k+1} \mathscr{m}_v^{n-1} N(f_1)_{\max}.$$

Thus, if

$$\pi: \mathscr{m}_{u,v}^l N(f_1)_{max} \to \mathscr{m}_{u,v} \cdot coker(\tau_{l-1}(F))/\mathscr{E}_{l-1}$$

denotes the projection, then by increasing induction on j we obtain

$$\pi(\mathscr{m}_v^j \cdot \mathscr{m}_u^{l-j} N(f_1)_{max}) \subset \pi(\mathscr{m}_u^l N(f_1)_{max}) + Im(\tau_l(F)) \subset Im(\tau_l(F)).$$

Thus, $\pi(\mathscr{m}_{u,v}^l N(f_1)_{max}) \subset Im(\tau_l(F))$ so $\tau_l(F)$ is surjective. A similar argument works for $\tau_l^+(F)$.   □

§8 Two examples

We consider here two examples of how the preceding results can be applied. Both of these examples are uni-modal singularities with simple moduli (recall from part I that such germs can only be deformed to other values of the modulus parameter or to simple singularities). Then, we already know by Proposition 4.2 of part I that our methods must apply.

Example 1 $f_0(x, y) = (x^2 + \varepsilon y^4, xy^3 + ty^5)$
where $\varepsilon = \pm 1$ (over $\mathbb{R}$) and t is the modulus parameter. This is the first non-simple $\sum_2$-singularity (see e.g. [3], [8, VI], or [10]). We also must have $\varepsilon + t^2 \neq 0$ to ensure that f_0 is finitely $\mathscr{K}$-determined. We assign weights $wt(x) = 2$, $wt(y) = 1$, and then $wt(\varepsilon_1) = -4$, $wt(\varepsilon_2) = -5$. In this case, the modulus occurs in weight zero and $N(f_0)_+ = 0$. Thus, $N(f_0)_{<max} = N(f_0)_-$. There is a basis for $N(f_0)_-$ given by: $y^i\varepsilon_2$, $1 \leq i \leq 4$; $x\varepsilon_2$, $xy\varepsilon_2$; and $y^i\varepsilon_1$, $1 \leq i \leq 3$. There is a nonsingular pairing on $\langle y^2, y^3, y^4, x, xy \rangle \cdot \varepsilon_2$ (dually paired to $\langle y, y^2, y^3, xy, x \rangle \subset \mathscr{m}_x Q(f_0))$ as long as $\varepsilon + t^2 \neq 0$.

Also, a direct calculation shows that as long as $\varepsilon + t^2 \neq 0$, τ_0 is surjective in weights 4 and 5 (we are using the weighting described at the

k	0	1	2	3	4	5
$\dim \theta(f_0)_k$	6	7	8	9	10	11
$\dim(\theta_s + \theta_t + \mathscr{C}_y\{\delta_i\})_k$	6	9	11	12	12	14
$\dim \ker(\tau_0)_k$	1	2	3	3	2	3
$\dim \mathscr{m}_x Q(f_0)_k$	0	1	2	2	2	1
$\dim \tau_1(E)_k$	0	1	2	2	0	0
$\dim N(f_0)_{-k}$	1	2	3	3	1	0

beginning of §3). Thus, $\mathscr{m}_y \cdot N(f_0)_0 \subset \operatorname{Im}(\tau_0)$ and lemma 3.1 implies that τ_0 is surjective in weights $\neq 0$. We then have the situation given by the chart.

In the chart we compute $\dim \ker(\tau_0)_k$ for $k > 0$ as the difference of the first two rows. Since for $k > 0$ $\dim((\mathscr{m}_u/\mathscr{m}_u^2)N(f_0)_0)_k = \dim N(f_0)_{-k}$, we see from the chart that numerically it is possible for τ_1 to be surjective in each weight.

We consider the unfolding versal in negative weight

$$(f_0 + u_1y + u_2y^2 + u_3y^3, v_1x + v_2xy + v_3y + v_4y^2 + v_5y^3 + v_6y^4), \mathbf{u}, \mathbf{v})$$

$$wt \quad 3 \quad 2 \quad 1 \quad 3 \quad 2 \quad 4 \quad 3 \quad 2 \quad 1$$

Here the wt denotes the weight of the unfolding parameters. We have used $\mathbf{u}$ and $\mathbf{v}$ to keep track of the terms in each coordinates. We know by lemma 6.5, that $\dim \tau_1(E)_k = \dim Q(f_0)_k$ when $0 < k < 4$. Then comparing with $\dim N(f_0)_{-k}$, we see that we still need an extra relation in each weight $1 \leq k \leq 4$. There is the following relation

$$36f_{01}^2\varepsilon_1 + 100\varepsilon tf_{02}y^3\varepsilon_1 + (-100t^2 - 60\varepsilon)f_{01}y^4\varepsilon_1 + (10txy^2 - 6f_{01})\zeta$$

where

$$\zeta = (3x + 5ty^2)\frac{\partial f_0}{\partial x} - y\frac{\partial f_0}{\partial y} = (6x^2 + 10txy^2 - 4\varepsilon y^4)\varepsilon_1.$$

Note that $y^4\varepsilon_1 \notin$ basis we choose for $N(f_0)_-$. However, by the Euler relation

$$f_{01}y^4\varepsilon_1 = \frac{1}{4}\left(y^42x\frac{\partial f_0}{\partial x} + y^4 \cdot y\frac{\partial f_0}{\partial y} - 5f_{02}y^4\varepsilon_2\right).$$

Thus, we can replace $f_{01}y^4\varepsilon_1$ by this term and obtain an element in $\ker(\tau_0)_4$. Applying τ_1 yields $60(t^2 + \varepsilon)v_3y^5\varepsilon_2$. This gives $v_3 \cdot y^5 \cdot \varepsilon_2$. For the remaining 3 relations, we let

$$\psi = 10tf_{01}y\varepsilon_1 - 3f_{01}y^2\varepsilon_2 - 10\varepsilon f_{02}\varepsilon_1 + (3\varepsilon y^3 - 5tx)\frac{\partial f_0}{\partial x} + x\frac{\partial f_0}{\partial y}.$$

Then, ψ, $y\psi$ and $y^2\psi$ are relations in weights 1, 2, and 3 respectively. Also, we can immediately choose the ψ_1 so that $\psi = \tau_0(\psi_1)$. Then, $y\psi$ and $y^2\psi$ are $\tau_0(y\psi_1)$ and $\tau_0(y^2\psi_1)$ respectively. When we compute

$\tau_1(y^i\psi_1)$, we obtain successively

$$(tv_6 - (\tfrac{9}{2} + 5\varepsilon t^2)u_3)y^5\varepsilon_2;$$

$$\{(-4 - 5\varepsilon t^2)u_2 + (7\varepsilon + 5t^2)v_2 + 2tv_5\}y^5\varepsilon_2; \text{ and}$$

$$\{(\tfrac{7}{2} - 5\varepsilon t^2)u_1 + (8\varepsilon + 5t^2)v_1 + 3tv_4\}y^5\varepsilon_2.$$

When $\varepsilon + t^2 \neq 0$, it is not too difficult to verify that these are independent of $\tau_1(E)$. Hence, we conclude τ_1 is surjective.

Secondly, for this example we also want to consider the situation of §7. A basis for $m_xQ(f_0)$ is given by $\{y^i, 1 \le i \le 6; xy^j, 0 \le j \le 2\}$. We consider any germ $f_1(x,y) = (x^2 + \varepsilon y^4, xy^3 + ty^5, y^6, 0, \ldots, 0)$ where the coordinates y_i with $y_i \circ f_1 = 0$ have weight 6. Then, we can apply proposition 7.2 if the associated map $\tilde{N}(f_0) \to \text{Hom}((m_xQ(f_0))_{\le}{}^5, N(f_0)_0)$ is surjective. As $N(f_0)_0 \simeq k$, we have a dual basis for $\text{Hom}(m_xQ(f_0)_{\le}{}^5, N(f_0)_0)$. In terms of this dual basis, the non-singular pairing on the elements $h\varepsilon_2$ guarantee that the image contains the dual basis to $m_xQ(f_0)_{\le}{}^3$. Also, ε_2 maps to the dual basis for y^5. This leaves $m_xQ(f_0)_4 = \langle y^4, xy^2 \rangle$. If we look at the image of ε_1 and $y\varepsilon_2$ applied to this basis $\{y^4, xy^2\}$ we obtain the matrix $\begin{pmatrix} -\tfrac{1}{2}t\varepsilon & -\tfrac{1}{2} \\ 1 & -t \end{pmatrix}$ with determinant $\dfrac{\varepsilon}{2}(t^2 + \varepsilon)$. Thus, these images span a subspace containing the dual elements to y^4 and xy^2. Thus, proposition 7.2 is applicable. We summarize all of these results with the following.

PROPOSITION 8.1: *If $t^2 + \varepsilon \neq 0$, the germs*

$$f_0(x,y) = (x^2 + \varepsilon y^4, xy^3 + ty^5) \text{ or}$$

$$f_1(x,y) = (x^2 + \varepsilon y^4, xy^3 + ty^5, y^6, \overbrace{0, \ldots, 0}^{r}), r \ge 0$$

have negative versal unfoldings which are finitely $\mathscr{A}$-determined and, in fact, of $\mathscr{A}$-codimension = 1.

Example 2 $f_0(x,y,z) = (2xz + y^2, 2yz, x^2 + 3gy^2 - cz^2)$
$(c \neq 0, c + 9g^2 \neq 0$; here c is fixed and g is the modulus). This is the first non-simple $\sum_3$-singularity (again see [3] or [8, VI]). The form of these equations are due to Wall [12]. However, the equi-dimensional case was considered independently (using a different form of the equations) by Ronga in [11]. In this case all of the coordinate functions are homogeneous quadratic, so the coordinates x, y, z have weight 1.

Again the modulus occurs in weight zero. A basis for $N(f_0)_-$ is given

by: $y\varepsilon_1, z\varepsilon_1, x\varepsilon_2, y\varepsilon_2$, and $y\varepsilon_3, z\varepsilon_3$. Then, $N(f_0)$ has a non-singular pairing on $\langle z\varepsilon_1, x\varepsilon_2, y\varepsilon_2 \rangle$ so $\dim_k \tau_1(E)_1 = 3$. We identify $N(f_0)_0 \simeq \langle y^2 \cdot \varepsilon_3 \rangle$. Again a direct calculation shows that (with $c \neq 0$, $c + 9g^2 \neq 0$), $\mathrm{Im}(\tau_0)$ contains all quartic polynomials (i.e. $\theta(f_0)_2$); thus $\mathscr{m}_y \cdot N(f_0)_0 \subset \mathrm{Im}(\tau_0)$. As $\dim \ker(\tau_0)_1 = 6$, there are 3 extra relations in addition to the Euler relations; rather than give them, we merely give their images under τ_1 for the unfolding

$$(f_0 + (u_1 y + u_2 z, v_1 y + v_2 x, w_1 y + w_2 z), \mathbf{u}, \mathbf{v}, \mathbf{w}).$$

We obtain: $(3gu_1 - w_1)y^2 \varepsilon_3$; $(-3gu_2 - (c + 9g^2)v_1 + w_2)y^2 \varepsilon_3$; and $\{(3c - 90g^2)u_1 + (39g)w_1 + 6g(c + 9g^2)v_2\}y^2 \varepsilon_3$. These span a subspace of dimension 3 complementary to $\tau_1(E)$. Thus, τ_1 is surjective.

Furthermore, a basis for $\mathscr{m}_x Q(f_0)$ is given by $\{x, y, z, y^2, z^2, xy, z^3\}$. We consider a germ $f_1(x, y, z) = (f_0(x, y, z), z^3, 0, \ldots, 0)$ where we assign weights 3 for coordinates y_i with $y_i \circ f_1 = 0$. Again to apply proposition 7.2, it is sufficient to show that the mapping

$$\tilde{N}(f_0) \to \mathrm{Hom}_k((\mathscr{m}_x Q(f_0))_{\leq}{}^2, N(f_0)_0)$$

is surjective. The non-singularity of the pairing gives the dual basis to $\{x, y, z\}$. Also, ε_2 maps to the dual basis element of xy. Lastly the images of $\{\varepsilon_1, \varepsilon_3\}$ take values on $\{y^2, z^2\}$ given by $\begin{pmatrix} -3g & (\frac{1}{2}) \\ 1 & (\frac{3}{2})gc^{-1} \end{pmatrix}$ with determinant $-(\frac{1}{2})c^{-1}(c + 9g^2)$. Thus, the image of $\langle \varepsilon_1, \varepsilon_3 \rangle$ contains the dual basis elements to z^2 and y^2. Thus, the map is surjective.

Again we can summarize these results with the following.

PROPOSITION 8.2: *If $c \neq 0$ and $c + 9g^2 \neq 0$, the germs*
$f_0(x, y, z) = (2xz + y^2, 2yz, x^2 + 3gy^2 - cz^2)$ *and*
$f_1(x, y, z) = (2xz + y^2, 2yz, x^2 + 3gy^2 - cz^2, z^3, \overbrace{0, \ldots, 0}^{r}), r \geq 0,$
have negative versal unfoldings which are finitely $\mathscr{A}$-determined and, in fact, have $\mathscr{A}$-codimension $= 1$.

Further consequences for these germs will follow from the results of the next section.

§9 Topological stability

In this section, we consider a uni-maximal germ $f_0 : \mathbb{R}^s, 0 \to \mathbb{R}^t, 0$ and an unfolding f versal in non-maximal weight. By the methods of this section, we will show that if f is finitely $\mathscr{A}$-determined (or has its com-

plexification infinitesimally stable off the subspace of non-positive weight) then f is a topologically stable germ. In fact, we will show that it is only a transversality condition which is needed to prove topological stability.

We consider the above f_0 with an unfolding versal in non-maximal weights which is either finitely $\mathscr{A}$-determined (or whose complexification is infinitesimally stable off the subspace of non-positive weight). If $N(f_0)_{max}$ is generated by v then by Theorem 1 or 2 of part I there is an open neighbourhood T of 0 in $\mathbb{R}$, so that the versal unfolding of f_0 is topologically trivial along the v-parameter-axis for parameter values in T. For l sufficiently large and $r \geq 0$, we let

$$\mathscr{S} = \mathscr{S}(s + r, t + r) = \{\mathscr{K}^l((f + tv) \times id_{\mathbb{R}^r}) : t \in T\}.$$

Then, the topological stability of germs transverse to $\mathscr{S}$ is contained in the following.

THEOREM 4: *Let $f : M \to N$ be a smooth mapping where $\dim M = m$, $\dim N = n$ and $m - n = s - t$. If $j^l f(x_0) \in S$ and $j^l(f)$ is transverse to S at x_0, then there is a neighbourhood $\mathscr{U}$ of f in the Whitney topology and compact neighbourhoods U' of x_0 and V' of $f(x_0) = y_0$ such that for $g \in \mathscr{U}$, there are homeomorphisms $\phi_g : U' \hookrightarrow M$ and $\psi_g : V' \hookrightarrow N$ depending continuously on g in the C^0-topology so that $\phi_f = id$, $\psi_f = id$ and on U'*

$$f = \psi_g^{-1} \circ g \circ \phi_g.$$

REMARK: Before beginning the proof, we want to emphasize a key point; namely, that modulo the method of Mather of using versal unfoldings, the proof only depends upon the existence of locally integrable (continuous) vector fields which give the topological triviality of the versal unfolding. Nowhere do we use the differentiability of these vector fields off a subspace, although this fact appears to be basic at this stage to extending this result for proving global topological stability.

PROOF: Note that the transversality condition implies $m \geq s + q$, where $q = \dim N(f_0)_{<max}$. We first consider the case $m = s + q$, and then indicate the modifications for the general case. Also, we shall see that by the form of the proof, we may as well assume $j^l(f)(x_0) \in \mathscr{K}^l(f_0 \times id)$.

Then, the versal unfolding of the germ of f at x_0 is C^∞-equivalent to the versal unfolding of f_0. Thus, we may choose a representative F for the versal unfolding of f_0 $F : U \to V$ such that: (i) F is infinitesimally stable, $F|\sum(F)$ is proper (and finite to one), and $F^{-1}(0) \cap \sum(F) = 0$, and

(ii) there are closed embeddings $i:V' \hookrightarrow V$ and $j:U' \hookrightarrow U$ so that i is transverse to F, $F^{-1}(i(V')) = j(U')$, and $F|:F^{-1}(i(V')) \to i(V')$ is C^∞-equivalent to $f|$ neighbourhood of x_0. Under this identification, we have the fiber square

$$
\begin{array}{ccc}
U & \xrightarrow{\;F\;} & V \\[4pt]
\curvearrowright j & & \curvearrowright i \\[4pt]
U' & \xrightarrow{\;f\;} & V'
\end{array}
$$

Lastly, we use the proof of Theorem 1 and 2 in part I. We let $L \subset U$ and $L \subset V$ denote the unfolding directions for maximal weight (say the u_{q+1}-axis).

Next we may assume there are continuous vector fields ξ on U and η on V, locally integrable on $U_1 \times [-\delta, \delta]$, $V_1 \times [-\delta, \delta]$, with U_1, V_1 compact neighbourhoods of 0 in U, V respectively so that the integral curves remain in U or V and

i) $dF(\xi) = \eta \circ F$,

ii) $\xi|L = \dfrac{\partial}{\partial u_{q+1}}$, $\eta|L = \dfrac{\partial}{\partial u_{q+1}}$, and

iii) V' is transverse to L and U' is transverse to L at 0.

We will use the methods of [9, §7], to reduce our problem to considering a neighbourhood of the mapping $i|V''$, for V'' a compact neighbourhood of y_0. To describe the neighbourhood V'', we make use of the following fact about intersecting submanifolds in an ambient space.

LEMMA 9.1: *Let* $i:X \hookrightarrow \mathbb{R}^n$ *and* $j:Y \hookrightarrow \mathbb{R}^n$ *be embeddings of compact submanifolds of complementary dimension in* $\mathbb{R}^n$. *Suppose* i *and* j *are transverse and* $i(X) \cap j(Y) = \{0\}$. *Then, there are neighbourhoods* $\mathcal{U}$ *of* i *and* $\mathcal{V}$ *of* j *in the Whitney* C^1 *topology so that the map* $\Psi: \mathcal{U} \times \mathcal{V} \to \mathbb{R}^n$ *given by* $(i', j') \mapsto i'(X) \cap j'(Y)$ *is a continuous well-defined map.*

PROOF: Given any $i':X \to \mathbb{R}^n$ and $j':Y \to \mathbb{R}^n$, we define $(i' - j'):X \times Y \to \mathbb{R}^n$ by $(x,y) \mapsto i'(x) - j'(y)$. For i and j, this map is a local diffeomorphism at (x_0, y_0) where $i(x_0) = j(y_0) = 0$. Hence we may choose compact neighbourhoods U of x_0 and V of y_0 so that $i - j|U \times V$ is a diffeomorphism. Thus, since $(i', j') \mapsto i' - j'$ is continuous for the Whitney topologies, as is restriction to a compact neighbourhood, we may choose neighbourhoods $\mathcal{U}$ of i and $\mathcal{V}$ of j so that $i' - j'|U \times V$ is a diffeomorphism for $(i', j') \in \mathcal{U} \times \mathcal{V}$ with $0 \in \operatorname{int}\{(i' - j')(U \times V)\}$ and $i'(X - U) \cap j'(Y - V) = \phi$. Then, $(i' - j')^{-1}(0)$ is a single point (x', y'), so $i'(X) \cap j'(Y) = i'(x') = j'(y')$; and the continuity of $i' - j'|U \times V$ implies that (x', y') and then $i'(x')$ depend continuously on (i', j'). $\qquad\square$

REMARK: The proof, in fact, shows that i' and j' are transverse.

For η or ξ we denote the integral curve passing through y or x at $t = 0$ by γ_y or γ_x. Although the γ_y and γ_x need not depend differentiably on y and x, the continuity and local integrability of η and ξ do imply that the maps $y \to \gamma_y : [-\delta, \delta] \to V$ and $x \to \gamma_x : [-\delta, \delta] \to U$ are continuous in the Whitney C^1-topology. Since $i(V')$ and $j(U')$ are transverse to L and L' at 0, we may choose compact neighbourhoods U'' of x_0 and V'' of y_0 so that $f(U'') \subset \text{int}(V'')$ and $j(U'') \cap L' = \{0\}$, $i(V'') \cap L = \{0\}$. Thus, we can apply lemma 9.1 to $i(V'')$ and γ_0 to conclude that there are neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of i and γ_0 in the Whitney C^1-topology and the mapping Ψ satisfying the conclusions of the lemma (we can also ensure that the images remain in V). Thus, there is a smaller neighbourhood of y_0 in $\text{int}(V'')$, V''', so that if $y \in V'''$, then $\gamma_{i(y)} \in \mathcal{V}$. Then, we define for $i' \in \mathcal{U}$

$$\psi'_{i'} : V''' \to i'(V'')$$

$$\psi'_{i'}(y) = \Psi(i', \gamma_{i(y)}).$$

By lemma 9.1, this is a continuous injection. As V''' is compact, this is a homeomorphism onto its image.

Similarly, we can apply lemma 9.1 to $L' = \gamma_0$ and U'' to find neighbourhoods $\mathcal{U}'$ and $\mathcal{V}'$ and a continuous map $\Phi : \mathcal{U}' \times \mathcal{V}' \to U$. Again there is a compact neighbourhood U''' of x_0 contained in $\text{int}(U'')$ so that $\gamma_x \in \mathcal{V}'$ for $x \in j(U''')$. By the identification of a neighbourhood $\mathcal{W}$ of $f | U''$ with pull-backs of imbeddings $i' : V'' \hookrightarrow V$, corresponding to g there are embeddings $i_g : V'' \hookrightarrow V$ and $j_g : U'' \hookrightarrow U$ so that $j_g : U'' \hookrightarrow F^{-1}(i_g(V''))$, and $F \circ j_g = i_g \circ g$. We may choose our neighbourhood $\mathcal{W}$ small enough that $i_g \in \mathcal{U}$, and $j_g \in \mathcal{U}'$ for $g \in \mathcal{W}$. Then we define $\phi_g : U'' \to U$ by $\phi_g(x) = j_g^{-1} \circ \Phi(j_g, \gamma_{j(x)})$. Again ϕ_g is a homeomorphism. By the condition $dF(\xi) = \eta \circ F$, it follows that $F(\gamma_x) = \gamma_{F(x)}$. Hence, by construction, if we set $\psi_g = \psi'_{i_g}$, then $f = \psi_g^{-1} \circ g \circ \phi_g$. The continuous dependence on g of (ϕ_g, ψ_g) follows again from Lemma 9.1. This completes the proof for the case $m = s + q$.

If $m > s + q$, then we must modify the proof as follows. The infinitesimally stable germ F has the form $F_1 \times id_{\mathbb{R}^r}$, where F_1 is the infinitesimally stable unfolding of f_0. Thus, $j^l(F)^{-1}(\mathcal{S}) \simeq L \times W \subset U \times W$ for W a neighbourhood of 0 in $\mathbb{R}^r$, and similarly for $F(j^l(F)^{-1}(\mathcal{S})) \simeq L \times W$.

We claim that we may assume that codim V' in $V \times W = 1$. We consider $K = T_0 i(V') \cap (L \oplus \mathbb{R}^r)$. By the transversality assumption, codim $K = $ codim V' in $V \times W$. If $K \supset L$, then the germ is, in fact, infinitesimally stable and the result follows from the C^∞-theory. Otherwise we can

choose a subspace K' complementary to the projection of K onto $\mathbb{R}^r$. If we project along K' in $\mathbb{R}^{t+q+r+1}$ onto $T_0 i(V')$, then, this restricts to a local diffeomorphism pr on $i(V')$. As F is constant along the fibers of this projection, the new germ obtained from F by restricting to $F^{-1}(pr \circ i(V'))$ is C^∞-equivalent to f. Finally, with codim V' in $V \times W$ $= 1$, we may pull-back ζ and η to $V \times W$ and $U \times W$ and apply the first part of the proof. $\qquad \square$

As an immediate corollary we have

COROLLARY 9.2: *If $f_0 : \mathbb{R}^s$, $0 \to \mathbb{R}^t$, 0 is a uni-maximal germ and f is an unfolding versal in non-maximal weight which is finitely $\mathscr{A}$-determined or whose complexification is infinitesimally stable off the subspace of non-positive weight, then f is a topologically stable germ.*

We obtain information about topological stability on the edge of the nice dimensions with

COROLLARY 9.3: *The germs transverse to any of the following strata are topologically stable*

(i) *if $\varepsilon + t^2 \not\Rightarrow 0$ the union of the $\mathscr{K}$-orbits of $(x^2 + \varepsilon y^4, xy^3 + ty^5)$ if*

$$s = t \text{ or } (x^2 + \varepsilon y^4, xy^3 + ty^5, y^6, \overbrace{0, \ldots, 0}^{r}), r \geq 0 \text{ if } s < t.$$

(ii) *if $c \not\Rightarrow 0$, $c + 9g^2 \not\Rightarrow 0$, the union of the $\mathscr{K}$-orbits of $(2xz + y^2, 2yz,*

$$x^2 + 3gy^2 - cz^2) \text{ if } s = t \text{ or } (2xz + y^2, 2yz, x^2 + 3gy^2 - cz^2, z^3, \overbrace{0, 0, \ldots, 0}^{r}),$$

$r \geq 0$ *if $s < t$.*

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