COMPOSITIO MATHEMATICA

HASSE CARLSSON

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Compositio Mathematica, tome 46, nº 2 (1982), p. 227-253
<http://www.numdam.org/item?id=CM_1982__46_2_227_0>

COMPOSITIO MATHEMATICA, Vol. 46, Fasc. 2, 1982, pag. 227–253

ERROR ESTIMATES IN D-DIMENSIONAL RENEWAL THEORY

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1. Introduction

Let $X_1, X_2, \ldots,$ be independent d-dimensional random vectors with a common distribution μ. We assume that μ is strictly d-dimensional, that is, μ is not concentrated on any hyperplane whose dimension is less than d. Let

$$\nu = \sum_{n=0}^{\infty} \mu^{n*}$$

be the renewal measure. Here μ^{n*} denotes n-fold convolution and μ^{0*} is the Dirac measure at 0. We are interested in the behavior of $\nu(A+x)$ for large values of x. Such results were obtained by Doney [1] and later refined by Stam [7, 8]. See also Nagaev [4].

We always assume that $E[X_1] \neq 0$ and to simplify the statements of our results, we assume that coordinates are chosen in such a way that $E[X_1] = (\mu_1, 0, \ldots, 0)$, $\mu_1 > 0$. Put $X_1 = (Y_1, \ldots, Y_d)$ and let B be the covariance matrix

$$B = (E[Y_i Y_j])_{i,j=2,\ldots,d}.$$

Let ω be the measure with density

$$w(x) = \begin{cases} \dfrac{\mu_1^{\rho-1}}{(\det B)^{1/2}(2\pi x_1)^{\rho}} \exp\left(-\dfrac{\mu_1 B^{-1}(x', x')}{2x_1}\right), & x_1 > 0 \\ 0, & x_1 \leq 0, \end{cases}$$

where $x = (x_1; x')$, $B^{-1}(x', x')$ is the quadratic form with matrix B^{-1}

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and $\rho = \frac{1}{2}(d-1)$. We say that μ has finite moments of order $(\alpha_1, \ldots, \alpha_d)$ if $E[|Y_i|^{\alpha_i}] < +\infty$, $i = 1, \ldots, d$.

We first consider the non-lattice case, that is, we assume that

$$f(t) = 1 \Leftrightarrow t = 0,$$

where

$$f(t) = \int e^{-itx} \, d\mu(x)$$

is the characteristic function of μ. (Unspecified integrations are always taken over the whole Euclidean space.)

THEOREM 1: *Assume that μ is a non-lattice measure with finite moments of order* $(1+\epsilon, 2)$ *if* $d = 2$ *and* $(\rho+\epsilon; 2+\epsilon)$ *if* $d \geq 3$ *for some* $\epsilon > 0$. *If A is a bounded measurable set with* $\text{Vol}(\partial A) = 0$, *then*

$$\nu(A+x) = \omega(A+x) + o(x_1^{-\rho}), \quad x_1 \to +\infty,$$

uniformly in x'.

REMARK: As $\partial w/\partial x_i = 0(x_1^{-(\rho+(1/2))})$, $x_1 \to +\infty$, $i = 1, \ldots, d$, uniformly in x', the conditions in Theorem 1 implies that

$$\nu(A+x) = w(x)\,\text{Vol}(A) + o(x_1^{-\rho}), \quad x_1 \to +\infty,$$

uniformly in x'. In particular, as $e^{-c/x_1} = 1 + 0(c/x_1)$, $x_1 \to +\infty$, we have

$$\nu(A+x) = \frac{\text{Vol}(A)}{(\det B)^{1/2}(2\pi x_1)^{\rho}} + o(x_1^{-\rho}), \quad x_1 \to +\infty,$$

uniformly for x' in bounded sets. (Similar remarks apply to Theorems 2–5 below.)

In [7, 8] Stam proved this result assuming μ to have finite moments of order $(\max(2, \rho); 2)$.

The proof of Theorem 1 is based on the fact that $\hat{\nu}$ and $\hat{\omega}$ have a similar behavior at the origin. If we assume that μ is strongly non-lattice, that is $\liminf_{|t|\to\infty} |1 - f(t)| > 0$, this method gives sharper estimates when further moments exist.

THEOREM 2: *Assume that μ is a strongly non-lattice measure with* *finite moments of order* $(\max(1,\rho)+\alpha;\ 2+2\alpha)$, $0<\alpha\le 1/2$. *If R is a parallelepiped we have*

$$\nu(R+x)=\omega(R+x)+o(x_1^{-(\rho+\lambda)}),\quad x_1\to+\infty,$$

for all $\lambda<\alpha$. The estimate is uniform in x' and for R in a fixed bounded set.

To get more information about ν we want estimates of $\nu(A+x)$ for 'arbitrary' sets A. We can not hope for a uniform estimate for all measurable sets A unless ν is non-singular with respect to Lebesgue measure. To see this, we observe that $\omega(A+x_1)\sim c(2\pi x_1)^{-\rho}$, $x_1\to+\infty$. Then, if we had such a uniform estimate, there would be an x_1 such that

$$|\nu(B+x_1)-\omega(B+x_1)|<\tfrac{1}{2}\omega(A+x_1)$$

for all B. If we apply this to the two subsets A_i of A where $(\nu-\omega)(\cdot+x_1)$ is positive or negative, we get

$$\|\nu-\omega\|(A+x_1)<\omega(A+x_1).$$

($\|\cdot\|$ denotes absolute variation.) If ν is singular,

$$\|\nu-\omega\|(A+x_1)=\nu(A+x_1)+\omega(A+x_1)\ge\omega(A+x_1),$$

which is a contradiction. (Compare Rogozin [5, p. 697].)

Put $(\partial A)_\epsilon=\{x;\,d(x,\partial A)<\epsilon\}$. We say that a set A in R^d has a K-regular boundary if $\mathrm{Vol}(\partial A)_\epsilon\le K\epsilon$, A is called regular if it is K-regular for some K.

THEOREM 3: *Assume that μ is a strongly non-lattice measure with finite moments of order* $(\max(1+\alpha,\rho+\alpha+\beta);\ 2+2\alpha)$ *where* $0\le\beta\le\alpha d$ *and* $0<\alpha\le 1/2$. *If A is a bounded measurable set with a regular boundary, then*

$$\nu(A+x)=\omega(A+x)+o(x_1^{-(\rho+(\lambda+\beta)(d+1)^{-1})}),\quad x_1\to+\infty,$$

for all $\lambda<\alpha$. The estimate is uniform in x' and for K-regular sets in a fixed bounded set.

Due to the uniform estimates in Theorems 1–3, it is possible to obtain estimates for other type of sets. Assume for instance that $\mu_1 = 1$, $B = I$ (the identity matrix) and consider $\nu(A(x_1))$, where

$$A(x_1) = (I_1 + x_1) \times x_1^{1/2} I_2 \times \cdots \times x_1^{1/2} I_d$$

and I_k are intervals. If we divide $A(x_1)$ into $[x_1^\rho]$ bounded boxes and apply Theorem 2 to each of them we get

$$\nu(A(x_1)) = \omega(A(x_1)) + o(x_1^{-\lambda}), \quad x_1 \to +\infty.$$

Now

$$\begin{aligned}\omega(A(x_1)) &= \int_{I_1+x_1} (2\pi y_1)^{-\rho} \, dy_1 \prod_{k=2}^{d} \int_{x_k^{1/2} I_k} \exp(-y_k^2/2y_1) \, dy_k \\ &= \int_{I_1+x_1} \prod_{k=2}^{d} \Phi((x_1/y_1)^{1/2} I_k) \, dy_1,\end{aligned}$$

where $\Phi(A)$ is the standard normal measure of A. Since

$$\Phi((x_1/y_1)^{1/2} I_k) = \Phi(I_k) + O(1/x_1), \quad x_1 \to +\infty,$$

if $y_1 \in I_1 + x_1$, we get

$$\nu(A(x_1)) = \mathrm{Vol}(I_1) \prod_{k=2}^{d} \Phi(I_k) + o(x_1^{-\lambda}), \quad x_1 \to +\infty,$$

if μ is a strongly non-lattice measure with finite moments of order $(\max(1, \rho) + \alpha; 2 + 2\alpha)$ and $\lambda < \alpha$.

We now consider the lattice case, that is, we assume that there exist a linear map Λ such that the support of μ is contained in the lattice $L_\Lambda = \Lambda(Z^d)$. We say that μ is distributed on L_Λ, if L_Λ is the minimal lattice that contains supp μ. In the lattice case we have the following analogues of Theorems 1–3:

THEOREM 4: *Assume that μ is distributed on the lattice L_Λ and has finite moments of order $(1 + \epsilon, 2)$ if $d = 2$ and $(\rho + \epsilon; 2 + \epsilon)$ if $d \geq 3$ for some $\epsilon > 0$. Then, for $x \in L_\Lambda$,*

$$\nu(x) = |\det \Lambda| w(x) + o(x_1^{-\rho}), \quad x_1 \to +\infty,$$

uniformly in x'.

THEOREM 5: *Assume that μ is distributed on the lattice L_Λ and has finite moments of order* $(\max(1,\rho)+\alpha;\ 2+2\alpha)$, $0<\alpha\leq 1/2$. *Then, for* $x\in L_\Lambda$,

$$\nu(x)=|\det\Lambda|w(x)+o(x_1^{-(\rho+\lambda)}),\quad x_1\to+\infty,$$

for all $\lambda<\alpha$. *The estimate is uniform in* x'.

2. Fourier transforms of ν and ω

Throughout Section 2-6, where we prove Theorems 1–3, μ is assumed to be a non-lattice measure.

To prove Theorems 1–3 we may assume that $\mu_1=1$ and $B=I$. Otherwise consider $\tilde{X}=\Lambda X$, where

$$\Lambda=\begin{pmatrix}\mu_1^{-1} & 0\ldots.0\\ 0 & \\ \cdot & \\ \cdot & \Lambda_1\\ \cdot & \\ 0 & \end{pmatrix}$$

and Λ_1 is chosen such that $\Lambda_1 B\Lambda_1^T=I$. Then $\tilde{\mu}_1=1$ and $\tilde{B}=I$. Furthermore, $B^{-1}=\Lambda_1^T\Lambda_1$ and $|\det\Lambda_1|=(\det B)^{-1/2}$. Hence

$$\begin{aligned}\nu(A+x)&=\tilde{\nu}(\Lambda(A+x))\\ &=\int_{\Lambda(A+x)}\tilde{w}(y)\,dy+o((\Lambda x_1)^{-\gamma})\\ &=\int_{A+x}\tilde{w}(\Lambda y)|\det\Lambda|\,dy+o(x_1^{-\gamma})\\ &=\mu_1^{\rho-1}(\det B)^{-1/2}\int_{A+x}(2\pi y_1)^{-\rho}\\ &\quad\times\exp(-\mu_1\Lambda_1^T\Lambda_1(y',y')/2y_1)\,dy+o(x_1^{-\gamma})\\ &=\omega(A+x)+o(x_1^{-\gamma}),\quad x_1\to+\infty.\end{aligned}$$

In the sequel we always assume that this normalization is made and thus ω has the density

$$w(x)=\begin{cases}(2\pi x_1)^{-\rho}\exp(-|x'|^2/2x_1), & x_1>0\\ 0, & x_1\leq 0.\end{cases}$$

We will now compute the Fourier transform of ν and ω. The Fourier transforms will be computed in the sense of distributions. For the theory of distributions and its standard notation we refer to Schwartz [6] and Gelfand–Shilov [3].

Put

$$\nu_N = \sum_{n=0}^{N-1} \mu^{n*}.$$

Then

$$\hat{\nu}_N(t) = \sum_{n=0}^{N-1} f^n(t) = \frac{1-f^N(t)}{1-f(t)}.$$

To examine the limit of $\hat{\nu}_N$ we need estimates of f at the origin. Put $\eta(t) = f(t) - 1 + \mathrm{i}t_1 + \frac{1}{2}|t'|^2$. Then

$$\eta(t) = \int \{e^{-\mathrm{i}tx} - 1 + \mathrm{i}t_1x_1 + \tfrac{1}{2}((t_2x_2)^2 + \cdots + (t_dx_d)^2)\}\, \mathrm{d}\mu(x)$$

(2.1)
$$= \int \{e^{-\mathrm{i}t_1x_1} - 1 + \mathrm{i}t_1x_1 + (e^{-\mathrm{i}t_1x_1} - 1)(e^{-\mathrm{i}t'x'} - 1) + (e^{-\mathrm{i}t'x'} - 1 + \mathrm{i}t'x' + \tfrac{1}{2}(t'x')^2)\}\, \mathrm{d}\mu(x).$$

From the Taylor expansion of the exponential function we get

$$\eta(t) = o(|t_1| + |t'|^2), \quad t \to 0,$$

if μ has finite moments of order (1; 2). If $|t|$ is sufficiently small we therefore get

$$|1 - f(t)| \geq |\tfrac{1}{2}|t'|^2 + \mathrm{i}t_1| - |\eta(t)| \geq c_d(|t_1| + |t'|^2) - o(1)(|t_1| + |t'|^2) \geq \tfrac{1}{2}c_d(|t_1| + |t'|^2).$$

Thus $(1-f)^{-1} \in L^1_{\mathrm{loc}}$ and by dominated convergence we get

$$\langle \hat{\nu}_N, \varphi \rangle = \int \frac{1-f^N}{1-f}\varphi \, \mathrm{d}t \to \int \frac{1}{1-f}\varphi \, \mathrm{d}t \quad \text{if } \varphi \in \mathcal{D},\ N \to \infty.$$

If μ is strongly non-lattice, this convergence also holds for $\varphi \in \mathcal{S}$ and thus $\nu_N \to \nu$, where ν is a positive measure with

(2.2)
$$\hat{\nu} = (1-f)^{-1}.$$

To see that this is true also if μ only is non-lattice, fix a non-negative $\psi \in \hat{\mathscr{D}} = \{\varphi ; \hat{\varphi} \in \mathscr{D}\}$ with $\psi(x) \geq 1$ if $|x_i| \leq 1$, $i = 1, \ldots, d$. Then $(\psi * \nu_N)^\wedge = \hat{\psi}(1 - f^N)(1 - f)^{-1}$ and

$$\|\psi * \nu_N\|_\infty \leq \|\hat{\psi}(1 - f^N)(1 - f)^{-1}\|_1 \leq 2\|\hat{\psi}(1 - f)^{-1}\|_1 \leq K.$$

Hence

$$K \geq \int \psi(x - y)\,\mathrm{d}\nu_N(y) \geq \int_{|y_i - x_i| \leq 1} \psi(x - y)\,\mathrm{d}\nu_N(y) \geq \int_{|y_i - x_i| \leq 1} \mathrm{d}\nu_N(y).$$

From this uniform bound we see that $\nu_N \to \nu$ in $\mathscr{S}'$ also in this case and

$$\int_{A+x} \mathrm{d}\nu(y) \leq C \tag{2.3}$$

if A is a bounded set.

To compute the Fourier transform of ω, we first observe that

$$\int e^{-\mathrm{i}t'x'} \exp(-|x'|^2/2x_1)\,\mathrm{d}x' = (2\pi x_1)^\rho \exp(-\tfrac{1}{2}x_1|t'|^2).$$

Thus

$$I_N(t) = \int_0^N \mathrm{d}x_1 \int_{-\infty}^{+\infty} e^{-\mathrm{i}tx} w(x)\,\mathrm{d}x' = \frac{1 - \exp(-N(\mathrm{i}t_1 + \frac{1}{2}|t'|^2))}{\mathrm{i}t_1 + \frac{1}{2}|t'|^2}.$$

Hence

$$\langle \hat{\omega}, \varphi \rangle = \langle \omega, \hat{\varphi} \rangle = \lim_{N \to \infty} \int_0^N \mathrm{d}x_1 \int_{+\infty}^{-\infty} w(x)\hat{\varphi}(x)\,\mathrm{d}x = \lim_{N \to \infty} \int \varphi(t) I_N(t)\,\mathrm{d}t,$$

where the last equality follows from Fubinis theorem. By dominated convergence we now get

$$\langle \hat{\omega}, \varphi \rangle = \int \varphi(t) \frac{1}{\mathrm{i}t_1 + \frac{1}{2}|t'|^2}\,\mathrm{d}t,$$

that is

$$\hat{\omega}(t) = (\mathrm{i}t_1 + \tfrac{1}{2}|t'|^2)^{-1}. \tag{2.4}$$

3. Derivatives of non-integral order

To estimate $\nu(A + x_1)$ we want to show that $x_1^\rho(\nu - \omega)$ has a locally integrable Fourier transform. Since multiplication by x_1 corresponds to differentiation of the transform, we want to examine derivatives of $(\nu - \omega)^\wedge$. As ρ is not necessarily an integer, we need an analogue of this for non-integral numbers.

Let $0 < \lambda < 1$. Then, according to Gelfand–Shilov [3, p. 173], $|x|^\lambda$ has the one-dimensional Fourier transform

$$(|x|^\lambda)^\wedge(t) = c_\lambda |t|^{-(1+\lambda)},$$

where $|t|^{-(1+\lambda)}$ is defined by

$$\langle |t|^{-(1+\lambda)}, \varphi\rangle = \int \frac{\varphi(t) - \varphi(0)}{|t|^{1+\lambda}}\,\mathrm{d}t.$$

On R^d we therefore have

$$(|x_1|^\lambda)^\wedge(t) = d_\lambda |t_1|^{-(1+\lambda)},$$

where $|t_1|^{-(1+\lambda)}$ is the distribution defined by

$$\langle |t_1|^{-(1+\lambda)}, \varphi\rangle = \int \frac{\varphi(t_1; 0) - \varphi(0; 0)}{|t_1|^{1+\lambda}}\,\mathrm{d}t_1.$$

Thus we want to examine

$$D_{t_1}^\lambda g(t) = |t_1|^{-(1+\lambda)} * g(t).$$

(Compare Gelfand–Shilov [3, Sect. 5.5].) Put $\Delta_{s_1} g(t) = g(t_1 - s_1; t') - g(t_1; t')$.

LEMMA 1: *Assume that g is a measurable function with compact support and*

$$\int \frac{|\Delta_{s_1} g(t)|}{|s_1|^{1+\lambda}}\,\mathrm{d}s_1 \in L^1_{\mathrm{loc}}(R^d).$$

Then

$$D_{t_1}^{\lambda} g(t) = \int \frac{\Delta_{s_1} g(t)}{|s_1|^{1+\lambda}} \, ds_1.$$

PROOF: If φ is a test function, then

$$D_{t_1}^{\lambda} \varphi(t) = \int \frac{\Delta_{s_1} \varphi(t)}{|s_1|^{1+\lambda}} \, ds_1$$

As g has compact support, $D_{t_1}^{\lambda} g$ is well-defined and characterized by

$$D_{t_1}^{\lambda} g * \varphi = |t_1|^{-(1+\lambda)} * (g * \varphi).$$

Hence

$$\begin{aligned}\langle D_{t_1}^{\lambda} g, \varphi \rangle &= D_{t_1}^{\lambda} g * \check{\varphi}(0) = |t_1|^{-(1+\lambda)} * (g * \check{\varphi})(0) \\ &= \int \frac{\Delta_{s_1}(g * \check{\varphi})(0)}{|s_1|^{1+\lambda}} \, ds_1 = \int |s_1|^{-(1+\lambda)} \, ds_1 \int \varphi(t) \Delta_{s_1} g(t) \, dt \\ &= \int \varphi(t) \Big(\int \frac{\Delta_{s_1} g(t)}{|s_1|^{1+\lambda}} \, ds_1 \Big) \, dt,\end{aligned}$$

where the last equality follows from Fubinis theorem.

4. Estimates of $(\nu - \omega)^{\wedge}$ and its derivatives

Throughout this section we assume that μ is a non-lattice measure with finite moments of order $(\max(1, \rho) + \alpha;\ 2 + 2\alpha)$, $0 < \alpha < \frac{1}{2}$. Put

$$\begin{aligned}g(t) = (\nu - \omega)^{\wedge}(t) &= \frac{1}{1 - f(t)} - \frac{1}{it_1 + \frac{1}{2}|t'|^2} \\ &= (f(t) - 1 + it_1 + \tfrac{1}{2}|t'|^2) \cdot \frac{1}{it_1 + \frac{1}{2}|t'|^2} \cdot \frac{1}{1 - f(t)} \\ &= \eta(t) \cdot \frac{1}{a(t)} \cdot \frac{1}{1 - f(t)}.\end{aligned}$$

By straightforward integration we see that

$$a^{-(1+\alpha)}(t) |t'|^{-2\beta} \in L^1_{\mathrm{loc}}(R^d) \tag{4.1}$$

if $\alpha+\beta<\rho$. By considering $\{|t_1|>|t'|^2\}$ and $\{|t_1|<|t'|^2\}$, we also get

$$t_1^{-\alpha}a^{-(1+\beta)}(t)\in L^1_{\mathrm{loc}}(R^d) \tag{4.2}$$

if $0\leq\alpha<1$ and $\alpha+\beta<\rho$.

By the Leibnitz formula,

$$\frac{\partial^n g}{\partial t_1^n}=\sum_{k_1+k_2+k_3=n} c_k D_{t_1}^{k_1}\eta(t)D_{t_1}^{k_2}\frac{1}{a(t)}D_{t_1}^{k_3}\frac{1}{1-f(t)}.$$

Now

$$D_{t_1}^{k_2}a^{-1}(t)=c_{k_2}a^{-(k_2+1)}(t)$$

and

$$D_{t_1}^{k_3}\frac{1}{1-f(t)}=\frac{P_{k_3}(f,D_{t_1}f,\ldots,D_{t_1}^{k_3}f)}{(1-f(t))^{k_3+1}}$$

for some polynomial P_{k_3}. Thus, with $\eta_k=D_{t_1}^k\eta$, we get

$$\begin{aligned}\frac{\partial^n g}{\partial t_1^n}(t)&=\sum_{k_1+k_2+k_3=n} c_k\eta_{k_1}(t)\frac{P_{k_3}(f,\ldots,D_{t_1}^{k_3}f)}{a^{k_2+1}(t)(1-f(t))^{k_3+1}}\\&=\sum_{k_1+k_2+k_3=n}A_{n,k}(t).\end{aligned}$$

Put $m=[\rho]$. Then $f,\ldots,D_{t_1}^m f$ are bounded. From the Taylor expansion of the exponential function and the inequality $|x_1||x'|^{2\alpha}\leq|x_1|^{1+\alpha}+|x'|^{2(1+\alpha)}$ (to estimate the middle term), we get from (2.1)

$$\eta_0(t)=o(1)a^{1+\alpha}(t),\quad t\to 0. \tag{4.3}$$

Furthermore,

$$\begin{aligned}\eta_1(t)&=\int -\mathrm{i}x_1\{(e^{-\mathrm{i}t_1x_1}-1)+e^{-\mathrm{i}t_1x_1}(e^{-\mathrm{i}t'x'}-1)\}\,\mathrm{d}\mu(x)\\&=o(1)a^{\alpha}(t),\quad t\to 0,\end{aligned} \tag{4.4}$$

and

$$\eta_k(t)=0(1)\quad\text{if } k\leq m,\quad t\to 0. \tag{4.5}$$

Hence

$$\frac{\partial^n g}{\partial t_1^n}(t) = o(1)a^{\alpha-(n+1)}(t), \quad t \to 0,$$

and $\partial^n g/\partial t_1^n$ is bounded for $t \neq 0, \infty$. Consequently

$$\frac{\partial^n g}{\partial t_1^n} \in L^1_{\mathrm{loc}}(R^d)$$

if $n \leq m$.

The rest of this section is devoted to the proof of the following proposition.

PROPOSITION 1: *Let* $\psi \in \mathcal{D}$ *and assume that* $\gamma < \alpha_0 = (\rho - m) + \alpha$ *and* $n \leq m$. *Then*

$$D^{\gamma}_{t_1}(\psi D^n_{t_1} g) \in L^1(R^d).$$

Put $G_n = \psi D^n_{t_1} g$. By Lemma 1 it is enough to prove that

$$\int \frac{|\Delta_{s_1} G_n(t)|}{|s_1|^{1+\gamma}} \,\mathrm{d}s_1 \in L^1(R^d).$$

We recall that $G_n(t) = o(a^{\alpha-(n+1)}(t))$, $t \to 0$, and that G_n is the sum of a number of terms of the form $\psi(t)A_{n,k}(t)$.

As

$$\Delta_{s_1} D^k_{t_1} f(t) = \int (-\mathrm{i}x_1)^k e^{-\mathrm{i}tx}(e^{-\mathrm{i}s_1x_1} - 1)\,\mathrm{d}\mu(x),$$

we get from the moment condition on μ that $|\Delta_{s_1} D^k_{t_1} f(t)| \leq c|s_1|^{\alpha_0}$ and

$$|\Delta_{s_1}\eta_k(t)| \leq c|s_1|^{\alpha_0} \tag{4.6}$$

if $k \leq m$. Thus, if $|t| \geq \delta$ and $|s_1| \leq \frac{1}{2}\delta$, we have, for an arbitrary factor F_i of $\psi A_{n,k}$, that F_i is bounded and $|\Delta_{s_1} F_i(t)| \leq c|s_1|^{\alpha_0}$. By repeated use of

$$\begin{aligned} |\Delta_{s_1} F_i F_j(t)| &\leq |F_i(t_1 - s_1; t')\Delta_{s_1} F_j(t)| \\ &\quad + |F_j(t)\Delta_{s_1} F_i(t)| \leq c|s_1|^{\alpha_0}, \end{aligned} \tag{4.7}$$

we get

$$|\Delta_{s_1} G_n(t)| \leq c|s_1|^{\alpha_0}.$$

Write

$$\int \frac{|\Delta_{s_1} G_n(t)|}{|s_1|^{1+\gamma}} \, ds_1 = \int_{|s_1| \leq (1/2)\delta} + \int_{|s_1| > (1/2)\delta} \frac{|\Delta_{s_1} G_n(t)|}{|s_1|^{1+\gamma}} \, ds_1$$
$$= A_1(t) + A_2(t).$$

If $|t| \geq \delta$, A_1 is bounded and has compact support. Hence

$$\int_{|t| \geq \delta} A_1(t) \, dt < +\infty. \tag{4.8}$$

By Fubini's Theorem

$$\int_{|t| \geq \delta} A_2(t) \, dt \leq \int_{|s_1| > (1/2)\delta} \frac{ds_1}{|s_1|^{1+\gamma}} \int_{|t| \geq \delta} |G_n(t_1 - s_1; t') - G_n(t)| \, dt$$
$$\leq c\|G_n\|_1 < +\infty. \tag{4.9}$$

as $G_n \in L^1(R^d)$.

To complete the proof of the proposition it is therefore enough to show that

$$\int \frac{|\Delta_{s_1} G_n(t)|}{|s_1|^{1+\gamma}} \, ds_1$$

is integrable at the origin. We divide the integral into two parts:

$$\int \frac{|\Delta_{s_1} G_n(t)|}{|s_1|^{1+\gamma}} \, ds_1 = \int_{|s_1| \leq 2|t_1|} + \int_{|s_1| > 2|t_1|} \frac{|\Delta_{s_1} G_n(t)|}{|s_1|^{1+\gamma}} \, ds_1$$
$$= B_1(t) + B_2(t).$$

If $|s_1| > 2|t_1|$, then $|s_1 - t_1| > |t_1|$. Hence

$$B_2(t) = 0(1)|G_n(t_1; t')| \int_{|s_1| > 2|t_1|} \frac{ds_1}{|s_1|^{1+\gamma}}$$
$$= 0(1) \frac{|G_n(t)|}{|t_1|^{\gamma}}, \quad t \to 0,$$

and thus (4.2) implies that B_2 is integrable at the origin since $n-\alpha+\gamma<\rho$.

To estimate B_1 we put

$$I_\gamma f(t)=\int_{|s_1|\leq 2|t_1|}\frac{|\Delta_{s_1}f(t)|}{|s_1|^{1+\gamma}}\,ds_1.$$

Recall that $G_n=\Sigma\ \psi A_{n,k}$. Now

$$\begin{aligned}I_\gamma(\psi A_{n,k})(t)&=0(1)(\|\psi\|_\infty I_\gamma A_{n,k}(t)+A_{n,k}(t)I_\gamma\psi(t))\\&=0(1)(I_\gamma A_{n,k}(t)+A_{n,k}(t)),\quad t\to 0.\end{aligned}$$

Since $A_{n,k}\in L^1_{\text{loc}}(R^d)$ it is enough to estimate $I_\gamma A_{n,k}$. As remarked above

$$\Delta_{s_1}P_{k_3}(f,\ldots,D_{t_1}^{k_3}f)=0(1)s_1^{\alpha_0},\quad s\to 0.$$

Hence, by using (4.7),

$$|I_\gamma A_{n,k}(t)|\leq\|P_{k_3}\|_\infty I_\gamma B_{n,k}(t)+B_{n,k}(t)|t_1|^{\alpha_0-\gamma},$$

where

$$B_{n,k}(t)=\eta_{k_1}(t)\cdot\frac{1}{a^{k_2+1}(t)}\cdot\frac{1}{(1-f(t))^{k_3+1}}.$$

As $B_{n,k}(t)=o(1)a^{\alpha-(m+1)}(t)$, $t\to 0$, we have

$$B_{n,k}(t)|t_1|^{\alpha_0-\gamma}\in L^1_{\text{loc}}(R^d).$$

To estimate $I_\gamma B_{n,k}(t)$ we first prove the following assertion:

$$\int_{|s_1|\leq 2|t_1|}\frac{ds_1}{a(t_1-s_1;t')|s_1|^\gamma}=0(1)\frac{\log|t'|}{a^\gamma(t)},\quad t\to 0. \tag{4.10}$$

To prove this we may assume that $t_1>0$ and estimate

$$\int_{-2t_1}^{2t_1}\frac{ds_1}{(|t_1-s_1|+|t'|^2)|s_1|^\gamma}.$$

It is easily seen that the integral over $[-2t_1,0)$ is bounded by a constant times $a^{-\gamma}(t)$. To estimate the integral over $[0,2t_1]$ we consider two cases:

(i) $t_1 \leq 2|t'|^2$

Then

$$\int_0^{2t_1} \frac{ds_1}{(|t_1 - s_1| + |t'|^2)s_1^\gamma} = \frac{0(1)}{|t'|^2} \int_0^{2t_1} \frac{ds_1}{s_1^\gamma}$$
$$= 0(1) \frac{t_1^{1-\gamma}}{|t'|^2} = 0(1) \frac{1}{a^\gamma(t)}, \quad t \to 0.$$

(ii) $t_1 > 2|t'|^2$

We make a further partition of the integral into the four intervals $[0, \frac{1}{2}t_1)$, $[\frac{1}{2}t_1, t_1 - |t'|^2)$, $[t_1 - |t'|^2, t_1 + |t'|^2)$ and $[t_1 + |t'|^2, 2t_1]$. It is now easy to see that we have the given bound, for instance

$$\int_{(1/2)t_1}^{t_1 - |t'|^2} \frac{ds_1}{(|t_1 - s_1| + |t'|^2)s_1^\gamma} = \frac{0(1)}{t_1^\gamma} \int_{(1/2)t_1}^{t_1 - |t'|^2} \frac{ds_1}{t_1 - s_1}$$
$$= \frac{0(1)}{t_1^\gamma} \log \frac{\frac{1}{2}t_1}{|t'|^2} = \frac{0(1)}{a^\gamma(t)} \log|t'|, \quad t \to 0,$$

as desired.

We return to the estimate of $I_\gamma B_{n,k}$.

$$\begin{aligned} \Delta_{s_1} B_{n,k}(t) &= \eta_{k_1}(t_1 - s_1; t')\Delta_{s_1}(a^{-(k_2+1)}(1-f)^{-(k_3+1)})(t) \\ &\quad + a^{-(k_2+1)}(t)(1 - f(t))^{-(k_3+1)}\Delta_{s_1}\eta_{k_1}(t) \\ &= C_{n,k}(s_1, t) + D_{n,k}(s_1, t). \end{aligned}$$

By (4.4) and the mean value theorem $|\Delta_{s_1}\eta_0(t)| \leq c|a^\alpha(t)s_1|$. Thus $|I_\gamma \eta_0(t)| \leq c|a^\alpha(t)t_1^{1-\gamma}|$ and for $k_1 = 0$ we get

$$\int_{|s_1| \leq 2|t_1|} \frac{|D_{n,k}(s_1, t)|}{|s_1|^{1+\gamma}} ds_1 = 0(1) \frac{a^\alpha(t)}{a^{n+2}(t)} t_1^{1-\gamma}, \quad t \to 0,$$

which by (4.2) is locally integrable if $n \leq m$ since $m + 1 - \alpha - (1 - \gamma) < \rho$. If $0 < k_1 \leq m$, (4.6) implies

$$\int_{|s_1| \leq 2|t_1|} \frac{|D_{n,k}(s_1, t)|}{|s_1|^{1+\gamma}} ds_1 = \frac{0(1)}{a^{n+1}(t)} t_1^{\alpha_0 - \gamma}, \quad t \to 0,$$

which is locally integrable since $n - (\alpha_0 - \gamma) < \rho$.

To estimate $C_{n,k}$ we write

$$\Delta_{s_1} \frac{1}{a^{k_2+1}(1-f)^{k_3+1}}(t) = \frac{1}{a^{k_2+1}(t_1-s_1;t')} \Delta_{s_1} \frac{1}{(1-f)^{k_3+1}}(t) + \frac{1}{(1-f(t))^{k_3+1}} \Delta_{s_1} \frac{1}{a^{k_2+1}}(t).$$

As $\Delta_{s_1}(1-f)^{-(k+1)}(t)$ and $\Delta_{s_1}a^{-(k+1)}(t)$ are bounded by a constant times

$$\frac{s_1}{a(t)a^{k+1}(t_1-s_1;t')}, \quad t \to 0,$$

we get, by using (4.3)–(4.5) at the point $(t_1-s_1;t')$, that

$$C_{n,k}(s_1,t) = 0(1)\frac{s_1}{a(t)|t'|^2 a(t_1-s_1;t')}, \quad t \to 0.$$

By (4.10) we now get

$$\int \frac{|C_{n,k}(s_1,t)|}{|s_1|^{1+\gamma}} \mathrm{d}s_1 = 0(1)\frac{1}{a(t)|t'|^{2(n-\alpha)}} \cdot \frac{\log|t'|}{a^{\gamma}(t)}, \quad t \to 0.$$

which by (4.1) is integrable at the origin since $n-\alpha+\gamma<\rho$ if $n \le m$.

5. Proof of Theorem 1

Let $\phi \in \hat{\mathscr{D}} = \{\phi; \hat{\phi} \in \mathscr{D}\}$. If d is odd, $m=\rho$ and by Proposition 1, there is an $\epsilon>0$ such that

$$(|x_1|^{\epsilon}(\phi * x_1^{\rho}(\nu-\omega)))^{\wedge} = cD_{t_1}^{\epsilon}\left(\frac{\partial^m g}{\partial t_1^m}\right) \in L^1(R^d).$$

Hence

$$|x_1|^{\epsilon}(\phi * x_1^{\rho}(\nu-\omega))(x) \in L^{\infty}(R^d)$$

and

$$\phi * (x_1^{\rho}(\nu-\omega))(x) \to 0, \quad x_1 \to +\infty,$$

uniformly in x'.

If $d>2$ is even, we first need a bound for $x_1^m\nu$. To get this, fix a non-negative $\phi \in \hat{\mathscr{D}}$ with $\phi(x) \ge 1$ if $|x_i| \le 1$, $i=1,\ldots,d$. Since the

Fourier transform of $\phi * x_1^m(\nu-\omega) \in L^1(R^d)$, $\phi * x_1^m(\nu-\omega) \in L^\infty(R^d)$. Also $\phi * x_1^m\omega \in L^\infty(R^d)$ and thus

$$\phi * x_1^m \nu \in L^\infty(R^d). \tag{5.1}$$

As $\phi \in \mathscr{S}$, we have for p large enough and $x_1 \geq 1$ that

$$\begin{aligned}
&\left|\int_{y_1\leq 0} \phi(x-y) y_1^m \, d\nu(y)\right| \\
&\leq c \frac{1}{(1+|x_1|)^p} \int_{y_1\leq 0} \frac{|y_1|^m}{(1+|x'-y'|)^p(1+|y_1|)^p} d\nu(y) \\
&\leq \frac{c_1}{(1+|x_1|)^p},
\end{aligned} \tag{5.2}$$

where the last inequality follows from (2.3). Thus (5.1) and (5.2) implies

$$K_1 \geq \int_{y_1\geq 0} \phi(x-y) y_1^m \, d\nu(y) \geq \int_{|x_i-y_i|\leq 1} y_1^m \, d\nu(y) \tag{5.3}$$

if $x_1 \geq 1$. By Proposition 1,

$$(|x_1|^{(1/2)+\epsilon}(\phi * x_1^m(\nu-\omega)))^\wedge = cD_{t_1}^{(1/2)+\epsilon}\left(\hat{\phi}\, \frac{\partial^m g}{\partial t_1^m}\right) \in L^1(R^d)$$

for some $\epsilon > 0$. Hence

$$|x_1|^{1/2}(\phi * x_1^m(\nu-\omega))(x) \to 0, \quad x_1 \to +\infty, \tag{5.4}$$

uniformly in x'. We also want to assert that

$$\phi * (x_1^\rho(\nu-\omega))(x) \to 0, \quad x_1 \to +\infty.$$

(x_1^ρ is interpreted as 0 if $x_1 < 0$ and ρ is not an integer.) To see this, write

$$\begin{aligned}
\phi * (x_1^\rho(\nu-\omega))(x) &= \int_{y_1\geq 0} \phi(x-y) y_1^{m+(1/2)} \, d(\nu-\omega)(y) \\
&= x_1^{1/2} \int \phi(x-y) y_1^m \, d(\nu-\omega)(y) \\
&\quad - x_1^{1/2} \int_{y_1\leq 0} \phi(x-y) y_1^m \, d(\nu-\omega)(y)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{y_1\geq 0}\phi(x-y)(y_1^{1/2}-x_1^{1/2})y_1^m\,\mathrm{d}(\nu-\omega)(y)\\
&=A_1(x)-A_2(x)+A_3(x).
\end{aligned}
$$

By (5.4), $A_1(x)\to 0$, $x_1\to+\infty$, and from (5.2) we get $A_2(x)\to 0$, $x_1\to+\infty$. For A_3 we have by (5.3),

$$
\begin{aligned}
|A_3(x)| &= \left|\int_{y_1\geq 0}\phi(x-y)(x_1-y_1)(x_1^{1/2}+y_1^{1/2})^{-1}y_1^m\,\mathrm{d}(\nu-\omega)(y)\right|\\
&\leq x_1^{-1/2}\int_{y_1\geq 0}|(x_1-y_1)\phi(x-y)|y_1^m\,\mathrm{d}(\nu+\omega)(y)\\
&\leq Cx_1^{-1/2}\int|(x_1-y_1)\phi(x-y)|\,\mathrm{d}y\leq C_1x_1^{-1/2}\to 0,\quad x_1\to+\infty.
\end{aligned}
$$

If $d=2$ and $\phi\in\hat{\mathscr{D}}$ we have by Fourier inversion

$$\phi*(\nu-\omega)(x)=\frac{1}{4\pi^2}\int e^{itx}g(t)\hat{\phi}(t)\,\mathrm{d}t.$$

Under the moment conditions in Theorem 1

$$g(t)=o(1)\left(\frac{1}{(|t_1|+t_2^2)^{1-\epsilon}}+\frac{t_2^2}{(|t_1|+t_2^2)^2}\right),\quad t\to 0,$$

and

$$\frac{\partial g}{\partial t_1}(t)=\frac{o(1)}{(|t_1|+t_2^2)^2},\quad t\to 0.$$

With $Q_\delta=\{t;|t_i|<\delta\}$ we get

$$\int_{Q_\delta}e^{itx}g(t)\hat{\phi}(t)\,\mathrm{d}t=o(\delta),\quad \delta\to 0.$$

For the integral over $R^2\smallsetminus Q_\delta$ we get by an integration by parts with respect to t_1

$$\int_{R^2\smallsetminus Q_\delta}e^{itx}g(t)\hat{\phi}(t)\,\mathrm{d}t=\frac{1}{x_1}o\left(\frac{1}{\delta}\right),\quad \delta\to 0.$$

Hence

$$\phi*(\nu-\omega)(x)=o(\delta+(x_1\delta)^{-1}),\quad \delta\to 0.$$

If we put $\delta = x_1^{-1/2}$, we get

$$x_1^{1/2}(\phi * (\nu - \omega))(x) \to 0, \quad x_1 \to +\infty,$$

and as above

$$\phi * (x_1^{1/2}(\nu - \omega))(x) \to 0, \quad x_1 \to +\infty.$$

Thus we have, for $\phi \in \hat{\mathscr{D}}$, that

$$\phi * (x_1^{\rho}(\nu - \omega))(x) \to 0, \quad x_1 \to +\infty,$$

for all d and uniformly in x'. This can be interpreted as

$$\int \phi \, d\sigma_x \to 0, \quad x_1 \to +\infty, \quad \phi \in \hat{\mathscr{D}},$$

where σ_x is the measure defined by

$$\sigma_x(A) = \int_{A+x} y_1^{\rho} \, d(\nu - \omega)(y).$$

If $\varphi \in C_c(R^d)$, supp $\varphi \subset K$, take $\phi \in \hat{\mathscr{D}}$ such that $\|\varphi - \phi\|_\infty < \epsilon$ and

$$\left| \int_{R^d \setminus K} \phi \, d\sigma_x \right| < \epsilon.$$

(Recall the bound (2.3).) Then

$$\begin{aligned} \left| \int \varphi \, d\sigma_x \right| &\le \left| \int \phi \, d\sigma_x \right| + \left| \int_K (\varphi - \phi) \, d\sigma_x \right| + \left| \int_{R^d \setminus K} \phi \, d\sigma_x \right| \\ &\le \left| \int \phi \, d\sigma_x \right| + C\epsilon. \end{aligned}$$

Consequently $\sigma_x \to 0$ weakly as $x_1 \to +\infty$. Since weak convergence to 0 of the measures σ_x is equivalent to $\sigma_x(A) \to 0$, $x_1 \to +\infty$, for all bounded measurable sets A with $\mathrm{Vol}(\partial A) = 0$, Theorem 1 follows since

$$\begin{aligned} (\nu - \omega)(A + x) &= \int_{A+x} d(\nu - \omega)(y) = x_1^{-\rho} \int_{A+x} (x_1/y_1)^{\rho} y_1^{\rho} \, d(\nu - \omega)(y) \\ &= 0(1) x_1^{-\rho} \sigma_x(A) = o(1) x_1^{-\rho}, \quad x_1 \to +\infty. \end{aligned}$$

6. Proof of Theorems 2 and 3

To prove Theorems 2 and 3 we will estimate

$$\phi_T * \chi_{-A} * (\nu - \omega)(x) = \phi_T * (\nu - \omega)(A + \cdot)(x)$$

where ϕ_T is an approximation of the identity. Thus fix a non-negative $\phi \in \hat{\mathscr{D}}$ such that $\int \phi \, \mathrm{d}x = 1$, $\operatorname{supp} \hat{\phi} \subset \{x; |x_i| \leq 1\}$ and put $\phi_T(x) = T^d \phi(Tx)$. As $\phi \in \mathscr{S}$ there are constants c_p such that

$$\left| \int_{|y|>\epsilon} \phi_T(y) \, \mathrm{d}y \right| \leq \int_{|y|>T\epsilon} |\phi(y)| \, \mathrm{d}y \leq c_p (T\epsilon)^{-p} \tag{6.1}$$

for all p.

Put $Q_T = \{t; |t_i| \leq T\}$. If $A = R$ is a parallelepiped we have

$$\int_{Q_T} |D_{t_1}^n \chi_{-R}(t)| \, \mathrm{d}t \leq C \log^d T, \tag{6.2}$$

where C can be chosen uniformly for R in bounded sets. To see this, write $-R = \Lambda Q_1$ for some linear map $\Lambda = (a_{ij})$. For R in a fixed bounded set we have $\max|a_{ij}| \leq M$ for some constant M. Now

$$\begin{aligned} \hat{\chi}_{-R}(t) &= \int_{-R} \mathrm{e}^{-\mathrm{i}tx} \, \mathrm{d}x = |\det \Lambda| \int_{Q_1} \mathrm{e}^{-\mathrm{i}t\Lambda y} \, \mathrm{d}y \\ &= |\det \Lambda| \int_{Q_1} \mathrm{e}^{-\mathrm{i}(\Lambda^T t) y} \, \mathrm{d}y = |\det \Lambda| \hat{\chi}_{Q_1}(\Lambda^T t). \end{aligned}$$

Since $\hat{\chi}_{Q_1}(t) = 2^d \Pi_{i=1}^d \sin t_i / t_i$, we have

$$|D^\alpha \hat{\chi}_{Q_1}(t)| \leq c \prod_{i=1}^d \frac{1}{1 + |t_i|} = A(t)$$

for all α. Thus

$$|D_{t_1}^n \hat{\chi}_{-R}(t)| \leq |\det \Lambda| (\mathrm{d}M)^n A(\Lambda^T t)$$

and

$$\begin{aligned} \int_{Q_T} |D_{t_1}^n \hat{\chi}_{-R}(t)| \, \mathrm{d}t &\leq |\det \Lambda| (\mathrm{d}M)^n \int_{Q_T} A(\Lambda^T t) \, \mathrm{d}t \\ &= |\det \Lambda| (\mathrm{d}M)^n |\det \Lambda^{-1}| \int_{\Lambda^T Q_T} A(y) \, \mathrm{d}y \\ &\leq (\mathrm{d}M)^n \int_{Q_{MT}} A(y) \, \mathrm{d}y \leq C \log^d T \end{aligned}$$

as desired.

As $\hat{\phi}_T\hat{\chi}_{-R} \in \mathcal{D}$, Proposition 1 implies

$$D_{t_1}^{\rho+\lambda}(\hat{\phi}_T(t)\hat{\chi}_{-R}(t)g(t)) \in L^1(R^d)$$

if $\lambda < \alpha$. We reconsider the estimates (4.8) and (4.9) of the terms A_1 and A_2. As μ is strongly non-lattice, $(1-f(t))^{-1}$ is bounded for $|t| \geq \delta$. Since G_m has support in Q_T, (6.2) renders

$$\|D_{t_1}^{\rho+\lambda}(\phi_T * \chi_{-R} * (\nu - \omega))^\wedge\|_1 \leq C \log^d T$$

and

$$\phi_T * (\nu - \omega)(R + \cdot)(x) = 0(1)x_1^{-(\rho+\lambda)} \log^d T, \quad x_1 \to +\infty. \tag{6.3}$$

The estimate is uniform in x' and for R in a fixed bounded set.

To estimate $\phi_T * (\nu - \omega)(A + \cdot)$ under the conditions in Theorem 3, we fix $\psi_i \in C^\infty(R^d)$, $i = 1, 2$, supp $\psi_1 \subset Q_1$, supp $\psi_2 \cap Q_{1/2} = \emptyset$ and $\psi_1 + \psi_2 = 1$. Then

$$\begin{aligned}\hat{\phi}_T(t)\hat{\chi}_{-A}(t)g(t) &= \psi_1(t)\hat{\phi}_T(t)\hat{\chi}_{-A}(t)g(t) \\ &\quad + \psi_2(t)\hat{\phi}_T(t)\hat{\chi}_{-A}(t)g(t) = g_1(t) + g_2(t).\end{aligned}$$

By Proposition 1, we have $D_{t_1}^{\rho+\lambda}g_1 \in L^1(R^d)$ if $\lambda < \alpha$. Since g_1 has support in Q_1, the L^1-norm of $D_{t_1}^{\rho+\lambda}g_1$ is bounded independently of T. Furthermore $D_{t_1}^{\rho+\lambda+\beta}g_2 \in L^1(R^d)$ if $\lambda < \alpha$, and again by considering the estimates (4.8) and (4.9), we see that the L^1-norm is bounded by a constant times T^d. Thus

$$\phi_T * (\nu - \omega)(A + \cdot)(x) = 0(1)(x_1^{-(\rho+\lambda)} + T^d x_1^{-(\rho+\lambda+\beta)}), \quad x_1 \to +\infty. \tag{6.4}$$

The estimate is uniform in x' and for A in a fixed bounded set.

We will now estimate

$$\phi_T * (\nu - \omega)(A + \cdot)(x) - (\nu - \omega)(A + x).$$

Put $A_\epsilon^- = \{x; x \in A \text{ and } d(x, \partial A) \geq \epsilon\}$ and $A_\epsilon^+ = \{x; d(x, A) < \epsilon\}$. (When $A = R$ is a parallelepiped we modify A_ϵ^+ so that it also is a parallelepiped.) Then $\nu(A_\epsilon^- + x - y) \leq \nu(A + x) \leq \nu(A_\epsilon^+ + x - y)$ if $|y| \leq \epsilon/2$. We recall the bound (2.3), $\|\nu(A + x)\|_\infty \leq C$ uniformly for A in bounded sets. Thus

$$\nu(A+x)=\int \nu(A+x)\phi_T(y)\,dy \le \int_{|y|\le(1/2)\epsilon} \nu(A_\epsilon^+ + x - y)\phi_T(y)\,dy$$
$$+\,C\int_{|y|>(1/2)\epsilon} \phi_T(y)\,dy \le \phi_T * \nu(A_\epsilon^+ + \cdot)(x) + c_p(T\epsilon)^{-p}.$$

There is a similar lower bound and we get

$$\phi_T * \nu(A_\epsilon^- + \cdot)(x) - c_p(T\epsilon)^{-p} \le \nu(A+x) \le \phi_T * \nu(A_\epsilon^+ + \cdot)(x) + c_p(T\epsilon)^{-p}.$$

In the same way we also obtain

$$\phi_T * \omega(A_\epsilon^- + \cdot)(x) - c_p(T\epsilon)^{-p} \le \omega(A+x) \le \phi_T * \omega(A_\epsilon^+ + \cdot)(x) + c_p(T\epsilon)^{-p}.$$

Furthermore,

$$\phi_T * (\omega(A_\epsilon^+ + \cdot) - \omega(A_\epsilon^- + \cdot))(x)$$
$$= \int_{|y|\le\epsilon} + \int_{|y|>\epsilon} (\omega(A_\epsilon^+ + x - y) - \omega(A_\epsilon^- + x - y))\phi_T(y)\,dy.$$

As $w \in L^1_{\text{loc}}$ and w is bounded for $x \neq 0$, we have

$$\omega(A_\epsilon^+ + x - y) - \omega(A_\epsilon^- + x - y) = \int_{(\partial A)_\epsilon + x - y} w(u)\,du = 0(1).$$

Also, since A has a regular boundary,

$$\omega(A_\epsilon^+ + x - y) - \omega(A_\epsilon^- + x - y) = 0(1)\epsilon x_1^{-\rho}, \quad x_1 \to +\infty,$$

if $|y| \le \epsilon$. Hence

$$\phi_T * (\omega(A_\epsilon^+ + \cdot) - \omega(A_\epsilon^- + \cdot))(x) = 0(1)(\epsilon x_1^{-\rho} + (T\epsilon)^{-p}), \quad x_1 \to +\infty,$$

and we get

$$\begin{aligned} &\phi_T * (\nu - \omega)(A_\epsilon^- + \cdot)(x) - 0(1)(\epsilon x_1^{-\rho} + (T\epsilon)^{-p}) \le (\nu - \omega)(A + x) \\ (6.5)\quad &\le \phi_T * (\nu - \omega)(A_\epsilon^+ + \cdot)(x) + 0(1)(\epsilon x_1^{-\rho} + (T\epsilon)^{-p}), \quad x_1 \to +\infty. \end{aligned}$$

(6.3) and (6.5) implies

$$(\nu - \omega)(R + x) = 0(1)(x_1^{-\rho} + (T\epsilon)^{-p} + x_1^{-(\rho+\lambda)} \log^d T), \quad x_1 \to +\infty.$$

If we put $\epsilon = x_1^{-1/2}$ and $T = x_1$, we get, since p is arbitrary, that

$$\nu(R+x) = \omega(R+x) + 0(1)x_1^{-(\rho+\lambda)} \log^d x_1, \quad x_1 \to +\infty,$$

and Theorem 2 is proved.

From (6.4) and (6.5) we get

$$(\nu-\omega)(A+x) = 0(1)(\epsilon x_1^{-\rho} + (T\epsilon)^{-p} + x_1^{-(\rho+\lambda)} + x_1^{-(\rho+\lambda+\beta)}T^d),$$
$$x_1 \to +\infty,$$

If we put $\epsilon = T^{\delta-1}$, δ small, and $T = x_1^{(\lambda+\beta)(d+1)^{-1}}$, we get if p is large enough that

$$\nu(A+x) = \omega(A+x) + 0(1)(x_1^{-(\rho+(1-\delta)(\lambda+\beta)(d+1)^{-1})} + x_1^{-(\rho+\lambda)}),$$
$$x_1 \to +\infty,$$

and since δ is arbitrary Theorem 3 is established.

REMARK: We see from the proof that the sharper estimate in Theorem 2 is due to the decrease of $\hat{\chi}_R$ at infinity and in fact Theorem 2 is true for any regular set with

$$\int_{Q_T} |D_{t_1}^n \hat{\chi}_A(t)| \, dt \leq C \log^d T.$$

7. The lattice case

In this section we will sketch the modifications needed to prove our results in the lattice case.

It is no restriction to assume that $\mu_1 = 1$ and $B = I$. Since μ is distributed on L_Λ, the Fourier transform of ν is defined on the torus $T^d = (\Lambda^T)^{-1}(\{t; -\pi < t_i \leq \pi\})$ and

$$\nu(\Lambda n) = (2\pi)^{-d} |\det \Lambda| \int_{T^d} (1-f(t))^{-1} e^{it\Lambda n} \, dt.$$

Let ϕ_T be an approximation of the identity as in Section 6 with $\hat{\phi} = 1$ on T^d. Then, for $x \in L_\Lambda$,

$$|\det \Lambda|^{-1}\nu(x) - \phi_T * w(x) = (2\pi)^{-d} \int_{T^d} (1-f(t))^{-1} e^{itx} \, dt$$
$$- (2\pi)^{-d} \int_{R^d} a^{-1}(t)\hat{\phi}_T(t) \, e^{itx} \, dt.$$

Fix a $\psi_1 \in \mathcal{D}$ with $0 \leq \psi_1 \leq 1$, supp $\psi_1 \subset T^d$ and $\psi_1 = 1$ in a neighborhood of the origin and put $\psi_2 = 1 - \psi_1$. Define two measures λ_i, $i = 1, 2$, on L_Λ by

$$\lambda_1(x) = (2\pi)^{-d} \int_{T^d} ((1 - f(t))^{-1} - a^{-1}(t))\psi_1(t) \, e^{itx} \, dt, \quad x \in L_\Lambda,$$

and

$$\lambda_2(x) = (2\pi)^{-d} \int_{T^d} (1 - f(t))^{-1} \psi_2(t) \, e^{itx} \, dt, \quad x \in L_\Lambda,$$

and let λ_3 be the density defined by

$$\lambda_3(x) = (2\pi)^{-d} \int_{R^d} a^{-1}(t) \hat{\phi}_T(t) \psi_2(t) \, e^{itx} \, dt, \quad x \in R^d.$$

To estimate $\lambda_i(\Lambda n)$, $i = 1, 2$, we want to integrate by parts with respect to t_1: Fix an even $\chi \in \mathcal{D}$ with $\chi = 1$ in a neighborhood of the origin and supp $\chi \subset T^d \cap \{t; t' = 0\}$ and let $|t_1|^{-(1+\lambda)}$ be the distribution on T^d defined by

$$\langle t_1^{-(1+\lambda)}, \varphi \rangle = \int \frac{\varphi(t_1; 0) - \varphi(0; 0)}{|t_1|^{1+\lambda}} \chi(t_1) \, dt_1.$$

Then for $x \in L_\Lambda$

$$\begin{aligned}
(|t_1|^{-(1+\lambda)})^\vee(x) &= (2\pi)^{-d} \langle |t_1|^{-(1+\lambda)}, e^{itx} \rangle \\
&= (2\pi)^{-d} \int \frac{e^{-it_1 x_1} - 1}{|t_1|^{1+\lambda}} \chi(t_1) \, dt_1 \\
&= 2(2\pi)^{-d} |x_1|^\lambda \int_0^{+\infty} \frac{\cos s - 1}{s^{1+\lambda}} \chi(s/x_1) \, ds = |x_1|^\lambda \theta(x_1),
\end{aligned}$$

where $\theta(x_1)$ is bounded away from zero and infinity as $x_1 \to +\infty$. Hence

$$(|(\Lambda n)_1|^\lambda \theta((\Lambda n)_1))^\wedge(t) = |t_1|^{-(1+\lambda)}.$$

As in the non-lattice case (compare Section 3) we get

$$\begin{aligned}
(|(\Lambda n)_1|^\lambda \theta((\Lambda n)_1) g(\Lambda n))^\wedge(t) &= |t_1|^{-(1+\lambda)} * \hat{g}(t) \\
&= \int \frac{\Delta_{s_1} g(t)}{|s_1|^{1+\lambda}} \chi(s_1) \, ds_1.
\end{aligned}$$

From Section 4 we see that if $\gamma < \alpha_0$, we can integrate by parts $m+\gamma$ times in the integral defining λ_i, $i = 1, 2$. Thus

$$x_1^{m+\gamma}\theta(x_1)\lambda_i(x) \in L^\infty, \quad x \in L_\Lambda, \quad i = 1, 2,$$

or

$$x_1^{\rho+\lambda}\lambda_i(x) = 0(1), \quad x_1 \to +\infty, \quad x \in L_\Lambda, \quad i = 1, 2,$$

for all $\lambda < \alpha$. Moreover (compare the estimate (6.4))

$$x_1^{\rho+\lambda}\lambda_3(x) = 0(1)T^d, \quad x_1 \to +\infty.$$

Thus, for $x \in L_\Lambda$,

$$\begin{aligned} |\det \Lambda|^{-1}\nu(x) - \phi_T * w(x) &= \lambda_1(x) + \lambda_2(x) + \lambda_3(x) \\ &= 0(1)x_1^{-(\rho+\lambda)}T^d, \quad x_1 \to +\infty. \end{aligned} \tag{7.1}$$

As $\partial w/\partial x_i(x) = 0(1)x_1^{-(\rho+(1/2))}$, $x_1 \to +\infty$, (uniformly in x') and $w \in L^1_{\text{loc}}$, we get

$$\begin{aligned} \phi_T * w(x) - w(x) &= \int_{|y|\le 1} + \int_{|y|>1} (w(x-y) - w(x))\phi_T(y)\,dy \\ &= 0(1)(x_1^{-(\rho+(1/2))} + T^{-p}), \quad x_1 \to +\infty, \end{aligned}$$

for all p. Hence, by (7.1),

$$\begin{aligned} \nu(x) = |\det \Lambda| w(x) + 0(1)(T^d x_1^{-(\rho+\lambda)} + x_1^{-(\rho+(1/2))} + T^{-p}), \\ x_1 \to +\infty, \quad x \in L_\Lambda. \end{aligned}$$

If we put $T = x_1^\delta$ for δ small enough and take p large enough, we get

$$\nu(x) = |\det \Lambda| w(x) + o(x_1^{-(\rho+\lambda)}), \quad x_1 \to +\infty, \quad x \in L_\Lambda,$$

for all $\lambda < \alpha$ as required.

8. Concluding remarks

The above method can also be used to obtain estimates of the renewal measure when $x \to \infty$ along other directions by integration by parts with respect to the $t_2, \ldots, t_d$-variables. In a similar way as in (4.3)–(4.4), we obtain $\eta(t) = o(1)a^{1+\alpha}(t)$, $D_{t_i}\eta(t) = o(1)a^{(1/2)+\alpha}(t)$, $D^2_{t_i}\eta(t) = o(1)a^{\alpha}(t)$ and $D_{t_i}f(t) = 0(1)a^{1/2}(t)$, $t \to 0$, $i = 2, \ldots, d$. Also $D_{t_i}a(t) = 0(1)a^{1/2}(t)$ and thus the singularity at the origin of $(\nu - \omega)\hat{}$ increases with a factor $a^{-1/2}(t)$ if we differentiate with respect to t_i, $i = 2, \ldots, d$, to be compared with the factor $a^{-1}(t)$ if we differentiate with respect to t_1. Hence it is possible to obtain a more rapid decrease of the remainder term in these directions. For instance we can prove the following results.

THEOREM 6: *Assume that μ is a non-lattice measure with finite moments of order $(1+\epsilon;\ 2\max(1,\rho)+\epsilon)$ for some $\epsilon > 0$. If A is a bounded measurable set with* $\mathrm{Vol}(\partial A) = 0$, *then*

$$\nu(A + x) = \omega(A + x) + o(|x'|^{-2\rho}), \quad |x'| \to \infty,$$

uniformly in x_1.

THEOREM 7: *Assume that μ is a strongly non-lattice measure with finite moments of order $(1+\alpha;\ 2(\max(1,\rho)+\alpha))$, $0 < \alpha \leq \frac{1}{2}$. If R is a parallelepiped we have*

$$\nu(R + x) = \omega(R + x) + o(|x'|^{-2(\rho+\lambda)}), \quad |x'| \to \infty,$$

for all $\lambda < \alpha$. The estimate is uniform in x_1 and for R in a fixed bounded set.

Theorem 1 and the result of Stam [7] suggest, as already conjectured by him, that finite moments of order $(\max(1,\rho); 2)$ should be sufficient in Theorem 1. This could perhaps be proved by more careful estimates of the integrals in Section 4.

In contrast to the one-dimensional case we do not get a stronger remainder term in Theorem 2 by prescribing more moments. In fact there are absolutely continuous measures with finite moments of all orders such that

$$\nu(A + (x_1; x_1^{1/2})) = \omega(A + (x_1; x_1^{1/2})) + r(x_1),$$

where $\limsup_{x_1 \to +\infty} |x_1^{\rho+(1/2)} r(x_1)| > 0$. We only prove this for $d = 2$, but the argument easily generalizes to any dimension.

Consider two measures $\mu_i = \sigma \times \tau_i$, $i = 1, 2$. We assume that σ and τ_i

are absolutely continuous, $\operatorname{supp}\sigma \subset [3/4, 5/4]$, τ_1 is normal measure with density $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ and τ_2 has first moment 0 and second and third moment 1. Then, if $A = I \times I$, $I = [0, 1]$, we have

$$\nu_i(A + (n, n^{1/2})) = \sum_{k=(1/2)n}^{(3/2)n} \sigma^{k*}(I + n)\tau_i^{k*}(I + n^{1/2}).$$

In particular,

$$\nu_1(A + (n, n^{1/2})) = \sum_{k=(1/2)n}^{(3/2)n} \sigma^{k*}(I + n) \int_{(n/k)^{1/2}}^{(n^{1/2}+1)/k^{1/2}} \phi(x)\,dx.$$

Let $Y_1, Y_2, \ldots$ be independent random variables with distribution τ_2 and put $S_k = Y_1 + \cdots + Y_k$. From the Edgeworth expansion in the central limit theorem (see Feller [2], p. 535), we have that the density f_k of $k^{-1/2}S_k$ satisfies

$$f_k(x) = \phi(x)(1 + ck^{-1/2}\mu_3(x^3 - 3x)) + 0(1/k), \quad k \to +\infty,$$

uniformly in x. Hence

$$\begin{aligned} \nu_2(A + (n, n^{1/2})) &= \nu_1(A + (n, n^{1/2})) \\ &+ \sum_{k=(1/2)n}^{(3/2)n} \sigma^{k*}(I + n)\Big(ck^{-1/2} \int_{(n/k)^{1/2}}^{(n^{1/2}+1)/k^{1/2}} (x^3 - 3x)\exp(-x^2/2)\,dx \\ &+ 0(k^{-3/2})\Big), \quad n \to +\infty. \end{aligned}$$

From the one-dimensional renewal theorem we get

$$\begin{aligned} &\left|\sum_{k=(1/2)n}^{(3/2)n} \sigma^{k*}(I + n)ck^{-1/2} \int_{(n/k)^{1/2}}^{(n^{1/2}+1)/k^{1/2}} (x^3 - 3x)\exp(-x^2/2)\,dx\right| \\ &\geq \sum_{k=(1/2)n}^{(3/2)n} C_0k^{-1}\sigma^{k*}(I + n) \geq C_0/3n, \quad n \geq N, \end{aligned}$$

and

$$\left|\sum_{k=(1/2)n}^{(3/2)n} \sigma^{k*}(I + n)0(k^{-3/2})\right| = 0(n^{-3/2}), \quad n \to +\infty.$$

Thus $\nu_1(A + (n, n^{1/2}))$ and $\nu_2(A + (n, n^{1/2}))$ differ by a factor C/n and for at least one of the remainders $r_i(n) = (\nu_i - \omega)(A + (n, n^{1/2}))$, we have $\limsup_{n\to\infty} |nr_i(n)| > 0$.

To obtain more refined estimates of the renewal measure, we must therefore compare it with a measure ω_N, that depends on the higher moments of μ. One possible such candidate is that measure ω_N whose Fourier transform is $(1-f_N)^{-1}$, where f_N is the Taylor polynomial of f of degree N.

Acknowledgements

This paper is based on the second half of my doctoral thesis and I wish to express my sincere gratitude to my advisor Tord Ganelius for his valuable guidance of my work. I am also grateful to Torgny Lindvall for stimulating discussions.

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(Oblatum 26-VI-1980 & 1-VII-1981)

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