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GREG W. ANDERSON

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LOGARITHMIC DERIVATIVES OF DIRICHLET L-FUNCTIONS AND THE PERIODS OF ABELIAN VARIETIES

Greg W. Anderson*

§0. Introduction

The purpose of this article is to show how the periods of abelian varieties with complex multiplication by an abelian extension of **Q** can be evaluated up to algebraic factors in terms of logarithmic derivatives at $s = 0$ of Dirichlet L-functions. Our principal result (stated in §2) generalizes the following consequence of the Kronecker limit formula: The complex number

$$P_K = \exp\left(\frac{1}{2}\frac{d}{ds}\log \zeta_K(s)\,|_{s=0}\right).$$

where ζ_K is the Dedekind zeta-function of the imaginary quadratic field K, has the property that for any elliptic curve

$$E: y^2 = 4x^3 - g_2x - g_3$$

defined over the algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$ in the complex numbers and admitting complex multiplication by K and for any 1-cycle c not homologous to zero on $E(\mathbf{C})$,

$$P_K \sim \int_c \frac{dx}{y}, \tag{0.1}$$

where here and elsewhere, given two complex numbers α and β we

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write $\alpha \sim \beta$ if $(\beta/\alpha) \in \bar{\mathbf{Q}}$. Notably, the Kronecker limit formula is not used in the proof of the principal result. Instead, we use results of Weil [W], Gross (and Rohrlich) [GR], Shimura [S], and Deligne. As the results of Deligne that we need (concerning the relationship of periods of abelian varieties and the Γ-function) are unpublished, they will be formulated and proved here in a form suited to our purposes. In the final section of the paper we give a p-adic analogue of the principal result.

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§1. Definition of the period distribution

The central concept of the paper is that of the period distribution. It is essential to the formulation of the principal result and provides an interpretation of crucial results of Shimura [S] and so we begin with its definition.

By a number field we always mean a subfield of $\mathbf{C}$ finite over $\mathbf{Q}$. We write $\bar{\mathbf{Q}}$ for the compositum of all number fields and G for the galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. We write x^ρ instead of $\bar{x}$ to denote the complex conjugate of a complex number x. A number field K will be called CM if $K^{\sigma\rho} = K^\sigma$ for all $\sigma \in G$ and the restriction of ρ to K is not the identity.

An abelian variety with complex multiplication is a triple (A, K, ι) consisting of an abelian variety A defined over $\bar{\mathbf{Q}}$, a CM number field K of degree $2\dim_{\mathbf{C}}(A)$ over $\mathbf{Q}$ and a representation $\iota: K \to \mathbf{Q} \otimes \mathrm{End}(A)$ which makes the first homology group of A with rational coefficients into a one-dimensional vector space over K. For each CM abelian variety (A, K, ι) we define the CM type $\Phi(A, K, \iota)$ to be the isomorphism class of $K \otimes_{\mathbf{Q}} \mathbf{C}$-modules containing $H^0(\Omega^1(A))$ (on which K acts via ι). We have

$$\Phi(A, K, \iota) \oplus \Phi(A, K, \iota \circ \rho) = K \underset{\mathbf{Q}}{\otimes} \mathbf{C}, \tag{1.1}$$

which is nothing but the Hodge decomposition of $H^1_{DR}(A)$ into holomorphic and antiholomorphic parts.

The period $P(A, K, \iota) \in \mathbf{C}^\times/\bar{\mathbf{Q}}^\times$ of a CM abelian variety is defined as follows: Since the de Rham cohomology groups of A have a natural purely algebraic definition, we can speak of a de Rham cohomology class as being $\bar{\mathbf{Q}}$-rational. So now, choosing any nonzero $\bar{\mathbf{Q}}$-rational one-dimensional de Rham cohomology class ω satisfying

$$\iota^*(x)\omega = x\omega$$

for all $x \in K$ and any 1-cycle c not homologous to zero on A we put

$$P(A, K, \iota) \sim \int_c \omega.$$

Since ω is defined up to a factor in $\bar{\mathbf{Q}}^\times$ and $K^\times$ acts transitively on the set of nonzero one-dimensional homology class with rational coefficients, $P(A, K, \iota)$ is well defined. By the Hodge-Riemann bilinear relations

$$(1.2) \qquad P(A, K, \iota)P(A, K, \iota \circ \rho) \sim 2\pi i;$$

in view of this, periods are computable in terms of the periods of $\bar{\mathbf{Q}}$-rational *holomorphic* 1-forms alone.

For each number field K let I_K denote the Grothendieck group generated by the finitely generated $K \otimes_{\mathbf{Q}} \mathbf{C}$-modules; it is the free $\mathbf{Z}$-module on the complex embeddings of K. Let $(\cdot\,,\cdot)_K$ denote the pairing

$$(\Phi, \Psi)_K = \dim_{\mathbf{C}} \operatorname{Hom}_{K \underset{\mathbf{Q}}{\otimes} \mathbf{C}}(\Phi, \Psi)$$

which is clearly bilinear, symmetric and nondegenerate. For each extension L/K of number fields we have a restriction map $\mathrm{Res}_{L/K} : I_L \to L_K$; the induction map $\mathrm{Ind}_{L/K} : I_K \to I_L$ is uniquely defined by requiring

$$(\mathrm{Ind}_{L/K}(\Phi), \Psi)_L = (\Phi, \mathrm{Res}_{L/K}(\Psi))_K$$

for all $\Phi \in I_K$, $\Psi \in I_L$.

For each CM number field K let I_K^0 be the subgroup of I_K generated by the CM types of abelian varieties with CM by K. I_K^0 can also be described as the subgroup of modules Φ satisfying

$$\Phi \oplus \Phi^\rho = n(K \underset{\mathbf{Q}}{\otimes} \mathbf{C})$$

for a suitable integer n; here ρ acts on I_K in such a way as to exchange each complex embedding of K for its complex conjugate. It is useful to note that for any pair $K \subseteq L$ of CM number fields

$$\mathrm{Ind}_{L/K}(I_K^0) \subseteq I_L^0$$

The main properties of the invariants P and Φ are summarized by

THEOREM 1.3:

(i) *If* (A, K, ι) *and* (B, L, κ) *are CM abelian varieties such that* $K \subseteq L$ *and* $\Phi(B, L, \kappa) = \mathrm{Ind}_{L/K}\Phi(A, K, \iota)$, *then* $P(B, L, \kappa) = P(A, K, \iota)$.

(ii) *Given a collection* $\langle (A_i, K, \iota_i)\rangle_{i=1}^s$ *of abelian varieties all with CM by a number field K, any relation*

$$0 = \Sigma\, n_i \Phi(A_i, K, \iota_i)$$

in I_K *implies a relation*

$$1 \sim \Pi\, P(A_i, K, \iota_i)^{n_i}.$$

Parts (i) and (ii) of Theorem 1.3 are proposition 1.4 and Theorem 1.3 of [S] respectively. As the notations of [S] and this paper differ somewhat we relate them for the reader's convenience: Given a CM abelian variety (A, K, ι), the CM type $\Phi(A, K, \iota)$ viewed as a formal linear combination of complex embeddings of K *is* the CM type Φ assigned by Shimura to (A, K, ι), while in terms of Shimura's symbol "$p_K(\tau, \Phi)$" we have

$$P(A, K, \iota) = \left.\begin{matrix} \pi p_K(id_k, \Phi) \\ p_K(\rho\,|_K, \Phi)^{-1} \end{matrix}\right\}$$

accordingly as id_K appears or does not appear in Φ.

Each $\Phi \in I_K^0$ can be regarded as a locally constant $\mathbf{Z}$-valued function on G by the rule

$$\Phi(\sigma) = (\sigma\,|_K, \Phi)_K$$

for all $\sigma \in G$. Now given an inclusion of CM number fields $K \subseteq L$ we have

$$\mathrm{Ind}_{L/K}(\Phi)(\sigma) = \Phi(\sigma).$$

Hence $\lim_{\rightarrow} I_K^0$ can be regarded as a subgroup of the group of locally constant $\mathbf{Z}$-valued functions on G. The reader will have little difficulty verifying

PROPOSITION 1.4: *A locally constant function $\varphi : G \to \mathbf{Z}$ belongs to $\lim_{\to} I^0_K$ if and only if for all $\sigma, \tau \in G$*

(i) $\varphi(\sigma\rho\sigma^{-1}\rho\tau) = \varphi(\tau)$,
(ii) $\varphi(\sigma\rho) + \varphi(\sigma) = \varphi(\mathrm{id}) + \varphi(\rho)$.

Let M denote the space of locally constant $\mathbf{Q}$-valued functions on G satisfying (i) and (ii) of Proposition 1.4. From Theorem 1.3 and Proposition 1.4 one easily deduces

PROPOSITION 1.5: *There exists a unique $\mathbf{Q}$-linear map $W : M \to \mathbf{C}^\times/\bar{\mathbf{Q}}^\times$ (which we call the period distribution) such that for all CM abelian varieties (A, K, ι)*

$$P(A, K, \iota) = W(\Phi(A, K, \iota)).$$

REMARK 1.6: We have, from (1.2),

$$W(1) \sim 2\pi i.$$

§2. Formulation of the principal result

Let $\hat{G}$ denote the group of continuous homomorphisms of G into $\mathbf{C}^\times$. For each rational prime p let $\mathrm{Frob}_p \in G$ be a fixed choice of *arithmetic* Frobenius element. In what follows we shall identify $\hat{G}$ and the set of primitive Dirichlet characters by rule

$$\chi(\mathrm{Frob}_p) = \chi(p)$$

for all sufficiently large rational primes p. Note that $\chi(\rho) = \chi(-1)$.

Let G_{ab} denote the galois group of the maximal abelian extension of $\mathbf{Q}$ in $\mathbf{C}$ and M^{ab} denote the subspace of M consisting of functions factoring through G_{ab}. The set

$$\{\chi \in \hat{G} \mid L(0, \chi) \neq 0\}$$

is a basis over $\mathbf{C}$ for $M^{ab} \otimes \mathbf{C}$ and so each $\varphi \in M^{ab}$ has a unique

Fourier expansion

$$\varphi = c_1 + \Sigma c_\chi \chi$$

where the summation need only be extended over a finite list of odd Dirichlet characters. We define

$$\frac{d}{ds} \log L(s, \varphi) \big|_{s=0} = c_1 \frac{d}{ds} \log \zeta(s) \big|_{s=0} + \Sigma c_\chi \frac{d}{ds} \log L(s, \chi) \big|_{s=0}.$$

For each $\varphi \in M$ we define

$$\varphi^*(\sigma) = \varphi(\sigma^{-1})$$

for all $\sigma \in G$. The principal result of this paper is

THEOREM 2.1: *For all $\varphi \in M^{ab}$,*

$$W(\varphi) \sim \exp\left(\frac{d}{ds} \log L(s, \varphi^*) \big|_{s=0}\right).$$

The proof of Theorem 2.1 is deferred to §5.

REMARK 2.2: Since $\exp(\zeta'(0)/\zeta(0)) = 2\pi$, Theorem 2.1 is consistent with our earlier observation that $W(1) \sim 2\pi i$.

Let us see how to recover relation (0.1) from Theorem 2.1. To make the elliptic curve E into an abelian variety with CM in the sense defined above, let $\iota : K \to \mathrm{End}(E) \otimes \mathbf{Q}$ be the unique representation such that

$$\iota^*(\alpha) \frac{dx}{y} = \alpha \frac{dx}{y}$$

for all $\alpha \in K$. Then $\Phi(E, K, \iota)$ is the characteristic function of $\mathrm{Gal}(\bar{\mathbf{Q}}/K) \subseteq G$, and by definition

$$P(E, K, \iota) \sim \int_c \frac{dx}{y}$$

for any 1-cycle on E not homologous to zero. By definition of the period distribution

$$W(\Phi(E, K, \iota)) = P(E, K, \iota).$$

Let χ be the quadratic Dirichlet character attached to K. We have

$$\Phi(E, K, \iota) = \tfrac{1}{2}(1+\chi),$$

and since $\zeta_K(s) = \zeta(s)L(s, \chi)$, we finally obtain, by Theorem 1.2,

$$W(\Phi(E, K, \iota)) \sim \exp\left(\frac{1}{2}\frac{d}{ds}\log \zeta_K(s)\,\Big|_{s=0}\right).$$

§3. A set of generators for M^{ab}

Let functions $\langle\ \rangle, \{\ \} : \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}$ be defined according to the rules

$$\langle a\rangle \equiv a = \{a\} \qquad \operatorname{mod} \mathbf{Z}$$
$$0 \leq \langle a\rangle < 1, \qquad 0 < \{a\} \leq 1,$$

and a function $\mathfrak{G} : \mathbf{Q}/\mathbf{Z} \to M^{ab}$ according to the rule

$$\mathfrak{G}(a)(\sigma) = \langle -b\rangle,$$

where σ runs through G and $\exp(2\pi ia)^{\sigma} = \exp(2\pi ib)$. The following theorem is due to Deligne.

THEOREM 3.1: *The image of $\mathfrak{G} : \mathbf{Q}/\mathbf{Z} \to M^{ab}$ spans M^{ab} over $\mathbf{Q}$.*

The key to the proof is

LEMMA 3.2: *Let $j < f$ be relatively prime positive integers. Then*

$$\mathfrak{G}(j/f) = \frac{1}{2} + \frac{1}{\varphi(f)}\sum_{\chi} (\chi(j)L(0, \chi^{\rho}) \prod_{p\mid f} (1-\chi^{\rho}(p)))\chi$$

where $\varphi(f) = \#((\mathrm{Z}/(f))^{*})$ and the summation is over odd Dirichlet characters χ of conductor dividing f.

As the case $f = 2$ is trivial, we may assume $f > 2$. Let μ denote normalized Haar measure on G. For each pair $\psi, \eta : G \to \mathbf{C}$ of locally constant functions we write

$$(\psi, \eta) = \int_G \psi^{\rho}(\sigma)\eta(\sigma)d\mu(\sigma).$$

It is easy to see that

$$\mathfrak{G}(j/f)=\frac{1}{2}+\sum_{\chi}(\chi,\mathfrak{G}(j/f))\chi,$$

where the summation is over odd Dirichlet characters of conductor dividing f. Fix an odd Dirichlet character χ of conductor dividing f. To prove the lemma it will be enough to compute $(\chi, \mathfrak{G}(j/f))$. Now the Hurwitz zeta function

$$H(x,s)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}$$

has the well-known Taylor expansion

$$H(x,s)=\left(\frac{1}{2}-x\right)+\left(\log\frac{\Gamma(x)}{\sqrt{2\pi}}\right)s+O(s^2), \tag{3.3}$$

a proof of which can be found in [*WW*, §13.21]. We have the easily verified identity

$$f^{-s}\sum_{\substack{0<k<f\\(k,f)=1}}\chi(k)H(k/f,s)=L(s,\chi)\prod_{p\mid f}(1-\chi(p)p^{-s}). \tag{3.4}$$

Finally, we have

$$\begin{aligned}(\chi,\mathfrak{G}(j/f))&=\frac{1}{\varphi(f)}\sum_{\substack{0<k\le f\\(k,f)=1}}\chi^{\rho}(k)\langle -kj/f\rangle\\&=\frac{1}{\varphi(f)}\sum_{\substack{0<k\le f\\(k,f)=1}}\chi^{\rho}(k)\left(\frac{1}{2}-\langle kj/f\rangle\right)\\&=\frac{\chi(j)}{\varphi(f)}L(0,\chi^{\rho})\prod_{p\mid f}(1-\chi^{\rho}(p)).\end{aligned}$$

This proves the lemma.

To prove the theorem, we first observe that it is equivalent to prove that the image of $\mathfrak{G}$ spans $M^{ab}\otimes\mathbf{C}$ over $\mathbf{C}$. Now as observed in §2, $\hat{G}\cap(M^{ab}\otimes\mathbf{C})$ is a basis over $\mathbf{C}$ for $M^{ab}\otimes\mathbf{C}$. Supposing the theorem false, let $\chi_0\in\hat{G}\cap(M^{ab}\otimes\mathbf{C})$ be a character not expressible as a linear combination of functions in the image of $\mathfrak{G}$ with smallest possible conductor f. Since $\mathfrak{G}(\frac{1}{2})\equiv\frac{1}{2}$, we can assume $\chi_0(-1)=-1$

(and $f > 2$). Let V_f be the subspace of $M^{ab} \otimes \mathbf{C}$ spanned by characters of conductor strictly less than f. By Lemma 3.2 we have

$$\varphi(f)\mathfrak{G}(j/f) = \sum_{\chi} \chi(j)L(0, \chi^{\rho})\chi \quad \bmod V_f,$$

where the summation extends over odd Dirichlet characters χ of conductor exactly f. Multiplying both sides of this relation by $\chi_0^{\rho}(j)/\varphi(f)$ and summing over j such that $0 < j < f$ and $(j, f) = 1$ we obtain

$$L(0, \chi_0^{\rho})\chi_0 = \sum_{\substack{0<j<f \\ (j,f)=1}} \chi_0(j)\mathfrak{G}(j/f) \bmod V_f;$$

now since $\chi_0(-1) = -1$, $L(0, \chi_0^{\rho}) \neq 0$. We have therefore obtained a contradiction. This proves Theorem 3.1.

§4. The Γ-function and the period distribution

This section is devoted to a brief exposition of an unpublished result of Deligne's characterizing the period distribution restricted to M^{ab} in terms of the values modulo $\bar{\mathbf{Q}}^{\times}$ of the classical Γ-function at rational values of its argument. The required explicit computation of the periods of Fermat curves in terms of special values of the Γ-function sketched here (perhaps too) briefly is given in more detail in [W] and [GR, appendix].

Let a function $\tilde{\Gamma} : \mathbf{Q}/\mathbf{Z} \to \mathbf{C}^{\times}/\bar{\mathbf{Q}}^{\times}$ be defined by the rule

$$\tilde{\Gamma}(a) = W(\mathfrak{G}(a)).$$

From the easily verified relations

$$\sum_{mb=a} \mathfrak{G}(b) = \frac{m-1}{2} + \mathfrak{G}(a),$$

$$\mathfrak{G}(a) + \mathfrak{G}(-a) \equiv 1 \qquad (a \neq 0),$$

$$\mathfrak{G}(0) = 0,$$

it follows that $\tilde{\Gamma}$ satisfies

$$\prod_{mb=a} \tilde{\Gamma}(b) \sim \tilde{\Gamma}(a)\pi^{(m-1)/2} \tag{4.1}$$

$$\tilde{\Gamma}(a)\tilde{\Gamma}(-a) \sim 2\pi i, \qquad (a \neq 0), \tag{4.2}$$

$$\tilde{\Gamma}(0) \sim 1, \tag{4.3}$$

relations closely analogous to those satisfied by the classical Γ-function:

$$\prod_{j=0}^{m-1} \Gamma\left(\frac{s+j}{m}\right) = (2\pi)^{(m-1)/2} m^{(1/2)-s}\Gamma(s), \tag{4.4}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{4.5}$$

$$\Gamma(1) = 1. \tag{4.6}$$

The following theorem is due to Deligne.

THEOREM 4.7: $\tilde{\Gamma}(a) \sim \Gamma(\{a\})$.
We require a

REDUCTION LEMMA: *If for all $a, b, c \in \mathbf{Q}/\mathbf{Z}$ such that*

$$\begin{gathered} a+b+c=0, \\ a, b, c \neq 0, \\ \langle a \rangle + \langle b \rangle < 1, \end{gathered} \tag{4.8}$$

we have

$$\tilde{\Gamma}(a)\tilde{\Gamma}(b)\tilde{\Gamma}(c) \sim \Gamma(\langle a \rangle)\Gamma(\langle b \rangle)\Gamma(\langle c \rangle),$$

then the theorem is true.

Let $e \in \mathbf{Q}/\mathbf{Z}$ of order m be fixed. Using the hypothesis of the lemma and (4.2, 3, 5, 6) it can be verified that

$$\begin{gathered} \tilde{\Gamma}(e)\tilde{\Gamma}(j/m - e)\tilde{\Gamma}(-j/m) \\ \sim \Gamma(\{e\})\Gamma(\{j/m - e\})\Gamma(\{-j/m\}) \end{gathered}$$

for $j = 0, \ldots, m-1$. By forming the product over j of these relations and applying (4.1, 4) we obtain

$$\tilde{\Gamma}(e)^m \pi^{m-1} \sim \Gamma(\{e\})^m \pi^{m-1}.$$

This concludes the proof of the reduction lemma.

Let $a, b, c \in \mathbf{Q}/\mathbf{Z}$ be given so as to satisfy (4.8). Let m be the order of the subgroup of $\mathbf{Q}/\mathbf{Z}$ generated by a, b, c. Let x be an indeterminate, $\bar{\mathbf{Q}}(x)$ the rational function field over $\bar{\mathbf{Q}}$ and $L = \bar{\mathbf{Q}}(x, x^a(1-x)^b)$ a Kummer extension of $\bar{\mathbf{Q}}(x)$ with cyclic galois group Z. Let $\psi \in \hat{Z}$ be the character defined by the relation

$$(x^a(1-x)^b)^\sigma = \psi(\sigma)x^a(1-x)^b$$

for all $\sigma \in Z$. The character ψ gives an isomorphism between Z and the mth roots of unity. Let K be the extension of $\mathbf{Q}$ obtained by adjoining the values of ψ. Since (4.8) forces $m > 2$, K is CM. Let X be the nonsingular complete curve defined over $\bar{\mathbf{Q}}$ to which L gives rise, which is a quotient of a Fermat curve. Let $J(X)$ be the Jacobian of X, choose a $\bar{\mathbf{Q}}$-rational basepoint ξ_0 on X and let $\varphi : X \to J(X)$ be the morphism sending $\xi \in X$ to the divisor class of $(\xi) - (\xi_0)$. $J(X)$ and φ are defined over $\bar{\mathbf{Q}}$. The group Z acts upon $J(X)$ in such a fashion that φ induces a Z-isomorphism between $H^0(\Omega^1(J(X)))$ and $H^0(\Omega^1(X))$. The action of Z extends in $\mathbf{Q}$-linear fashion to a representation

$$\kappa : \mathbf{Q}[Z] \to \mathrm{End}(J(X)) \otimes \mathbf{Q}.$$

The character ψ extends in $\mathbf{Q}$-linear fashion to a surjective homomorphism $\psi : \mathbf{Q}[Z] \to K$ which has a unique section $\epsilon : K \to \mathbf{Q}[Z]$ embedding K as an ideal. For a suitable positive integer N, $N\kappa \circ \epsilon(1)$ belongs to $\mathrm{End}(J(X))$. To abbreviate we write η instead of $N\kappa \circ \epsilon(1)$. Let A be the image of η, an abelian variety defined over $\bar{\mathbf{Q}}$, and let $\iota : K \to \mathrm{End}(A) \otimes \mathbf{Q}$ be the unique representation such that for all $\alpha \in K$, $\eta \circ (\kappa \circ \epsilon(\alpha)) = \iota(\alpha) \circ \eta$. Note that $i(1) = \mathrm{id}_A$. For each $\sigma \in G$ we have

$$\begin{aligned}
&\dim_{\mathbf{C}}\{\omega \in H^0(\Omega^1(A)) \mid \iota(\alpha)^*\omega = \alpha^\sigma\omega \text{ for all } \alpha \in K\} \\
&= \dim_{\mathbf{C}}\{\omega \in H^0(\Omega^1(J(X))) \mid \tau^*\omega = \psi(\tau)^\sigma\omega \text{ for all } \tau \in Z\} \\
&= \dim_{\mathbf{C}}\{\omega \in H^0(\Omega^1(X)) \mid \tau^*\omega = \psi(\tau)^\sigma\omega \text{ for all } \tau \in Z\} \\
&= (\mathfrak{G}(a) + \mathfrak{G}(b) + \mathfrak{G}(c))(\sigma) - 1,
\end{aligned}$$

the last step being justified by an application of the Weil–Chevalley formula [CW]. By summing the relation immediately above over cosets of $\mathrm{Gal}(\bar{\mathbf{Q}}/K)$ in G we find that $\dim(A) = \dim_{\mathbf{C}} H^0(\Omega^1(A)) = \frac{1}{2}[K : \mathbf{Q}]$. Hence (A, K, ι) is an abelian variety with complex multiplication, $\Phi(A, K, \iota) = \mathfrak{G}(a) + \mathfrak{G}(b) + \mathfrak{G}(c) - 1$, and we have

$$\pi P(A, K, \iota) = \tilde{\Gamma}(a)\tilde{\Gamma}(b)\tilde{\Gamma}(c). \tag{4.9}$$

Now the differential of the first kind

$$\omega_{abc} = x^{\langle a\rangle - 1}(1-x)^{\langle b\rangle - 1}\, dx$$

forms a basis over $\bar{\mathbf{Q}}$ of the space

$$\{\omega \in H^1_{DR}(X) \mid \tau^*\omega = \psi(\tau)\omega \text{ for all } \tau \in Z \text{ and } \omega \text{ is } \bar{\mathbf{Q}}\text{-rational}\}.$$

To compute $P(A, K, \iota)$ directly, we have only to compute the integral of ω_{abc} on any integral cycle such that the value of the integral is nonzero. For a cycle C constructed by lifting a commutator of loops around 0 and 1 on $\mathbf{P}^1 - \{0, 1, \infty\}$ to X (the "Pochhammer contour" discussed in [WW, §12.43]) we have

$$\int_C \omega_{abc} = (e^{2\pi i a} - 1)(e^{2\pi i b} - 1)\frac{\Gamma(\langle a\rangle)\Gamma(\langle b\rangle)}{\Gamma(\langle a\rangle + \langle b\rangle)},$$

and hence (after applying (4.6) once)

$$\pi P(A, K, \iota) \sim \Gamma(\langle a\rangle)\Gamma(\langle b\rangle)\Gamma(\langle c\rangle). \tag{4.10}$$

The comparison of (4.9) and (4.10) concludes the proof of Theorem 4.7.

§5. Proof of the principal result

In view of Theorem 3.1 and Theorem 4.7 it will suffice to verify that

$$\exp\left(\frac{d}{ds}\log L(s, \mathfrak{G}(a)^*)\,\Big|_{s=0}\right) \sim \Gamma(\{a\}). \tag{5.1}$$

Since (5.1) is readily verified for $a = 0, \frac{1}{2}$, we may assume that $a = (j/f)$ with j and f relatively prime integers satisfying $0 < j < f$ and $f > 2$. In the calculations that follow, χ runs through the odd primitive Dirichlet characters of conductor dividing f and k runs through the integers satisfying $0 < k < f$, $(k, f) = 1$. We have

$$\mathfrak{G}(j/f)^* = \frac{1}{2} + \sum_{\chi} (\chi^{\rho}, \mathfrak{G}(j/f))\chi,$$

where the summation is over odd characters χ with conductor dividing f, and where by Lemma 3.2

$$(\chi^{\rho}, \mathfrak{G}(j/f)) = \frac{\chi^{\rho}(j)}{\varphi(f)} L(0, \chi) \prod_{p \mid f} (1 - \chi(p))$$

Hence

$$\frac{d}{ds} \log L(s, \mathfrak{G}(j/f)^* \mid_{s=0}) - \frac{1}{2} \log 2\pi = \frac{1}{\varphi(f)} \sum_{\chi} \chi^{\rho}(j) L'(0, \chi) \prod_{p \mid f} (1 - \chi(p)). \tag{5.2}$$

Differentiation of (3.4) at $s = 0$ and an application of (3.3) yield the relation

$$\begin{aligned} L'(0, \chi) \prod_{p \mid f} (1 - \chi(p)) = \sum_{k} \chi(k) \log \frac{\Gamma(k/f)}{\sqrt{2\pi}} - L(0, \chi)\{\log f \prod_{p \mid f} (1 - \chi(p)) \\ + \sum_{q \mid f} \chi(q) \log q \prod_{\substack{p \mid f \\ p \neq q}} (1 - \chi(p))\}. \end{aligned} \tag{5.3}$$

Multiplication of both sides of (5.3) by $\chi^{\rho}(j)/\varphi(f)$ and summation over χ yield

$$\begin{aligned} \frac{d}{ds} \log L(s, \mathfrak{G}(j/f)^*) \mid_{s=0} - \frac{1}{2} \log 2\pi = \frac{1}{2} \log \frac{\Gamma(j/f)}{\sqrt{2\pi}} - \frac{1}{2} \log \frac{\Gamma(1 - j/f)}{\sqrt{2\pi}} \\ + (\text{element of } \mathbf{Q} \log \mathbf{Q}^{\times}). \end{aligned} \tag{5.4}$$

Exponentiation of both sides of (5.4) and an application of (4.5) yield the desired relation. This completes the proof of Theorem 2.1.

§6. p-Adic complements

In this section we will try to show how Theorem 2.1 fits into a larger pattern involving not only the Dirichlet L-functions but the p-adic L-functions of Kubota and Leopoldt as well. Fix an odd rational prime p and an isomorphism $\lambda : \mathbf{C}_p \xrightarrow{\sim} \mathbf{C}$ under which p-adic and complex numbers are henceforth identified; in particular, we regard $\bar{\mathbf{Q}}$ as a subfield of both $\mathbf{C}$ and $\mathbf{C}_p$. Following Honda [H] we say that an algebraic α is of type A_0 if for a suitable nonnegative integer f

and all $\sigma \in G$,

$$|\alpha^\sigma| = p^{f/2}.$$

Let $\mathfrak{A}_p$ be the smallest subgroup of $\bar{\mathbf{Q}}^\times$ containing all the numbers of type A_0 and closed under root extraction. Let $|\cdot|_p$ denote the p-adic absolute value (normalized so that $|p|_p = p^{-1}$). We define

$$D_p = \{\sigma \in G \mid |x^\sigma|_p = |x|_p \text{ for all } x \in \bar{\mathbf{Q}}\}$$

and

$$M_p = \{\varphi \in M \mid \varphi(\sigma\tau) = \varphi(\sigma) \text{ for all } \sigma \in G \text{ and } \tau \in D_p\}.$$

To each $\alpha \in \mathfrak{A}_p$ we assign a locally constant function $\Phi_\alpha : G \to \mathbf{Q}$ by the rule

$$|\alpha|^2 |\alpha^\sigma|_p = p^{\Phi_\alpha(\sigma)}.$$

The following proposition serves to define the p-adic analogue of the period distribution.

PROPOSITION 6.1:

(i) *For all $\alpha \in \mathfrak{A}_p$, $\Phi_\alpha \in M_p$.*

(ii) *There exists a unique* $\mathbf{Q}$*-linear map $W_p : M_p \to \mathfrak{A}_p/\boldsymbol{\mu}_\infty$ with the property that for all $\alpha \in \mathfrak{A}_p$,*

$$W_p(\Phi_\alpha) = \alpha \mod \boldsymbol{\mu}_\infty.$$

Part (i) is readily checked. Part (ii) is equivalent to the assertion that the sequence

$$0 \to \boldsymbol{\mu}_\infty \to \mathfrak{A}_p \to M_p \to 0$$

is exact, where the second arrow is the map $\alpha \mapsto \Phi_\alpha$ defined above. Exactness at the middle follows from the theorem of Kronecker asserting that a global unit in a number field of absolute value 1 at all complex embeddings is a root of unity. Exactness at the right is essentially Theorem 1 of [H].

REMARK 6.2: We have

$$W_p(1) = p \qquad \mod \boldsymbol{\mu}_\infty.$$

Let $\log_p : \mathbf{C}_p^\times \to \mathbf{C}_p$ be the Iwasawa logarithm (defined by the usual power series on the principal units and extended to $\mathbf{C}_p^\times$ by requiring $\log_p xy = \log_p x + \log_p y$ and $\log_p p = 0$). Let $\omega : \mathbf{Z} \to \mathbf{C}_p$ be the Teichmüller character (the unique primitive Dirichlet character of conductor p satisfying $|\omega(n) - n|_p < 1$ for all integers n). The $\mathbf{Q}$-linear map $\log_p W_p : M_p \to \mathbf{C}_p$ has a unique $\mathbf{C}_p$-linear extension to $M_p \otimes \mathbf{C}_p$ which we again call $\log_p W_p$. Now $M_p \otimes \mathbf{C}_p \cap \hat{G} = \{1\} \cup \{\chi \mid \chi(-1) = -1$ and $\chi(p) = 1\}$ and forms a basis of $M_p^{ab} = M_p \cap M^{ab}$. Therefore the following theorem, a p-adic analogue of Theorem 2.1, determines the restriction of $\log_p W_p$ to M_p^{ab}.

THEOREM 6.3: *For all* $1 \neq \chi \in M_p \otimes \mathbf{C}_p \cap \hat{G}$ *we have*

$$\log_p W_p(\chi) = \frac{L_p'(0, \chi^\rho \omega)}{L(0, \chi^\rho)}.$$

The proof of Theorem 6.3 will take up the rest of this section, proceeding along lines parallel to those followed in the proof of Theorem 2.1. The analogue (and immediate consequence) of Theorem 3.1 is

PROPOSITION 6.4: *Let* $a \in (\mathbf{Q} \cap \mathbf{Z}_p)/\mathbf{Z}$ *and a positive integer* m *be given so that* $(p^m - 1)a = 0$. Then

$$\sum_{\nu=1}^{m} \mathfrak{G}(p^\nu a) \in M_p^{ab}$$

and the set of all such functions spans M_p^{ab} *over* $\mathbf{Q}$.

Let ζ be a primitive pth root of unity in $\mathbf{C}_p$. Fix a and m as above and put $q = p^m$. We define the Gauss sum

$$g(a, q) = -\sum_{\epsilon^{q-1}=1} \epsilon^{-a(q-1)} \zeta^{\epsilon + \epsilon^p + \cdots + \epsilon^{p^{m-1}}}.$$

Let $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ be the Morita p-adic gamma function. For its definition and properties we refer to [B]. The formulas of which have particular need are

$$\text{(6.8)} \qquad \sum_{\nu=0}^{n} \log_p \Gamma_p\left(\frac{\langle a\rangle + \nu}{n}\right) = \log_p \Gamma_p(\langle a\rangle) + (\langle p^{m-1}a\rangle - \langle a\rangle) \log_p n,$$

$$\text{(6.9)} \qquad \log_p \Gamma_p(\langle a\rangle) + \log_p \Gamma_p(\langle -a\rangle) = 0,$$

where a and m are as above.

The following theorem provides a p-adic analogue of Theorem 4.7.

THEOREM 6.5: *Let a, m and q be as above. Then*

$$\log_p W_p\left(\sum_{\nu=1}^{m} \mathfrak{G}(p^\nu, a)\right) = \log_p g(a, q) = \sum_{\nu=1}^{m} \log_p \Gamma_p(\langle p^\nu a\rangle).$$

The first equality is a restatement of the factorization of Gauss sums due to Stickelberger, while the second is obtained by taking the Iwasawa logarithm of both sides of the formula proved in [GK].

Let f be a positive integer prime to p and χ a primitive Dirichlet character of conductor f such that $\chi(-1) = -1$ and $\chi(p) = 1$. Then according to [FG]

$$L_p'(0, \chi) = \sum_{0<j\le f} \chi(j) \log_p \Gamma_p(j/f). \tag{6.6}$$

A mild generalization of (6.6) is required, namely

LEMMA 6.7: *Let f be a positive integer prime to p and χ a primitive Dirichlet character of conductor g dividing f, such that $\chi(-1) = -1$ and $\chi(p) = 1$. Then*

$$L_p'(0, \chi\omega) \prod_{q\mid f} (1 - \chi(q)) = \sum_{\substack{0<j\le f\\(j,f)=1}} \chi(j) \log_p \Gamma_p(j/f).$$

Choose a positive integer m so that $f \mid (p^m - 1)$, and let μ denote the usual Möbius function. We compute

$$\begin{aligned}\sum_{\substack{0<j\le f\\(j,f)=1}} \chi(j) \log_p \Gamma_p(j/f) &= \sum_{d\mid f} \mu(f/d)\left\{\sum_{0<j\le d} \chi(jf/d) \log_p \Gamma_p(j/d)\right\}\\
&= \sum_{n\mid (f/g)} \mu(f/gn)\chi(f/gn)\left\{\sum_{0<j\le gn} \chi(j) \log_p \Gamma_p(j/gn)\right\}\\
&= \sum_{n\mid (f/g)} \mu(f/gn)\chi(f/gn)\left\{\sum_{0<j\le g} \chi(j)\right.\\
&\qquad \left.\times\left(\sum_{\nu=0}^{n-1} \log_p \Gamma_p(j/gn + \nu/n)\right)\right\}\\
&= \sum_{n\mid (f/g)} \mu(f/gn)\chi(f/gn)\left\{\sum_{0<j\le g} \chi(j) \log_p \Gamma_p(j/g)\right.\\
&\qquad \left.+ \log_p n \sum_{0<j\le g} \chi(j)(\langle p^{m-1}j/g\rangle - \langle j/g\rangle)\right\}\end{aligned}$$

$$= \Big(\prod_{q \mid f} (1-\chi(q))\Big) \sum_{0<j\leq g} \chi(j) \log_p \Gamma_p(j/g)$$

$$= \Big(\prod_{q \mid f} (1-\chi(q))\Big) L_p'(0, \chi\omega).$$

This calculation concludes the proof of Lemma 6.7.

Let $\eta : M_p^{ab} \otimes \mathbf{C}_p \to \mathbf{C}_p$ be the unique $\mathbf{C}_p$-linear map such that $\eta(1) = 0$ and for all $1 \neq \chi \in M_p^{ab} \otimes \mathbf{C}_p \cap G$,

$$\eta(\chi) = \frac{L_p'(0, \chi^\rho \omega)}{L(0, \chi^\rho)}.$$

To finish the proof of Theorem 6.3 we will show that η and $\log_p W_p$ coincide by checking on the generating set provided by Proposition 6.4. Fix a pair $j < f$ of relatively prime positive integers with f prime to p. Let m be a positive integer such that $f \mid (p^m - 1)$. By Lemma 3.2 we have

$$\sum_{\nu=0}^{m-1} \mathfrak{G}(p^\nu j/f) = \frac{m}{2} + \frac{m}{\varphi(f)} \sum_{\chi} (\chi(j) L(0, \chi^\rho) \prod_{q \mid f} (1 - \chi^\rho(q)))\chi$$

where the summation is over the set of odd primitive Dirichlet characters χ of conductor dividing f and satisfying $\chi(p) = 1$.

We have

$$\eta\Big(\sum_{\nu=0}^{m-1} \mathfrak{G}(p^\nu, j/f)\Big) = \frac{m}{\varphi(f)} \sum_{\chi} \chi(j) L_p'(0, \chi^\rho \omega) \prod_{q \mid f} (1 - \chi^\rho(q))$$

$$= \frac{m}{\varphi(f)} \sum_{\chi} \chi(j) \Big(\sum_{\substack{0<k<f \\ (k,f)=1}} \chi^\rho(k) \log_p \Gamma_p(k/f) \Big)$$

$$= \sum_{\nu=0}^{m-1} (\tfrac{1}{2} \log_p \Gamma_p(\langle p^\nu j/f \rangle) - \tfrac{1}{2} \log_p \Gamma_p(\langle -p^\nu j/f \rangle))$$

$$= \sum_{\nu=0}^{m-1} \log_p \Gamma_p(\langle p^\nu j/f \rangle)$$

$$= \log_p W_p \Big(\sum_{\nu=0}^{m-1} \mathfrak{G}(p^\nu a) \Big).$$

This concludes the proof of Theorem 6.3.

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The Institute for Advanced Study
Princeton, New Jersey 08540