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GEODESIC CYCLES AND THE WEIL REPRESENTATION I; QUOTIENTS OF HYPERBOLIC SPACE AND SIEGEL MODULAR FORMS

Stephen S. Kudla* and John J. Millson**

Introduction

In a series of papers we intend to give a general method of constructing harmonic forms on locally symmetric spaces as special values of certain families of forms depending on complex parameters, somewhat analogous to Eisenstein series. These harmonic forms will be the Poincaré duals of certain totally geodesic cycles in the locally symmetric space. On the other hand, for the locally symmetric spaces which arise from the action of certain arithmetic groups on the symmetric spaces associated to $SO(p, q)$ and $SU(p, q)$, the global Weil representation provides a method of constructing automorphic forms, and, in particular, harmonic forms in some cases. We will show that, in certain cases, there is a coincidence of the duals of geodesic cycles and the harmonic forms coming from the Weil representation (Corollary 10.1). As a consequence of this coincidence, we will show that the results of Hirzebruch and Zagier [7] for the Hilbert modular surfaces and of Kudla [12] for certain arithmetic quotients of the complex 2-ball, relating intersection numbers of cycles and Fourier coefficients of elliptic modular forms, are special cases of a general formula relating intersection numbers of cycles in certain arithmetic quotients of the symmetric spaces of $SO(p, q)$ and

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$SU(p, q)$ to the Fourier coefficients of Siegel modular and Hermitian modular forms respectively.

In this paper we carry out the above program for certain discrete subgroups of $SO(n, 1)$.

In the first part of this paper, we generalize the results of [13] to quotients of hyperbolic n-space. More precisely, let M be a compact, oriented, Riemannian n-manifold with constant negative curvature, so that we may realize M as a quotient $M = \Gamma\backslash D$ of hyperbolic n-space D by a torsion free group Γ of orientation-preserving isometries. Suppose that N is a totally geodesic, oriented $(n-k)$-dimensional submanifold of M, and let $[N] \in H_{n-k}(M, \mathbb{Z})$ be the corresponding homology class. We then obtain an explicit formula for the harmonic k-form which is the Poincaré dual to N by the following procedure: We realize N as a quotient, $N = \Gamma_1\backslash D_1$, where $D_1 \subset D$ is an $(n-k)$-dimensional hyperbolic space, totally geodesic in D, and $\Gamma_1 \subset \Gamma$ is the fundamental group of N (Section 1). Then we reduce the problem of finding a Poincaré dual ω to N in M to the problem of constructing a smooth closed k-form ψ, having certain properties (Lemma 2.1), on the 'tube' $\Gamma_1\backslash D$. A Poincaré dual form is then obtained by averaging:

$$\omega = \sum_{\gamma \in \Gamma_1\backslash\Gamma} \gamma^* \psi.$$

In Section 3 we construct a family ψ_s of such k-forms, depending on a complex parameter s, such that the series

$$\omega_s = \sum_{\gamma \in \Gamma_1\backslash\Gamma} \gamma^* \psi_s$$

is absolutely convergent, and hence defines a Poincaré dual form, provided s lies in the half-plane $\mathrm{Re}(s) > \frac{1}{2}(k-1)$. In Section 4 we show that the family of forms ω_s satisfy the 'shift' equation:

$$\Delta\omega_s = 2s(2s + m - 2k)(\omega_{s+1} - \omega_s).$$

As a consequence, we obtain a meromorphic analytic continuation of ω_s, which is analytic at $s = s_0 = \max\{0, k - \frac{1}{2}m\}$. The form ω_{s_0} is then the desired harmonic dual form to N (Theorem 4.3).

In the second part of the paper, we consider the case where Γ is a congruence subgroup of the unit group of an anisotropic quadratic form over a totally real number field $\mathscr{k}$. In this case, we establish a relationship between the harmonic dual forms ω_{s_0} to totally geodesic cycles of codimension k in M and the harmonic forms on M which arise from the Weil representation for the dual reductive pair $\mathrm{Sp}(k)\times O(n,1)$ in the sense of Howe [8], [4].

More precisely, let V be a $\mathscr{k}$ vector space with $\dim_{\mathscr{k}} V = m$, and let $(,)$ be a nondegenerate, symmetric, $\mathscr{k}$-bilinear form on V such that $(V(\mathscr{k}),(,))$ is anisotropic, and

$$\operatorname{sig}(V_\lambda,(,)) = \begin{cases} (n,1) & \text{if } \lambda = 1 \\ (m,0) & \text{if } \lambda > 1, \end{cases}$$

where $\mathscr{k}_\lambda$, $\lambda = 1,\ldots, r = |\mathscr{k}:\mathbb{Q}|$, are the archimedian completions of $\mathscr{k}$, and $V_\lambda = V\otimes_{\mathscr{k}} \mathscr{k}_\lambda$.

Let $G = SO(V)$ be the special orthogonal group of V, viewed as an algebraic group over $\mathscr{k}$. Let $\mathcal{O}$ be the ring of integers of $\mathscr{k}$, and let $L\subset V$ be an $\mathcal{O}$-lattice such that the dual lattice

$$L^* = \{Y\in V(\mathscr{k}) \mid \operatorname{tr}_{\mathscr{k}}(X, Y)\in\mathbb{Z}, \forall X\in L\}$$

contains L. We then let Γ be a torsion free congruence subgroup of

$$G(L) = \{\gamma\in G(\mathscr{k}) \mid \gamma L = L \text{ and } \gamma \text{ acts trivially in } L^*/L\},$$

and let

$$M = \Gamma\backslash D,$$

where we view D as the space of negative lines in V_1.

Then, in Section 6, we associate to each frame $X = (X^1,\ldots, X^k)\in V^k$, with $(X,X) = ((X^i, X^j))$ totally positive definite, an oriented, totally geodesic cycle $\iota_X: N_X\to M$ of codimension k, and hence a homology class $C_X = (\iota_X)_*(1_X)\in H_{n-k}(M,\mathbb{Z})$.

In Sections 7 and 8 we construct a certain type of theta-function. Specifically, for $\tau = (\tau_1,\ldots,\tau_r)\in\mathfrak{H}_k^r$, $\tau_\lambda = u_\lambda + iv_\lambda$; $Z\in D$, $W = (W^1,\ldots, W^k)\in V_1^k$ with $W^i\in Z^\perp$ and $\mu\in(L^*)^k/L^k$, we define

$$\theta_\mu(\tau, Z, W) = \det v_1^{1/2} \sum_{X\equiv\mu(L^k)} \det(X_1, W) e_*((X, X)_{\tau,Z})$$

where

$$(X, X)_{\tau,Z} = (u_1(X_1, X_1) + iv_1(X_1, X_1)_Z, \tau_2(X_2, X_2),\ldots, \tau_r(X_r, X_r))\in M_k(\mathbb{C})^r$$

with $(,)_Z$ the majorant of $(,)$ associated to Z, and, for $Y \in M_k(\mathbb{C})^r$,

$$e_*(Y) = e\Big(\sum_\lambda \mathrm{tr}(Y_\lambda)\Big).$$

If we identify the tangent space $T_Z(D) \simeq Z^\perp \subset V_1$, then as a function of Z and $W \in T_Z(D)^k$, θ_μ determines a k-form on D which is easily seen to be Γ-invariant. On the other hand, we prove (Proposition 8.4) that, as a function of $\tau \in \mathfrak{H}_k^r$, θ_μ has a transformation law like a Hilbert–Siegel modular form of weight $\frac{m}{2}$ and a certain θ-multiplier with respect to a certain congruence subgroup $\check{\Gamma} \subset \mathrm{Sp}(k, \mathcal{O})$.

In Section 9 we use the theta-kernel θ_μ to define a lifting $\mathcal{L}_k$ from $S_{m/2}(\check{\Gamma})$, the space of holomorphic Hilbert–Siegel cusp forms which transform as θ_μ does, to differential k-forms on M, and we give an explicit formula (Theorem 9.1) for the lifting $\mathcal{L}_k$ of the (generalized) Poincaré series, $P^*_{\beta,s}$. In fact, if $\beta = {}^t\beta \in M_k(\mathscr{k})$ with $\beta \gg 0$ (totally positive definite), we let

$$C(\beta;\, \mu) = \sum_{\substack{X \equiv \mu(L^k) \\ (X,X) = 2\beta \\ \mathrm{mod}\, \Gamma}} C_X$$

so that $C(\beta;\, \mu) \in H_{n-k}(M, \mathbb{Z})$, and we let

$$\omega_s(\beta;\, \mu) = \sum_X \omega_{X,s}$$

where $\omega_{X,s}$ is the family of Poincaré dual forms to C_X constructed in part 1. Then we find that

$$\mathcal{L}_k(P^*_{\beta,s}) = 2^{k/2}\omega_s(\beta,\, \mu).$$

In Section 10 we obtain several consequences of this identity. In particular, if $k < \frac{1}{4}m$, we obtain (Corollary 10.1)

$$\mathcal{L}_k(P^*_{\beta,0}) = 2^{k/2}\omega_0(\beta;\, \mu)$$

which shows that, in this case, the image of the lifting $\mathcal{L}_k$ is the complex span of the harmonic dual forms to the cycles $C(\beta;\mu)$. We also show (Corollary 10.4) that if $C \in H_k(M, \mathbb{C})$, then the generating function for the intersection numbers

$$I_k(\tau,\, C) = \sum_{\substack{\beta \in \mathcal{L}^* \\ \beta \gg 0}} C \cdot C_\beta\, e_*(\beta\tau),$$

where $\mathscr{L}^*$ is a certain lattice, lies in $S_{m/2}(\check{\Gamma})$. This generalizes the main result of Hirzebruch and Zagier [7].

Finally, in Section 11, we use a slight generalization of the non-vanishing theorem of Millson–Raghunathan [15] to prove that, for suitable L, μ, and Γ the cycle $C(\beta; \mu) \neq 0$ (Corollary 11.3). Thus our mappings $\mathscr{L}_k$ and I_k^0 will, in general, be nontrivial. The required generalization of [15] is proved in the appendix.

Our initial interest in this question was sparked by the work of Hirzebruch and Zagier [7] and Zagier [24]. Our approach is different in that we find a formula for the Poincaré dual of a geodesic cycle, then prove the intersection formula. Also, we believe that our use of the Weil representation results in a better understanding of the nature of the intersection number formula. This use of the Weil representation was inspired by Shintani [21], who constructed the map I_1 in the split case for $n = 2$, although he stated his result in terms of periods rather than intersection numbers. A similar mapping occurs in the work of Oda [18].

We have greatly profited from the ideas of Roger Howe about the Weil representation.

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CONTENTS

§1. Totally geodesic cycles in quotients of hyperbolic n-space

Let V be a real vector space, $\dim_{\mathbb{R}} V = m$, and let (,) be a non-degenerate symmetric $\mathbb{R}$-bilinear form on V such that

$$\mathrm{sig}(V, (,)) = (n, 1),$$

with $m = n + 1$. Let

$$G = SO^0(V)$$

be the connected component of the special orthogonal group of V, (,), and let

$$D = \{\ell \in \mathbb{P}(V) | (,)_{\ell} < 0\}$$

be the space of all negative lines in V with respect to (,). Then G acts transitively on D, and the isotropy subgroup of any point of D is a maximal compact subgroup of G.

We identify D with one sheet of the hyperboloid of two sheets in V:

$$D \simeq \{Z \in V \mid (Z, Z) = -1\}^0,$$

and thus we have a natural identification

$$T_Z(D) \simeq Z^{\perp}$$

where $T_Z(D)$ is the tangent space of D at Z, and

$$Z^{\perp} = \{W \in V \mid (Z, W) = 0\}.$$

The bilinear form (,) induces a positive definite inner product on each $T_Z(D)$, and hence defines a G-invariant Riemannian metric on D. The geodesics in D with respect to this metric are just the intersections of D with hyperplanes in V, and, for $Z, Z' \in D$, the geodesic distance $d(Z, Z')$ is given by

$$\cosh d(Z, Z') = |(Z, Z')|.$$

Also D has constant curvature -1.

Let $\Gamma \subset G$ be a discrete, torsion free subgroup such that the quotient space

$$M = \Gamma \backslash D$$

is compact. Thus M is a compact orientable Riemannian n-manifold with constant curvature -1.

We now consider totally geodesic cycles in M. For $k \in \mathbb{Z}$ with $1 \leq k < n$, let $U \subset V$ be a subspace with $\dim_{\mathbb{R}} U = k$ and such that $(,)_U > 0$.

Define an involution $\sigma = \sigma_U \in O(V)$ by

$$\sigma = \begin{cases} -1 \text{ on } U \\ +1 \text{ on } U^{\perp}. \end{cases}$$

Let

$$G_\sigma = \{g \in G \mid \sigma g \sigma = g\}$$

be the centralizer of σ in G, and let

$$D_\sigma = \{z \in D \mid \sigma z = z\}$$

be the fixed point set of σ in D. Note that $D_\sigma = D \cap U^{\perp}$, so that D_σ is a totally geodesic submanifold of D with $\dim D_\sigma = n - k$.

Let $\Gamma_\sigma = \Gamma \cap G_\sigma$, and let $N = \Gamma_\sigma \backslash D_\sigma$. We then obtain the diagram

$$\begin{array}{ccc} D_\sigma & \hookrightarrow & D \\ \downarrow & & \downarrow \\ N = \Gamma_\sigma \backslash D_\sigma & \xrightarrow{\iota} & \Gamma \backslash D = M \end{array}$$

where the map ι is induced by the inclusion of D_σ into D. In general, Γ_σ will be the identity, and the image $\iota(N)$ of D_σ in M will be dense.

Thus to obtain nice cycles we assume the *compatibility condition*:

$$(*) \qquad \sigma\Gamma\sigma = \Gamma.$$

The following lemma is then proved in Millson–Raghunathan [15].

LEMMA 1.1: *Assume that the involution σ defined above satisfies* (*). *Then*

(i) *the quotient $N = \Gamma_\sigma \backslash D_\sigma$ is compact, and*
(ii) (*Jaffee*) *the map ι is an embedding with locally finite image.*

Now let G_σ^0 be the connected component of G_σ, and assume that

$$(**) \qquad \Gamma_\sigma \subset G_\sigma^0,$$

so that N is orientable. We then obtain a compact embedded totally geodesic $n-k$ cycle

$$\iota : N \hookrightarrow M$$

and hence, choosing an orientation $1_N \in H_{n-k}(N, \mathbb{Z})$, a homology class

$$[N] = \iota_*(1_N) \in H_{n-k}(M, \mathbb{Z}).$$

When Γ is a group of units of an indefinite quadratic form over a totally real number field, we will construct a large family of subspaces $U \subset V$ for which the corresponding involution σ_U satisfies (*) and (**). In this case Millson and Raghunathan [15]—see also Prop. 11.1 and Appendix—have shown that, after passing to a suitable congruence subgroup, we can obtain nonvanishing classes $[N]$ in all dimensions. We do not know, however, if it is possible to obtain similar results for other types of discrete subgroups $\Gamma \subset G$.

§2. Poincaré dual forms

In this section we begin the construction of a Poincaré dual form to the cycle N considered in §1.

Recall that if M is a compact, smooth oriented n-manifold and if $\iota : N \to M$ is a smooth singular cycle with N oriented of dimension $n-k$, then a Poincaré dual form to N is a smooth closed k-form ω on

M such that, for all smooth closed $n-k$ forms η on M we have

$$\int_M \omega \wedge \eta = \int_N \iota^* \eta.$$

Any two Poincaré dual forms to N are cohomologous, and, by Hodge theory, there exists a unique harmonic dual form which we call *the* Poincaré dual form to N. If N' is any smooth oriented k-cycle in M and ω is a Poincaré dual of N, then

$$\int_{N'} \omega = [N'] \cdot [N]$$

where $[N'] \cdot [N]$ is the intersection number of the classes $[N] \in H_{n-k}(M, \mathbb{Z})$ and $[N'] \in H_k(M, \mathbb{Z})$.

We now return to the situation of §1, and we assume that σ satisfies conditions (*) and (**). We then consider the partial quotient $E = \Gamma_\sigma \backslash D$ and obtain the diagram

$$\begin{array}{ccc} D_\sigma & \hookrightarrow & D \\ \downarrow & & \downarrow \\ N = \Gamma_\sigma \backslash D_\sigma & \hookrightarrow & \Gamma_\sigma \backslash D = E \\ \| & & \downarrow \\ N & \overset{\iota}{\hookrightarrow} & M \end{array} .$$

Note that the projection $E \to M$ is a non-normal covering.

For convenience we will identify differential forms on M (resp. E) with Γ(resp. Γ_σ) invariant forms on D.

LEMMA 2.1: *Suppose that ψ is a smooth closed k-form on E such that*

(i) *ψ is integrable on E, i.e.,* $\int_E \|\psi\| dv < \infty$,
(ii) *for any closed, bounded, $n-k$ form η on E*

$$\int_E \psi \wedge \eta = \int_N \eta,$$

and

(iii) the series

$$\sum_{\gamma \in \Gamma_\sigma \backslash \Gamma} \gamma^* \psi$$

is absolutely convergent, uniformly on compact subsets of E. Then the Γ-invariant form

$$\omega = \sum_{\gamma \in \Gamma_\sigma \backslash \Gamma} \gamma^* \psi$$

is a Poincaré dual form to N in M.

Proof: By (iii) ω is closed, smooth, and Γ-invariant. Moreover, if F is a measurable fundamental domain for Γ in D, we have

$$\begin{aligned}
\int_M \omega \wedge \eta &= \int_F \omega \wedge \eta \\
&= \int_F \left(\sum_\gamma \gamma^* \psi\right) \wedge \eta \\
&= \int_F \sum_\gamma \gamma^*(\psi \wedge \eta) \\
&= \sum_\gamma \int_F \gamma^*(\psi \wedge \eta) \\
&= \sum_\gamma \int_{\gamma F} \psi \wedge \eta \\
&= \int_E \psi \wedge \eta \\
&= \int_N \eta
\end{aligned}$$

where we observe that $\bigcup_{\gamma \in \Gamma_\sigma \backslash \Gamma} \gamma F$ is a fundamental domain for Γ_σ in D. By (iii) the interchange of summation and integration is justified, and the lemma is proved.

§3. A family of dual forms

In this section we will construct a family of Γ_σ-invariant forms ψ_s depending on a complex parameter s, such that, for $\mathrm{Re}(s)$ sufficiently large, ψ_s satisfies the conditions of Lemma 2.1. Thus we obtain a family of Poincaré dual forms to N.

First observe that there is a natural projection

$$\pi : D \to D_\sigma$$

defined as follows. Write $V = U + U^{\perp}$ where $U^{\perp}$ is the orthogonal complement to U in V, and let $\pi: V \to U^{\perp}$ be the orthogonal projection. Then the image of $\ell \in D$ under π is again a negative line in $U^{\perp}$; hence we have

$$\pi : \ell \mapsto \pi(\ell) \in D_{\sigma}.$$

Moreover, if $g \in G_{\sigma}$, then

$$\pi(g\ell) = g\pi(\ell),$$

i.e., π is G_{σ} equivariant. Therefore π induces a fibration:

$$\begin{array}{ccc} D & \xrightarrow{\pi} & D_{\sigma} \\ \downarrow & & \downarrow \\ E = \Gamma_{\sigma}\backslash D & \xrightarrow{\pi} & \Gamma_{\sigma}\backslash D_{\sigma} = N \end{array}$$

which we also denote by π.

Observe that if $\ell' \in D_{\sigma}$, then

$$\pi^{-1}(\ell') = \{\ell \in D \mid \ell \subset \ell' + U\},$$

and so the fibers of π are totally geodesic hyperbolic subspaces of D of dimension k.

REMARK: In fact, E is diffeomorphic to the normal bundle of N in M, and we want to construct, in effect, a form φ on E representing the Thom class of this bundle, i.e., the dual form to the zero section. However, such a form is usually taken to have compact support. Since we want to construct a *harmonic* form, we must proceed in a different way.

We may describe the fibration $\pi: E \to N$ a little more explicitly. If $Z_1 \in D_{\sigma}$, $Z_2 \in U$ with $(Z_2, Z_2) = 1$, and $t \in (0, \infty)$, then

$$Z = ch(t)Z_1 + sh(t)Z_2 \in D$$

and $\pi(Z) = Z_1$. Conversely, every $Z \in \pi^{-1}(Z_1)$ has this form, so we obtain a parametrization:

$$f: D_{\sigma} \times S^{k-1} \times (0, \infty) \xrightarrow{\sim} D - D_{\sigma}$$

$$(Z_1, Z_2, t) \mapsto ch(t)Z_1 + sh(t)Z_2 = Z$$

where $S^{k-1} \subset U$ is the unit sphere. Note that

$$\cosh d(Z, Z_1) = ch(t)$$

so that t is the geodesic distance from $f(Z_1, Z_2, t)$ to D_σ. To compute the metric in these coordinates, we first make identifications:

$$T_{Z_1}(D_\sigma) \simeq Z_1^\perp \cap U^\perp$$

and

$$T_{Z_2}(S^{k-1}) \simeq Z_2^\perp \cap U.$$

Then it is easy to check that if $w_1 \in T_{Z_1}(D_\sigma)$ and $w_2 \in T_{Z_2}(S^{k-1})$, then

$$df_{(Z_1, Z_2, t)}(w_1) = ch(t)w_1$$
$$df_{(Z_1, Z_2, t)}(w_2) = sh(t)w_2$$

and

$$df_{(Z_1, Z_2, t)}\left(\frac{d}{dt}\right) = sh(t)Z_1 + ch(t)Z_2.$$

Thus we have:

LEMMA 3.1: *Let ds^2, ds_1^2, and ds_2^2 be the metrics induced by (,) on D, D_σ and S^{k-1}. Then, in the coordinates given by f,*

$$ds^2 = ch(t)^2 ds_1^2 + sh(t)^2 ds_2^2 + dt^2,$$

and the corresponding volume form is

$$dv = ch(t)^{n-k} sh(t)^{k-1} dv_1 \wedge dv_2 \wedge dt.$$

We may now construct the required family of forms. Let μ be the volume form on D_σ, and let

$$\varphi = {*}\pi^*\mu$$

where $*$ is the Hodge $*$ operator with respect to our metric. Then φ is a k-form on D which is, in fact, G_σ-invariant since π is G_σ-equivariant. Now for $s \in \mathbb{C}$ we define

$$\varphi_s = \|\varphi\|^{2s/(n-k)}\varphi.$$

LEMMA 3.2: *In the coordinates given above on $D - D_\sigma$, we have*:

$$\varphi = ch(t)^{-(n-k)} sh(t)^{k-1} dv_2 \wedge dt$$
$$\|\varphi\| = ch(t)^{-(n-k)}$$

and so

$$\varphi_s = ch(t)^{-(n-k+2s)} sh(t)^{k-1} dv_2 \wedge dt.$$

Moreover, $d\varphi_s = 0$ for all s.

PROOF: First we have

$$\pi^*\mu = dv_1.$$

Since $ch(t)^{n-k} dv_1$ is the volume form on the 'horizontal' subspace at distance t from D_σ, we must have

$$\varphi = *\pi^*\mu = ch(t)^{-(n-k)} sh(t)^{k-1} dv_1 \wedge dt$$

where $sh(t)^{k-1} dv_1 \wedge dt$ is the volume form on the fiber at distance t. Of course, φ and φ_s are G_σ-invariant, so we may view them as forms on E.

REMARK: The form φ_s is square integrable on E provided

$$4\,\mathrm{Re}(s) > 2k - n - 1.$$

In particular, φ is itself square integrable provided

$$k < \tfrac{1}{2}m.$$

We next want to normalize φ_s so that it has integral 1 over the fibers of $\pi : E \to N$.

LEMMA 3.3: *If* $\mathrm{Re}(s) > k - \frac{1}{2}m$, *then φ_s is integrable over the fibers* *of* $\pi : E \to N$. *Explicitly, if F is such a fiber, and we let*

$$\kappa(s) = \int_F \varphi_s$$

then

$$\kappa(s) = \tfrac{1}{2}\operatorname{vol}(S^{k-1}) \frac{\Gamma(\frac{1}{2}k)\Gamma\left(s + \frac{m}{2} - k\right)}{\Gamma\left(s + \frac{m}{2} - \frac{k}{2}\right)}.$$

PROOF: We compute

$$\kappa(s) = \int_0^\infty \int_{S^{k-1}} ch(t)^{-(n-k+2s)} sh(t)^{k-1} dv_1 dt$$

$$= \operatorname{vol}(S^{k-1}) \int_0^\infty ch(t)^{-(n-k+2s)} sh(t)^{k-1} dt,$$

and this integral converges for $\operatorname{Re}(s) > k - \frac{1}{2}(n+1)$ and is easily evaluated, yielding the value above.

For $\operatorname{Re}(s) > k - \frac{1}{2}m$, define

$$\psi_s = \kappa(s)^{-1}\varphi_s$$

so that ψ_s is G-invariant, closed for all s and has fiber integral 1.

PROPOSITION 3.4: *If* $\operatorname{Re}(s) > \frac{1}{2}(k-1)$, *then* ψ_s *satisfies conditions* (*i*), (*ii*) *and* (*iii*) *of Lemma* 2.1 *and hence determines a dual form*

$$\omega_s = \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \gamma^* \psi_s$$

to N in M.

PROOF: Since

$$\int_E \|\varphi_s\| dv = \int_{N \times S^{k-1} \times (0,\infty)} ch(t)^{-(n-k+2r)} ch(t)^{n-k} sh(t)^{k-1} dv_1 dv_2 dt$$

is finite provided $r = \operatorname{Re}(s) > \frac{1}{2}(k-1)$, condition (i) is satisfied in this range. To check condition (ii) we note that

$$\int_E \psi_s \wedge \eta = \int_{E-N} \psi_s \wedge \eta.$$

But now, if we let

$$F = S^{k-1} \times (0, \infty),$$

then

$$E - N \simeq N \times F,$$

and we may consider the projection onto F:

$$\pi' : E - N \to F.$$

Note that, if $(Z_2, t) \in F$, then

$$(\pi')^{-1}(Z_2, t) = \psi(N \times \{Z_2\} \times \{t\}).$$

We write

$$N_{(Z_2, t)} = (\pi')^{-1}(Z_2, t),$$

and observe that, in E the cycles $N_{(Z_2, t)}$ are all homotopic to N via

$$N \times [0, 1] \to E$$
$$(Z_1, \lambda) \mapsto ch(t\lambda)Z_1 + sh(t\lambda)Z_2.$$

Now we apply 'fiber integration' to π'. Since the fibers of π' are compact, fiber integration gives a mapping [5, Chapt. VII, §5],

$$I : \Omega^n(E) \to \Omega^k(F).$$

Then it is easily checked that

$$I(\psi_s \wedge \eta)_{(Z_2, t)} = \psi_s \cdot \left(\int_{N(Z_2, t)} \eta \right).$$

Since the $N_{(Z_2,t)}$'s are homotopic to N and η is closed, we have simply,

$$I(\psi_s \wedge \eta)_{(Z_2, t)} = \left(\int_N \eta\right)(\psi_s)_{(Z_2, t)}$$

and so

$$\int_{E-N} \psi_s \wedge \eta = \int_F I(\psi_s \wedge \eta)$$

$$= \left(\int_F \psi_s\right)\left(\int_N \eta\right)$$

$$= \int_N \eta.$$

This proves (ii). Finally, (iii) follows by a standard argument like that of [13, p. 199, Lemma 2.3].

§4. The shift equation, analytic continuation, and the harmonic dual form

We now want to construct the harmonic dual form to N.

First we compute the Laplacian Δ applied to φ_s. Note that by Lemma 3.1

$$*(ch(t)^{n-k}dv_1) = sh(t)^{k-1}dv_2 \wedge dt$$

$$*(sh(t)^{k-1}dv_2 \wedge dt) = (-1)^{nk+k}ch(t)^{n-k}dv_1$$

and

$$*(ch(t)^{n-k}dv_1 \wedge dt) = (-1)^{k-1}sh(t)^{k-1}dv_2.$$

LEMMA 4.1:

$$\Delta\varphi_s = 2s((2s+m-k)\varphi_{s+1} - (2s+m-2k)\varphi_s)$$

where $m = n+1$.

PROOF: We compute

$$\begin{aligned}
\Delta\varphi_s &= d\delta\varphi_s \\
&= (-1)^{nk+n+1} d*d*(ch(t)^{-(n-k+2s)} sh(t)^{k-1} dv_2 \wedge dt) \\
&= (-1)^{n+k+1} d*d(ch(t)^{-2s} dv_1) \\
&= (2s) d*(ch(t)^{-2s-1} sh(t) dv_1 \wedge dt) \\
&= (-1)^{k-1}(2s) d(ch(t)^{-(n-k+2s+1)} sh(t)^k dv_2) \\
&= (-1)^{k-1}(2s)[-(n-k+2s+1) ch(t)^{-(n-k+2s+2)} sh(t)^{k+1} \\
&\quad + k\, ch(t)^{-(n-k+2s)} sh(t)^{k-1}] dt \wedge dv_2 \\
&= (2s)[(2s+m-k)\varphi_{s+1} - (2s+m-2k)\varphi_s]
\end{aligned}$$

as claimed.

COROLLARY 4.2: (*the shift equation*).

$$\Delta\psi_s = 2s(2s+m-2k)[\psi_{s+1} - \psi_s].$$

PROOF: We observe that

$$\frac{\kappa(s+1)}{\kappa(s)} = \frac{2s+m-2k}{2s+m-k}$$

and so

$$\begin{aligned}
\Delta\psi_s &= (2s)\left[(2s+m-k)\frac{\kappa(s+1)}{\kappa(s)}\psi_{s+1} - (2s+m-2k)\psi_s\right] \\
&= (2s)(2s+m-2k)[\psi_{s+1} - \psi_s]
\end{aligned}$$

as claimed.

Thus we see that at $s = 0$ and $s = k - \frac{1}{2}m$ the form ψ_s will be harmonic. Neither of these values is in the range allowed in Proposition 3.4 and so we must obtain the harmonic dual form by first averaging ψ_s over $\Gamma_1\backslash\Gamma$ and then constructing an analytic continuation. The analytic continuation will follow from the shift equation. This method of continuation was inspired by Selberg [20] and exploited in [13].

THEOREM 4.3: *Let*

$$\omega_s = \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \gamma^* \psi_s.$$

Then:

(i) *This series is absolutely convergent for* $\mathrm{Re}(s) > \frac{1}{2}(k-1)$ *and defines a holomorphic family of closed k-forms dual to N.*

(ii) ω_s *has a meromorphic analytic continuation to the s-plane and satisfies the 'shift equation'*:

$$\Delta\omega_s = 2s(2s+m-2k)[\omega_{s+1} - \omega_s].$$

(iii) *if* $k \leq \frac{1}{2}m$, *then the function* $s \mapsto \omega_s$, *valued in* $\Omega^k(M)$ *is holomorphic at* $s = 0$, *and* ω_0 *is the harmonic dual form to N. If* $n > k \geq \frac{1}{2}m$, *then the function* $s \mapsto \omega_s$ *is holomorphic at* $s = k - \frac{1}{2}m$, *and* $\omega_{k-1/2m}$ *is the harmonic dual form to N.*

PROOF: Part (i) is just a restatement of Proposition 3.4. To obtain an analytic continuation of ω_s we expand in terms of eigenforms. Let $0 \leq \lambda_1 \leq \lambda_2, \ldots$, be the eigenvalues of Δ on $\Omega^k(M)$, repeated according to their multiplicity, and let $\{f_j\}_{j=1,2,\ldots}$ be a corresponding orthonormal basis for $\Omega^k(M)$. Thus $\Delta f_j = \lambda_j f_j$. For $\mathrm{Re}(s) > \frac{1}{2}(k-1)$ we have the expansion

$$\omega_s = \sum_{j=1}^{\infty} a_j(s) f_j$$

where

$$a_j(s) = (\omega_s, f_j).$$

In the half plane of absolute convergence, the shift equation for ψ_s given in Corollary 4.2 implies that

$$\Delta\omega_s = 2s(2s+m-2k)[\omega_{s+1} - \omega_s],$$

and therefore

$$a_j(s) = \frac{2s(2s+m-2k)}{\lambda_j + 2s(2s+m-2k)} a_j(s+1).$$

We may write this as

$$a_j(s) = [1 - \lambda_j P_j(s)^{-1}] a_j(s+1)$$

with

$$P_j(s) = \lambda_j + 2s(2s + m - 2k).$$

Iterating this formula, we obtain

$$a_j(s) = \prod_{\ell=0}^{r-1} [1 - \lambda_j P_j(s+\ell)^{-1}] a_j(s+r).$$

First observe that, if $\lambda_j = 0$, we obtain simply

$$a_j(s) = a_j(s+r),$$

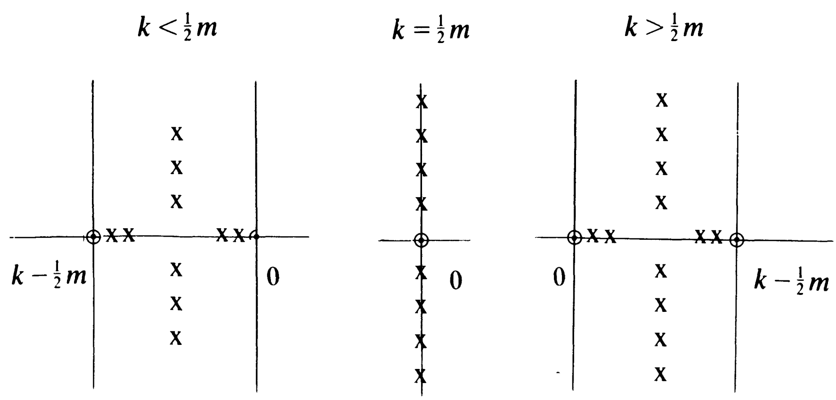
and so $a_j(s)$ is an entire, periodic function of s in this case. Next, if $\lambda_j \neq 0$, we see that $a_j(s)$ has a meromorphic analytic continuation to the whole s-plane with possible poles at points

$$\{x - \ell \mid \ell \in \mathbb{Z}_{\geq 0}, \text{ and } P_j(x) = 0\}.$$

Now the roots of $P_j(x) = 0$ have the form

$$x = \frac{2k - m \pm \sqrt{(2k-m)^2 - 4\lambda_j}}{4},$$

and these occur in pairs: (a) symmetric about the line $\mathrm{Re}(s) = \frac{1}{2}(k - \frac{1}{2}m)$ and strictly inside the strip $0 < |\mathrm{Re}(s)| < |k - \frac{1}{2}m|$ (these occur if $\lambda_j < \frac{1}{4}(2k-m)^2$), or (b) complex conjugates lying on the line $\mathrm{Re}(s) = \frac{1}{2}(k - \frac{1}{2}m)$. Moreover, since we have taken care of the case $\lambda_j = 0$ above, there is no pole at the point $s = \frac{1}{2}(k - \frac{1}{2}m)$. Thus we have the following picture depending on the codimension k:



where the line of possible poles repeats at integer translates to the left.

REMARK: The condition $k < \frac{1}{2}m$ is precisely that required for square integrability of the form ψ_0 on E.

Thus we have obtained an analytic continuation of the coefficients $a_j(s)$ of the eigenfunction expansion. We now want to show that the series $\Sigma_j a_j(s) f_j$ is absolutely convergent uniformly on any compact set in the s-plane from which we have deleted the possible poles of the $a_j(s)$'s. If K is any such compact set, then there exists an $\epsilon > 0$ such that K does not meet any ϵ-disk about any of the points $\{x_j - \ell \mid x_j$ is a root of $P_j(x) = 0$ and $\ell \in \mathbb{Z}$, $\ell \geq 0\}$. Moreover, we may choose $r \in \mathbb{Z}$, $r \geq 0$, such that the set $K + r$ lies within the half plane $\mathrm{Re}(s) > \frac{1}{2}(k-1) + \epsilon$. Applying the above shift formula for $a_j(s)$ we obtain

$$\begin{aligned} |a_j(s)| &\leq \sum_{\ell=0}^{r-1} (1 + \lambda_j |P_j(s+\ell)|^{-1}) |a_j(s+r)| \\ &\leq (1 + \epsilon^{-2}\lambda_j)^r |a_j(s+r)| \\ &\leq C\lambda_j^r |a_j(s+r)| \end{aligned}$$

where, since $\lambda_j \to \infty$ as $j \to \infty$, we may choose C uniformly with respect to j. Now to obtain convergence in the Hilbert space norm on $\Omega^k(M)$ we observe the following:

LEMMA 4.4: *Suppose that s lies in some compact set K' in the half plane* $\mathrm{Re}(s) > \frac{1}{2}(k-1)$. *Then*

1) $\forall \ell \in \mathbb{Z}_{>0}$, $\exists C_\ell > 0$ such that

$$|a_j(s)| < C_\ell \lambda_j^{-\ell}$$

where C_ℓ is uniform on K'.

2) (*Gaffney* [3]) *asymptotically*

$$\lambda_j \sim C' j^{2/n}$$

for some constant C'.

PROOF: For $\ell \in \mathbb{Z}$, $\ell > 0$, let

$$A_{j,\ell}(s) = \langle \Delta^\ell \omega_s, f_j \rangle.$$

Then

$$\begin{aligned} A_{j,\ell}(s) &= \langle \omega_s, \Delta^\ell f_j \rangle \\ &= \lambda_j^\ell a_j(s), \end{aligned}$$

and by the Schwarz inequality for forms

$$\begin{aligned} |A_{j,\ell}(s)| &= |\langle \Delta^\ell \omega_s, f_j \rangle| \\ &\leq \|\Delta^\ell \omega_s\| \, \|f_j\| \\ &= \|\Delta^\ell \omega_s\|. \end{aligned}$$

Thus we may take $C_\ell = \max_{s \in K'} \|\Delta^\ell \omega_s\|$.

Now on our original set K we obtain uniform convergence in the Hilbert space norm as follows:

$$\begin{aligned} \sum_j |a_j(s)|^2 &\leq C^2 \sum_j \lambda_j^{2r} |a_j(s+r)|^2 \\ &\leq C^2 C_\ell^2 \sum_j \lambda_j^{2r-2\ell} \end{aligned}$$

which is, by 2) of the lemma, convergent provided $\frac{4(\ell - r)}{n} > 1$. We choose such an ℓ and obtain convergence of ω_s to a holomorphic $\Omega^k(M)$ valued function on K. Of course, ω_s continues to satisfy the shift equation, so we have proved (ii).

Finally, from what was shown above about the possible poles of the $a_j(s)$'s, it is clear that ω_s is holomorphic at $s = 0$ if $k \leq \frac{1}{2}m$ and at $s = k - \frac{1}{2}m$. Moreover, there are no poles to the right of these values, i.e., at $s = 1$ (resp. $s = k - \frac{1}{2}m + 1$), and so via the shift equation, ω_0 (resp. $\omega_{k-1/2\,m}$) is the harmonic dual form to N.

§5. Additional remarks

5.1. In this section we make two variations in our previous constructions which will be needed in part II.

First we want to weaken condition (*) of §1. Suppose that M is a smooth, compact, oriented n-manifold, and let

$$\pi : M' \to M$$

be a finite covering. Suppose that

$$\iota' : N' \to M'$$

is a smooth $(n-k)$-cycle in M' with N' compact and oriented.

LEMMA 5.1: *Suppose that ω' is a Poincaré dual form to (N', ι') in M', and let*

$$\omega = \pi_* \omega',$$

so that

$$\omega_x = \sum_{x' \in \pi^{-1}(x)} \omega'_{x'}.$$

Then ω is a Poincaré dual form to $(N', \pi \circ \iota')$ in M.

PROOF: Let r be the degree of the covering π. Then for any closed $n-k$ form η on M we have:

$$\int_M \omega \wedge \eta = r^{-1} \int_{M'} \pi^*(\omega \wedge \eta).$$

But it is easy to check that

$$\begin{aligned} \int_{M'} \pi^* \omega \wedge \pi^* \eta &= r \int_{M'} \omega' \wedge \pi^* \eta \\ &= r \int_{N'} (\iota')^* \pi^* \eta \end{aligned}$$

as required.

Now suppose that there exists a compact oriented $n-k$ manifold N, a finite covering $\pi_1 : N' \to N$ of degree d and a smooth mapping $\iota : N \to M$ such that the diagram

$$\begin{array}{ccc} N' & \xrightarrow{\iota'} & M' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\iota} & M \end{array}$$

commutes.

Corollary 5.2: *If* ω' *is a Poincaré dual form to* (N', ι') *in* M', *then*

$$\omega = d^{-1}\pi_*\omega'$$

is a Poincaré dual form to (N, ι) *in* M.

Proof: If η is any $n-k$ form on M, we have

$$\int_{N'} (\iota')^*\pi^*\eta = \int_{N'} \pi_1^*\iota^*\eta$$
$$= d\int_N \iota^*\eta,$$

and so Lemma 5.1 gives the result.

Corollary 5.2 allows us to extend our construction of Poincaré dual forms to any cycle (N, ι) which is 'covered' by a cycle (N', ι') of the type considered in §1. In particular, we may replace condition (*) of §1 with the weaker condition:

$$(*)' \qquad \sigma\Gamma\sigma \sim \Gamma$$

where '$\sim$' means commensurable. If (*)′ holds for Γ, then (*) holds for the subgroup

$$\Gamma' = \Gamma \cap \sigma\Gamma\sigma$$

which has finite index in Γ.

5.2: We now want to find a formula for the function φ_s which will allow us to vary U and which will be essential in making the link with θ-functions. We will use the following notation: If $X = (X_1, \dots, X_k) \in V^k$ and $Y = (Y_1, \dots, Y_\ell) \in V^\ell$, then

$$(X, Y) = ((X_i, Y_j)) \in M_{k,\ell}(\mathbb{R}).$$

Note that if $A \in GL(k, \mathbb{R})$ and $B \in GL(\ell, \mathbb{R})$, then

$$(XA, YB) = {}^tA(X, Y)B.$$

Also, if $X \in V^k$, we let

$$\|X\| = \det(X, X)^{1/2}.$$

Now choose a k-frame $X \in V^k$ such that $\operatorname{span} X = \operatorname{span}\{X_1, \ldots, X_k\} = U$. We will now give a formula for φ_s in terms of X, and we write D_X for D_σ and $\varphi_{s,X}$ for φ_s to emphasize this dependence. If $Z \in D$, let

$$\tilde{X} = X - Z(Z, Z)^{-1}(Z, X)$$

be the projection of the k-frame X into the tangent space $T_Z(D) = Z^\perp$, and let

$$\|X\|_Z = \|\tilde{X}\|.$$

PROPOSITION 5.3: In the notation above, if $W = (W_1, \ldots, W_k)$ is a k-tuple in $T_Z(D)$, then

$$(\varphi_{s,X})_Z(W) = \|X\|^{n-k+2s}\|X\|_Z^{-(n-k+2s+1)}\det(X, W).$$

Also, if $g \in G$, then

$$g^* \varphi_{s,X} = \varphi_{s,g^{-1}X}$$

and finally,

$$(\omega_{s,X})_Z(W) = \kappa(s)^{-1} \sum_{X' \in \Gamma Z} \|X'\|^{n-k+2s}\|X'\|_Z^{-(n-k+2s+1)}\det(X', W).$$

PROOF: Recall that the fiber of $\pi : D \to D_X$ through Z was

$$\begin{aligned}\pi^{-1}(\pi(Z)) &= D \cap \operatorname{span}\{X, \pi(Z)\}\\ &= D \cap \operatorname{span}\{X, Z\}.\end{aligned}$$

In particular, the tangent space to the fiber of π through Z is

$$\begin{aligned}T_Z(D) \cap \operatorname{span}\{X, Z\} &= Z^\perp \cap \operatorname{span}\{X, Z\}\\ &= \operatorname{span} \tilde{X}.\end{aligned}$$

If we view the components $\tilde{X}_i$ of $\tilde{X}$ as vector fields on D and let η_i be the dual 1-form to $\tilde{X}_i$, then the volume form to the fibers of π is

$$\|\eta_1 \wedge \cdots \wedge \eta_k\|^{-1}\eta_1 \wedge \cdots \wedge \eta_k.$$

This was the form $sh(t)^{k-1}dv_2 \wedge dt$ in the coordinates of §3. Moreover, we have

$$\|\eta_1 \wedge \cdots \wedge \eta_k\| = \|\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_k\|$$
$$= \|\tilde{X}\|.$$

Next observe that if

$$Z' = Z - X(X, X)^{-1}(X, Z)$$

is the component of Z orthogonal to X, then

$$\pi(Z) = |(Z', Z')|^{-1/2} Z',$$

and so

$$\cosh t = |(Z, \pi(Z))|$$
$$= |(Z', Z')|^{1/2}.$$

Also

$$(Z', Z') = -[(Z, Z) - (Z, X)(X, X)^{-1}(X, Z)].$$

But it is easy to check that

$$(Z, Z) \det(\tilde{X}, \tilde{X}) = \det(X, X)(Z', Z')$$

and so

$$\cosh^2 t = \|\tilde{X}\|^2 \|X\|^{-1}.$$

Putting these facts together we find that

$$\varphi_{s,X} = \|X\|^{n-k+2s} \|\tilde{X}\|^{-(n-k+2s)} \|\tilde{X}\|^{-1} \eta_1 \wedge \cdots \wedge \eta_k,$$

and since, for $W = (W_1, \ldots, W_k)$, $W_i \in T_Z(D)$, we have

$$\eta_1 \wedge \cdots \wedge \eta_k(W) = \det(X, W),$$

the first part of the proposition is proved. The rest of the proposition is immediate.

Part II

§6. Families of cycles

In this section we will construct, for a certain type of Γ, families of cycles to which our previous results may be applied.

Let k be a totally real number field with $|k:\mathbb{Q}| = r$, and let k_λ, $\lambda = 1, \ldots, r$, be the real completions of k. Let V be a k vector space, $\dim_k V = m$, and let $(,)$ be a nondegenerate, symmetric, k-bilinear form on V. We assume that

$$\text{signature}\,(V_\lambda, (,)) = \begin{cases} (n, 1) & \text{if } \lambda = 1 \\ (m, 0) & \text{if } \lambda > 1, \end{cases}$$

where $V_\lambda = k_\lambda \otimes_k V$.

Let $G = SO(V, (,))$ be the special orthogonal group of V, $(,)$, viewed as an algebraic group over k, and let $G_\lambda = G(k_\lambda)$ and $G_\infty = \Pi_{\lambda=1}^r G_\lambda$. Let

$$D = \{Z \in V_1 \mid (Z, Z) = -1\}^0$$

be the symmetric space associated to G_1, so that G_∞ acts transitively on D via projection on the first factor.

Let $\mathcal{O}$ be the ring of integers of k, let $L \subset V(k)$ be an $\mathcal{O}$-lattice, and let

$$G^*(L) = \{\gamma \in G(k) \mid \gamma L = L\}$$

be the group of units of L. We then take a congruence subgroup $\Gamma \subset G^*(L)$ such that

1) Γ is torsion free,

and

2) the image of Γ in G_1 lies in G_1^0, the connected component of the identity in G_1.

The existence of such a congruence subgroup follows from:

PROPOSITION 6.1: (*Millson–Raghunathan* [15]): *There exist infinitely many prime ideals* $\mathfrak{p}$ *of* $\mathcal{O}$ *such that the principal congruence subgroup*

$$G(L; \mathfrak{p}) = \{\gamma \in G^*(L) \mid \gamma \text{ acts trivially in } L^*/\mathfrak{p}L\}$$

lies in the kernel of the spinor norm, and hence has image in G_1^0.

Having chosen such a Γ we obtain an orientable n-manifold $M = \Gamma\backslash D$ as in §1. We further assume that either

i) $\mathscr{k} = \mathbb{Q}$, $n \leq 3$ and $V(\mathbb{Q})$, (,) is anisotropic,

or

ii) $|\mathscr{k} : \mathbb{Q}| = r > 1$.

These conditions guarantee that M is compact.

We next construct families of cycles in each codimension. For $k \in \mathbb{Z}$, $1 \leq k < n$, we choose an auxiliary $\mathscr{k}$-vector space E with $\dim_{\mathscr{k}} E = k$, and we let

$$\tilde{V} = V \otimes_{\mathscr{k}} E.$$

For $X \in \tilde{V}$ we define a subspace

$$\text{span } X \subset V$$

as follows: Choose any basis $e_1, \ldots, e_k$ for E, and write

$$X = \sum_{i=1}^{k} X_i \otimes e_i.$$

Then let

$$\text{span } X = \text{span}\{X_1, \ldots, X_k\}.$$

Of course, span X is independent of the choice of basis for E. Let

$$\text{Fr}_k(V) = \{X \in \tilde{V} \mid \dim_{\mathscr{k}} \text{span } X = k\},$$

and le

$$\text{Fr}_k^+(V) = \{X \in \text{Fr}_k(V) | \, (,)|_{(\text{span } X)_1} > 0\}$$

where $(\text{span } X)_1$ is the span of $\{X_1, \ldots, X_k\}$ in V_1. For $X \in \text{Fr}_k^+(V)$ we let

$$D_X = \{Z \in D \mid (Z, X) = 0\},$$
$$G_X = \{g \in G \mid gX = X\}$$

and

$$\Gamma_X = \Gamma \cap G_X(\mathscr{k}),$$

where we let G act on $\tilde{V}$ via its action on V; we view G_X as an algebraic subgroup of G defined over $\mathscr{k}$, and we let $(Z, X) = \Sigma_i (Z, X_i) \otimes e_i \in E$. We then obtain a cycle, as in §1,

$$\begin{array}{ccc} D_X & \hookrightarrow & D \\ \downarrow & & \downarrow \\ N_X = \Gamma_X \backslash D_X & \xrightarrow{\iota_X} & \Gamma \backslash D = M. \end{array}$$

Since we must now consider the whole family of cycles parameterized by $X \in \mathrm{Fr}_k^+(V)$, it is important to orient these cycles in a coherent way.

Fix an orientation of V_1 and of E, and define orientations of D and D_X as follows:

(i) Let $Z \in D$. Then an n-frame $W = (W_1, \ldots, W_n)$, $W_i \in T_Z(D) = Z^\perp$ is properly oriented if the m-frame $\{W, Z\}$ is properly oriented for V_1.

(ii) Choose a properly oriented basis $e_1, \ldots, e_k$ for E, and for $X \in \tilde{V}$ write

$$X = \sum_{i=1}^{k} X_i \otimes e_i.$$

Let $Z \in D_X$. Then an $(n-k)$-frame $W = (W_1, \ldots, W_{n-k})$, $W_i \in T_Z(D_X) = Z^\perp \cap X^\perp$ is properly oriented if the m-frame $\{W, X_1, \ldots, X_k, Z\}$ is properly oriented for V_1.

The orientations thus defined have the following properties:

1) The orientation of D_X depends on the orientation of the frame $\{X_1, \ldots, X_k\}$ in $(\mathrm{span}\, X)_1$ but not on the choice of properly oriented basis for E.

2) The group $G_X(\mathscr{k}_1) \cap G_1^0$, and hence Γ_X, preserves the orientation of D_X, and we may define

$$[N_X] = (\iota_X)_*(1_X) \in H_{n-k}(M, \mathbb{Z}).$$

3) Suppose that there exists $\gamma \in \Gamma$ such that $\gamma(\mathrm{span}\, X) = \mathrm{span}\, X$,

but the orientation of γX is opposite to that of X. Then we have a commutative diagram

$$\begin{array}{ccc} N_X & \xrightarrow{\iota_X} & M \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle id} \\ N_X & \xrightarrow{\iota_X} & M \end{array}$$

and so

$$\begin{aligned}[N_X] &= (\iota_X)_*(1_X) \\ &= (\iota_X \circ \gamma)_*(1_X) \\ &= -(\iota_X)_*(1_X) \\ &= -[N_X],\end{aligned}$$

hence

$$[N_X] = 0.$$

4) Fix some $X \in \mathrm{Fr}_k^+(V)$, and let

$$G_{\text{span } X} = \{g \in G \mid g(\text{span } X) = \text{span } X\}.$$

This is the analogue of G_σ of §1. Then Proposition 6.1 guarantees the existence of a congruence subgroup $\Gamma' \subset \Gamma$ such that

$$\Gamma' \cap G_{(\text{span } X)} = \Gamma' \cap G_X = \Gamma'_X.$$

For such a Γ', $[N'_X]$ doesn't vanish for the trivial reason 3), and, in fact, we shall see—Proposition 11.1—that the result of Millson–Raghunathan [15] provides the existence of a further $\Gamma'' \subset \Gamma'$ such that $[N''_X] \neq 0$.

§7. Theta functions: local constructions

7.1: We begin by recalling a few facts about the oscillator representation.

Let V be a real vector space, $V^* = \mathrm{Hom}(V, \mathbb{R})$, and let

$$[,] : V \times V^* \to \mathbb{R}$$

be the natural pairing. We let $\dim_{\mathbb{R}} V = m$, and let dx and dx^* be Haar measures on V and V^*, dual with respect to [,]. Define an alternating form A on the vector space $V \times V^*$ by

$$A(z, z') = [x, y'] - [x', y]$$

where $z = (x, y)$ and $z' = (x', y')$. Note that the subspaces $V \times \{0\}$ and $\{0\} \times V^*$ determine a complete polarization of $V \times V^*$ with respect to A.

Let $\mathrm{Sp}(V) = \mathrm{Sp}(V \times V^*, A)$ be the symplectic group of $V \times V^*$, A, where we let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(V)$, with $a \in \mathrm{End}(V)$, $b \in \mathrm{Hom}(V, V^*)$, $c \in \mathrm{Hom}(V^*, V)$ and $d \in \mathrm{End}(V^*)$, act on $V \times V^*$ by right multiplication.

It is well known that there is a projective unitary representation—the oscillator representation—of $\mathrm{Sp}(V)$ on $L^2(V)$, and this projective representation lifts to a unitary representation of the metaplectic group $\mathrm{Mp}(V)$. Moreover, the action of this group preserves the Schwartz space $\mathcal{S}(V)$. For $g \in \mathrm{Sp}(V)$ we let $\mathrm{R}(g)$ be some choice of the unitary operator on $L^2(V)$ associated to g.

Let

$$\mathfrak{H}(V) = \{\tau \in \mathrm{Hom}(V, V^*_{\mathbb{C}}) \mid \tau^* = \tau \text{ and } \mathrm{Im}(\tau) = (2i)^{-1}(\tau - \bar{\tau}) > 0\}$$

be the Siegel space associated to $\mathrm{Sp}(V)$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(V)$ and $\tau \in \mathfrak{H}(V)$, we have

$$g(\tau) = (a\tau + b)(c\tau + d)^{-1},$$

and we let

$$J(g, \tau) = \det(c\tau + d)$$

where $c\tau + d \in \mathrm{GL}(V^*_{\mathbb{C}})$. Note that, if we write $v = v(\tau) = \mathrm{Im}(\tau) \in \mathrm{Hom}(V, V^*)$ and let $|v|$ be the module of v with respect to the measures dx and d^*x^*, i.e.,

$$d^*(xv) = |v| dx,$$

then

$$|v(g(\tau))| = |J(g, \tau)|^{-2} |v|.$$

We define a mapping

$$f : \mathfrak{H}(V) \to \mathcal{S}(V)$$
$$\tau \mapsto f_\tau$$

by

$$f_\tau(x) = e(\tfrac{1}{2}[x, x\tau])$$

for $x \in V$. The following lemma is then well known:

LEMMA 7.1: *For each $g \in \mathrm{Sp}(V)$, choose a continuous branch of $J(g, \tau)^{1/2}$ on $\mathfrak{H}(V)$. Then the unitary operator $R(g)$ on $L^2(V)$ can be chosen so that*

$$R(g)f_\tau = J(g, \tau)^{-1/2} f_{g\tau}.$$

PROOF: See [9, p. 175].

For this choice of operators $R(g)$ we have

$$R(g_1 g_2) = c(g_1, g_2) R(g_1) R(g_2)$$

where the cocycle $c(g_1, g_2)$ is given by

$$c(g_1, g_2) = J(g_1 g_2, \tau)^{1/2} J(g_1, g_2\tau)^{-1/2} J(g_2, \tau)^{-1/2}.$$

In particular, the twofold covering of $\mathrm{Sp}(V)$ is

$$\mathrm{Mp}(V) = \mathrm{Sp}(V) \times \{\pm 1\}$$

with multiplication

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)),$$

and the unitary representation of $\mathrm{Mp}(V)$ in $L^2(V)$ is given by

$$R(\tilde{g}) = \epsilon R(g)$$

where $\tilde{g} = (g, \epsilon)$ and $R(g)$ is the operator normalized as in Lemma 7.1.

7.2: Now suppose that V is a real vector space with $\dim V = m$, and let (,) be a nondegenerate, symmetric bilinear form on V. Assume, for the moment, that

$$\text{sig } V, (,) = (p, q).$$

Later V will be one of the completions of our k-vector space of §6, so that (p, q) will be either $(m, 0)$ or $(n, 1)$. Let

$$G = SO(V),$$

and let

$$Q : V \xrightarrow{\sim} V^*$$

be the symmetric isomorphism corresponding to (,).

For a positive integer k, let E be a real vector space with $\dim E = k$. Let $E^* = \text{Hom}(E, \mathbb{R})$, and let

$$[,]_E : E \times E^* \to \mathbb{R}$$

be the natural pairing. Then let

$$\tilde{V} = V \otimes_{\mathbf{R}} E$$

so that

$$\tilde{V}^* \simeq V \otimes_{\mathbf{R}} E^*$$

via Q, and

$$[,] = (,) \otimes [,]_E$$

where $[,] : \tilde{V} \times \tilde{V}^* \to \mathbb{R}$ is the natural pairing.

We define an alternating form A_E on $E \times E^*$ as in the first part of this section, and we obtain an alternating form

$$A = (,) \otimes A_E$$

on

$$\tilde{V} \times \tilde{V}^* \simeq V \otimes (E \times E^*).$$

Then we have a natural homomorphism

$$\rho : O(V) \times \text{Sp}(E) \to \text{Sp}(\tilde{V})$$

$$(g, h) \mapsto g \otimes h$$

whose image is a dual reductive pair in the sense of Howe [4], [8].

Let

$$\mathfrak{H}(E) = \{\tau \in \mathrm{Hom}(E, E^*_{\mathbb{C}}) \mid \tau^* = \tau \text{ and } \mathrm{Im}(\tau) > 0\},$$

and let

$$D = \{Z \in \mathrm{Gr}_q(V) \mid (,)|_Z < 0\}.$$

Then

$$D \simeq \{I \in \mathrm{GL}(V) \mid 1)\ (X, IY) \text{ is symmetric and positive definite, and } 2)\ I^2 = 1\}$$

where the involution I_Z corresponding to Z is defined by

$$I_Z = \begin{cases} -1 \text{ on } Z \\ +1 \text{ on } Z^\perp, \end{cases}$$

and we have an embedding

$$\eta : D \times \mathfrak{H}(E) \to \mathfrak{H}(\tilde{V})$$
$$(Z, \tau) \mapsto 1_V \otimes \tfrac{1}{2}(\tau + \bar{\tau}) + I_Z \otimes \tfrac{1}{2}(\tau - \bar{\tau}).$$

which is equivariant:

$$\rho(g, h)\eta(Z, \tau) = \eta(gZ, h\tau)$$

for $g \in O(V)$ and $h \in \mathrm{Sp}(E)$.

Now, via ρ, the oscillator representation of $\mathrm{Sp}(\tilde{V})$ restricts to a projective representation—the Weil representation—of $O(V) \times \mathrm{Sp}(E)$ on $L^2(\tilde{V})$. Explicitly, for $g \in G \subset O(V)$ we have

$$R(\rho(g))f(X) = f(g^{-1}X)$$

Also, if $\tau \in \mathfrak{H}(\tilde{V})$ and if $f_\tau \in \mathscr{S}(\tilde{V})$ is as in the first part of this section,

$$R(\rho(g, h))f_\tau = J(\rho(g, h), \tau)^{-1/2} f_{\rho(g, h)\tau}.$$

LEMMA 7.2: *For $h \in \mathrm{Sp}(E)$,*

$$J(\rho(h), \eta(Z, \tau)) = J(h, \tau)^p J(h, \bar{\tau})^q$$

and for $g \in O(V)$

$$J(\rho(g), \eta(Z, \tau)) = \det g.$$

PROOF: An easy calculation which we omit.

For $\tau = u + iv \in \mathfrak{H}(E)$ and $Z \in D$ we let

$$f_{\tau,Z} = |v|^{q/2} f_{\eta(Z,\tau)}$$

and we find that, for a suitable choice of branch of $J(h, \tau)^{1/2}$:

COROLLARY 7.3:

$$R(\rho(g, h)) f_{\tau,Z} = J(h, \tau)^{(q-p)/2} f_{h\tau, gZ}$$

for $g \in G$ *and* $h \in \mathrm{Sp}(E)$.

Therefore, if we let $C_{\tilde{V}}$ and C_E be the 2-cocycles defining $\mathrm{Mp}(\tilde{V})$ and $\mathrm{Mp}(E)$, we have:

$$c_{\tilde{V}}(\rho(h), \rho(h')) = (c_E(h, h'))^{p-q},$$

and so ρ lifts to a homomorphism

$$G \times \mathrm{Mp}(E) \to \mathrm{Mp}(\tilde{V})$$
$$(g, \tilde{h}) \mapsto (g, 1)(\rho(h), \epsilon^{p-q})$$

where $\tilde{h} = (h, \epsilon) \in \mathrm{Mp}(E)$. We then obtain a representation of $\mathrm{Mp}(E)$ in $L^2(\tilde{V})$ given by

$$R(\tilde{h}) = \epsilon^{p-q} R(h)$$

where $\tilde{h} = (h, \epsilon) \in \mathrm{Mp}(E)$. If we let

$$j(\tilde{h}, \tau) = \epsilon^{-1} J(h, \tau)^{1/2},$$

then

$$R(\tilde{h}) f_{\tau,Z} = j(\tilde{h}, \tau)^{q-p} f_{\tilde{h}(\tau),Z}.$$

The particular Schwartz function of interest to us is defined as follows: There is a natural pairing

$$\tilde{V} \times \tilde{V}^* \to E \otimes E^* = \mathrm{End}(E)$$

given by

$$(v \otimes e, v' \otimes e^*) \mapsto (v, v')e \otimes e^*.$$

Then for $\tau \in \mathfrak{H}(E)$, $Z \in D$, and $W \in Z^\perp \otimes E^* \subset \tilde{V}^*$ we define

$$f^*_{\tau,Z,W}(X) = \det(X, W) f_{\tau,Z}(X)$$

where $(X, W) \in \mathrm{End}(E)$.

PROPOSITION 7.4: *For* $\tilde{h} \in \mathrm{Mp}(E)$,

$$R(\tilde{h}) f^*_{\tau,Z,W} = j(\tilde{h}, \tau)^{q-p-2} f^*_{h(\tau),Z,W}.$$

PROOF: Since the automorphy factor $j(\tilde{h}, \tau)$ satisfies

$$j(\tilde{h}\tilde{h}', \tau) = j(\tilde{h}, \tilde{h}'\tau) j(\tilde{h}', \tau)$$

it is sufficient to check the required relation for a set of generators of $\mathrm{Mp}(E)$. For the central subgroup $\{(1, \epsilon) \mid \epsilon = \pm 1\}$ the relation is immediate, and so we need only check that it holds for elements of the form $\tilde{h} = (h, 1)$ where h runs over a set of generators for $\mathrm{Sp}(E)$.

For convenience we choose a basis $e_1, \ldots, e_k$ for E and let $e_1^*, \ldots, e_k^*$ be the dual basis for E^*. Then $\tilde{V} \simeq V^k$ and $\tilde{V}^* \simeq V^k$. Also we have $\mathrm{Sp}(E) \simeq \mathrm{Sp}(k, \mathbb{R})$, so that we may take generators $\begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}$, $a \in GL(k, \mathbb{R})$, $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$, $b = {}^t b \in M_k(\mathbb{R})$, and $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. For $f \in L^2(\tilde{V})$ we find that

$$R\left(\begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}\right) f(X) = (\det a)^{m/2} f(Xa)$$

$$R\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right) f(X) = e(\tfrac{1}{2}\operatorname{tr}((X, X)b)) f(X)$$

and

$$R\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) f(X) = \gamma \int_{\tilde{V}} e(\operatorname{tr}(X, Y)) f(Y)\, dY$$

where dY is the self-dual measure on $V^k \simeq \tilde{V} \simeq \tilde{V}^*$. Also $J\left(\begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}\right)$,

$\tau)^{1/2} = (\det a)^{-1/2}$ determines the branch of $(\det a)^{1/2}$, and γ is a scalar of absolute value 1. Note that we are using the 'matrix notation' $(X, X) \in M_k(\mathbb{R})$, etc., as in §5 and $\check{a} = {}^t a^{-1}$.

The relation of the proposition is then obvious for the first two types of generators, so we need only check it for $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$.

For $W \in \tilde{V}^*$ we view $\det(X, W)$ as a homogeneous polynomial of degree k in $X \in \tilde{V}$ and let

$$\nabla(W) = \det\left(\frac{\partial}{\partial X}, W\right)$$

be the corresponding constant coefficient differential operator of order k on $\tilde{V}$. If $W \in Z^{\perp} \otimes E^* \subset \tilde{V}^*$, then by a straightforward calculation we have

$$\nabla(W)\{f_{\tau,z}\}(X) = (2\pi i)^k \det(\tau) f_{\tau,Z,W}(X),$$

and

$$\nabla(W)\{e(\mathrm{tr}(X, Y))\} = (2\pi i)^k \det(Y, W) e(\mathrm{tr}(X, Y)).$$

Recalling that

$$R(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right))\{f_{\tau,Z}\}(X) = \det(-\tau)^{(q-p)/2} f_{-\tau^{-1},Z}(X),$$

we have, on the one hand,

$$\nabla(W)\{R(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)) f_{\tau,Z}\}(X) = (2\pi i)^k \det(-\tau)^{(q-p)/2} \det(-\tau^{-1}) f^*_{-\tau^{-1},Z,W}(X),$$

while applying $\nabla(W)$ to the integral gives

$$\begin{aligned}\nabla(W)\{R(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)) f_{\tau,Z}\}(X) &= \gamma \int_{\tilde{V}} (2\pi i)^k \det(Y, W) e(\mathrm{tr}(X, Y)) f_{\tau,Z}(Y)\,dY \\ &= (2\pi i)^k R(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)) f^*_{\tau,Z,W}(X).\end{aligned}$$

Thus

$$R(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)) f^*_{\tau,Z,W} = \det(-\tau)^{(q-p)/2-1} f^*_{-\tau^{-1},Z,W}$$

as claimed, and the proposition is proved.

§8. Theta functions: global constructions

8.1: We now return to the global situation of §6 and the notation defined there. For $k \in \mathbb{Z}$ with $1 \le k < n$ we have a $\mathscr{k}$-vector space E with $\dim_{\mathscr{k}} E = k$. Let $E^* = \mathrm{Hom}_{\mathscr{k}}(E, \mathscr{k})$ be the dual space of E, and let $[,]_E : E \times E^* \to \mathscr{k}$ be the natural pairing. We have

$$\tilde{V} = V \otimes_{\mathscr{k}} E$$

and

$$\tilde{V}^* = V \otimes_{\mathscr{k}} E^*,$$

and the natural pairing $[,]: \tilde{V} \times \tilde{V}^* \to \mathscr{k}$ is given by

$$[,] = (,) \otimes [,]_E$$

where we view

$$\tilde{V} \times \tilde{V}^* \simeq V \otimes_{\mathscr{k}} (E \times E^*).$$

Then for the alternating forms A_E on $E \times E^*$ and A on $\tilde{V} \times \tilde{V}^*$ defined as in §7, we have

$$A = (,) \otimes A_E$$

and, viewing $\mathrm{Sp}(E) = \mathrm{Sp}(E \times E^*, A_E)$ and $\mathrm{Sp}(\tilde{V}) = \mathrm{Sp}(\tilde{V} \times \tilde{V}^*, A)$ as algebraic groups over $\mathscr{k}$, we obtain a $\mathscr{k}$-homomorphism

$$\rho : G \times \mathrm{Sp}(E) \to \mathrm{Sp}(\tilde{V}),$$

just as in §7. For convenience we let $H = \mathrm{Sp}(E)$ and $\tilde{H} = \mathrm{Sp}(\tilde{V})$.

We now restrict scalars from $\mathscr{k}$ to $\mathbb{Q}$ and obtain real vector spaces:

$$V' = (R_{\mathscr{k}/\mathbb{Q}} V)(\mathbb{R})$$

$$E' = (R_{\mathscr{k}/\mathbb{Q}} E)(\mathbb{R}),$$

and

$$\tilde{V}' = (R_{\mathscr{k}/\mathbb{Q}} \tilde{V})(\mathbb{R}).$$

Also let

$$G' = (R_{\mathscr{k}/\mathbb{Q}} G)(\mathbb{R}),$$

$$H' = (R_{\mathscr{k}/\mathbb{Q}} H)(\mathbb{R}),$$

and

$$\tilde{H}' = (R_{k/\mathbb{Q}}\tilde{H})(\mathbb{R})$$

so that ρ induces a homomorphism

$$\rho' : G' \times H' \to \tilde{H}'.$$

Moreover, there is a natural inclusion

$$\rho'' : \tilde{H}' \to \mathrm{Sp}(\tilde{V}')$$

where $\mathrm{Sp}(\tilde{V}') = \mathrm{Sp}(\tilde{V}' \times \tilde{V}'^*, A')$ and

$$A'(Z_1, Z_2) = [X_1, Y_2]' - [X_2, Y_1]'$$

for $[,]' : \tilde{V}' \times \tilde{V}'^* \to \mathbb{R}$ the natural pairing. Note that

$$\tilde{V}' \simeq \prod_\lambda \tilde{V}_\lambda$$

and

$$\tilde{V}'^* \simeq \prod_\lambda \tilde{V}^*_\lambda$$

where $\tilde{V}_\lambda = \tilde{V} \otimes_k k_\lambda$ is the λ-th completion of $\tilde{V}$, and that

$$[X, Y]' = \sum_\lambda [X_\lambda, Y_\lambda]$$

$$= \mathrm{tr}_k[X, Y].$$

Also $\tilde{H}_\lambda = \mathrm{Sp}(\tilde{V}_\lambda)$ and

$$\tilde{H}' \simeq \prod_\lambda \tilde{H}_\lambda.$$

There is an obvious ρ''-equivariant embedding

$$\eta'' : \prod_\lambda \mathfrak{H}(\tilde{V}_\lambda) \to \mathfrak{H}(\tilde{V}')$$

where for $X=(X_1,\dots,X_r)\in\tilde V'$ and $\tau=(\tau_1\dots,\tau_r)\in\prod_\lambda \mathfrak{H}(\tilde V_\lambda)$,

$$X\eta''(\tau)=(X_1\tau_1,\dots,X_r\tau_r).$$

For this we have

$$J(\rho''(h),\ \eta''(\tau))=\prod_\lambda J(h_\lambda,\ \tau_\lambda).$$

Now, as in §7.1, there is a projective unitary representation of $\mathrm{Sp}(\tilde V)$ on $L^2(\tilde V)$ which lifts to a representation of the twofold covering $\mathrm{Mp}(\tilde V')$ if we choose normalized operators $R(h)$ as in Lemma 7.1. If for each λ we choose a continuous branch of $J(h_\lambda,\tau_\lambda)^{1/2}$ on $\mathfrak{H}(\tilde V_\lambda)$ for all $h_\lambda\in\mathrm{Sp}(\tilde V_\lambda)$; then, for each $h\in\tilde H'$, there exists a unique continuous branch of $J(\rho''(h),\tau)^{1/2}$ on $\mathfrak{H}(\tilde V')$ such that

$$J(\rho''(h),\ \eta''(\tau))^{1/2}=\prod_\lambda J(h_\lambda,\ \tau_\lambda)^{1/2}.$$

Then, by Lemma 7.1, there is a unique operator $R(\rho''(h))$ such that

$$R(\rho''(h))f_\tau=J(\rho''(h),\ \tau)^{-1/2}f_{\rho''(h)\tau}$$

for $\tau\in\mathfrak{H}(\tilde V')$. On the other hand, we know that, on the dense subspace $\mathscr{S}(\tilde V_1)\otimes\cdots\otimes\mathscr{S}(\tilde V_r)\subset\mathscr{S}(V')$, the subgroup $\rho''(\tilde H')$ acts componentwise by the corresponding projective representations of $\mathrm{Sp}(\tilde V_\lambda)$ on $\mathscr{S}(\tilde V_\lambda)$; i.e.,

$$R(\rho''(h))f=R_1(h_1)f_1\otimes\cdots\otimes R_r(h_r)f_r$$

for $h=(h_1,\dots,h_r)\in\tilde H'$ and $f=f_1\otimes\cdots\otimes f_r\in\otimes_\lambda\mathscr{S}(\tilde V_\lambda)$, where $R_\lambda(h_\lambda)$ is some unitary operator on $L^2(\tilde V_\lambda)$ associated to $h_\lambda\in\mathrm{Sp}(\tilde V_\lambda)$. In particular, if $\tau=(\tau_1,\dots,\tau_r)\in\Pi_\lambda\ \mathfrak{H}(\tilde V_\lambda)$, and we let

$$\begin{aligned} f''_\tau &= f_{\eta''(\tau)} \\ &= f_{\tau_1}\otimes\cdots\otimes f_{\tau_r}, \end{aligned}$$

then

$$\begin{aligned} R(\rho''(h))f''_\tau &= J(\rho''(h),\ \eta''(\tau))^{-1/2}f''_{h_\tau} \\ &= \bigotimes_\lambda J(h_\lambda,\ \tau_\lambda)^{-1/2}f_{h_\lambda\tau_\lambda}. \end{aligned}$$

Thus we conclude that, on the dense subspace $\otimes_\lambda \mathscr{S}(\tilde{V}_\lambda)$

$$R(\rho''(h)) = \bigotimes_\lambda R_\lambda(h_\lambda)$$

where $R_\lambda(h_\lambda)$ is, in fact, the normalized operator determined by our choice of $J(h_\lambda, \tau_\lambda)^{1/2}$.

Now for $\tau = (\tau_1, \ldots, \tau_r) \in \Pi_\lambda \mathfrak{H}(E_\lambda)$, $Z \in D$, and $W \in Z^\perp \otimes E_1^* \subset \tilde{V}_1^*$ define a function $\underline{f}^*_{\tau,Z,W} \in \mathscr{S}(\tilde{V}')$ by

$$\underline{f}^*_{\tau,Z,W} = f^*_{\tau_1,Z,W} \otimes f_{\tau_2} \otimes \cdots \otimes f_{\tau_r}$$

where $f^*_{\tau_1,Z,W}$ is the Schwartz function defined in §7.

LEMMA 8.1: *If $h = (h_1, \ldots, h_r) \in H' \simeq \Pi_\lambda \operatorname{Sp}(E_\lambda)$, then*

$$R(\rho'' \circ \rho(h))\underline{f}^*_{\tau,Z,W} = N_k(J(h, \tau)^{-m/2})\underline{f}^*_{h\tau,Z,W}$$

where

$$N_k(J(h, \tau)^{m/2}) = \prod_\lambda J(h_\lambda, \tau_\lambda)^{m/2},$$

and the branch of $J(h_\lambda, \tau_\lambda)^{1/2}$ is determined as in Corollary 7.3.

PROOF: As observed above,

$$\begin{aligned} R(\rho'' \circ \rho(h))\underline{f}^*_{\tau,Z,W} &= R_1(\rho(h_1))f^*_{\tau_1,Z,W} \otimes \cdots \otimes R_r(\rho(h_r))f_{\tau_r} \\ &= J(h_1, \tau_1)^{-m/2} f^*_{h_1\tau_1,Z,W} \otimes \cdots \otimes J(h_r, \tau_r)^{-m/2} f_{h_r\tau_r} \end{aligned}$$

by Proposition 7.4 and Corollary 7.3, respectively.

8.2: We now want to construct certain theta-functions.

For convenience we fix a k-basis $e_1, \ldots, e_k$ for E and let $e_1^*, \ldots, e_k^*$ be the dual basis for E^*. Thus we obtain identifications

$$\tilde{V} \cong V^k \cong \tilde{V}^*$$

with

$$[X, Y] = \operatorname{tr}(X, Y)$$

in the matrix notation of §5.2.

Let $\mathcal{O}$ be the ring of integers of k, and let $L \subset V$ be an $\mathcal{O}$-lattice such that the dual lattice

$$L^* = \{Y \in V \mid \operatorname{tr}_k(X, Y) \in \mathbb{Z}, \forall X \in L\}$$

contains L, i.e., $L^* \supset L$. Let

$$\tilde{L} = L^k \subset V^k \cong \tilde{V}$$

so that

$$\tilde{L}^* = (L^*)^k \subset V^k \cong \tilde{V}^*,$$

and we have

$$\tilde{L}^* \supset \tilde{L}$$

under the identification of $\tilde{V}$ and $\tilde{V}^*$.

By restriction of scalars, we obtain $\mathbb{Z}$-lattices

$$\tilde{L}' \subset (\tilde{L}^*)' \subset \tilde{V}'.$$

Now for any $\mu \in (\tilde{L}^*)'/\tilde{L}'$ we define a theta-distribution θ_μ on $\mathcal{S}(\tilde{V}')$ by

$$\theta_\mu(f) = \sum_{X \in \tilde{L}'} f(X + \mu).$$

We recall the generalized Poisson summation formula for the θ_μ's as formulated in Shintani [21]. For $h \in \operatorname{Sp}(\tilde{V}')$ and $z = (x, y) \in \tilde{V}' \times (\tilde{V}')^*$, let

$$F_h(z) = e(\tfrac{1}{2}[x', y']' - \tfrac{1}{2}[x, y]')$$

where $(x', y') = zh$. Then let

$$\operatorname{Sp}(\tilde{V}', \tilde{L}') = \{h \in \operatorname{Sp}(\tilde{V}') \mid (\tilde{L}' \times \tilde{L}')h = \tilde{L}' \times \tilde{L}' \text{ and } F_h(z) = 1, \forall z \in \tilde{L}' \times \tilde{L}'\}.$$

For any $h \in \operatorname{Sp}(\tilde{V}', \tilde{L}')$ and $f \in \mathcal{S}(\tilde{V}')$, Proposition 1.1 of [21] implies that

$$\theta_\mu(R(h)f) = \sum_{\nu \in (\tilde{L}')^*/\tilde{L}'} C_h(\mu, \nu)\theta_\nu(f)$$

with a certain unitary matrix $C_h = (C_h(\mu, \nu))$, independent of f and satisfying

$$C_{hh'} = c(h, h')C_hC_{h'}$$

where

$$c(h, h') = J(hh', \tau)^{1/2}J(h, h'\tau)^{-1/2}J(h', \tau)^{-1/2}$$

as in §7.1.

LEMMA 8.2: *Consider the subgroup* $\mathrm{Sp}(k, \mathcal{O}) \subset H(\mathscr{k})$, *and let*

$$\Gamma_L = \{\gamma \in \mathrm{Sp}(k, \mathcal{O}) \mid \textit{the image of } \gamma \textit{ in } \mathrm{Sp}(\tilde{V}') \textit{ lies in } \mathrm{Sp}(\tilde{V}', \tilde{L}')\}.$$

Then

$$\Gamma_L = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(k, \mathcal{O}) \mid \mathrm{tr}(Xa, Xb) \in 2\mathscr{D}^{-1}, \mathrm{tr}(Yc, Yd) \in 2\mathscr{D}^{-1}, \forall X, Y \in \tilde{L}\right\},$$

where $\mathscr{D}^{-1}$ *is the inverse different of* $\mathscr{k}/\mathbb{Q}$.

PROOF: This follows easily from the definition of $\mathrm{Sp}(\tilde{V}', \tilde{L}')$ and of $F_{h'}$ and the fact that

$$[X, Y]' = \mathrm{tr}_{\mathscr{k}}\, \mathrm{tr}(X, Y).$$

We can give a more explicit formula for C_h on a certain congruence subgroup of Γ_L.

PROPOSITION 8.3: *Choose* $N \in \mathbb{Z}_{>0}$ *such that*

(i) $NL^* \subset L$
and (ii) $N(X, Y) \in 2\mathscr{D}^{-1}, \forall X, Y \in L^*$,

and let

$$\Gamma_0(N) = \{\gamma \in \Gamma_L \mid c \equiv 0(NM_k(\mathcal{O}))\}.$$

Then for $\gamma \in \Gamma_0(N)$ *and for the choice of* $R(\rho'' \circ \rho'(\gamma))$ *made in* §8.1,

$$C_\gamma(\mu, \nu) = \psi(\gamma)\delta_{\mu,\nu{}^t d}\, e(\tfrac{1}{2}\mathrm{tr}_{\mathscr{k}}\, \mathrm{tr}(\mu, \mu)b\,{}^t a)\mathfrak{H}(b, d; L)$$

where

$$\mathfrak{H}(b, d; L) = |\det d|^{-m/2} \sum_{X \in \tilde{L}/\tilde{L}^t d} e(\tfrac{1}{2} \operatorname{tr}_{k} \operatorname{tr}(X, X) b d^{-1})$$

and

$$\delta_{\mu,\nu} = \begin{cases} 0 \text{ if } \mu \not\equiv \nu \bmod \tilde{L} \\ 1 \text{ if } \mu \equiv \nu \bmod \tilde{L} \end{cases}.$$

Finally $\psi(\gamma)$ *is a root of unity determined as follows:*

$$\psi(\gamma) = N_{k}(\epsilon(\gamma\omega)^{-1}\epsilon(\omega^{-1})^{-1}c(\gamma\omega, \omega^{-1})^m)$$

where $\omega = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \in \mathrm{Sp}(k, \mathcal{O})$, *and, for* $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(k, k)$ *with* $\det c \neq 0$, *we let*

$$\epsilon_\lambda(h_\lambda) = (i^k \operatorname{sgn} \det c_\lambda)^{1/2(p_\lambda - q_\lambda)}$$

with $(p_\lambda, q_\lambda) = \operatorname{sgn}(V_\lambda'(,))$.*Finally, for* $h, h' \in \mathrm{Sp}(k, k)$, $c_\lambda(h_\lambda, h'_\lambda)$ *is the cocycle for* $\mathrm{Mp}(E_\lambda)$ *as in* §7.1. *Note that the choice of root in* $\epsilon_\lambda(h_\lambda)$ *and* $c_\lambda(h_\lambda, h'_\lambda)$ *is determined by our choice of operators for* $\tilde{h} \in \tilde{H}'$ *in* §8.1.

PROOF: If $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_\lambda$ with $\det c \neq 0$, then for $f \in \mathcal{S}(\tilde{V}_\lambda)$

$$R_\lambda(h)f(X) = \epsilon_\lambda(h)^{-1}|\det c|^{-m/2} \int_{V_\lambda} e(\tfrac{1}{2} \operatorname{tr}((X, Yac^{-1}) - 2(X, Yc^{-1}) + (Y, Yc^{-1}d)))f(Y)d^*Y$$

where d^*Y is the self-dual measure on $\tilde{V}_\lambda \cong V_\lambda^k \cong \tilde{V}_\lambda^*$,

$$\epsilon_\lambda(h) = (i^k \operatorname{sgn} \det c)^{1/2(p_\lambda - q_\lambda)}$$

with $(p_\lambda, q_\lambda) = \operatorname{sig} V_\lambda(,)$ and the choice of root depends on our choice of $J(h, \tau_\lambda)^{1/2}$.

Next suppose that $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_L$ with $\det c \neq 0$. Then by Proposition 1.1(iii) of [21], we find that

$$C_h(\mu, \nu) = \operatorname{vol}(\tilde{V}'/\tilde{L}')^{-1/2}|c|^{-1/2}\epsilon(h)^{-1} \times \sum_{\ell \in \tilde{L}/\tilde{L}^t c} e(\tfrac{1}{2} \operatorname{tr}_{k} \operatorname{tr}((\ell + \mu, \ell + \mu)ac^{-1} - 2(\ell + \mu, \nu)c^{-1} + (\nu, \nu)c^{-1}d))$$

with $|c| = \Pi_\lambda |\det c_\lambda|^m$ and $\epsilon(h) = \Pi_\lambda \epsilon_\lambda(h_\lambda)$. Observe that if $\gamma \in \Gamma_0(N)$, then the condition $a^t d - b^t c = 1_k$ implies that $\det d \neq 0$. Write $\omega = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \in \mathrm{Sp}(k, \mathcal{O})$, and write

$$\gamma = \gamma\omega\omega^{-1}$$

so that, by Lemma 8.1

$$C_\gamma = N_{\not{k}}(c(\gamma\omega, \omega^{-1})^m) C_{\gamma\omega^{-1}} C_{\omega^{-1}}.$$

Then, by the same argument as in [21, p. 96], we obtain:

$$C_\gamma(\mu, \nu) = \epsilon(\gamma\omega)^{-1}\epsilon(\omega^{-1})^{-1} N_{\not{k}}(c(\gamma\omega, \omega^{-1})^m) \operatorname{vol}(\tilde{V}'/\tilde{L}')^{-2} |\tilde{L}^*/\tilde{L}| \times$$
$$\times |d|^{-1/2} \sum_{\ell \in \tilde{L}/\tilde{L}^t d} e(\tfrac{1}{2} \operatorname{tr}_{\not{k}} \operatorname{tr}(\ell + \mu, \ell + \mu) b d^{-1}) \delta_{\mu, \nu^t d}.$$

If $\gamma = 1$ we obtain

$$\epsilon(\omega)^{-1}\epsilon(\omega^{-1})^{-1} N_{\not{k}}(c(\omega, \omega^{-1})^m) \operatorname{vol}(\tilde{V}'/\tilde{L}')^{-2} |\tilde{L}^*/\tilde{L}| = 1,$$

and so $\epsilon(\omega)^{-1}\epsilon(\omega^{-1})^{-1} N_{\not{k}}(c(\omega, \omega^{-1})^m) = 1$ and $\operatorname{vol}(\tilde{V}'/\tilde{L}')|\tilde{L}^*/\tilde{L}| = 1$. Finally, as in [21, p. 97],

$$\sum_{\ell \in \tilde{L}/\tilde{L}^t d} e(\tfrac{1}{2} \operatorname{tr}_{\not{k}} \operatorname{tr}(\ell + \mu, \ell + \mu) b d^{-1}) =$$
$$= e(\tfrac{1}{2} \operatorname{tr}_{\not{k}} \operatorname{tr}(\mu, \mu) b^t a) \sum_{\ell \in \tilde{L}/\tilde{L}^t d} e(\tfrac{1}{2} \operatorname{tr}_{\not{k}} \operatorname{tr}(\ell, \ell) b d^{-1}),$$

which completes the proof.

We can now construct the theta-functions which we need. For $\tau \in \prod_\lambda \mathfrak{H}(E_\lambda)$, $Z \in D$, and $W \in Z^\perp \otimes E_1^*$, define

$$\theta_\mu(\tau, Z, W) = \theta_\mu(\underline{f}^*_{\tau, Z, W})$$

where $\mu \in \tilde{L}^*/\tilde{L}$ and $\underline{f}^*_{\tau, Z, W}$ is as in §8.1.

Proposition 8.4: *Let $N \in \mathbb{Z}_{>0}$ and $\Gamma_0(N) \subset \mathrm{Sp}(k, \mathcal{O})$ be as in Proposition* 8.3. *Then* $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$\theta_\mu(\gamma\tau, Z, W) = \psi(\gamma) \mathfrak{H}(b, d; L) N_{\not{k}}(J(\gamma, \tau)^{m/2})$$
$$\times e(\tfrac{1}{2} \operatorname{tr}_{\not{k}} \operatorname{tr}(\mu, \mu) b^t a) \theta_{\mu a}(\tau, Z, W).$$

PROOF: We have, via Lemma 8.1 and Proposition 8.3,

$$\begin{aligned}\theta_\mu(\gamma\tau, Z, W) &= \theta_\mu(\underline{f}^*_{\gamma\tau,Z,W}) \\ &= N_{\mathscr{k}}(J(\gamma, \tau)^{m/2})\theta_\mu(R(\rho''\circ\rho(\gamma))\underline{f}^*_{\tau,Z,W}) \\ &= \psi(\gamma)\mathfrak{H}(b, d; L)N_{\mathscr{k}}(J(\gamma, \tau)^{m/2}) \\ &\quad \times e(\tfrac{1}{2}\operatorname{tr}_{\mathscr{k}}\operatorname{tr}(\mu, \mu)b\,{}^t a)\theta_{\mu a}(\underline{f}^*_{\tau,Z,W})\end{aligned}$$

since $\mu a\,{}^t d \equiv \mu \bmod \tilde{L}$.

§9. The main theorem

In this section we will show that the theta-functions constructed in §8 provide a link between Hilbert–Siegel modular forms and certain collections of the cycles constructed in §6.

We use the same notation as in §8.2. Since we have chosen a basis for E, we have an isomorphism

$$\prod_\lambda \mathfrak{H}(E_\lambda) \simeq \mathfrak{H}_k^r$$

where $\mathfrak{H}_k$ is the Siegel space of genus k.

For $N \in \mathbb{Z}_{>0}$ as in Proposition 8.3, let

$$\Gamma(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_L \subset \mathrm{Sp}(k, \mathcal{O}) \mid a \equiv 1(N),\ c \equiv 0(N),\ b \equiv 0(N)\}.$$

Then for $\gamma \in \Gamma(N)$, Proposition 8.4 gives:

$$\theta_\mu(\gamma\tau, Z, W) = \psi(\gamma)\mathfrak{H}(b, d;\ L)N_{\mathscr{k}}(J(\gamma, \tau)^{m/2})\theta_\mu(\tau, Z, W).$$

Set

$$\lambda(\gamma) = \psi(\gamma)\mathfrak{H}(b, d; L).$$

Let $S_{m/2}(\Gamma(N))$ be the space of Hilbert–Siegel cusp forms of weight $\frac{1}{2}$m and multiplier $\lambda(\gamma)$ with respect to $\Gamma(N)$. Let

$$\mathcal{S}_k(\mathscr{k}) = \{\beta \in M_k(\mathscr{k}) \mid {}^t\beta = \beta\},$$

let

$$\mathscr{L} = \mathcal{S}_k(\mathscr{k}) \cap M_k(N\mathcal{O}),$$

and let

$$\mathcal{L}^* = \{\beta' \in \mathcal{S}_k(\mathscr{k}) \mid \mathrm{tr}_{\mathscr{k}}\, \mathrm{tr}(\beta\beta') \in \mathbb{Z},\ \forall \beta \in \mathcal{L}\}.$$

Then any function $\varphi \in S_{m/2}(\Gamma(N))$ has a Fourier expansion

$$\varphi(\tau) = \sum_{\substack{\beta \in \mathcal{L}^* \\ \beta \gg 0}} a(\beta) e_*(\beta\tau)$$

where, for convenience, we write

$$e_*(\beta\tau) = e(\mathrm{tr}_{\mathscr{k}}\, \mathrm{tr}(\beta\tau)).$$

Also the condition $\beta \gg 0$ means that $\beta_\lambda \in \mathcal{S}_k(\mathscr{k}_\lambda)$ is positive definite for all λ.

Now for $\beta \in \mathcal{L}^*$ with $\beta \gg 0$ and for $s \in \mathbb{C}$, define the Poincaré series:

$$P_{\beta,s}(\tau) = c^{-1} \sum_{\gamma \in \Gamma(N)_\infty \backslash \Gamma(N)} \lambda(\gamma)^{-1} N_{\mathscr{k}}(J(\gamma, \tau)^{-m/2} \det v(\gamma\tau)^s) e_*(\beta\gamma\tau)$$

where $\tau = (\tau_1, \ldots, \tau_r) \in \mathfrak{H}_k^r$, $\tau_\lambda = u_\lambda + iv_\lambda$; $c = \mathrm{vol}(\mathcal{S}_k(\mathbb{R})^r/\mathcal{L})$, and

$$\Gamma(N)_\infty = \Gamma(N) \cap \{\left(\begin{smallmatrix} \pm 1 & \beta \\ & \pm 1 \end{smallmatrix}\right) \mid \beta \in \mathcal{S}_k(\mathscr{k})\}.$$

By the argument of Klingen [10], this series is absolutely convergent for $\mathrm{Re}(s) > k - \frac{1}{4}m$, and, if $k < \frac{1}{4}m$, the holomorphic functions $\{P_{\beta,0} \mid \beta \in \mathcal{L}^*, \beta \gg 0\}$ span the space $S_{m/2}(\Gamma(N))$. In particular, for φ_1, $\varphi_2 \in S_{m/2}(\Gamma(N))$, we let

$$\langle \varphi_1, \varphi_2 \rangle = \int_F \varphi_1(\tau)\overline{\varphi_2(\tau)} N_{\mathscr{k}}(\det v^{m/2}) d\mu(\tau)$$

be the Petersson inner product, where F is a fundamental domain for $\Gamma(N)$ in $\mathfrak{H}_k^r$ and $d\mu(\tau) = N_{\mathscr{k}}(\det v^{-k-1} du dv)$. Then for $\varphi \in S_{m/2}(\Gamma(N))$,

$$\langle \varphi, P_{\beta,\bar{s}} \rangle = \Lambda_k(\alpha)^r N_{\mathscr{k}}(\det \beta)^{-\alpha} a(\beta)$$

where

$$\alpha = s + \frac{n}{2} - \frac{k}{2}.$$

and

$$\Lambda_k(\alpha) = (4\pi)^{-k\alpha}\pi^{1/4k(k-1)}\Gamma(\alpha)\Gamma(\alpha-\tfrac{1}{2})\cdots\Gamma\left(\alpha-\frac{k-1}{2}\right).$$

Note that the first pole of $\Lambda_k(\alpha)$ occurs at $s = k - \frac{1}{2}m$. We normalize $P_{\beta,s}$ by setting:

$$P^*_{\beta,s} = \Lambda_k(\alpha)^{-1}N_k(\det\beta)^{\alpha}P_{\beta,s}$$

so that

$$\langle\varphi, P^*_{\beta,s}\rangle = a(\beta).$$

If Re(s) is sufficiently large, $P^*_{\beta,s}$ is rapidly decreasing on a fundamental domain for $\Gamma(N)$, and so we may define the inner product $\langle P^*_{\beta,s}, \theta_\mu\rangle$. Our main result is an explicit formula for this inner product in terms of the dual forms constructed in §6.

THEOREM 9.1: *Let $\theta_\mu(\tau, Z, W)$ be the theta-function defined in* §8.2, *and let $P^*_{\beta,s}$ be the normalized Poincaré series. Then if*

$$\mathrm{Re}(s) > \begin{cases} k+\dfrac{1}{r} & \text{for } mr \text{ even} \\ k+\dfrac{1}{2r} & \text{for } mr \text{ odd,} \end{cases}$$

$$\langle P^*_{\beta,s}, \theta_\mu\rangle = 2^{-k/2}\kappa(s)^{-1}\sum_{\substack{X\equiv\mu(\tilde{L})\\(X,X)=2\beta}}\|X_1\|^{2\alpha}\|X_1\|_Z^{-1-2\alpha}\det(X_1, W)$$

where $\kappa(s)$ is as in Lemma 3.3, *$\|\;\|$ and $\|\;\|_Z$ are as in Proposition* 5.3, *X_1 denotes the image of the frame X in $\tilde{V}_1 \cong V_1^k$, and $\alpha = s + \frac{n}{2} - \frac{k}{2}$. In particular, we have*

$$\langle P^*_{\beta,s}, \theta_\mu\rangle = 2^{-k/2}\sum_{\substack{X\equiv\mu(\tilde{L})\\(X,X)=2\beta\\ \mathrm{mod}\,\Gamma}}\omega_{X,s}$$

where, in the notation of §6, *$\Gamma\subset G(L)$ is any congruence subgroup* *satisfying conditions* 1) *and* 2) *of* §6 *and such that*, $\forall \gamma \in \Gamma$,

$$\gamma\mu \equiv \mu(\tilde{L}),$$

and $\omega_{X,s}$ *is the dual form to the cycle* N_X *defined in* §6.

PROOF: By the usual unfolding argument we have:

$$\langle P_{\beta,s}, \theta_\mu\rangle = \int_{\mathcal{F}_\infty} c^{-1} e_*(\beta\tau)\overline{\theta_\mu(\tau)} N_{\not k}(\det v^{s+(m/2)-k-1} du dv),$$

where $\mathcal{F}_\infty$ is a fundamental domain for $\Gamma(N)_\infty$ in $\mathfrak{H}_k^r$. Taking

$$\mathcal{F}_\infty = \{\tau \in \mathfrak{H}_k^r \mid \operatorname{Re}(\tau) \in \mathcal{S}_k(\mathbb{R})^r/\mathcal{L}\}$$

we obtain:

$$\int_{v \gg 0} N_{\not k}(\det v^{s+(m/2)-k-1}) \int_{\mathcal{S}_k(\mathbb{R})^r/\mathcal{L}} c^{-1} e_*(\beta\tau)\overline{\theta_\mu(\tau)} N_{\not k}(du dv).$$

Now we have

$$\theta_\mu(\tau) = (\det v_1)^{1/2} \sum_{X \equiv \mu(\tilde{L})} \det(X_1, W) e_*(\tfrac{1}{2}(X, X)_{\tau,Z})$$

where

$$(X, X)_{\tau,Z} = (u_1(X_1, X_1) + i v_1(X_1, X_1)_Z, \tau_2(X_2, X_2), \ldots, \tau_r(X_r, X_r))$$

with X_λ the image of X in $\tilde{V}_\lambda \simeq V_\lambda^k$, and $(,)_Z$ the majorant of $(,)$ associated to $Z \in D$. Therefore, computed term-by-term, the inside integral is just

$$(\det v_1)^{1/2} e_*(i\beta v) \sum_{\substack{X \equiv \mu(\tilde{L}) \\ (X, X) = 2\beta}} \det(X_1, W) \exp(-\pi \operatorname{tr}(v_1(X_1, X_1)Z + \sum_{\lambda > 1} v_\lambda(X_\lambda, X_\lambda))).$$

For $\operatorname{Re}(s)$ as stated in the theorem we can compute the remaining integral termwise. This gives a sum of terms each of which is a product over λ of the integrals:

$$\int_{v_1 > 0} \det v_1^{s+(m/2)-k-(1/2)} \exp(-\pi \operatorname{tr}(v_1((X_1, X_1) + (X_1, X_1)_Z))) dv_1$$

for $\lambda = 1$, and

$$\int_{v_\lambda > 0} \det v_\lambda^{s+(m/2)-k-1} \exp(-4\pi \operatorname{tr}(\beta_\lambda v_\lambda)) dv_\lambda$$

for $\lambda > 1$. Now recalling Siegel's formula:

$$\int_{v>0} \det v^{s-(k+1/2)} \exp(-2\pi \operatorname{tr}(vv')) dv = 2^{ks} \Lambda_k(s) (\det v')^{-s}$$

we find that these factors become

$$2^{k\alpha_1} \Lambda_k(\alpha_1) \det(\tfrac{1}{2}(X_1, X_1) + \tfrac{1}{2}(X_1, X_1)_Z)^{-\alpha_1}$$

for $\lambda = 1$, with $\alpha_1 = s + \frac{m}{2} - \frac{k}{2}$, and

$$\Lambda_k(\alpha) \det \beta_\lambda^{-\alpha}$$

for $\lambda > 1$, with $\alpha = s + \frac{n}{2} - \frac{k}{2}$. Here $\Lambda_k(\alpha)$ is the product of Γ-factors above. Observe that $\alpha_1 = \alpha + \frac{1}{2}$ and that

$$2^{k(\alpha+(1/2))} \Lambda_k(\alpha + \tfrac{1}{2}) = 2^{k/2} (4\pi)^{-k/2} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma\left(\alpha - \frac{k-1}{2}\right)} 2^{k\alpha} \Lambda_k(\alpha).$$

Also recall that

$$\begin{aligned} \det(\tfrac{1}{2}(X_1, X_1) + \tfrac{1}{2}(X_1, X_1)_Z) &= \det(\tilde{X}_1, \tilde{X}_1) \\ &= \|X_1\|_Z^2 \end{aligned}$$

in the notation of Proposition 5.3. Therefore, we obtain

$$2^{k\alpha} \Lambda_k(\alpha)^r (2\pi)^{-k/2} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma\left(\alpha - \frac{k-1}{2}\right)} N_k (\det \beta)^{-\alpha} \times$$

$$\times \sum_{\substack{X \equiv \mu(\tilde{L}) \\ (X, X) = 2\beta}} 2^{-k\alpha} \|X_1\|^{2\alpha} \|X_1\|_Z^{-2\alpha-1} \det(X_1, W)$$

as claimed.

Using reduction theory for $R_{k/\mathbb{Q}}GL(k)$ and arguments similar to those of Maass [14], p. 285, we find that the above calculation is valid for $\mathrm{Re}(s) > k + \frac{1}{r}$ if mr is even, and for $\mathrm{Re}(s) > k + \frac{1}{2r}$ if mr is odd, and the theorem is proved.

§10. Consequences of the main theorem

In this section we show that Theorem 9.1 implies generalizations and analogues of results of Hirzebruch and Zagier [7], Zagier [24], Kudla [12], Shintani [21] and Oda [18]. We retain the notation of §9.

By Theorem 4.3 the form $\omega_{X,s}$ has a meromorphic analytic continuation to the whole plane, and, if $k \leq \frac{1}{2} m$, the value $\omega_{X,0}$ is the harmonic Poincaré dual form to the cycle C_X. On the other hand, if $k < \frac{1}{4} m$, then the series $P^*_{\beta,s}$ is a holomorphic function of s in the half plane $\mathrm{Re}(s) > k - \frac{1}{4} m$ whose value at $s = 0$ lies in $S_{m/2}(\Gamma(N))$. For $\varphi \in S_{m/2}(\Gamma(N))$ and for $\mu \in \tilde{L}^*/\tilde{L}$ let

$$\mathscr{L}_k(\varphi) = \langle \varphi, \theta_\mu \rangle,$$

and, for $\beta \in \mathscr{L}^*$ with $\beta \gg 0$, define a 'composite' cycle

$$C_\beta = \sum_{\substack{X \equiv \mu(\tilde{L}) \\ (X, X) = 2\beta \\ \mathrm{mod}\ \Gamma}} C_X$$

so that $C_\beta \in H_{n-k}(M, \mathbb{Z})$.

COROLLARY 10.1: *If $k < \frac{1}{4} m$, $\mathscr{L}_k$ defines a lifting*

$$\mathscr{L}_k : S_{m/2}(\Gamma(N)) \to \mathscr{H}^k(M)_{\mathbb{C}}$$

where $\mathscr{H}^k(M)$ is the space of harmonic k-forms on M, and the image of $\mathscr{L}_k$ is precisely the span of the harmonic dual forms to the totally geodesic cycles C_β for $\beta \in \mathscr{L}^*$ with $\beta \gg 0$.

PROOF: By continuation to $s = 0$, Theorem 9.1 yields

$$\mathscr{L}_k(P^*_{\beta,0}) = 2^{-k/2} \omega_{\beta,0}$$

where $\omega_{\beta,0}$ is the harmonic dual form to C_β.

Now consider the series

$$\Omega(\tau, Z, W) = \sum_{\substack{\beta \in \mathcal{L}^* \\ \beta \gg 0}} \omega_\beta(Z, W) e_*(\beta\tau).$$

LEMMA 10.2: *Assume that* $k < \frac{1}{4}m$. *Let*

$$K(\tau, \tau') = \sum_{\substack{\beta \in \mathcal{L}^* \\ \beta \gg 0}} P^*_{\beta,0}(\tau) \overline{e_*(\beta\tau')}.$$

Then $K(\tau, \tau')$ *is the reproducing kernel for the space* $S_{m/2}(\Gamma(N))$ *and*

$$\langle K(\cdot, \tau'), \theta_\mu \rangle = 2^{-k/2} \overline{\Omega(\tau', Z, W)}.$$

Finally, if $\varphi \in S_{m/2}(\Gamma(N))$, *then*

$$\langle \varphi, \Omega \rangle = 2^{k/2} \langle \varphi, \theta_\mu \rangle.$$

PROOF: The fact that $\langle \varphi, K(\cdot, \tau') \rangle = \varphi(\tau')$ follows from the fact that $\langle \varphi, P^*_{\beta,0} \rangle = a(\beta)$. The relation between $K(\tau, \tau')$ and Ω then follows from Theorem 9.1. Finally,

$$\begin{aligned}
\langle \varphi, \Omega \rangle &= 2^{k/2} \langle \varphi, \overline{\langle K(\tau, \tau'), \theta_\mu \rangle_\tau} \rangle_{\tau'} \\
&= 2^{k/2} \langle \langle \varphi, K(\tau', \tau) \rangle_{\tau'}, \theta_\mu \rangle_\tau \\
&= 2^{k/2} \langle \varphi, \theta_\mu \rangle_\tau
\end{aligned}$$

as claimed.

REMARK: As a function of $\tau \in \mathfrak{H}^r_k$, $\Omega(\tau, Z, W)$ belongs to $S_{m/2}(\Gamma(N))$, and this function is an analogue of the function $\Omega(\tau, z_1, z_2)$ considered by Zagier [23], [24].

Observe that Ω defines a $\mathbb{C}$-valued, Γ-invariant harmonic k-form on D, hence a harmonic k-form on M. For $\eta \in \mathcal{H}^{n-k}(M)_{\mathbb{C}}$, define

$$I_k(\eta) = \int_M \Omega \wedge \eta.$$

COROLLARY 10.3: *If* $k < \frac{1}{4}m$, I_k *defines a mapping*

$$I_k : \mathcal{H}^{n-k}(M)_{\mathbb{C}} \to S_{m/2}(\Gamma(N))$$

and

$$I_k(\eta)(\tau) = \sum_{\substack{\beta \in \mathcal{L}^* \\ \beta \gg 0}} \left(\int_{C_\beta} \eta \right) e_*(\beta\tau).$$

In particular, the Fourier coefficients of the Hilbert–Siegel modular forms $I_k(\eta)$ are the periods of the harmonic form η over the composite cycle C_β.

PROOF: This follows immediately from the fact that

$$\overline{\omega_\beta} = \omega_\beta$$

and the defining property of the dual form.

REMARK: Corollary 10.3 is the analogue of a theorem of Shintani [21]. An analogous result was obtained by Oda [18] for certain type $(p, 0)$ forms for $SO(2, p)$.

By Poincaré duality, $H_k(M, \mathbb{C}) \simeq \mathcal{H}^{n-k}(M)_{\mathbb{C}}$. For $C \in H_k(M, \mathbb{C})$ let η be the corresponding harmonic $(n-k)$-form. Then Corollary 10.3 may be reformulated as follows:

COROLLARY 10.4: *Assume that $k < \frac{1}{4}m$. Then there is a mapping*

$$I_k : H_k(M, \mathbb{C}) \to S_{m/2}(\Gamma(N))$$

$$C \mapsto \sum_{\substack{\beta \in \mathcal{L}^* \\ \beta \gg 0}} C \cdot C_\beta \, e_*(\beta\tau)$$

where $C \cdot C_\beta$ is the intersection number of $C \in H_k(M, \mathbb{C})$ and $C_\beta \in H_{n-k}(M, \mathbb{C})$. In particular, the generating function for the intersection numbers is a Hilbert–Siegel modular form.

PROOF: *Apply Corollary* 10.3 *to η, the harmonic dual of C.*

REMARK: Corollary 10.4 is the analogue of the results of Hirzebruch and Zagier [7] and Kudla [12]. Note that if $C \in H_k(M, \mathbb{Z})$, then the Fourier coefficients of $I_k(C)$ are integers.

For $\eta \in \mathcal{H}^{n-k}(M)_{\mathbb{C}}$ and $\eta' \in \mathcal{H}^k(M)_{\mathbb{C}}$, let

$$\langle \eta, \eta' \rangle_M = \int_M \eta' \wedge \bar{\eta}.$$

COROLLARY 10.5: *Assume that* $k < \frac{1}{4}m$. *Then for* $\varphi \in S_{m/2}(\Gamma(N))$ *and* $\eta \in \mathcal{H}^{n-k}(M)_{\mathbb{C}}$,

$$\langle \varphi, I_k(\eta)\rangle = 2^{k/2}\langle \mathcal{L}_k(\varphi), \eta\rangle_M.$$

Thus I_k *and* $\mathcal{L}_k$ *are adjoints with respect to the appropriate inner products (up to the constant* $2^{k/2}$*).*

PROOF: *Immediate.*

REMARK: Corollary 10.5 is the analogue, in our situation, of the result conjectured by Hirzebruch and Zagier [7] and proved by Zagier [24].

COROLLARY 10.6: *Assume that* $k < \frac{1}{4}m$, *and let* $C \in H_k(M, \mathbb{R})$ *and* $\varphi \in S_{m/2}(\Gamma(N))$.

$$\int_C \mathcal{L}_k(\varphi) = (-1)^{k(n-k)}2^{-k/2}\langle \varphi, I_k(C)\rangle.$$

In particular, the periods of the lift $\mathcal{L}_k(\varphi)$ *are given by a Petersson inner product.*

PROOF: Let $\eta \in \mathcal{H}^{n-k}(M)$ *be the Poincaré dual form of C. Then*

$$\int_C \mathcal{L}_k(\varphi) = \int_M \eta \wedge \mathcal{L}_k(\varphi)$$
$$= (-1)^{k(n-k)}\int_M \mathcal{L}_k(\varphi) \wedge \bar{\eta}$$
$$= (-1)^{k(n-k)}2^{-k/2}\langle \varphi, I_k(\eta)\rangle$$

by Corollary 10.5 and the fact that $\eta = \bar{\eta}$.

§11. Non-vanishing

In this section we will use the results of Millson–Raghunathan [15] to show that the lifting $\mathcal{L}_k$, and hence its adjoint I_k, is nonzero for a suitable congruence subgroup. Actually we will use a slight generalization of their result, which will be proved in an appendix. We continue to use the notation of §9.

We assume that $\dim V \geq 4$, and, as in §6, we fix an $\mathcal{O}$-lattice

$L \subset L^* \subset V$ and let

$$G^*(L) = \{\gamma \in G(\mathscr{k}) \mid \gamma L = L\}.$$

Also let

$$G(L) = \{\gamma \in G^*(L) \mid \gamma \text{ acts trivially in } L^*/L\}.$$

For any $\mathcal{O}$-ideal, $\mathfrak{A} \subset \mathcal{O}$, let

$$G(L; \mathfrak{A}) = \{\gamma \in G(L) \mid \gamma \text{ acts trivially in } L^*/\mathfrak{A}L\}.$$

We now choose an ideal $\mathfrak{A}_0 \subset \mathcal{O}$ such that $\Gamma = G(L, \mathfrak{A}_0)$ satisfies:
1) Γ is torsion free,

and

2) Γ is in the kernel of the spinor norm $\theta : G(\mathscr{k}) \to \mathscr{k}^\times/(\mathscr{k}^\times)^2$.
For any integral $\mathcal{O}$-ideal $\mathfrak{A}$, let

$$R_{\mathfrak{A}} : L^* \to L^*/\mathfrak{A}L$$

be the reduction map. Also let

$$\Gamma(\mathfrak{A}) = \Gamma \cap G(L; \mathfrak{A}),$$

and let

$$M(\mathfrak{A}) = \Gamma(\mathfrak{A})\backslash D.$$

For $X \in L^*$ with $(X, X) \gg 0$, let

$$N_X(\mathfrak{A}) = \Gamma(\mathfrak{A})_X \backslash D_X,$$

and let

$$C(X; \mathfrak{A}) = (\iota_X)_*(1_{X,\mathfrak{A}})$$

where $1_{X,\mathfrak{A}}$ is the orientation class of $N_X(\mathfrak{A})$. Thus $C(X; \mathfrak{A}) \in H_{n-k}(M(\mathfrak{A}), \mathbb{Z})$. Finally, for $\beta \in \mathscr{S}_k(\mathscr{k})$ with $\beta \gg 0$ and for $\mu \in \tilde{L}^*/\mathfrak{A}\tilde{L}$, we let

$$C(\beta; \mu; \mathfrak{A}) = \sum_{\substack{X \in \tilde{L} \\ R_{\mathfrak{A}}(X) = \mu \\ (X, X) = 2\beta \\ \text{mod } \Gamma(\mathfrak{A})}} C(X; \mathfrak{A})$$

so that $C(\beta; \mu; \mathfrak{A}) \in H_{n-k}(M(\mathfrak{A}), \mathbb{Z})$.

PROPOSITION 11.1: *For any* $X \in \tilde{L}$ *with* $(X, X) \gg 0$, *there exists an integral* $\mathcal{O}$*-ideal* $\mathfrak{A}$ *such that*

$$C(X; \mathfrak{A}) \neq 0.$$

PROOF: This is a slight generalization of a result of [15], and we will give its proof in the appendix below.

On the other hand, Theorem 9.1 says that, for $k < \frac{1}{4}m$ and for the theta-kernel θ_μ associated to $\mu \in \tilde{L}^*/\mathfrak{A}\tilde{L} \subset \mathfrak{A}^{-1}\tilde{L}^*/\mathfrak{A}\tilde{L}$, the lift of the normalized Poincaré series $2^{k/2}P^*_{\beta,0}$ is the harmonic dual form to the 'composite' cycle $C(\beta; \mu; \mathfrak{A})$. Therefore, to show the nonvanishing of $\mathcal{L}_k$ it will be sufficient to show the following:

Assume that dim $V \geq 4$ and that $1 \leq k < n-1$. Then,

THEOREM 11.2: *For any* $X \in \tilde{L}^*$ *with* $(X, X) = \beta \gg 0$, *there exists an integral* $\mathcal{O}$*-ideal* $\mathfrak{b}$ *such that*

$$(\mathrm{pr}_\mathfrak{b})_*\Bigg(\sum_{\mu \in R_\mathfrak{b}(\Gamma \cdot X)} C(\beta; \mu; \mathfrak{b})\Bigg) = d(X, \mathfrak{b})C(X; 1)$$

where $d(X; \mathfrak{b}) \in \mathbb{Z}_{>0}$ *and where*

$$\mathrm{pr}_\mathfrak{b} : M(\mathfrak{b}) \to M$$

is the natural projection.

COROLLARY 11.3: *Assume that* dim $V \geq 4$ *and that* $1 \leq k < \frac{1}{4}m$. *Let* $\beta \in \mathcal{S}_k(\mathscr{k})$ *with* $\beta \gg 0$ *such that there exists* $X \in \tilde{L}^*$ *with* $(X, X) = \beta$. *Then there exists an ideal* $\mathfrak{b}$ *and an element* $\mu \in \tilde{L}^*/\mathfrak{b}\tilde{L} \subset \mathfrak{b}^{-1}\tilde{L}^*/\mathfrak{b}\tilde{L}$ *such that*

$$\mathcal{L}_k(P^*_{\beta,0}) \neq 0$$

where $\mathcal{L}_k$ *is the lifting defined with respect to* θ_μ.

The proof of Theorem 11.2 is based on the following lemma:

LEMMA 11.4: *For* $\beta \in \mathcal{S}_k(\mathscr{k})$ *with* $\beta \gg 0$, *let*

$$\tilde{L}^*_\beta = \{X \in \tilde{L}^* \mid (X, X) = 2\beta\}.$$

Then there exists an integral $\mathcal{O}$*-ideal* $\mathfrak{b}$ *such that the reduction map* $R_\mathfrak{b}$

induces an injection on Γ*-orbits*:

$$R_{\mathfrak{b}}: \Gamma\backslash \tilde{L}^*_\beta \hookrightarrow \Gamma\backslash \tilde{L}^*/\mathfrak{b}L.$$

PROOF: It is sufficient to show that for any two elements X_1 and X_2 in $\tilde{L}^*_\beta$ such that $\Gamma \cdot X_1 \cap \Gamma \cdot X_2 = \phi$, there exists an ideal $\mathfrak{b}$ such that

$$R_{\mathfrak{b}}(\Gamma X_1) \cap R_{\mathfrak{b}}(\Gamma X_2) = \phi.$$

Suppose that no such ideal exists, so that we have

$$\forall \mathfrak{b},\ \exists \gamma \in \Gamma \text{ such that} \qquad (1)$$
$$R_{\mathfrak{b}}(\gamma X_1) = R_{\mathfrak{b}}(X_2).$$

Moreover, since $(X_1, X_1) = (X_2, X_2) = \beta$, there exists an element $\eta \in G(k)$ such that

$$\eta X_1 = X_2. \qquad (2)$$

Let $H = G_{X_1}$ viewed as an algebraic group over k. Since $\dim X_1^\perp \geq 3$, we may apply the result of §101.8 of O'Meara [17] and conclude that

$$\theta(H(k)) = \theta(G(k)) \subset k^\times/(k^\times)^2.$$

Thus we may assume that $\eta \in G(k)' = \ker \theta$.

Let $S \subset G(\mathbb{A}_f)$ be a compact open subgroup of the finite adeles of G such that

$$\Gamma = G(k) \cap G_\infty \cdot S.$$

Since we have taken $\Gamma = G(L; \mathfrak{A}_0)$ for some $\mathfrak{A}_0$, we may assume that

$$S = \prod_{\mathfrak{p}} S_{\mathfrak{p}}$$

for compact open subgroups $S_{\mathfrak{p}} \subset G(k_{\mathfrak{p}})$. Then let

$$\sum = \{\mathfrak{p} \mid \eta \notin S_{\mathfrak{p}}\}$$

where $\eta \in G(k)'$ is as in (2) above, and let

$$\mathfrak{b} = \prod_{\mathfrak{p} \in \Sigma} \mathfrak{p}.$$

Then, by (1), for each $m \in \mathbb{Z}_{>0}$, there exists an element $\gamma_m \in \Gamma$ such that

$$R_{\mathfrak{b}^m}(\gamma_m X_1) = R_{\mathfrak{b}^m}(X_2).$$

The set $\{\gamma_m\} \subset S' = S \cap G(\mathbb{A}_f)'$ has a limit point $\mu \in S'$. Here $G(\mathbb{A}_f)' = \ker \theta$ where $\theta : G(\mathbb{A}_f) \to (k_{\mathbb{A}}^{\times})/(k_{\mathbb{A}}^{\times})^2$ is the spinor norm. Then

$$\mu_{\mathfrak{p}} X_1 = X_2$$

for each $\mathfrak{p} \in \Sigma$. Define $\mu' \in S'$ by

$$\mu'_{\mathfrak{p}} = \begin{cases} \mu_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \Sigma \\ \eta & \text{if } \mathfrak{p} \notin \Sigma \end{cases}$$

so that

$$\mu'_{\mathfrak{p}} X_1 = X_2$$

for all $\mathfrak{p}$.

Now consider the element

$$\eta^{-1}\mu' \in G(\mathbb{A}_f)' \cap H(\mathbb{A}_f) = H(\mathbb{A}_f)'.$$

By the strong approximation theorem applied to H, there exist

$$\beta \in H(\mathbb{A}_f) \cap S$$

and

$$\nu \in H(k)$$

such that

$$\eta^{-1}\mu' = \nu\beta.$$

Thus

$$\eta\nu = \beta(\mu')^{-1} \in G(k) \cap S,$$

and so

$$\eta\nu \in \Gamma$$

and

$$\eta\nu X_1 = \eta X_1 = X_2.$$

This contradicts the assumption that $\Gamma \cdot X_1 \cap \Gamma \cdot X_2 = \phi$, and the lemma is proved.

COROLLARY 11.5: *For any $X \in \tilde{L}^*_\beta$ there exists an ideal $\mathfrak{b}$ such that*

$$\Gamma \cdot X = \bigcup_{\mu \in R_{\mathfrak{b}}(\Gamma \cdot X)} \{Y \in \tilde{L}^*_\beta \mid R_{\mathfrak{b}}(Y) = \mu\}.$$

PROOF: The inclusion "$\subset$" is clear for any $\mathfrak{b}$. If we take $\mathfrak{b}$ as in Lemma 11.4, then for $Y \in \tilde{L}^*_\beta$ the condition $R_{\mathfrak{b}}(Y) \in R_{\mathfrak{b}}(\Gamma \cdot X)$ implies that $Y \in \Gamma \cdot X$.

PROOF OF THEOREM 11.2: Let $(X, X) = \beta$, and take an ideal $\mathfrak{b}$ such that $R_{\mathfrak{b}}$ is injective on Γ-orbits in $\tilde{L}^*_\beta$ as in Lemma 11.4. Consider the cycle

$$C = \sum_{\mu \in R_{\mathfrak{b}}(\Gamma \cdot X)} C(\beta;\, \mu;\, \mathfrak{b})$$

so that, by Corollary 11.5,

$$C = \sum_{\substack{Y \in \Gamma \cdot X \\ \mathrm{mod}\, \Gamma(\mathfrak{b})}} C(Y;\, \mathfrak{b}).$$

On the other hand, if $Y = \gamma X$ with $\gamma \in \Gamma$, then γ induces an orientation-preserving isomorphism

$$\gamma : N_Y(\mathfrak{b}) \xrightarrow{\sim} N_X(\mathfrak{b}).$$

Thus, if

$$\mathrm{pr}_{\mathfrak{b}} : M(\mathfrak{b}) \to M$$

is the natural projection,

$$\begin{aligned}(\mathrm{pr}_{\mathfrak{b}})_*(C(Y;\, \mathfrak{b})) &= (\mathrm{pr}_{\mathfrak{b}})_*(\iota_Y)_*(1_{Y,\mathfrak{b}}) \\ &= (\mathrm{pr}_{\mathfrak{b}})_*(\gamma_*)(\iota_Y)_*(1_{Y,\mathfrak{b}}) \\ &= (\mathrm{pr}_{\mathfrak{b}})_*(\iota_X)_*(\gamma)_*(1_{Y,\mathfrak{b}}) \\ &= (\mathrm{pr}_{\mathfrak{b}})_*(C(X;\, \mathfrak{b})).\end{aligned}$$

But

$$(\mathrm{pr}_{\mathfrak{b}})_*(C(X;\,\mathfrak{b})) = |\Gamma_X : \Gamma_X(\mathfrak{b})|\,C(X;\,1),$$

so that

$$(\mathrm{pr}_{\mathfrak{b}})_*(C) = d(X;\,\mathfrak{b})C(X;\,1)$$

with

$$d(X;\,\mathfrak{b}) = |\Gamma_X : \Gamma_X(\mathfrak{b})|\;|\Gamma(\mathfrak{b})\backslash\Gamma/\Gamma_X|,$$

and the theorem is proved.

Appendix

In this appendix we will give a proof of Proposition 11.1. In idea this proof is essentially that given in [15] for certain diagonal forms over $\mathbb{Q}(\sqrt{d})$, $d > 0$, but, as we are only concerned with the case of $SO(n, 1)$, we may use rather elementary arguments. Moreover, we obtain the stronger result as stated in Proposition 11.1, and we can give more precise information about eliminating degenerate intersections, Lemma A.3.

We retain the notation of Section 11.

We have, then, a frame $X \in \tilde{L}^*$ with $(X, X) \gg 0$. Choose a frame $Y \in (L^*)^{n-k} \subset V^{n-k}$ such that

$$(X,\, Y) = 0$$

and

$$(Y,\, Y) \gg 0,$$

so that we have an orthogonal decomposition:

$$V = V_1 + V_2 + V_3$$

where $V_1 = \mathrm{span}\, X$, $V_2 = \mathrm{span}\, Y$, and $V_3 = V_1^{\perp} \cap V_2^{\perp}$. Note that $\dim_k V_3 = 1$. We will also let

$$G_1 = G_X$$

and

$$G_2 = G_Y,$$

and we note that

$$G_1 \cap G_2 = \{1_V\}.$$

We assume that our congruence subgroup $\Gamma \subset G(L)$ is obtained as

$$\Gamma = G(k) \cap G_\infty \cdot S$$

where

$$S = \prod_{\mathfrak{p}} S_{\mathfrak{p}}$$

is a compact open subgroup of $G(\mathbb{A}_f)$, the finite adèle group of G. We will then prove the following:

THEOREM A.1: *There exists an ideal* $\mathfrak{A} \subset \mathcal{O}$ *such that the intersection number*

$$C(X;\mathfrak{A}) \cdot C(Y;\mathfrak{A}) \neq 0$$

where $C(X;\mathfrak{A}) \in H_{n-k}(M(\mathfrak{A}), \mathbb{Z})$ *and* $C(Y;\mathfrak{A}) \in H_k(M(\mathfrak{A}), \mathbb{Z})$ *as in Section* 11.

PROOF: Let

$$\Delta = \{\gamma \in \Gamma \mid \gamma(D_X) \cap D_Y \neq \phi\},$$

and, for any ideal $\mathfrak{A}$, let

$$\Delta(\mathfrak{A}) = \Delta \cap \Gamma(\mathfrak{A}).$$

Then $\Delta = \Gamma_2 \Delta \Gamma_1$ where

$$\Gamma_1 = \Gamma \cap G_1(k)$$

and

$$\Gamma_2 = \Gamma \cap G_2(k),$$

and, as observed in [15, p. 16], the double coset space $\Gamma_2\backslash\Delta/\Gamma_1$ is in one-to-one correspondence with the set of components of $N_X \cap N_Y$. Also write

$$\Delta = \Delta_+ \cup \Delta_- \cup \Delta_0$$

where

$$\Delta_\pm = \{\gamma \in \Delta \mid \gamma(D_X) \cap D_Y \text{ is a point and } \gamma(D_X) \cdot D_Y = \pm 1\}$$

and

$$\Delta_0 = \{\gamma \in \Delta \mid \dim\, \gamma(D_X) \cap D_Y > 0\}.$$

Since D_X and D_Y are totally geodesic, all intersections will be proper. Also note that each of the sets $\Delta_\pm$ and Δ_0 is itself a finite union of double cosets, and there are natural inclusions:

$$\Delta(\mathfrak{A}) \subset \Delta$$

$$\Delta_1(\mathfrak{A}) \subset \Delta_\pm$$

and

$$\Delta_0(\mathfrak{A}) \subset \Delta_0.$$

Now letting

$$\Delta' = \{X' \in \tilde{L}^* \mid X' = \gamma' X \text{ for } \gamma' \in \Delta\}$$

we have:

LEMMA A.2: *Let $X' \in \Delta'$, and suppose that $\mathfrak{A} \subset \mathcal{O}$ is an ideal such that*

$$R_{\mathfrak{A}}(\Gamma_2 \cdot X') \cap R_{\mathfrak{A}}(\Gamma_2 \cdot X) = \phi.$$

Then

$$\Delta(\mathfrak{A}) \cap \Gamma_2 \gamma' \Gamma_1 = \phi,$$

where $X' = \gamma' X$, $\gamma' \in \Delta$.

PROOF: If $\gamma \in \Delta(\mathfrak{A}) \cap \Gamma_2\gamma'\Gamma_1$, then

$$\begin{aligned} R_{\mathfrak{A}}(X) &= R_{\mathfrak{A}}(\gamma X) \\ &= R_{\mathfrak{A}}(\gamma_2\gamma'\gamma_1 X) \\ &= R_{\mathfrak{A}}(\gamma_2\gamma' X) \\ &= R_{\mathfrak{A}}(\gamma_2 X'). \end{aligned}$$

According to this lemma, if we can find an ideal $\mathfrak{A}$ such that $R_{\mathfrak{A}}(\Gamma_2 X') \cap R_{\mathfrak{A}}(\Gamma_2 X) = \phi$, whenever $\Gamma_2 X' \cap \Gamma_2 X = \phi$, then

$$\Delta(\mathfrak{A}) \subset \Gamma_2\Gamma_1 \subset \Delta_+,$$

so that all intersections of $N_X(\mathfrak{A})$ and $N_Y(\mathfrak{A})$ will be transverse and positive, and Theorem A.1 follows.

PROOF OF PROPOSITION 11.1:
For $X' \in \Delta'$, let

$$B(X') = (X',\ Y) \in M_{k,n-k}(\mathscr{k})$$

in the matrix notation of Section 5.2. Then, for $\gamma_2 \in \Gamma_2$

$$B(\gamma_2 X') = B(X'),$$

and also

$$B(X) = 0.$$

Thus B is constant on Γ_2-orbits in Δ', and if $X' \in \Delta'$ with $B(X') \neq 0$, then we can find an ideal $\mathfrak{A}$ such that $R_{\mathfrak{A}}(\Gamma_2 \cdot X') \cap R_{\mathfrak{A}}(\Gamma_2 \cdot X) = \phi$. On the other hand, if $X' = \gamma' X \in \Delta'$ and $B(X') = B(X) = 0$, then there exists $g \in G(\mathscr{k})$ such that

$$g\{X,\ Y\} = \{\gamma X,\ Y\}$$

as n-frames. Since $gY = Y$, we have $g = g_2 \in G_2(\mathscr{k})$. But then $g_2^{-1}\gamma X = X$, so that $g_2^{-1}\gamma = g_1 \in G_1(\mathscr{k})$. Thus, if $B(X') = 0$, we can write

$$\gamma' = g_2 g_1 \in G_2(\mathscr{k})G_1(\mathscr{k}).$$

Note that such a decomposition is unique.

Similarly, suppose that $X' \in \Delta'$ is such that, for each prime $\mathfrak{p}$ and for each $j \in \mathbb{Z}_{>0}$, there exists $\gamma_2(j) = \gamma_2(j, \mathfrak{p}) \in \Gamma_2$ with

$$R_{\mathfrak{p}^j}(\gamma_2(j)X') = R_{\mathfrak{p}^j}(X).$$

Then the infinite set $\{\gamma_2(j)\} \subset \Gamma_2 \subset S_\mathfrak{p} \cap G_2(k_\mathfrak{p})$, and so we can conclude that there exists a limit point $\nu_\mathfrak{p}^{-1} \in S_\mathfrak{p} \cap G_2(k_\mathfrak{p})$, and that

$$\nu_\mathfrak{p}^{-1}X' = X.$$

But then

$$\mu_\mathfrak{p} = \nu_\mathfrak{p}^{-1}\gamma' \in G_1(k_\mathfrak{p}),$$

and we obtain a decomposition for every $\mathfrak{p}$:

$$\gamma' = \nu_\mathfrak{p}\mu_\mathfrak{p}$$

with $\nu_\mathfrak{p} \in S_\mathfrak{p} \cap G_2(k_\mathfrak{p})$ and $\mu_\mathfrak{p} \in S_\mathfrak{p} \cap G_1(k_\mathfrak{p})$. But since $G_1 \cap G_2 = \{1\}$, we must have

$$g_2 = \nu_\mathfrak{p}$$

and

$$g_1 = \mu_\mathfrak{p}$$

for all $\mathfrak{p}$! Hence $g_2 \in G_2(k) \cap G_\infty \cdot S = \Gamma_2$ and $g_1 \in G_2(k) \cap G_\infty \cdot S = \Gamma_1$ and

$$\gamma' \in \Gamma_2\Gamma_1.$$

Thus we have shown that if $\Gamma_2X' \cap \Gamma_2X = \phi$, then there must exist an ideal such that $R_\mathfrak{A}(\Gamma_2 \cdot X') \cap R_\mathfrak{A}(\Gamma_2 \cdot X) = \phi$, and the proof of Proposition 11.1 is complete.

The degenerate intersections, which correspond to double cosets in Δ_0, can be very easily eliminated as follows: Let Z be a nonzero vector in $V_3 \cap L^*$, and let

$$L' = \mathcal{O} \cdot X + \mathcal{O} \cdot Y + \mathcal{O} \cdot Z$$

be the $\mathcal{O}$-sublattice of L^* spanned by the frame $\{X, Y, Z\}$. Now we have $\gamma \in \Delta_0$ if and only if

$$\text{span}\{\gamma X,\ Y\} \geqslant 0$$

for (,), but

$$\dim_k \text{span}_k \{\gamma X,\ Y\} < n.$$

LEMMA A.3: *Let $\mathfrak{p}$ be any prime ideal such that*

$$L^*_{\mathfrak{p}} = L'_{\mathfrak{p}} = L_{\mathfrak{p}}.$$

Then

$$\Delta_0(\mathfrak{p}) = \phi.$$

PROOF: Let $\gamma \in \Gamma(\mathfrak{p})$ so that $R_{\mathfrak{p}}(\gamma X) = R_{\mathfrak{p}}(X)$. By our choice of $\mathfrak{p}$, the $\mathcal{O}/\mathfrak{p}$ vector space $R_{\mathfrak{p}}(\mathcal{O} \cdot \gamma X + \mathcal{O} \cdot Y) = R_{\mathfrak{p}}(\mathcal{O} \cdot X + \mathcal{O} \cdot Y) \subset L^*/\mathfrak{p}L$ has dimension n over $\mathcal{O}/\mathfrak{p}$, and this implies that

$$\dim_k \text{span}_k \{\gamma X,\ Y\} = n.$$

Thus $\Delta_0 \cap \Gamma(\mathfrak{p}) = \Delta_0(\mathfrak{p}) = \phi$ as claimed.

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