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M. VAN DER PUT

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COHOMOLOGY ON AFFINOID SPACES

M. van der Put

Introduction

An affinoid space X over a complete, non-archimedean, valued field k has a natural Grothendieck-topology. There are several sheaves (with respect to this topology) of special interest. For instance $\mathcal{O}$, $\mathcal{O}^0$, $\mathcal{O}(r)$, $\mathcal{O}^*$ and A. The sheaf $\mathcal{O}$ of holomorphic functions on X has trivial cohomology groups, i.e. $H^i(X, \mathcal{O}) = 0$ for $i \neq 0$, according to a result of J. Tate [8]. The sheaf $\mathcal{O}^0$, defined by $\mathcal{O}^0(U) = \{f \in \mathcal{O}^0(U) \mid |f(a)| \leq 1 \text{ for all } u \in U\}$, is far more complicated than $\mathcal{O}$. Its cohomology groups are in general not zero.

If X has dimension one and if X has a stable reduction Z then one can show (under some conditions) that $H^1(X, \mathcal{O}^0) \otimes \bar{k} = H^1(Z, \mathcal{O}_Z)$. Here $\bar{k}$ denotes the residue field of k. The stable reduction Z is an algebraic variety over $\bar{k}$ with structure sheaf $\mathcal{O}_Z$. The cohomology group $H^1(Z, \mathcal{O}_Z)$ is taken with respect to the Zariski-topology on Z. For higher dimensions almost nothing is known.

For a positive real number r, the sheaf $\mathcal{O}(r)$, is defined by $\mathcal{O}(r)(U) = \{f \in \mathcal{O}(U) \mid |f(U)| < r \text{ for all } u \in U\}$. This sheaf is very much like $\mathcal{O}^0$ but seems more manageable in higher dimensions. The results of W. Bartenwerfer [1] can be stated as $H^i(X, \mathcal{O}(r)) = 0$ for $i \neq 0$ and for a polydisk X. From this he derives the solution of the Corona-problem [2, 7]. We will show for a wider class of affinoid spaces that $\mathcal{O}(r)$ has no cohomology (3.15).

The interest of $\mathcal{O}^*$, the sheaf of invertible holomorphic functions, lies mainly in the group $H^1(X, \mathcal{O}^*)$. This group classifies the line bundles on X and coincides with the usual class group if X is regular (see also [6]). According to L. Gruson [5] and L. Gerritzen [3] the

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group $H^1(X, \mathcal{O}^*)$ is trivial if X is a polydisk (or some generalization of that). In (3.25) we will extend the result to a wider class of affinoid spaces.

Finally A, the constant sheaf with stalk A, is of interest because $H^i(X, A)$ tells something about the reductions of X. For an algebraic curve X over k (seen as a k-analytic space) the cohomology group $H^1(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $h \leq g =$ genus of X. One can show that $h = g$ holds if and only if X is a Mumford curve. The equality $h = 0$ holds if and only if the Jacobian variety of X has good reduction.

Our main results on the cohomology of the constant sheaves are stated in (3.10), (3.13).

The method of calculation of the H^i is based upon a base change theorem (2.3) for morphisms $\varphi : X \to Y$ of affinoid spaces. For a "point" p of Y a fibre $X \times p$ of the map φ is constructed. It turns out that the information of the fibres $X \times p$ is not sufficient to evaluate the cohomology groups $H^i(X, S)$ if one takes only the ordinary points p of Y. In fact one has to consider geometric points p of Y (introduced in (1.1)). Only for closed geometric points (1.1) one can give $X \times p$ the structure of an affinoid space. It is an affinoid space over some field extension K_p of k.

Our base change theorem will therefore only work for a certain class of sheaves, the constructible sheaves (1.4). A constructible sheaf is determined by its stalks at the closed geometric points.

The constant sheaf A is constructible. The sheaves $\mathcal{O}(r)$ and $\mathcal{O}^*$ are approximated by constructible sheaves.

Another feature of our method is that we have to do a careful analysis in dimension one over a field k which is not algebraically closed.

We remark that our approach works also for sheaves of non-abelian groups. In particular one can show for a class of affinoids X that $H^1(X, Gl_m(\mathcal{O})) = 0$. This means that any analytic vectorbundle on X is trivial. Finally, we refer to [4] for more details on affinoid spaces and analytic spaces.

§1. Geometric points and constructible sheaves

The field is supposed to be complete with respect to a non-archimedean valuation. Let A be an affinoid algebra over k. The set $X = \mathrm{Sp}(A)$ of all maximal ideals of A is called an affinoid space over

k. It is a topological space for a topology derived from the topology of the field k.

A subset Y of X is called a *rational subset of* X if there are $f_0, \ldots, f_s \in A$, generating the unit ideal of A, such that

$$Y = \{x \in X \mid |f_0(x)| \geq \max_{i=1,\ldots,s} |f_i(x)|\}.$$

We write $Y = R(f_0, \ldots, f_s)$. We associate with Y the affinoid algebra $A\langle T_1, \ldots, T_s\rangle/(f_1 - f_0T_1, \ldots, f_s - f_0T_s)$. On X we introduce a Grothendieck topology by:

(1) the allowed subsets are the rational subsets of X

(2) the allowed coverings are the finite coverings by allowed subsets

The structure sheaf $\mathcal{O}_X$ or $\mathcal{O}$ on X is given by

$$\mathcal{O}(R(f_0, \ldots, f_s)) = A\langle T_1, \ldots, T_s\rangle/(f_1 - f_0T_1, \ldots, f_s - f_0T_s).$$

This is not the only possibility for a Grothendieck-topology on X. From time to time it is easier to use the finite unions of rational subsets of X as allowed subsets. For more details we refer to [4] Ch. III.

For any sheaf (of abelian groups) S on X and for any point $x \in X$ we can form the stalk $S_x = \varinjlim\{S(U) \mid x \in U,\ U \text{ allowed}\}$. One can easily construct sheaves $S \neq 0$ such that all $S_x = 0$. This means that X does not have enough points to "separate" sheaves. We introduce a notion of generalized point.

(1.1) *Definitions*

A *geometric point* of X is a family p of allowed subsets of X such that (i) $X \in p$, $\phi \notin p$; (ii) if $Y_1, Y_2 \in p$ then $Y_1 \cap Y_2 \in p$; (iii) if $Y_1 \in p$ and $Y_1 \subset Y_2$ then $Y_2 \in p$; (iv) if $\{Y_1, \ldots, Y_n\}$ is an allowed covering of $Y \in p$ then $Y_i \in p$ for some i. The geometric point p is called *closed* if p is maximal among the collection of all geometric points (with respect to inclusion).

For any sheaf (of abelian groups always) S on X and any geometric point p the stalk S_p is defined as $\varinjlim\{S(U) \mid U \in p\}$.

(1.2) An ordinary point $x \in X$ can be regarded as geometric point $\tilde{x}$ in the following way: $\tilde{x} = \{U \mid x \in U,\ U \text{ allowed subset of } X\}$. Of course $S_{\tilde{x}} = S_x$. So we have indeed generalized the notion of point X. We show now a number of results on geometric points. The results imply that we have chosen the correct family of generalized points.

(1.2.1) Let S be a presheaf (of abelian groups) on X. The presheaf S^+ is defined by $S^+(Y) = \check{H}^0(Y, S) = \varinjlim \check{H}^0(\mathcal{U}, S)$, the limit is taken over all allowed coverings of Y. As usual, S^{++} is a sheaf and it is the sheaf associated with the presheaf S.

LEMMA: $S_p \xrightarrow{\sim} S_p^+$.

PROOF: Let $U \in p$ and let $\varphi_U : S(U) \to S^+(U)$ denote the canonical map. For any element $\xi \in \ker \varphi_U$ there is an allowed covering $\{U_1, \ldots, U_n\}$ of U such that the image of ξ in each $S(U_i)$ is equal to zero. Some U_i belongs to p. Hence the image of ξ in S_p is zero. This shows that $S_p \to S_p^+$ is injective. A similar argument shows that $S_p \to S_p^+$ is also surjective.

(1.2.2) LEMMA: *For any sheaf S on X, and any allowed $U \subset X$ the map $S(U) \to \Pi_{p \ni U} S_p$ is injective.*

PROOF: It suffices to show the lemma for $U = X$. The family $\tilde{X}$ of all geometric points of X is seen as a subset of $2^\tau = \{$all maps $\varphi : \tau \to \{0, 1\}\}$, where τ denotes the family of all allowed subsets of X. A geometric point $p \in \tilde{X}$ is identified with $\varphi : \tau \to \{0, 1\}$ given by $\varphi(A) = 1$ if and only if $A \in p$. The subset $\tilde{X}$ is closed in 2^τ with its product topology since $\tilde{X}$ consists of the $\varphi \in 2^\tau$ satisfying:

(a) if $A \subset B$ and $\varphi(A) = 1$ then $\varphi(B) = 1$.

(b) if $\varphi(A) = \varphi(B) = 1$ then $\varphi(A \cap B) = 1$.

(c) $\varphi(\phi) = 0$, $\varphi(X) = 1$.

(d) if $\varphi(A) = 1$ and $\{A_1, \ldots, A_n\}$ is an allowed covering of A then $\varphi(A_i) = 1$ for some i.

Let $\xi \in S(X)$ have image zero in each S_p. For each $p \in \tilde{X}$ there exists a $U \in p$ such that the image of ξ in $S(U)$ is zero. Put $\tilde{U} = \{q \in \tilde{X} \mid U \in q\}$. Then $\tilde{U}$ is an open neighbourhood of p in $\tilde{X}$ (with respect to its topology induced by $\tilde{X} \subset 2^\tau$). Since $\tilde{X}$ is compact, we find that $\tilde{X}$ is covered by finitely many such $\tilde{U}$; say $\tilde{X} = \tilde{U}_1 \cup \cdots \cup \tilde{U}_d$. Then also $X = U_1 \cup \cdots \cup U_d$ since any point of x corresponds to a geometric point $\tilde{x} \in \tilde{X}$. Since S is a sheaf we must have $\xi = 0$.

(1.2.3) LEMMA: *A sequence of sheaves $0 \to S' \to S \to S'' \to 0$ on X is exact if and only if for every geometric point p the sequence $0 \to S_p' \to S_p \to S_p'' \to 0$ is exact.*

PROOF: This follows in the usual way from (1.2.2) and (1.2.1).

(1.2.4) Let p be a geometric point. Define the sheaf S by: $S(U) = \mathbb{Z}$ if $U \in p$ and $S(U) = 0$ if $U \notin p$. Then one easily calculates $S_q = \mathbb{Z}$ if $q \subseteq p$ and $S_q = 0$ otherwise. This means that our collection of geometric point is not too big.

(1.3) *Description of the closed geometric points*

Let p be a geometric point on X. We associate with p a semi-norm $|\ |_p$ on $\mathcal{O}(X)$ as follows: $|f|_p = \inf\{\|f\|_U \mid U \in p\}$ where $\|f\|_U$ denotes $\max\{\|f(x)\| \mid x \in U\}$.

(1.3.1) LEMMA: $|\ |_p$ *has the properties*

(i) $|\ |_p \leq \|\ \|_X$.

(ii) $|f + g|_p \leq \max(|f|_p, |g|_p)$.

(iii) $|fg|_p = |f|_p|g|_p$.

(iv) $|\lambda|_p = |\lambda|$ *for every* $\lambda \in k$.

PROOF: Only (iii) is non-trivial. If $|f|_p = 0$ or $|g|_p = 0$ then certainly $|fg|_p = 0$ and (iii) follows. Let $|f|_p \neq 0$ and choose $\rho_1, \rho_2 \in \sqrt{|k^*|} = \{r > 0 \mid r^n \in |k^*| \text{ for some } n \geq 1\}$ such that $0 < \rho_1 < |f|_p < \rho_2$. Define the rational subsets

$$X_1 = \{x \in X \mid |f(x)| \leq \rho_1\}; \quad X_2 = \{x \in X \mid \rho_1 \leq |f(x)| \leq \rho_2\}$$

and

$$X_3 = \{x \in X \mid |f(x)| \geq \rho_2\}.$$

Certainly $X_1, X_3 \notin p$ and since $\{X_1, X_2, X_3\}$ is an allowed covering of X we have $X_2 \in p$. Similarly $Y_2 = \{x \in X \mid \rho_1' \leq |g(x)| \leq \rho_2'\} \in p$ where $0 < \rho_1' < |g|_p < \rho_2'$ and $\rho_2', \rho_2' \in \sqrt{|k^*|}$. Since $X_2 \cap Y_2 \in p$ we find

$$\rho_1|f|_p^{-1}\rho_1'|g|_p^{-1} \leq |fg|_p|f|_p^{-1}|g|_p^{-1} \leq \rho_2|f|_p^{-1}\rho_2'|g|_p^{-1}.$$

The equality (iii) follows because $\sqrt{|k^*|}$ is a dense subset of $\mathbb{R}_{>0}$.

(1.3.2) A map $|\ |: \mathcal{O}(X) \to \mathbb{R}_{\geq 0}$ satisfying the properties (i), (ii), (iii) and (iv) of (1.3.1) is called a *valuation* (of rank 1) on $\mathcal{O}(X)$. We can associate with every valuation $|\ |$ on $\mathcal{O}(X)$ a geometric point q given by: U belongs to q if there are $f_0, f_1, \ldots, f_n \in \mathcal{O}(X)$, without common zero's, such that

(i) $U \supseteq \{x \in X \mid |f_0(x)| \geq |f_i(x)| \text{ for } i = 1, \ldots, n\}$

(ii) $|f_0| \geq |f_i|$ $(i = 1, \ldots,)$ ($|\ |$ denotes the valuation on $\mathcal{O}(X)$).

(1.3.3) LEMMA: *Let $|\ |$ be a valuation (of rank 1) on $\mathcal{O}(X)$ and let q be the family of rational subsets of X defined above.*

(1) *q is a closed geometric point.*
(2) *If p is a geometric point such that $|\ | = |\ |_p$ then q is the unique closed geometric point containing p.*
(3) *Let p be any geometric point. The unique closed geometric point q containing p is given by: $R(f_0, f_1, \ldots, f_n) \in q$ if and only if for every $\rho \in \sqrt{|k^*|}$, $0 < \rho < 1$, the rational set $R(f_0, \rho f_1, \ldots, \rho f_n)$ belongs to p.*

PROOF:

(1) We write $R(f_0, f_1, \ldots, f_n)$ for the rational subset $\{x \in X \mid |f_0(x)| \geq |f_i(x)| \text{ for } i = 1, \ldots, n\}$. Suppose that $R(f_0, f_1, \ldots, f_n) = \emptyset$. Then $\mathcal{O}(X)\langle T_1, \ldots, T_n\rangle/(f_1 - f_0T_1, \ldots, f_n - f_0T_n)$ is 0 and hence $1 = \Sigma_{i=1}^n g_i(f_i - f_0T_i)$ for some $g_1, \ldots, g_n$ belonging to $\mathcal{O}(X)\langle T_1, \ldots, T_n\rangle$. Put $g_i = \Sigma\, g_{i,\alpha} T_1^{\alpha_1} \ldots T_n^{\alpha_n}$ with $g_{i,\alpha} \in \mathcal{O}(X)$ and $\lim|g_{i,\alpha}| = 0$.

After collecting the terms of total degree ℓ in $T_1, \ldots, T_n$ and substituting f_i for T_i ($i = 1, \ldots, n$), one obtains (for $\ell > 0$)

$$\sum_{|\alpha| = \ell} \sum_{i=1}^{n} f_i f_1^{\alpha_1} \ldots f_n^{\alpha_n} g_{i,\alpha} = f_0 \sum_{|\alpha| = \ell - 1} f_i f_1^{\alpha_1} \ldots f_n^{\alpha_n} g_{i,\alpha}.$$

Repeated use of this yields

$$f_0^\ell = \sum_{|\alpha| = \ell} f_1^{\alpha_1} \ldots f_n^{\alpha_n} \sum_{i=1}^{n} g_{i,\alpha} f_i \quad \text{for all } \ell \geq 1.$$

Since $\lim|g_{i,\alpha}| = 0$ one finds for $\ell \gg 0$:

$$f_0^\ell = \sum_{|\alpha|\ell} f_1^{\alpha_1} \ldots f_n^{\alpha_n} c_\alpha \quad \text{with all } |c_\alpha| < 1.$$

So the valuation $|\ |$ on $\mathcal{O}(X)$ must satisfy $|f_0| < |f_i|$ for some i. This proves that $\phi \notin q$. Certainly $X \in q$. So we have verified property (i) of the definition (1.1) of geometric point.

Property (ii) follows from $R(f_0, \ldots, f_n) \cap R(g_0, \ldots, g_m) = R(f_0g_0, \ldots, f_ng_m)$.

Property (iii) is trivial.

PROOF OF PROPERTY (iv): It suffices to consider a covering of X by rational sets $R_i = R(a_0^{(i)}, \ldots, a_n^{(i)})$ with $1 \leq i \leq \ell$. Consider all the products $a_{\lambda_1}^{(1)} \ldots a_{\lambda_\ell}^{(\ell)}$, with $0 \leq \lambda_i \leq n$, and denote this set by $\{f_0, f_1, \ldots, f_t\}$. Then X has a covering $\{R(f_i, f_0, \ldots, f_t) \mid 0 \leq i \leq t\}$ such that each R_i is a finite union of elements in that covering. Hence for some subset $J \subset \{0, \ldots, t\}$ the covering $\{R(f_i, f_0, \ldots, f_t) \mid i \in J\}$ refines the given covering $\{R_i \mid 1 \leq i \leq \ell\}$ of X. Suppose now that $0 \notin J$, then $|f_0(x)| \leq \max_{i \in J} |f_i(x)|$ holds for every $x \in X$. Then $R(f_i, f_0, \ldots, f_t) = R(f_i, f_1, \ldots, f_t)$ for all $i \in J$ and we can skip f_0 everywhere.

By induction on t we may assume that $J = \{0, \ldots, t\}$. Let $|f_i| \geq |f_j|$ for all j. Then $R(f_i, f_0, \ldots, f_t) \in q$. This proves (iv).

Finally we want to show that the geometric point q is maximal. Let $R(a_0, a_1, \ldots, a_n) \notin q$. Then $\max|a_i| = |a_1| > |a_0|$ can be assumed. Take $\rho_1, \rho_2 \in \sqrt{|k^*|}$ with $|a_0| < \rho_1 < \rho_2 < |a_1|$. Then $\{x \in X \mid |a_0(x)| \leq \rho_1$ and $|a_1(x)| \geq \rho_2\} = U \in q$. Clearly $U \cap R(a_0, a_1, \ldots, a_n) = \emptyset$. This means that q is maximal. The rest of statement (1) is easily verified.

(2) Let $R(f_0, f_1, \ldots, f_n)$ belong to p. Then clearly $|f_0|_p \geq |f_i|_p$ for $i = 1, \ldots, n$ and so $R(f_0, \ldots, f_n) \in q$. Hence $p \subseteq q$. Let q' be a closed geometric point containing p. Then for any $f \in \mathcal{O}(X)$ one has $|f|_p \geq |f|_{q'}$. Suppose that $|f|_p > \rho_1 > \rho_2 > |f|_{q'}$ is possible for some $f \in \mathcal{O}(X)$ and $\rho_1, \rho_2 \in \sqrt{|k^*|}$. Then X is the union of $\{x \in X \mid |f(x)| \geq \rho_1\}$ and $\{x \in X \mid |f(x)| \leq \rho_1\}$. From the definition of geometric point it follows that $\{x \in X \mid |f(x)| \geq \rho_1\} \in p \subseteq q'$. Similarly $\{x \in X \mid |f(x)| \leq \rho_2\} \in q'$. One obtains the contradiction $\phi = \{x \in X \mid |f(x)| \geq \rho_1\} \cap \{x \in X \mid |f(x)| \leq \rho_2\} \in q'$. Thus $\|_p = \|_q = \|_{q'}$ and as before one finds $q' \subseteq q$. Hence $q' = q$.

(3) This follows easily from the definition of the closed geometric point associated with $\|_p$. We note that the notation $R(f_0, \rho f_1, \ldots, \rho f_n)$ is slightly incorrect. $R(f_0, \rho f_1, \ldots, \rho f_n)$ denotes the rational set $\{x \in X \mid |f_0(x)| \geq \rho |f_i(x)|$ for $i = 1, \ldots, n\}$.

(1.3.3) COROLLARY: *There is a one-to-one correspondence between the closed geometric points on X and the valuations* (*of rank* 1) *on* $\mathcal{O}(X)$.

REMARK: Non-maximal geometric points of X correspond to valuations on $\mathcal{O}(X)$ with rank >1. The description is more complicated and we will not use it in the sequel.

(1.4) *Constructible sheaves*

(1.4.1) DEFINITIONS: Let U be a rational subset of the affinoid space X, given by $U = \{x \in X \mid |f_0(x)| \geq |f_i(x)|, \quad i = 1, \ldots, n\} = R(f_0, \ldots, f_n)$. A *neighbourhood* U' of U in X is a rational subset which contains $R(f_0, \rho f_1, \ldots, \rho f_n) = \{x \in X \mid |f_0(x)| \geq \rho |f_i(x)|$ for $i = 1, \ldots, n\}$ for some $\rho \in \sqrt{|k^*|}$, $\rho < 1$.

(1.4.2) The definition above does not depend on the chosen representation of $U = R(f_0, \ldots, f_n)$. This follows from:

LEMMA: *Let $R(a_0, \ldots, a_n) = R(b_0, \ldots, b_m)$. Then for any $\rho \in \sqrt{|k^*|}$, $\rho < 1$ there exists a $\rho^* \in \sqrt{|k^*|}$, $\rho^* < 1$ with*

$$R(a_0, \rho a_1, \ldots, \rho a_n) \supseteq R(b_0, \rho^* b_1, \ldots, \rho^* b_m).$$

PROOF: For ρ and ρ^* close to 1, $R(a_0, \rho a_1, \ldots, \rho a_n)$ and $R(b_0, \rho^* b_1, \ldots, \rho^* b_m)$ are contained in some rational subset of X where a_0 and b_0 are invertible. After replacing X by this rational subset we may suppose $a_0 = b_0 = 1$.

Let a denote any of the a_i $(i = 1, \ldots, n)$. The rational subset $Z = \{x \in X \mid |a(x)| \geq \rho^{-1}\}$ satisfies $Z \cap R(1, b_1, \ldots, b_m) = \emptyset$. As in the proof of lemma (1.3.2) we find

$$1 = \sum_{|\alpha| = \ell} c_\alpha b_1^{\alpha_1} \ldots b_m^{\alpha_m} \quad \text{for some } \ell \gg 0 \quad \text{and}$$

$$c_\alpha \in \mathcal{O}(Z) \quad \text{and} \quad \|c_\alpha\| < 1.$$

This implies that $Z \cap R(1, \rho^* b_1, \ldots, \rho^* b_m) = \emptyset$ for ρ^* close enough to 1. In another formulation $|a(x)| \leq \rho^{-1}$ for $x \in R(1, \rho^* b_1^*, \ldots, \rho^* b_m)$ and ρ^* close to 1. Since a can be any of the a_i $(i = 1, \ldots, n)$ it follows that

$$R(a_0, \rho a_1, \ldots, \rho a_n) \supseteq R(b_0, \rho^* b_1, \ldots, \rho^* b_m).$$

(1.4.3) We write $U \subset\subset U'$ (or better $U \underset{X}{\subset\subset} U'$) to denote that U' is a neighbourhood of U within X. A (pre-)sheaf S of abelian groups on X is said to be *constructible* if $S(U) = \varinjlim \{S(U') \mid U \subset\subset U'\}$ for every allowed $U \subset X$.

We gather now some properties of sheaves. First of all we prove that Čech-cohomology coincides with sheaf-cohomology on reasonable analytic spaces.

(1.4.4) PROPOSITION: *Let X be an analytic space over k which has an allowed covering $(X_i)_{i \in I}$ such that*

(i) *every X_i is an affinoid space.*

(ii) *$X_i \cap X_j$ is an affinoid subspace of X_i.*

(iii) *I is at most countable.*

Then for every sheaf S on X the canonical maps $\check{H}^i(X, S) \to H^i(X, S)$ are isomorphisms.

PROOF: We consider the most difficult case: $I = \mathbb{N}$. A geometric point p of X is defined to be a geometric point of some X_n. Let for any geometric point p an injective group $A(p)$ be given. Define the sheaf G on X by $G(U) = \Pi_{p \ni U} A(p)$. The sheaf G on X is easily seen to be an injective sheaf. Moreover for any sheaf S on X there is a choice of the groups $A(p)$ such that S is a subsheaf of G. Define the presheaf P on X be the exactness of $0 \to S(U) \to G(U) \to P(U) \to 0$ for every allowed U in X. The map $P(U) \to P^+(U) = \check{H}^0(U, P)$ is seen to be injective. This implies that $P^+ = P^{++}$ and P^+ is the sheaf associated with P. Consider the exact sequence of presheaves $0 \to P \to P^+ \to K \to 0$. Then $K^+ = 0$. Suppose that we have shown that $\check{H}^i(X, K) = 0$ for all $i \geq 0$. Then $\check{H}^i(X, P) = \check{H}^i(X, P^+)$ for all i. So we find an exact sequence $0 \to \check{H}^0(X, S) \to \check{H}^0(X, G) \to \check{H}^0(X, P^+) \to \check{H}^1(X, S) \to 0$ and isomorphisms $\check{H}^i(X, P^+) \simeq \check{H}^{i+1}(X, S)$ for $i \geq 1$. A comparison with the sheaf cohomology yields $\check{H}^1(X, S) \cong H^1(X, S)$. Induction on i proves that all $\check{H}^i(X, S) \cong H^i(X, S)$. So the proposition follows from the following lemma.

(1.4.5) LEMMA: *Let X be as in* (1.4.4) *and let K be a presheaf on X with $K^+ = 0$. Then $\check{H}^i(X, K) = 0$ for all $i \geq 0$.*

PROOF: Let ξ be an element (of degree $k-1$) in the Čech-complex of K with respect to an allowed covering $\mathscr{X}$ of X. It suffices to find a refinement $\mathscr{X}''$ of $\mathscr{X}$ in which ξ is mapped to zero.

The covering $\mathscr{X}$ is refined by $\cap_{n \geq 1} (\mathscr{X} \cap X_n)$ and each $\mathscr{X} \cap X_n$ admits a finite subcovering of X_n. Hence $\mathscr{X}$ can be refined by some allowed covering $\mathscr{X}^1 = (X_n^1)_{n \geq 1}$ such that each X_n^1 is an affinoid subspace of X. The element ξ is mapped to $\xi' \in \Pi K(X_{i_1}^1 \cap \cdots \cap X_{i_\ell}^1)$. For each $(i_1, \ldots, i_\ell)$ there exists a covering $\mathscr{Y}_{i_1, \ldots, i_\ell}$ of $X_{i_1}^1 \cap \cdots \cap X_{i_\ell}^1$ such that the component of ξ' in $K(X_{i_1}^1 \cap \cdots \cap X_{i_\ell}^1)$ is mapped to zero and the sets of $\mathscr{Y}_{i_1, \ldots, i_\ell}$. We construct $\mathscr{X}''$ by replacing each X_n^1 by the elements of a suitable finite covering $\mathscr{X}_n$ of X_n^1. The covering $\mathscr{X}_n$ of X_n^1 is chosen such that $\mathscr{X}_n \cap X_{i_1}^1 \cap \cdots \cap X_{i_\ell}^1$ is finer than $\mathscr{Y}_{i_1, \ldots, i_\ell}$ for all $i_1, \ldots, i_\ell$ with $\max(i_1, \ldots, i_\ell) = n$. This is possible since one can easily show the following: Let U be an affinoid subspace of an affinoid space V and let $\mathscr{Y}$ be an allowed covering of U. Then there exists an allowed covering $\mathscr{Y}'$ of V such that $\mathscr{Y}' \cap U$ is finer than $\mathscr{Y}$.

Finally, one easily verifies that the image of ξ' in the Čechcomplex with respect to $\mathscr{X}''$ is zero.

(1.4.6) *If S is constructible then so is S^+.*

(1.4.7) *If S is a constructible sheaf, then $U \mapsto H^i(U, S)$ are constructible presheaves.*

(1.4.8) *If $u: S_1 \to S_2$ is a morphism of constructible sheaves, then* $\ker(u)$, $\operatorname{coker}(u)$, $\operatorname{im}(u)$ *are constructible sheaves.*

(1.4.9) *If $\varphi: X \to Y$ is a morphism of affinoid spaces and if S is a constructible sheaf on Y then $\varphi^* S$ is constructible.*

(1.4.10) *Let $\varphi: X \to Y$ be a morphism of affinoid spaces and let S be a constructible sheaf on X then $\varphi_* S$ and all $R^i\varphi_* S$ are constructible sheaves.*

(1.4.11) *Let S be a constructible sheaf on an affinoid space X and let p be a geometric point of X. Then $S_p \xrightarrow{\sim} S_q$ where q is the closed geometric point with $|\ |_p = |\ |_q$.*

(1.4.12) *For every constructible sheaf S on an affinoid space X there is a sheaf G such that*

(i) *S is a subsheaf of G.*

(ii) *$H^i(U, G) = 0$ for $i \geq 1$ and any allowed $U \subset X$.*

(iii) *G is constructible.*

Proofs: Most of the proofs are straight forward (and somewhat tedious to carry out). We mention only some steps in the proofs.

(1.4.7) It is rather obvious that $U \mapsto H^i(U, S)$ is a constructible presheaf. So (1.4.7) follows from (1.4.4).

(1.4.10) $R^i\varphi_* S$ is the sheaf associated with the presheaf $U \mapsto H^i(\varphi^{-1}U, S)$. Those presheaves are constructible according to (1.4.7). Using (1.4.6) one finds (1.4.10).

(1.4.11) Let $U \in q$ then it can easily be verified that any $U' \supset\supset U$ belongs to p.

(1.4.12) S is a subsheaf of the sheaf $\mathscr{H}$ defined by $\mathscr{H}(U) = \Pi_{p \ni U} S_p$. The sheaf $\mathscr{H}$ has trivial cohomology, but is in general not constructible. The sheaf G defined by $G(U) = \varinjlim\{\mathscr{H}(U') \mid U' \supset\supset U\}$, contains S as a subsheaf, and is constructible. It is easily seen that G has trivial cohomology.

(1.4.13) For later use we insert the following:

Lemma: *Let X be an affinoid space of dimension n and let S be a sheaf on X. Then $H^i(X, S) = 0$ for $i > n$.*

Sketch of the Proof: According to (1.4.4), $H^i(X, S) \cong \check{H}^i(X, S)$. Any allowed covering $\mathscr{U}'$ of X can be refined to a pure covering $\mathscr{U}$ (see [4], p. 116, 117). Let $r = r_{\mathscr{U}}: X \to \bar{X}_{\mathscr{U}}$ denote the reduction w.r.t. this pure covering. It follows that $H^i(X, S)$ is isomorphic to $\varinjlim \check{H}^i(\bar{X}_{\mathscr{U}}, r_* S)$ where the limit is taken over all pure coverings $\mathscr{U}$ of

X and where $\check{H}^i(\bar{X}_{\mathcal{U}}, r_*S)$ denotes the Čech-cohomology groups of the sheaf r_*S on $\bar{X}_{\mathcal{U}}$ provided with the Zariski-topology. The affine variety $\bar{X}_{\mathcal{U}}$ over $\bar{k}$ (i.e. the residue field of k) has dimension n and it is well known that *)$H^i(\bar{X}_{\mathcal{U}}, -) = 0$ for $i > n$. In general $\check{H}^i(\bar{X}_{\mathcal{U}}, -) \neq H^i(\bar{X}_{\mathcal{U}}, -)$, but a proof similar to that of *) yields also $\check{H}^i(\bar{X}_{\mathcal{U}}, -) = 0$ for $i > n$. This proves the lemma.

(1.5) *Examples*

(1.5.1) *Constant sheaves.*

Let A be an abelian group. The constant presheaf P associated with A is defined by $P(U) = A$ for every $U \neq \emptyset$, U allowed in X. The sheaf associated with P is denoted by A_X. It is a constructible sheaf according to (1.4.4).

(1.5.2) Let $0 < r < s \leq \infty$, r a real number and s a real number or ∞. We define some sheaves on X by:

$$\mathcal{O}_X(s)(U) = \mathcal{O}(s)(U) = \{f \in \mathcal{O}(U) \mid |f(x)| < s \text{ for all } x \in U\}.$$

The sheaf $\mathcal{O}_X(r, s) = \mathcal{O}(r, s)$ is defined by the exactness of the sequence $0 \to \mathcal{O}_X(r) \to \mathcal{O}_X(s) \to \mathcal{O}_X(r, s) \to 0$.

LEMMA: *$\mathcal{O}_X(r, s)$ is a constructible sheaf.*

PROOF: Let P be the presheaf defined by the exactness of $0 \to \mathcal{O}(r)(U) \to \mathcal{O}(s)(U) \to P(U) \to 0$ for every allowed U in X. Let $U = R(f_0, f_1, \ldots, f_n)$ and let $\xi \in P(U)$. Then ξ is the image of an element

$$g = \sum a_\alpha \left(\frac{f_1}{f_0}\right)^{\alpha_1} \cdots \left(\frac{f_n}{f_0}\right)^{\alpha_n};$$

$a_\alpha \in \mathcal{O}(X)$ and $\lim \|a_\alpha\| = 0$, with $|g(x)| < s$ for all $x \in U$. We may replace g by a finite sum

$$h = \sum_{|\alpha| \leq \ell} a \left(\frac{f_1}{f_0}\right)^{\alpha_1} \cdots \left(\frac{f_n}{f_0}\right)^{\alpha_n}.$$

Still h has image ξ in $P(U)$, h is defined on some neighbourhood of U, $|h(x)| < s$ holds on some neighbourhood of U. (Compare the proof of (1.4.2)).

This shows that the map $\varinjlim\{P(U') \mid U' \supset\supset U\} \to P(U)$ is surjective. Further let $U' \supset\supset U$ and let $\xi \in P(U')$ be represented by an element $f \in \mathcal{O}(s)(U')$. If ξ has image zero in $P(U)$ then $|f(x)| < r$ for all $x \in U$. This property holds then also for some neighbourhood $U'' \supset\supset U$. Hence the image of ξ in $P(U' \cap U'') = 0$. So the map $\varinjlim\{P(U') \mid U' \supset\supset U\} \to P(U)$ is also injective.

(1.5.3) $\mathcal{O}^*$ (or $\mathcal{O}_X^*$) denotes the sheaf given by $\mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid f(u) \neq 0$ for all $u \in U\}$. This sheaf is not constructible.

Its subsheaf $\mathcal{O}^*(1)$, given by $\mathcal{O}^*(1)(U) = \{f \in \mathcal{O}(U) \mid |f(u) - 1| < 1$ for all $u \in U\}$ is not constructible. We define the sheaf S by the exact sequence $0 \to \mathcal{O}^*(1) \to \mathcal{O}^* \to S \to 0$. As in (1.5.2) one shows that S is a constructible sheaf. The sheaf $S = S_X$ has a subsheaf the constant sheaf A_X where $A = k^*/\{1 + h \mid h \in k, |h| < 1\}$. The sheaf $T = T_X$ defined by the exact sequence

$$0 \to A_x \to S_X \to T_X \to 0.$$

is again a constructible sheaf.

REMARK: Our main object in §3 will be to show that the sheaves in (1.5.1) and (1.5.3) have trivial cohomology on a polydisk. For dimension 1 this is not too difficult to show. For dimension >1, we will use a "Base-Change theorem" in order to make induction on the dimension. This base-change theorem seems only to work for closed geometric points. This is the reason why we have introduced a family of sheaves, the constructible sheaves, which are determined by their stalks at closed geometric points.

§2. Base change

Let $\varphi : X \to Y$ be a morphism of affinoid spaces over k and let p be a closed geometric point of Y. We associate with p a complete valued field extension K_p of k in the following way: $|\ |_p$ on $\mathcal{O}(X)$ induces a valuation (in the ordinary sense) on the field of fractions L_p of $\mathcal{O}(X)/\{f \in \mathcal{O}(X) \mid |f|_p = 0\}$. The completion of L_p is denoted by K_p..

(2.1) LEMMA: *$\mathcal{O}(X) \hat{\otimes}_{\mathcal{O}(Y)} K_p$ is an affinoid algebra over K_p.*

PROOF: $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} K_p$ is given the usual norm of the tensor product. The completion with respect to this norm is denoted by $\mathcal{O}(X) \hat{\otimes}_{\mathcal{O}(Y)} K_p$. The map $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ can be extended to a surjective map $\psi : \mathcal{O}(Y)\langle T_1, \ldots, T_n\rangle \to \mathcal{O}(X)$. We find an induced surjective

K_p-linear map

$$\psi \hat{\otimes} \mathrm{id}_{K_p} : \mathcal{O}(Y)\langle T_1, \ldots, T_n\rangle \hat{\otimes}_{\mathcal{O}(Y)} K_p = K_p\langle T_1, \ldots, T_n\rangle \to \mathcal{O}(X) \hat{\otimes}_{\mathcal{O}(Y)} K_p.$$

As a consequence $\mathcal{O}(X) \hat{\otimes}_{\mathcal{O}(Y)} K_p$ is an affinoid algebra over K_p.

(2.2) DEFINITIONS: We write $Z = X \times_Y p$ for the K_p-affinoid space with algebra $\mathcal{O}(X) \hat{\otimes}_{\mathcal{O}(Y)} K_p$. The natural map $\mathcal{O}(X) \to \mathcal{O}(Z) = \mathcal{O}(X) \hat{\otimes}_{\mathcal{O}(Y)} K_p$ does not induce a morphism $Z \to X$ since the image of a point in Z would, in general, be a geometric point of X. This problem is solved by replacing X (and likewise Y, Z) by $\hat{X}$ = the family of *closed* geometric points of X. According to corollary (1.3.2) we find maps $\alpha : \hat{Z} \to \hat{X}$ and $\tilde{\varphi} : \hat{X} \to \hat{Y}$. The set $\hat{X}$ (and in the same manner $\hat{Y}, \hat{Z}$) is given a topology and a Grothendieck topology as follows:

(i) For every rational domain $U \subset X$, we denote $\{p \in \hat{X} \mid U \in p\}$ by $\hat{U}$.

(ii) $\{\hat{U} \mid U \subset X,\ U \text{ rational}\}$ is a base for the topology on $\hat{X}$.

(iii) $\{\hat{U} \mid U \subset X,\ U \text{ rational}\}$ is the family of the allowed subsets of $\hat{X}$.

(iv) $\{\hat{U}_i\}_{i \in I}$ is an allowed covering of $\hat{U}$ if I is finite and $\cup \hat{U}_i = \hat{U}$.

Since X and $\hat{X}$ have the "same" Grothendieck topology they also have the same collection of sheaves and pre-sheaves. We will in the sequel identify any sheaf S on X with the corresponding sheaf on $\hat{X}$. Our aim is to prove the following result.

(2.3) THEOREM (Base Change): *Let $\varphi : X \to Y$ be a morphism of affinoid spaces over k, and let p be a closed geometric point of Y. Let Z denote the affinoid space over K_p, defined in* (2.2) (*i.e.* $Z = X \times p$). *Then we have a diagram*:

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{\varphi}} & \hat{Y}. \\ \uparrow{\scriptstyle\alpha} & & \\ \hat{Z} & & \end{array}$$

For any sheaf S on $\hat{X}$, there are canonical maps

$$H^i(Z, \alpha^* S) \leftarrow (R^i \varphi_* S)_p, \quad (i = 0, 1, \ldots).$$

If S is a constructible sheaf then the maps are isomorphisms.

PROOF: The sheaf $\alpha^*\bar{S}$ is associated with the presheaf P, $P(\hat{A}) = \varinjlim\{S(\hat{U}) \mid \alpha^{-1}(\hat{U}) \supset \hat{A}\}$. The natural map $(\varphi_* S)_p = \varinjlim_{V \in p} S(\varphi^{-1}V) \to P(Z)$ extends to a natural map $(\varphi_* S)_p \to H^0(Z, \alpha^* S)$. Using a suitable resolution of S (see also (2.8)) one finds canonical morphisms $(R^i\varphi_* S)_p \to H^i(Z, \alpha^* S)$.

We start the proof with some lemma's.

(2.4) LEMMA: *Let V be a rational subset of X. Then $\alpha^{-1}(\hat{V}) = \hat{A}$ for some rational subset A of Z. Moreover $\mathcal{O}_Z(A) \cong \mathcal{O}_X(V) \hat{\otimes}_{\mathcal{O}(Y)} K_p$.*

PROOF: Put $V = R(f_0, f_1, \ldots, f_n)$ and let $\tilde{f}_0, \ldots, \tilde{f}_n$ denote the images of the elements f_i in $\mathcal{O}(Z)$. Put $A = R(\tilde{f}_0, \ldots, \tilde{f}_n)$. Then $\alpha^{-1}(\hat{V}) = \hat{A}$;

$$\mathcal{O}(V) = \mathcal{O}(X)\langle T_1, \ldots, T_n\rangle/(f_1 - f_0T_1, \ldots, f_n - f_0T_n)$$

and

$$\mathcal{O}(A) = \mathcal{O}(Z)\langle T_1 \ldots T_n\rangle/(\tilde{f}_1 - \tilde{f}_0T_1, \ldots, \tilde{f}_n - \tilde{f}_0T_n).$$

Since $\mathcal{O}(X)\langle T_1, \ldots, T_n\rangle \hat{\otimes}_{\mathcal{O}(Y)} K_p \simeq \mathcal{O}(Z)\langle T_1, \ldots, T_n\rangle$ the result follows.

(2.5) LEMMA: *Let $f \in \mathcal{O}(X)$ have image $\tilde{f} \in \mathcal{O}(Z)$ such that $\max\{|\tilde{f}(z)| \mid z \in Z\} < 1$. Then there exists a $V \in p$ such that $\|f\|_{\varphi^{-1}V} = \max\{|f(x)| \mid x \in \varphi^{-1}V\} < 1$.*

PROOF: Let the surjective map $\mathcal{O}(Y)\langle T_1, \ldots, T_n\rangle \xrightarrow{\alpha} \mathcal{O}(X)$ induce the norm on $\mathcal{O}(X)$. The kernel of α is an ideal generated by some elements $f_1, \ldots, f_s$. For any $V \in p$ we have an induced surjective map $\alpha_V : \mathcal{O}(V)\langle T_1, \ldots, T_n\rangle \to \mathcal{O}(\varphi^{-1}V)$ and a surjective map $\alpha_p : K_p\langle T_1, \ldots, T_n\rangle \to \mathcal{O}(Z)$. For each of those maps the kernel is an ideal generated by $f_1, \ldots, f_s$.

On $K_p\langle T_1, \ldots, T_n\rangle$ we use the spectral norm as norm and on $\mathcal{O}(Z)$ the norm induced by the map α_p. The spectral norm of $\tilde{f}$ on Z is <1. So a suitable power $\tilde{f}^N$ has norm <1 and is the image of some $G \in K_p\langle T_1, \ldots, T_n\rangle$ with $\|G\| < 1$. Take an element F with $\alpha(F) = f^N$ and let F_p denote the image of F in $K_p\langle T_1, \ldots, T_n\rangle$ and write F_V for the image $\mathcal{O}(V)\langle T_1, \ldots, T_n\rangle$.

Then

$$(*) \qquad F_p - G = \sum_{i=1}^{s} f_i \sum_{\alpha} a_{\alpha,i} T_1^{\alpha_1} \ldots T_n^{\alpha_n}$$

with $a_{\alpha,i} \in K_p$ and $\varinjlim |a_{\alpha,i}| = 0$.

We note that the image of $\varinjlim\{\mathcal{O}(V) \mid V \in p\}$ in K_p is a dense subfield

M_p. Further for any $m \in M_p$ and any $\epsilon > 1$ there exists an element $V \in p$ and a $\xi \in \mathcal{O}(V)$ with image m and such that ξ has spectral *norm* $\leq \epsilon |m|_p$ on V.

In the expression (*) we truncate G and the infinite sums. The remaining coefficients in K_p are approximated by elements in M_p and then replaced by elements in a suitable $\mathcal{O}(V)$. This yields

$$F_V = \sum_{i=1}^{s} f_i \sum_{|\alpha|\leq \ell} b_{i,\alpha} T_1^{\alpha_1} \dots T_n^{\alpha_n} + \sum r_\alpha T_1^{\alpha_1} \dots T_n^{\alpha_n}$$

with all $b_{i,\alpha}$, $r_\alpha \in \mathcal{O}(V)$ and all r_α with spectral norm <1 on V. The image $\alpha_V(F_V) \in \mathcal{O}(\varphi^{-1}V)$ has spectral norm <1 and is equal to the restriction of f^N on $\varphi^{-1}V$. It follows that the spectral norm of f on $\varphi^{-1}V$ is less than one.

(2.6) Key Lemma: *Let A be a rational subset of Z. There exists a rational subset B of X such that*

(i) $\alpha^{-1}(\hat{B}) = \hat{A}$.

(ii) *For any rational $U \subset X$ with $\alpha^{-1}\hat{U} \supset \hat{A}$ and for any $U' \supset\supset U$, there exists a $V \in p$ with $U' \supset \varphi^{-1}(V) \cap B$.*

Proof: There are $f_0, \dots, f_{n-1} \in \mathcal{O}(Z)$ with $A = R(f_0, \dots, f_{n-1})$. A small change of the $f_0, \dots, f_{n-1}$ does not affect A. We may then assume that each f_i is the image in $\mathcal{O}(Z)$ of an element $g_i \in \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} L_p$. There exists an element $h \in \mathcal{O}(Y)$ with $|h|_p \neq 0$ such that hg_i is the image of some element $F_i \in \mathcal{O}(X)$. The elements $F_0, \dots, F_{n-1} \in \mathcal{O}(X)$ may have a common zero in X. Take $F_n = \rho \in k$ with $|\rho| > 0$ and small. Let $\tilde{F}_0, \dots, \tilde{F}_n$ denote the images of $F_0, \dots, F_n$ in $\mathcal{O}(Z)$. Then $A = R(\tilde{F}_0, \dots, \tilde{F}_n)$ and $B = R(F_0, \dots, F_n)$ satisfies $\alpha^{-1}(\hat{B}) = \hat{A}$. This proves (i).

Using lemma (2.4) we may replace X by B in the proof of (ii). So we may assume that $X = B$ and $A = Z$. Let $U = R(f_0, \dots, f_n)$ satisfy $\alpha^{-1}(\hat{U}) \supset \hat{Z}$. The image $\tilde{f}_0$ of f_0 in $\mathcal{O}(Z)$ has an inverse h. This element can be approximated by a suitable $h' \in \mathcal{O}(\varphi^{-1}V)$. So $h'f_0 = 1 + \delta$ holds on $\varphi^{-1}V$, where $\delta \in \mathcal{O}(\varphi^{-1}V)$ has a spectral norm <1 on Z. Using (2.5) one obtains a $V' \in p$ with $V' \subset V$ such that δ has also on V' a spectral norm smaller than one. This means that f_0 is invertible on V'. Upon replacing X by V' we may suppose that $f_0 = 1$. Take some $\pi \in k$, $0 < |\pi| < 1$, and take an integer $N \geq 1$. The elements (or rather their images in $\mathcal{O}(Z)$) $\pi f_1^N, \dots, \pi f_n^N$ have spectral norms <1 on Z. So by (2.5) we find a $V \in p$ such that $\pi f_1^N, \dots, \pi f_1^N$ have spectral norms <1 on $\varphi^{-1}V$. This shows that any neighbourhood of $R(1, f_1, \dots, f_N)$ contains some $\varphi^{-1}V$.

(2.7) LEMMA: *For any constructible sheaf S on X, the presheaf P on $\hat{Z}$ is given by $P(\hat{A}) = \varinjlim\{S(\hat{U}) \mid \alpha^{-1}\hat{U} \supset \hat{A}\}$ is a constructible sheaf.*

PROOF: P is clearly a constructible presheaf. Using (2.4) one sees that it suffices to show that $P(\hat{Z}) = \check{H}^0(\mathcal{A}, P)$ where $\mathcal{A} = \{\hat{A}_1, \ldots, \hat{A}_n\}$ is a finite covering by rational subsets. Choose $B_1, \ldots, B_n$ as in (2.6). For any $(i_0, \ldots, i_p)$ one gets from (2.6) $P(\hat{A}_{i_0} \cap \cdots \cap \hat{A}_{i_p}) = \varinjlim S(B_{i_0} \cap \cdots \cap B_{i_\ell} \cap \varphi^{-1}V)$ where the limit is taken over all $V \in p$. For V small enough, $V \in p$ the sets $\{B_1 \cap \varphi^{-1}V, \ldots, B_n \cap \varphi^1 V\}$ form an allowed covering of $\varphi^{-1}V$. So $0 \to S(\varphi^{-1}V) \to \bigoplus S(B_i \cap \varphi^{-1}V) \to \bigoplus_{i<j} S(B_i \cap B_j \cap \varphi^{-1}V)$ is exact. Since $\varinjlim$ preserves exactness one finds indeed $P(\hat{Z}) = \check{H}^0(\mathcal{A}, P)$.

(2.8) END OF THE PROOF OF THEOREM (2.3): According to (2.7) the sheaf α^*S is equal to P. Using (2.6) and the constructibility of P one obtains $P(\hat{Z}) = \varinjlim\{S(\varphi^{-1}V) \mid V \in p\}$ and the last group equals $(\varphi_*S)_p$. So we have shown (2.3) for $i = 0$.

Following (1.4.12) we have an exact sequence $0 \to S \to G \to S' \to 0$, with G and S' constructible; G with trivial cohomology on every allowed subset of X.

Then $0 \to \varphi_*S \to \varphi_*G \to \varphi_*S' \to R^1\varphi_*S \to 0$ is exact and $R^i\varphi_*S' \simeq R^{i+1}\varphi_*S$. The same holds for their stalks at p. On Z (or $\hat{Z}$) we have an exact sequence $0 \to \alpha^*S \to \alpha^*G \to \alpha^*S' \to 0$. The sheaf α^*G on $\hat{Z}$ is according to (2.7) equal to the presheaf $P : \hat{A} \mapsto \varinjlim\{G(\hat{U}) \mid \alpha^{-1}\hat{U} \supset \hat{A}\}$. Following the argument of (2.7) one sees that P has trivial cohomology (on any allowed subset of $\hat{Z}$).

Hence $0 \to H^0(Z, \alpha^*S) \to H^0(\hat{Z}, \alpha^*G) \to H^0(\hat{Z}, \alpha^*S') \to \hat{H}^1(\hat{Z}, \alpha^*S) \to 0$ is exact. This implies $H^1(Z, \alpha^*S) \cong (R^1\varphi_*S)_p$. Induction on $i \geq 1$ ends the proof.

(2.9) COROLLARY: *Let $\varphi : X \to V$ be a morphism of affinoid spaces over k and let S be a sheaf on X.*

(1) *If $R^i\varphi_*S = 0$ for all $i \neq 0$ then $H^i(X, S) \cong H^i(Y, \varphi_*S)$ for all i.*

(2) *If S is constructible and if $H^i(X \times p, \alpha^*S) = 0$ for $i \neq 0$ and every closed geometric point p of Y, then*

$$H^i(X, S) \cong H^i(Y, \varphi_*S) \quad \textit{for all } i.$$

PROOF: A sheaf G on X is called flabby if the map $G(X) \to G(U)$ is surjective for every allowed $U \subset X$. A flabby sheaf has trivial cohomology. Moreover every sheaf S is a subsheaf of a flabby sheaf. Indeed the sheaf $U \mapsto \Pi S_p$ (where the product is taken over all geometric points p, which contain U) is flabby and contains S as a subsheaf.

Using a flabby resolution of S, the result (1) follows Part (2) follows from (1), theorem (2.3) and the fact that $\varphi_* S$ is a constructible sheaf.

§3. Cohomology on polydisks

In this section we study the cohomology groups of the sheaves A_X, $\mathcal{O}_X(r)$ etc. on polydisks and on some generalizations of polydisks. We will use §2, in particular (2.3) to decrease the dimension. Even if one starts with an algebraically closed field one will find in the induction process complete fields which are not algebraically closed. So we have to consider the sheaves mentioned above on E_k^1 or the projective line $\mathbb{P}_k^1$ over a complete field k which is not necessarily algebraically closed. We start with an investigation on E_k^1 and $\mathbb{P}_k^1$.

(3.1) *Dimension one*

We denote $\mathrm{Sp}(k\langle T\rangle)$ by E_k^1. A rational subset of E_k^1 given by inequalities: $|t-a|\leq\rho$ and $|t-a_i|\geq\rho_i$ for $i=1,\ldots,s$, in which we assume that

$$a, a_1, \ldots, a_s \in k;\quad \rho, \rho_1, \ldots, \rho_s \in |k^*|;\quad |a|\leq 1;\quad \rho\leq 1;$$
$$|a-a_i|\leq\rho;\quad \rho_i\leq\rho;\quad |a_i-a_j|\geq\rho_i\quad \text{for } i\neq j,$$

is called a *standard subset* of E_k^1. We write $\mathrm{st}(k)$ for the family of all standard subsets and we write $\tau(k)$ for the family of all finite unions of elements in $\mathrm{st}(k)$. We observe that for any $R_1, R_2 \in \mathrm{st}(k)$ either $R_1\cap R_2\in\mathrm{st}(k)$ or $R_1\cap R_2=\emptyset$. If $R_1\cap R_2\neq\emptyset$ then also $R_1\cup R_2\in\mathrm{st}(k)$. So any $R\in\tau(k)$, $R\neq\emptyset$, can uniquely be written as a union $R_1\cup\cdots\cup R_s$ with $R_i\in\mathrm{st}(k)$ and $R_i\cap R_j=\emptyset$ for $i\neq j$.

If k happens to be algebraically closed then it is well known that $\tau(k)$ is the family of all rational subsets of E_k^1 and that $\mathrm{st}(k)$ is the family of all connected rational subsets.

For a field k which is not algebraically closed the situation is more complicated. We consider a finite Galois-extension K of k with Galois group G. It gives a finite map of k-affinoid spaces $\pi: E_K^1\to E_k^1$ and there is a natural action of G on E_K^1 considered as a k-affinoid space. Any $\sigma\in G$ permutes the elements of $\mathrm{st}(K)$ and also permutes the elements of $\tau(K)$.

For any $R\in\tau(K)$, $R\neq\emptyset$, which is invariant under the action of G, we can consider $R'=\pi(R)$. This is a rational subset of E_k^1 and its algebra of holomorphic functions $\mathcal{O}(R')$ is equal to $\mathcal{O}(R)^G$. Further $\pi^{-1}(R') = R$.

Conversely, for any rational subset R' of E^1_k there exists a finite Galois-extension K of k such that $R = \pi^{-1}(R')$ is a G-invariant element of $\tau(K)$. Moreover $\mathcal{O}(R) = \mathcal{O}(R') \otimes_k K$.

Let R, a G-invariant element of $\tau(K)$, decompose into the disjoint union $R_1 \cup \cdots \cup R_s$ of elements of $\mathrm{st}(K)$. Then G permutes $\{R_1, \ldots, R_s\}$. The rational subset $R' = \pi(R)$ of E^1_k is connected if and only if G acts transitively on $\{R_1, \ldots, R_s\}$. The proof of the statements above is left to the reader.

(3.2) Lemma: *Let D' and E' be connected rational subsets of E^1_k such that $D' \cap E' \neq \emptyset$. Then also $D' \cup E'$ and $D' \cap E'$ are connected rational subsets of E^1_k.*

Proof: As in (3.1) we take a suitable finite Galois extension K of k with Galoisgroup G such that $\pi^{-1}(D') = D_1 \cup \cdots \cup D_s$ and $\pi^{-1}(E') = E_1 \cup \cdots \cup E_t$ are decompositions into disjoint standard subsets of E^1_k. The union $\pi^{-1}(D') \cup \pi^{-1}(E')$ is a G-invariant element of $\tau(K)$ and consequently $D \cup E$ is a rational subset of E^1_k. We may suppose that $D_1 \cap E_1 \neq \emptyset$. Then set $\cup_{\sigma\in G}\, \sigma(D_1 \cup E_1)$ is clearly equal to $\pi^{-1}(D') \cup \pi^{-1}(E')$, moreover $D_1 \cup E_1 \in \mathrm{st}(K)$ and so $D' \cup E'$ is connected. It is more difficult to see that $D' \cap E'$ is also connected. We will show that $\cup_{\sigma\in G}\, \sigma(D_1 \cap E_1) = \pi^{-1}(D' \cap E')$. Suppose that D_1 is given by the inequalities $|t - a| \leq \rho$ and $|t - a_i| \geq \rho_i$ for $i = 1, \ldots, \alpha$ (with the additional conditions stated in (3.1)).

Since any $\sigma \in G$ is an isometry $\sigma(D_1)$ is given by the inequalities:

$$|t - \sigma(a)| \leq \rho \quad \text{and} \quad |t - \sigma(a_i)| \geq \rho_i \quad (i = 1, \ldots, \alpha).$$

Since $D_1 \cap E_1 \neq \emptyset$ the set E_1 is described by inequalities:

$$|t - a| \leq \mu \quad \text{and} \quad |t - b_j| \geq \mu_j \quad (j = 1, \ldots, \beta).$$

Moreover we assume that $\rho \geq \mu$. One draws from this the following conclusion: if $\sigma(D_1) \cap E_1 \neq \emptyset$ (or if $D_1 \cap \sigma^{-1}E_1 \neq \emptyset$) then $\sigma D_1 = D_1$. Let now $D_i \cap E_j \neq \emptyset$ for some i and j. There is a $\sigma \in G$ with $D_i = \sigma(D_1)$. Now $D_1 \cap \sigma^{-1}E_j \neq \emptyset$ and $\sigma^{-1}E_j = \tau E_1$ for some $\tau \in G$ with $\tau D_1 = D_1$. This implies $D_i \cap E_j = \sigma\tau(D_1 \cap E_1)$. As a consequence $\cup_{\sigma\in G}\, \sigma(D_1 \cap E_1) = \pi^{-1}(D') \cap \pi^{-1}(E')$.

(3.3) Corollary: *The constant sheaf A_X has trivial cohomology for any rational $X \subset E^1_k$.*

PROOF: Let $D_1, \ldots, D_m$ be rational connected domains in E_k^1 with union X. Then we want to show that the Čech complex with respect to this covering of X has trivial H^i $(i > 0)$. For $m = 2$ this follows at once from (3.2). For $m > 2$ a combinatorical argument and induction on m yields the result. For $m = 3$ we will only sketch the proof and we will leave the remaining steps to the reader. For $m = 3$ the result is trivial if $D_i \cap D_j = \emptyset$ for all $i \neq j$. Suppose now that $D_1 \cap D_2 \neq \emptyset$. Then $D_1 \cup D_2$ is again connected and we have a commutative diagram

$$\begin{tikzcd}
 & & 0 & 0 & 0 & \\
 & 0 \arrow[r] & A(D_1 \cap D_2) \arrow[r] \arrow[u] & A(D_1 \cap D_2) \oplus A(D_1 \cap D_2 \cap D_3) \arrow[r] \arrow[u] & (A(D_1 \cap D_2 \cap D_3) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A(X) \arrow[r] \arrow[u] & A(D_1) \oplus A(D_2) \oplus A(D_3) \arrow[r] \arrow[u] & A(D_1 \cap D_2) \oplus A(D_1 \cap D_3) \oplus A(D_2 \cap D_3) \arrow[r] \arrow[u] & A(D_1 \cap D_2 \cap D_3) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A(X) \arrow[r] \arrow[u] & A(D_1 \cup D_2) \oplus A(D_3) \arrow[r] \arrow[u] & A((D_1 \cup D_2) \cap D_3) \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The columns in the diagram are exact and two of the rows are exact. Hence the middle row, which is the augmented Čech-complex for the covering $\{D_1, D_2, D_3\}$ is also exact.

(3.4) REMARK: Instead of working on E_k^1 we might work on $\mathbb{P}_k^1$. A rational subset of $\mathbb{P}_k^1$ is defined by a finite number of inequalities $|f| \leq \rho$ where $f \in k(z)$ is a non-constant function and $\rho \in |k^*|$. In analogy with (3.1) we can define standard subsets of $\mathbb{P}_k^1$, etc. In particular the corollary (3.3) above becomes:

For every rational subset X of $\mathbb{P}_k^1$, the cohomology groups $H^i(X, A_X)$ are zero for $i \neq 0$.

(3.5) PROPOSITION: *Let $X_1, \ldots, X_n$ be rational subsets of E_k^1 with union X. The augmented Čech-complex of $\mathcal{O}^0$ with respect to the covering $\{X_1, \ldots, X_n\}$ of X is homotopic to zero with a homotopy which is k^0-linear and continuous.*

PROOF: As in the proof of (3.3) the general case follows from the case $n = 2$. So we have to show that the map

$$d : \mathcal{O}^0(X_1) \otimes \mathcal{O}^0(X_2) \to \mathcal{O}^0(X_1 \cap X_2)$$

has a right-inverse which is k^0-linear and continuous.

For any rational X in $\mathbb{P} = \mathbb{P}_k^1$, the complement $\mathbb{P} - X$ is the union of (infinitely many) rational subsets. It is given the Grothendieck topology defined by the family of all its rational subsets (Compare [4] Ch. IV, §1). For this Grothendieck topology $\mathbb{P}-X$ splits as a disjoint union of finitely many analytic subspaces $V_1, \ldots, V_s$ of $\mathbb{P}^1_k$. For every V_i the complement $\mathbb{P}-V_i$ is again a rational subset of $\mathbb{P}^1_k$. We note that V_i need not stay connected if we enlarge the field k.

Let now X be rational in E^1_k; put $\mathbb{P}-X = V_1 \cup \cdots \cup V_s$ where V_1 is the component which contains ∞. Using the decomposition of Mittag–Leffler, every $f \in \mathcal{O}(X)$ can uniquely be written as $f_1 + \cdots + f_s$ where $f_i \in \mathcal{O}(\mathbb{P}-V_i)$ and $f_i(\infty)=0$ for $i \neq 1$. Moreover $\|f\|_X = \max(\|f_i\|_{\mathbb{P}-V_i})$.

In the special case that $X \subset E^1_k$ is connected one easily verifies that V_1 remains connected after any field extension and that V_i $(i \neq 1)$ decomposes in a field extension of k into a finite union of open disks.

In proving the case $n=2$ we may suppose that X_1 and X_2 are connected and that $X_1 \cap X_2 \neq \emptyset$. We decompose into components:

$$\mathbb{P}-X_1 = U_1 \cup \cdots \cup U_a; \quad \mathbb{P}-X_2 = V_1 \cup \cdots \cup V_b$$

and

$$\mathbb{P}-(X_1 \cap X_2) = W_1 \cup \cdots \cup W_c.$$

In a suitable Galois extension K of k (with group G) each U_i and V_j $(i \neq 1 \neq j)$ decomposes into a union of open disks on which the group G acts transitively. Using this one can easily verify the following statements:

(a) If $U_i \cap V_j \neq \emptyset$ and $i \neq 1 \neq j$) then $U_i \subseteq V_j$ or $U_i \supseteq V_j$.

(b) If $U_1 \cap V_j \neq \emptyset$ (and $j \neq 1$) then $U_1 \supseteq V_j$.

It follows that $W_1 = U_1 \cup V_1$ and in fact $W_1 = U_1$ or $W_1 = V_1$.

Any W_h with $h \neq 1$ is equal to some U_i with $i \neq 1$ or to V_j with $j \neq 1$. Using the Mittag Leffler decomposition of $\mathcal{O}(X_1)^0$, $\mathcal{O}(X_2)^0$ and $\mathcal{O}(X_1 \cap X_2)^0$ the desired right-inverse of d is constructed on each factor $\mathcal{O}(\mathbb{P}-W_i)^0$ of $\mathcal{O}(X_1 \cap X_2)^0$. This ends the proof.

(3.6) Remarks:

(3.6.1) The proposition (3.5) remains valid if one replaces $\mathcal{O}^0$ by the sheaf $\mathcal{O}(r)$.

(3.6.2) A more natural way to prove (3.5) would be to extend k to a field K such that all $\pi^{-1}(X_i)$ belong to $\tau(K)$ and to prove the splitting of the Čech-complex in that situation. There are however two series difficulties involved in that. First of all $\mathcal{O}^0(X \times K)$ is not isomorphic to $\mathcal{O}^0(X) \otimes_{k^0} K^0$ in general. Secondly, the $G = \mathrm{Gal}(K \mid k)$-module $\mathcal{O}^0(X \times K)$ can have a $H^1(G, \mathcal{O}^0(X \times K)) \neq 0$.

(3.7) PROPOSITION: *Let $X_1, \ldots, X_n$ be rational subsets of E^1_k with union of X. The augmented Čech-complex of $\mathcal{O}^*$ with respect to the covering $\{X_1, \ldots, X_n\}$ of X is exact.*

PROOF: We may suppose that all the X_i's are connected. As in the proof of (3.3) we have only to consider the case $n = 2$. So we have to verify that

$$d : \mathcal{O}^*(X_1) \oplus \mathcal{O}^*(X_2) \to \mathcal{O}^*(X_1 \cap X_2)$$

is surjective, where X_1 and X_2 are connected and $X_1 \cap X_2 \neq \emptyset$.

Let X be any rational subset of $\mathbb{P}^1_K$ (or E^1_k). An invertible function f on X can be approximated by a rational function g. Then g has no poles or zeros on X and $f = g(1 + h)$ where $h \in \mathcal{O}(X)$ has $\|h\| < 1$. Let $\mathbb{P} - X = V_1 \cup \cdots \cup V_s$ then g can be written as $g_1 \ldots g_s$ where g_i has all its poles and zeros in V_i. So $g_i \in \mathcal{O}^*(\mathbb{P} - V_i)$. Using Mittag–Leffler we decompose $h = h_1 + \cdots + h_s$ with $h_i \in \mathcal{O}(\mathbb{P} - V_i)$ and $\|h_i\| \leq \|h\|$. Then $(1 + h) = (1 + h_1) \ldots (1 + h_s)(1 + h^*)$ where $h^* \in \mathcal{O}(X)$ has $\|h^*\| \leq \|h\|^2$. We can continue with a decomposition $(1 + h^*) = (1 + h^*_1) \ldots (1 + h^*_s)(1 + h^{**})$ and $\|h^{**}\|_X \leq \|h^*\|^2_X$.

In the limit of this process we find a decomposition

$$(1 + h) = (1 + h'_1) \ldots (1 + h'_s) \text{ with } h'_1 \in \mathcal{O}(\mathbb{P} - V_i) \text{ and } \|h'_i\| \leq \|h\|.$$

This shows that the map $\bigoplus_{i=1}^{s} \mathcal{O}^*(\mathbb{P} - V_i) \to \mathcal{O}^*(X)$ given by $(f_1, \ldots, f_s) \mapsto f_1 \ldots f_s$ is surjective.

Now we copy the final part of the proof of (3.5). It yields that

$$\bigoplus_{i=1}^{a} \mathcal{O}^*(\mathbb{P} - V_i) \times \bigoplus_{j=1}^{b} \mathcal{O}^*(\mathbb{P} - V_j) \to \bigoplus_{n=1}^{c} \mathcal{O}^*(\mathbb{P} - W_n) \to \mathcal{O}^*(X_1 \cap X_2)$$

is surjective. Hence also d is surjective.

(3.8) We conclude our discussion of the dimension one case with:

COROLLARY: *The sheaves A_X, $\mathcal{O}^0_X$, $\mathcal{O}_X(r)$, $\mathcal{O}^*_X$ have trivial cohomology groups for any rational subset X of $\mathbb{P}^1_k$.*

(3.9) Our aim in the rest of this section is to extend the results (3.8) to some affinoid spaces of higher dimension.

A *generalized polydisk* over k is a rational subspace of $E^n_k = \mathrm{Sp}(k\langle z_1, \ldots, z_n\rangle)$ of the form $D_1 \times \cdots \times D_n$ where each D_i is a standard subset of E^1_k.

A rational subset X of E_k^n is said to be *monomially convex* if X is given by a finite number of inequalities $|z^\alpha| \le r_\alpha$, $\alpha \in A$, A a finite subset of $\mathbb{N}_0^n$ and $r_\alpha \in \sqrt{|k^*|}$. We start to investigate a monomially convex X. With X we associate a convex subset $[X]$ in $\mathbb{R}_{\ge 0}^n$. Let $(x_1, \ldots, x_n)$ denote the coordinates of x in $\mathbb{R}_{\ge 0}^n$ then $[X]$ is given by the inequalities $\Sigma_{i=1}^n \alpha_i x_i \ge -\log r_\alpha$ for all $\alpha \in A$. A point $z = (z, \ldots, z_n) \in E_k^n$ with all its coordinates $\neq 0$ belongs to X if and only if $(-\log|z_1|, \ldots, -\log|z_n|)$ belongs to $[X]$. The finite sums $f = \Sigma\, a_\beta z^\beta$ lie dense in $\mathcal{O}(X)$. Only easily calculates that $-\log\|z^\beta\|_X = \min\{\Sigma_{i=1}^n \beta_i x_i \mid (x_1, \ldots, x_n) \in [X]\}$.

A small calculation shows that

$$\|f\|_X = \max|a_\beta|\|z^\beta\|_X.$$

It follows that $\{z^\beta \mid \beta \in \mathbb{N}_0^n\}$ is an orthogonal base of the Banach space $\mathcal{O}(X)$ (with respect to the spectral norm $\|\ \|_X$ on X). The natural injection $\mathcal{O}(X) \to k[\![z_1, \ldots, z_n]\!]$ shows that $\mathcal{O}(X)$ has no zero-divisors. In particular X is connected.

(3.10) *The constant sheaves*

THEOREM: *If $X \subset E_k^n$ is monomially convex or if X is a generalized polydisk then $H^i(X, A_X) = 0$ for $i \neq 0$ and for any constant sheaf AX.*

PROOF: Suppose first that X is monomially convex. Let $\varphi : X \to D$ denote the projection on the last coordinate. The image D is a rational subset of E_k^1 given by $|z_n| \le r$ (where $0 < r \le 1$).

We apply (2.3) to the map φ. For any closed geometric point p of D, the space $X \times p \subset E_{K_p}^{n-1}$ is again monomially convex. By induction $R_{\varphi_*}^i A_X = H^i(X \times p, \alpha^* A_X) = 0$ for $i \neq 0$, since the natural map $A_{X\times p} \to \alpha^* A_X$ is an isomorphism. The natural map $A_D \to \varphi_* A_X$ is an isomorphism since $(\varphi_* A_X)_p = H^0(X \times p, A_{X\times p}) = A$. ($X \times p$ is again connected!)

By (2.9) and (3.3) we have $H^i(X, A_X) \cong H^i(D, \varphi_* A_X) = 0$ for $i \neq 0$. In the case where X is a generalized polydisk, $X = D_1 \times \cdots \times D_n$, we use the same method for the projection $\varphi : X \to D_n$. Again $H^i(X, A_X) \cong H^i(D_n, \varphi_* A_X)$ follows. We need to see that the natural map $A_{D_n} \to \varphi_* A_X$ is an isomorphism. However for a connected rational subset $E \subset D_n$ we have that $\varphi_* A_X(E) = A_X(D_1 \times \cdots \times D_{n-1} \times E) = A$ according to the next lemma.

(3.11) LEMMA: *Let $D_1, \ldots, D_n$ be connected rational subsets of E_k^1. Then $D_1 \times \cdots \times D_n$ is also connected.*

PROOF: For connected affinoid spaces X and Y over an algebraically closed field it is well known (and easily verified) that the product $X \times Y$ is also connected.

Let $K \supset k$ be a finite Galois extension with Galois group G and let $\pi : E_K^1 \to E_k^1$ denote the map induced by $k\langle T\rangle \to K\langle T\rangle$.

For a suitable $K \supset k$ each $\pi^{-1}(D_i) = \cup_{j=1}^{m(i)} D_{ij}$, is a disjoint union of standard subsets of E_K^1. In any further extension of K standard subsets remain standard subsets. In particular standard subsets are absolutely connected. This implies the following: choose for every i, $1 \le i \le n$, an integer $\alpha(i)$ with $1 \le \alpha(i) \le m(i)$, then $D_{1,\alpha(1)} \times D_{2,\alpha(2)} \times \cdots \times D_{n,\alpha(n)}$ is connected.

We note further that $\mathcal{O}(D_1 \times \cdots \times D_n) = \mathcal{O}(D_1) \hat{\otimes}_k \cdots \hat{\otimes}_k \mathcal{O}(D_n)$ and that $(\mathcal{O}(D_1 \times \cdots \times D_n) \otimes_k K)^G = \mathcal{O}(D_1 \times \cdots \times D_n)$. From the above it follows that any idempotent $\ell \in \mathcal{O}(D_1 \times \cdots \times D_n)$ has considered as element of $\mathcal{O}(D_1 \times \cdots \times D_n) \otimes_k K$ uniquely the form $\ell_1 \otimes \cdots \otimes \ell_n$ where each ℓ_i is an idempotent of $\mathcal{O}(D_i) \otimes_k K$. Since ℓ is invariant under G also each "component" ℓ_i is invariant under G. Since $\mathcal{O}(D_i) = (\mathcal{O}(D_i) \otimes_k K)^G$ contains no other idempotents that 0 and 1 it follows that $\ell = 0$ or 1. This proves the lemma.

3.12 *Examples. Not every rational subset of a polydisk has trivial cohomology for the constant sheaves. Indeed,*

EXAMPLE 1: X = the rational subset of E_k^2 defined by the inequality $|y^2 - x(x-\pi)(x-1)(x-1-n)| \le |\pi|^2$ in which $\pi \in k$ satisfies $0 < |\pi| < 1$. Then $H^i(X, A_X) = A$ for $i = 0$, 1 and 0 for $i \ge 2$ (if $\bar{k}$ has characteristic $\neq 2$).

PROOF: We apply theorem (2.3) to the surjective map $\varphi : X \to E_k^1$ given by $\varphi(x, y) = x$. For any closed geometric point p of E_K^1 one has again $X \times p \subset E_{K_p}^1$ and $\alpha^* A_X = A_{X\times p}$ has trivial cohomology. So $H^i(X, A_X) \simeq H^i(E_K^1, \varphi_* A_X)$. The sheaf $\varphi_* A_X$ on E_k^1 is however not the constant sheaf. Let $\rho \in \sqrt{|k^*|}$ satisfy $1 \ge \rho > |\pi|$ and let U_ρ denote the subset of E_k^1 given by the inequalities $|x| \ge \rho$ and $|x-1| \ge \rho$.

A small computation shows that $x(x-\pi)(x-1)(x-1-\pi)$ is the square of some $f \in \mathcal{O}(U_\rho)$. Further one computes that $\varphi^{-1}(U_\rho)$ has two components, namely:

$$\left\{(x, y) \in X \;\middle|\; x \in U_\rho \text{ and } |y - f(x)| \le \frac{|\pi^2|}{\rho}\right\}$$

and

$$\left\{(x, y) \in X \;\middle|\; x \in U_\rho \text{ and } |y + f(x)| \le \frac{|\pi^2|}{\rho}\right\}.$$

Hence $\varphi_* A_X(U_\rho) = A^2$. A further investigation shows that for any connected rational $U \subset E_k^1$ one has either $\varphi_* A_X(U) = A^2$ and $U \supset U_\rho$ for some ρ or $\varphi_* A_X(U) = A$ and U contains no U_ρ.

From this one calculates (for instance by using the covering U_ρ, $\{x \in E_k^1 \mid |x| \leq \rho\}$, $\{x \in E_k^1 \mid |x-1| \leq \rho\}$) that $H^i(E_k^1, \varphi_* A_X)$ is equal to A for $i = 0, 1$ and to 0 for $i \geq 2$.

EXAMPLE 2: Let X be the subset of E_k^2 given by the inequality $|xy| \leq |\pi|$ where $\pi \in k$ and $0 < |\pi| < 1$. Then $H^i(X, A_X) = A$ for $i = 0$ and 0 for $i \neq 0$. (This is a special case of (3.10)).

PROOF: We apply again (2.3) to the map $\varphi : X \to E_k^1$ given by $\varphi(x, y) = x$. One follows the same arguments as in the example 1. But now $\varphi_* A_X = A_{E^1}$. The easiest way to show this is to verify that for any closed geometric point p of E_k^1 the space $X \times p \subset E_{K_p}^1$ is connected. But $X \times p$ is given by the single inequality $|x|_p |y| \leq |\pi|$ where $|\ |_p$ denotes the valuation of K_p.

EXAMPLE 3: Let X be the subset of E_k^2 given by the inequalities $|x(x-1)y| \leq |\pi|^2$ and $|(x-\pi)(x-1)y| \leq |\pi|^2$ in which $\pi \in k$ satisfies $0 < |\pi| < 1$. Then $H^i(X, A_X) = A$ for $i = 0, 1$ and $= 0$ for $i \geq 2$.

PROOF: Applying (2.3) to the map $\varphi : X \to E_k^1$ given by $\varphi(x, y) = y$ one finds $H^i(X, A_X) \cong H^i(E_k^1, \varphi_* A_X)$. The sheaf $\varphi_* A_X$ is not a constant sheaf. For a closed geometric point p of E_k^1 one finds $(\varphi_* A_X)_p = A$ if $|y|_p \leq |\pi|^2$ or if $|\pi| < |y|_p \leq 1$ $(\varphi_* A_X)_p = A^2$ if $|\pi|^2 < |y|_p \leq |\pi|$.

Let $\rho_1, \rho_2 \in \sqrt{|k^*|}$ satisfy $|\pi|^2 < \rho_1 < \rho_2 < |\pi|$. We produce the following covering $\mathcal{U}$ of E_k^1:

$$U_1 = \{y \in E_k^1 \mid |y| \leq \rho_1\}; \quad U_2 = \{y \in E_k^1 \mid \rho_1 \leq |y| \leq \rho_2\};$$

$$U_3 = \{y \in E_k^1 \mid \rho_2 \leq |y|\}.$$

An easy inspection yields that $\varphi^{-1}(U_1)$ and $\varphi^{-1}(U_3)$ are connected and that $\varphi^{-1}(U_2)$, $\varphi^{-1}(U_1 \cap U_2)$ and $\varphi^{-1}(U_2 \cap U_3)$ have each two components. It follows that $\check{H}^i(\mathcal{U}, \varphi_* A_X) = A$ for $i = 0, 1$ and $= 0$ for $i \geq 2$.

One can further verify that $\varphi_* A_X$ has no cohomology on U_1, U_2, U_3 and $U_1 \cap U_2$, $U_2 \cap U_3$. With Leray's theorem it follows that $H^i(E_k^1, \varphi_* A_X)$ is isomorphic to $\check{H}^i(\mathcal{U}, \varphi_* A_X)$.

(3.13) *Another result on the cohomology to constant sheaves is*:

THEOREM: *Let X be an affinoid space over k and let D be a standard subset of E_k^1. Then, for all $i \geq 0$, $H^i(X \times D, A_{X \times D}) \simeq H^i(X, A_X)$.*

PROOF: We apply (2.3) to the projection $\varphi : X \times D \to X$. For any closed geometric point p of X the space $p \times D$ is a standard subset of $E_{K_p}^1$. In particular $p \times D$ is connected and its cohomology for the constant sheaf is trivial. This implies that $\varphi_* A_{X \times D} \simeq A_X$ and the theorem follows in the usual way.

(3.14) REMARK: One would like to calculate the cohomology groups of $X \times Y$ in terms of the groups on X and Y. A difficulty in a direct approach with (2.3) is the following *problem*:

Suppose that the affinoid space X (over an algebraically closed field k) has the property

$$H^i(X, A_X) = \begin{cases} A & \text{for } i = 0 \\ 0 & \phantom{\text{for }} i \neq 0 \end{cases}$$

for every constant sheaf A_X. Let $K \supset k$ be a valued field extension. Does the affinoid space $X \times_k K$ over K, still have the same property?

(3.15) Our next subject is the *cohomology of the sheaves $\mathcal{O}(r)$ and $\mathcal{O}(r, s)$*. The main result is:

THEOREM: *Let $X \subset E_k^n$ be either a generalized polydisk or a monomially convex subset. Then*

$$H^i(X, \mathcal{O}_X(r)) = H^i(X, \mathcal{O}_X(r, s)) = 0 \quad \textit{for } i \neq 0.$$

PROOF: We will first of all make the induction step. Let $\varphi : X \to E_k^1$ denote the projection on the last coordinate. We fix a closed geometric point p of E_k^1. Then $X \times p \subset E_{K_p}^{n-1}$ is either a generalized polydisk or a monomially convex subset. The exact sequence of sheaves $0 \to \mathcal{O}_X(r) \to \mathcal{O}_X(\infty) \to \mathcal{O}_X(r, \infty) \to 0$ induces an exact sequence on $X \times p$:

$$0 \to \alpha^* \mathcal{O}_X(r) \to \alpha^* \mathcal{O}_X(\infty) \to \alpha^* \mathcal{O}_X(r, \infty) \to 0.$$

The next lemma will imply $\alpha^* \mathcal{O}_X(r, \infty) \simeq \mathcal{O}_{X \times p}(r, \infty)$.

(3.16) LEMMA: *Let $\varphi : X \to Y$ be a morphism of affinoid spaces over k. Let p_0 be a geometric point of Y with corresponding closed geometric point p of Y. Then*

(1) *The kernel and the cokernel of the natural map $(\varphi_* \mathcal{O}_X(r))_{p_0} \to \mathcal{O}_{X\times p}(r)(X \times p)$ do not depend on r.*

(2) $\alpha^* \mathcal{O}_X(r, s) \simeq \mathcal{O}_{X\times p}(r, s)$.

PROOF: (1) As seen in (2.5) the map $\tau : \varinjlim\{\mathcal{O}_X(\varphi^{-1}V) \mid V \in p_0\} \to \mathcal{O}(X \times p)$ has a dense image. An element $f \in \mathcal{O}_{X\times p}(r)(X \times p)$ can be written as $f = f_1 + f_2$ where

(a) f_1 is the image of some $g \in \mathcal{O}_X(\varphi^{-1}V)$; $V \in p_0$.

(b) $f_2 \in \mathcal{O}_{X\times p}(\delta)(X \times p)$ with δ positive and small.

Using (2.5) and the fact that any neighbourhood of an element in p lies in p_0 one finds that actually g can be chosen in $\mathcal{O}_X(r)(\varphi^{-1}W)$ for a suitable $W \in p_0$. This shows that the cokernel of the map in (3.16.1) does not depend on r. A similar argument shows that also the kernel does not depend upon r.

(2) Let $P(r)$ denote the presheaf on $X \times p$ defined by $P(r)(A) = \varinjlim\{\mathcal{O}_X(r)(U) \mid \alpha\hat{A} \subset \hat{U}\}$. As in (1) one shows that the kernel $K(A)$ and the cokernel $C(A)$ of the canonical map $P(r)(A) \to \mathcal{O}_{X\times p}(r)(A)$ do not depend on r. So we find an exact sequence of sheaves:

$$0 \to K^{++} \to P(r)^{++} \to \mathcal{O}_{X\times p}(r) \to C^{++} \to 0$$

and we note that the sheaf $P(r)^{++}$ is equal to $\alpha^* \mathcal{O}_X(r)$. From the exactness of the sequence

$$0 \to \alpha^* \mathcal{O}_X(r) \to \alpha^* \mathcal{O}_X(s) \to \alpha^* \mathcal{O}_X(r, s) \to 0$$

we draw the conclusion $\alpha^* \mathcal{O}_X(r, s) \cong \mathcal{O}_{X\times p}(r, s)$.

(3.17) We continue now our proof of (3.15). The exact sequence

$$0 \to \mathcal{O}_X(r) \to \mathcal{O}_X(\infty) \to \mathcal{O}_X(r, \infty) \to 0$$

induces a long exact sequence of sheaves on E_k^1:

$$0 \to \varphi_* \mathcal{O}_X(r) \to \varphi_* \mathcal{O}_X(\infty) \to \varphi_* \mathcal{O}_X(r, \infty) \to R^1\varphi_* \mathcal{O}_X(r) \to R^1\varphi_* \mathcal{O}_X(\infty) \to \cdots.$$

The sheaf $\mathcal{O}_X(\infty) = \mathcal{O}_X$ has no cohomology on the affinoid subsets of X, so $R^i\varphi_* \mathcal{O}_X(\infty) = 0$ for $i \neq 0$. Further the constructible sheaf $R^i\varphi_* \mathcal{O}_X(r, \infty)$ has stalks $\cong H^i(X \times p, \mathcal{O}_{X\times p}(r, \infty))$. By induction we have that $H^i(X \times p, \mathcal{O}_{X\times p}(r, \infty)) = 0$ for $i \neq 0$. This implies that $R^i\varphi_* \mathcal{O}_X(r, \infty) = 0$ for $i \neq 0$. As a consequence $R^i\varphi_* \mathcal{O}_X(r) = 0$ for $i \geq 2$.

In order to show that $R^1\varphi_*\mathcal{O}_X(r)=0$ we have to see that for *every* geometric point p_0 of D (with p as unique closed geometric point containing it) the map $(\varphi_*\mathcal{O}_X(\infty))_{p_0}\xrightarrow{\delta}(\varphi_*\mathcal{O}_X(r,\infty))_{p_0}$ is surjective. Consider the following commutative and exact diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & (\varphi_*\mathcal{O}_X(r))_{p_0} & \longrightarrow & (\varphi_*\mathcal{O}_X(\infty))_{p_0} & \xrightarrow{\delta} & (\varphi_*\mathcal{O}_X(r,\infty))_{p_0} & \\
& \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} & \\
0 \rightarrow & \mathcal{O}_{X\times p}(r)(X\times p) & \rightarrow & \mathcal{O}_{X\times p}(\infty)(X\times p) & \rightarrow & \mathcal{O}_{X\times p}(r,\infty)(X\times p) & \rightarrow 0.
\end{array}$$

According to (3.16.1) the maps γ_1, γ_2 have the same kernel and cokernel. The map γ_3 is bijective since $(\varphi_*\mathcal{O}_X(r,\infty))_{p_0}=(\varphi_*\mathcal{O}_X(r,\infty))_p=H^0(X\times p,\ \alpha^*\mathcal{O}_X(r,\infty))$ and $\alpha^*\mathcal{O}_X(r,\infty)=\mathcal{O}_{X\times p}(r,\infty)$. It follows that δ is surjective. So we have verified that also $R^1\varphi_*\mathcal{O}_X(r)=0$. As a consequence we have $H^i(X,\mathcal{O}_X(r))\simeq H^i(E_k^1,\varphi_*\mathcal{O}_X(r))$. Our next step in the proof will be to identify and analyse the sheaf $\varphi_*\mathcal{O}_X(r)$.

(3.18) Lemma: *Let X be a reduced affinoid space over k such that $\mathcal{O}(X)$ has an orthonormal base $\{e_n\}$ with respect to the spectral norm on $\mathcal{O}(X)$. Suppose that for any finite field extension K of k the set $\{e_n\otimes 1\}$ remains an orthonormal base with respect to the spectral norm on $\mathcal{O}(X\times_k K)=\mathcal{O}(X)\otimes_k K$.*

Then for any affinoid space Y over k and any $r>0$ one has $\mathcal{O}(r)(X\times Y)=\mathcal{O}^0(X)\hat{\otimes}_{k^0}\mathcal{O}(r)(Y)$.

Proof: $f\in\mathcal{O}(X\times Y)=\mathcal{O}(X)\hat{\otimes}_k\mathcal{O}(Y)$ can be written as $f=\Sigma_{n=1}^\infty e_n\otimes a_n$ where $a_n\in\mathcal{O}(Y)$ and $\lim\|a_n\|=0$. The spectral norm of f is

$$\|f\|_{sp}=\sup_{y\in Y}\left\|\Sigma\, a_n(y)e_n\right\|_{sp}=\max_n(\|a_n\|_{sp}).$$

From this (3.18) follows.

(3.19) We can now finish the proof of (3.15) for a generalized polydisk $X=D_1\times\cdots\times D_n$. Using (3.18) it follows that $\varphi_*\mathcal{O}_X(r)$ is isomorphic to the sheaf $U\mapsto\mathcal{O}^0(D_1\times\cdots\times D_{n-1})\hat{\otimes}_{k^0}\mathcal{O}(r)(U)$ on D_n. Indeed, one easily verifies that $\mathcal{O}(D)$ has an orthonormal base, which remains orthonormal after any field extension. The same holds for $\mathcal{O}(D_1\times\cdots\times D_{n-1})$. That $\varphi_*\mathcal{O}_X(r)$ has trivial cohomology groups follows finally from (3.5) and its variant (3.6).

(3.20) COROLLARY: *Let Y be an affinoid space over k and let D be a standard subset of E_k^1. Then*

$$H^i(Y \times D, \mathcal{O}_{Y\times D}(r)) \simeq H^i(Y, \mathcal{O}_y(r)) \hat{\otimes}_{k^0} \mathcal{O}^0(D).$$

PROOF: We apply our machinery to $\varphi: Y \times D \to Y$. It implies that $H^i(Y \times D, \mathcal{O}_{Y\times D}(r) \cong H^i(Y, \varphi_*\mathcal{O}_{Y\times D}(r))$. Using (3.18) one finds that $\varphi_*\mathcal{O}_{Y\times D}(r)$ is isomorphic to the sheaf $U \mapsto \mathcal{O}_Y(r)(U) \hat{\otimes}_{k^0} \mathcal{O}^0(D)$.

Since $\mathcal{O}^0(D)$ is a flat k^0-module with respect to $\hat{\otimes}_{k^0}$, the result follows.

(3.21) REMARK: Suppose that the rational domain $D \subset E_k^1$ has the properties: (i) $\mathcal{O}(D)$ has an orthogonal base; (ii) For any finite field extension $K \supset k$ the spectral norm on $\mathcal{O}(D \times_k K)$ is equal to the tensor product norm on $\mathcal{O}(D) \otimes_k K$. (An example of this situation is $D = \{z \in E_k^1 \mid |z| = \rho\}$ with $\rho \in \sqrt{|k^*|}$ and possibly $\rho \notin |k^*|$). An easy variant of (3.18) and (3.20) will prove the following: "*Suppose that the affinoid space Y over k satisfies $H^i(Y, \mathcal{O}_Y(r)) = 0$ for all $i \neq 0$ and all $r > 0$. Then $H^i(Y \times D, \mathcal{O}_{Y\times D}(r)) = 0$ for all $i \neq 0$ and all $r > 0$.*"

(3.22) *The proof of* (3.15) *for a monomially convex subset X of E_k^n.*
According to the end of (3.17) we have to show that $H^i(E_k^1, \varphi_*\mathcal{O}_X(r)) = 0$ for $i \neq 0$. A difficulty here is that $\varphi_*\mathcal{O}_X(r)$ is in general not isomorphic to a sheaf of the form $U \mapsto M \hat{\otimes}_{k^0} \mathcal{O}(r)(U)$. By (1.4.13) it suffices to consider $i = 1$. Using the exactness of

$$0 \longrightarrow \varphi_*\mathcal{O}_X(r) \longrightarrow \varphi_*\mathcal{O}_X(\infty) \xrightarrow{\beta} \varphi_*\mathcal{O}_X(r, \infty) \longrightarrow 0$$

one finds that $U \mapsto H^1(U, \varphi_*\mathcal{O}_X(r))$ is a constructible presheaf on E_k^1 and that all its stalks are zero. Further $H^1(U, \varphi_*\mathcal{O}_X(r)) = 0$ is equivalent with $\beta(U): H^0(U, \varphi_*\mathcal{O}_X(\infty)) \to H^0(U, \varphi_*\mathcal{O}_X(r, \infty))$ is surjective.

Let $\rho \in \sqrt{|k^*|}$, $0 < \rho \leq 1$, then $X_\rho = \{(z_1, \ldots, z_n) \in X \mid |z_n| = \rho\}$ is a product $X'_\rho \times \{z_n \in E_k^1 \mid |z_n| = \rho\}$ where $X'_\rho \subset E_k^{n-1}$ is again monomially convex. Using the variant (3.21) of (3.20) and using induction on n it follows that $H^1(\{z \in E_k^1 \mid |z| = \rho\}, \varphi_*\mathcal{O}_X(r)) = 0$. In a similar way $H^1(\{z \in E_k^1 \mid |z| \leq \rho\}, \varphi_*\mathcal{O}_X(r)) = 0$ for small enough ρ. Take now a $\rho \in \mathbb{R}$, $0 < \rho < 1$ and $\rho \notin \sqrt{|k^*|}$ (if possible!). We associate with ρ the closed geometric point p corresponding to the valuation on $\mathcal{O}(E_k^1)$ given by $|\Sigma\, a_n z^n|_p = \max |a_n| \rho^n$. One easily computes that every $V \in p$ contains some $\{z \in E_k^1 \mid \lambda_1 \leq |z| \leq \lambda_2\}$ for suitable $\lambda_1, \lambda_2 \in \sqrt{|k^*|}$ with $\lambda_1 < \rho < \lambda_2$. Let now $\xi \in H^1(E_k^1, \varphi_*\mathcal{O}_X(r))$ and let $f \in$

$H^0(E_k^1, \varphi_* \mathcal{O}_X(r,\infty))$ have image ξ. From the above it follows that E_k^1 has a covering $\{V_0, \ldots, V_a\}$ given by a sequence $0 < r_1 < r_2 \ldots r_a < 1$ of elements in $\sqrt{|k^*|}$ in the following way:

$$V_0 = \{z \mid |z| \le r_1\}, \quad V_1 = \{z \mid r_1 \le |z| \le r_2\}, \ldots, V_a = \{z \mid r_a \le |z| \le 1\},$$

such that the restriction of ξ with respect to each V_i is zero. Let $f_i \in H^0(V_i, \varphi_* \mathcal{O}_X(\infty))$ have image $f|_{V_i}$ in $H^0(V_i, \varphi_* \mathcal{O}_X(r,\infty))$. We have to glue the $f_0, \ldots, f_a$ to an element of $H^0(E_k^1, \varphi_* \mathcal{O}_X(\infty))$ in order to show that $\xi = 0$. The obstruction to that is the 1-cocycle $(f_i - f_j)$ of $\varphi_* \mathcal{O}_X(r)$ with respect to $\{V_0, \ldots, V_a\}$. For $a = 1$ the next lemma shows that the 1-cocycle above is trivial. For $a > 1$ a repeated use of the lemma shows the triviality of the 1-cocycle. So the next lemma ends the proof of (3.15).

(3.23) LEMMA: *Let $X \subset E_k^n$ by monomially convex; let $\rho \in \sqrt{|k^*|}$, $0 < \rho < 1$. Then the map from*

$$\mathcal{O}(r)(\{z \in X \mid |z_n| \le \rho\}) \oplus \mathcal{O}(r)(\{z \in X \mid |z_n| \ge 0\})$$
$$\text{to } \mathcal{O}(r)(\{z \in X \mid |z_n| = \rho\})$$

given by $(f_1, f_2) \mapsto f_1 - f_2$ is surjective.

PROOF: Let $[X] \subset \mathbb{R}^n_{\ge 0}$ be the convex subset associated with X (see (3.9)). This set is divided into two convex sets C_1 and C_2

$$C_1 = \{x = (x_1, \ldots, x_n) \in [X] \mid x_n \ge -\log \rho\}$$

and

$$C_2 = \{x \in [X] \mid 0 \le x_n \le -\log \rho\}.$$

Further $C_3 = C_1 \cap C_2 = \{x \in [X] \mid x_n = -\log \rho\}$. For each of the three affinoid spaces a subset of $\{z^\beta \mid \beta \in \mathbb{Z}^n\}$ is an orthogonal base. The lemma amounts to the following:

For $\beta \in \mathbb{Z}^n$ we can define three norms:

$$\|z^\beta\|_1 = \text{the supremumnorm of } z^\beta \text{ on } \{z \in X \mid |z_n| \le \rho\}$$

$$\|z^\beta\|_2 = \text{the supremumnorm of } z^\beta \text{ on } \{z \in X \mid |z_n| \ge \rho\}$$

$$\|z^\beta\|_3 = \text{the supremumnorm of } z^\beta \text{ on } \{z \in X \mid |z_n| = \rho\}.$$

The values are allowed to be ∞. Then $\|z^\beta\|_3 = \min(\|z^\beta\|_1, \|z^\beta\|_2)$. Let $L : \mathbb{R}^n \to \mathbb{R}$ be the linear map $L(x) = \Sigma_{i=1}^n \beta_i x_i$. Then $-\log\|z^\beta\|_i =$

$\inf\{L(x) \mid x \in C_i\}$ for $i = 1, 2, 3$. So we have to show $\inf L(C_3) = \max(\inf L(C_1), \inf L(C_2))$. The inequality $\geq$ is trivial because $C_3 \subset C_1$ and $C_3 \subset C_2$. Further for $e_1 \in C_1$ and $e_2 \in C_2$ there is a $t, \in [0, 1]$ with $e_3 = te_1 + (1-t)e_2 \in C_3$.

$$L(e_3) = tL(e_1) + (1-t)L(e_2) \leq \max(L(e_1), L(e_2)).$$

This shows $\inf L(C_3) \leq \max(\inf L(C_1), \inf L(C_2))$.

(3.24) REMARK: The theorem (3.15) does not exhaust all the possible cases where a rational subspace X of E_k^n has trivial cohomology for the sheaves $\mathcal{O}_X(r)$, $\mathcal{O}_X(r, s)$. However one can not expect that every rational X in E_k^n has trivial cohomology. We will give an example.

Let Z be the 1-dimensional affinoid space over an algebraically closed field k given by:

$$Z = \{(x, y) \in E_k^2 \mid y^2 = x(x - \pi^2)(x - 2\pi^2)\} \quad \text{(where } |2| = 1 \text{ is assumed).}$$

Let $Z_1 = \{(x, y) \in Z \mid |x| \leq |\pi|^2\}$ and $Z_2 = \{(x, y) \in Z \mid |x| > |\pi|^2\}$. One can show that $H^i(T, \mathcal{O}^0) = 0$ for $i \neq 0$ and $T = Z_1, Z_2$ or $Z_1 \cap Z_2$. It follows that

$$0 \to \mathcal{O}^0(Z) \to \mathcal{O}^0(Z_1) \oplus \mathcal{O}^0(Z_2) \to \mathcal{O}^0(Z_1 \cap Z_2) \to H^1(Z, \mathcal{O}^0) \to 0$$

is exact. A computation gives $H^1(Z, \mathcal{O}^0) = k^0/\pi k^0$. In a similar way one finds $H^1(Z, \mathcal{O}(1)) = k^{00}/\pi k^0 \neq 0$.

For $n \in \mathbb{N}$ sufficiently big the rational domain

$$X_n = \{(x, y) \in E_k^2 \mid |y^2 - x(x - \pi^2)(x - 2\pi^2)| \leq |\pi|^n\}$$

in E_k^2 has then also a $H^1(X_n, \mathcal{O}(1)) \neq 0$.

(3.25) We conclude this paragraph by a calculation of the cohomology groups of $\mathcal{O}^*$. The main result will be:

THEOREM: *If $X \subset E_k^n$ is monomially convex or if X is a generalized polydisk then*

$$H^i(X, \mathcal{O}_X^*) = 0 \quad \textit{for } i \neq 0.$$

(3.26) We start with some lemmata. Let D be a standard subset of E_k^1 given by the inequalities $|z - a| \leq \rho$ and $|z - a_i| \geq \rho_i$ $(i = 1, \ldots, s)$ as in (3.1). Then every $f \in \mathcal{O}^*(D)$ has the form *)

$\lambda(1+h)(z-a_1)^{n_1}\dots(z-a_s)^{n_s}$, where $n_1,\dots,n_s \in \mathbb{Z}$; $\lambda \in k^*$ and $h \in \mathcal{O}(D)$ has norm $\|h\| < 1$.

Indeed f can be approximated by a rational function g such that $f = (1+h_1)g$ and $h_1 \in \mathcal{O}(D)$ has norm <1. For g one easily computes the form *). We note that the integers $n_1,\dots,n_s$ are uniquely determined by f. The λ and the $(1+h)$ are not unique since one can change λ by a constant $\mu \in k$ with $|\mu - 1| < 1$. We have to change then $(1+h)$ in $(1+h)\mu^{-1}$.

We will suppose for convenience that D contains the rational point a. In that case we can normalize the expression *) by imposing $h(a) = 0$.

LEMMA: *Let Y be a connected affinoid space over k. Then every $f \in \mathcal{O}^*(Y \times D)$ has the form*

$$f = f_1(1+h)(z-a_1)^{n_1}\dots(z-a_s)^{n_s}$$

in which $f_1 \in \mathcal{O}^*(Y)$; $n_1,\dots,n_s \in \mathbb{Z}$ and $h \in \mathcal{O}(Y \times D)$ has norm <1.

PROOF: For every $y \in Y$ the element $f(y,.)$ is an invertible function on $D \times_k k(y)$. This function can be decomposed as in *) and we find integers $n_1(y),\dots,n_s(y)$. The connectedness of Y implies that $n_1(y),\dots,n_s(y)$ do not depend on $y \in Y$. We define f_1 by $f_1(y) = f(y,a)(a-a_1)^{-n_1}\dots(a-a_s)^{-n_s} \in \mathcal{O}(Y)$ and the result follows.

DEFINITIONS: On any affinoid space we define sheaves $\mathcal{O}^*(1)$, S and T in the following way:

$\mathcal{O}^*(1)$ is the subsheaf of $\mathcal{O}^*$ consisting of the functions f with $|f(x)-1| < 1$ for all x. Let A denote the group $k^*/\{1+h \mid h \in k, |h| < 1\}$ and let A also denote the constant sheaf with stalk A.

The sheaves S and T are defined by the exact sequences

$$0 \to \mathcal{O}^*(1) \to \mathcal{O}^* \to S \to 0 \quad \text{and} \quad 0 \to A \to S \to T \to 0.$$

REMARKS: If the residue field of k has characteristic zero then the sheaf $\mathcal{O}^*(1)$ is isomorphic to $\mathcal{O}(1)$ (use the logarithm). In any case one easily verifies that $\mathcal{O}^*(1)$ has trivial cohomology groups on some X if for all $r > 0$, $\mathcal{O}(r)$ has trivial cohomology groups.

In the two cases of the theorem (3.25), $\mathcal{O}(1)^*$ and A have trivial cohomology. So (3.25) will follow from: $H^i(X, T_X) = 0$ for $i \neq 0$.

(3.27) LEMMA: *Let Y be any affinoid space and let D denote a standard subset of E_k^1. The map $\varphi : Y \times D \to Y$ is the projection on the first factor. Then $\varphi_* T_{Y\times D} \cong T_Y \oplus \mathbb{Z}_Y^s$.*

PROOF: For a connected affinoid $U \subset Y$ we have a map $\mathcal{O}(U)^* \times \mathbb{Z}^s \to \mathcal{O}(U \times D)^*$ given by

$$(f, n_1, \ldots, n_s) \mapsto f(u)(z-a_1)^{n_1} \ldots (z-a_s)^{n_s}.$$

This induces a map

$$(\mathcal{O}(U)^*/\mathcal{O}(1)^*(U)) \oplus \mathbb{Z}^s \to \mathcal{O}^*(U \times D)/\mathcal{O}(1)^*(U \times D) \subset S_{Y\times D}(U \times D)$$

and finally a map $\beta : S_Y \oplus \mathbb{Z}_Y^s \to \varphi_* S_{Y\times D}$. In order to show that β is an isomorphism we have to verify that for each closed geometric point p the map β_p is an isomorphism. We know that $(\varphi_* S_{Y\times D})_p = H^0(p \times D, \alpha^* S_{Y\times D})$. As in the proof of (3.16) one can show that $\alpha^* S_{Y\times D} \cong S_{p\times D}$. Then $(\varphi_* S_{Y\times D})_p = \mathcal{O}^*(p \times D)/\mathcal{O}(1)^*(p \times D)$ and every element of this group has a unique expression $\lambda(z-a_1)^{n_1} \ldots (z-a_s)^{n_s}$ with $n_1, \ldots, n_s \in \mathbb{Z}$ and $\lambda \in K_p/\{1+h \mid h \in K_p, |h|<1\}$. The stalk $S_{Y,p}$ equals $\mathcal{O}_{Y,p}^*/\mathcal{O}(1)_{Y,p}^*$. Further $\mathcal{O}_{Y,p}$ is a dense subfield M_p of K_p and $\mathcal{O}_{Y,p}^* = M_p^*$; $\mathcal{O}(1)_{Y,p}^* = \{1+h \mid h \in M_p;\ |h|<1\}$. So β_p is an isomorphism. The exact sequence $1 \to A_{Y\times D} \to S_{Y\times D} \to T_{Y\times D} \to 0$ induces

$$1 \to \varphi_* A_{Y\times D} \to \varphi_* S_{Y\times D} \to \varphi_* T_{Y\times D} \to R^1\varphi_* A_{Y\times D} \ldots.$$

From $(R^i\varphi_* A_{Y\times D})_p = H^i(p \times D, \alpha^* A_{Y\times D})$ and $\alpha^* A_{Y\times D}$ is the constant sheaf on $p \times D$ with stalk $A = k^*/\{1+h \mid h \in k |h|<1\}$ (N.B. not K_p but k in this case!) it follows at once that $T_Y \oplus \mathbb{Z}_Y^s \cong \varphi_* T_{Y\times D}$.

(3.28) COROLLARY: *Let D be a standard subset of E_k^1. Then for any affinoid space Y over k one has:*

$$H^i(Y \times D, T_{Y\times D}) = H^i(Y, T_Y) \oplus H^i(Y, \mathbb{Z}^s).$$

(3.29) COROLLARY: *Suppose that the affinoid space Y over k has trivial cohomology groups for $\mathcal{O}(r)$ and the constant sheaves. Then for any standard subset D of E_k^1 one has:*

$$H^i(Y \times D, \mathcal{O}^*) \cong H^i(Y, \mathcal{O}^*) \quad \textit{for } i \neq 0.$$

(3.30) COROLLARY: *For a generalized polydisk X the groups $H^i(X, \mathcal{O}_X^*)$ are zero for $i \neq 0$.*

(3.31) We investigate now the *case where X is monomially convex.* An element $f \in \mathcal{O}(X)$, $f = \Sigma\, a_\beta z^\beta$, is invertible if and only if $|a_0| >$

$|a_\beta|\,\|z^\beta\|_X$ for all $\beta \neq 0$. This means that $T(X) = 0$. As before we consider the projection $\varphi: X \to E_k^1$ on the last coordinate. By induction one has $H^i(X, T_X) \simeq H^i(E_k^1, \varphi_* T_X)$. The sheaf $\varphi_* T_X$ can be rather complicated. Arguments similar to those in (3.22) imply that it suffices to show that $\check{H}^i(\mathcal{U}, \varphi_* T_X) = 0$ $(i \neq 0)$ for covering $U = \{U_1, U_2\}$ for E_k^1 given by $U_1 = \{z \in E_k^1 \mid |z| \leq r\}$ and $U_2 = \{z \in E_k^1 \mid |z| \geq r\}$, where $r \in \sqrt{|k^*|}$ and $0 < r < 1$. This amounts to showing that the map $T(\varphi^{-1}U_1) \oplus T(\varphi^{-1}U_2) \to T(\varphi^{-1}(U_1 \cap U_2))$ is surjective. We note that $\varphi^{-1}(U_1 \cap U_2) = X'_r \times \{z_n \in E^1(k) \mid |z_n| = r\}$ for some monomially convex $X'_r \subset E_k^{n-1}$. From (3.26) it follows that $T(\varphi^{-1}(U_1 \cap U_2)) = \mathbb{Z}$ and this group is generated by the image of $z_n \in \mathcal{O}^*(\varphi^{-1}(U_1 \cap U_2))$. Since $z_n \in \mathcal{O}^*(\varphi^{-1}U_2)$ the map $T(\varphi^{-1}U_2) \to T(\varphi^{-1}(U_1 \cap U_2))$ is already surjective.

This finishes the proof of (3.25).

(3.32) We end the paper with a more detailed version of (3.29). This version can be compared with [9].

PROPOSITION: *Let Y be an affinoid space over k, let D be a standard subset of E_k^1 having a rational point a and let $\varphi: Y \times D \to Y$ be the projection. Then*

(1) *$H^i(Y \times D, \mathcal{O}^*_{Y\times D}) \cong H^i(Y, \varphi_* \mathcal{O}^*_{Y\times D})$ for all i.*

(2) *$\varphi_* \mathcal{O}^*_{Y\times D} \cong \mathcal{O}^*_Y \oplus \mathbb{Z}^s_Y \oplus G$ for some sheaf G depending on D. In case $D = E_k^1$ we write G_0 for this sheaf. One has $G \cong G_0^{s+1}$.*

(3) *$H^i(Y \times D, \mathcal{O}^*_{Y\times D}) \cong H^i(Y, \mathcal{O}^*_Y) \oplus H^i(Y, \mathbb{Z}^s_Y) \oplus H^i(Y, G_0)^{s+1}$.*

(4) *If the characteristic of $\bar{k}$ is 0 then $H^i(Y, G_0)$ is the completion of a countable direct sum of copies of $H^i(Y, \mathcal{O}_Y(1))$.*

PROOF: Using $1 \to \mathcal{O}^*_{Y\times D}(1) \to \mathcal{O}^*_{Y\times D} \to S_{Y\times D} \to 0$ and $R^i\varphi_* S_{Y\times D} = 0$ for $i \neq 0$ (see (3.27)) one sees that it suffices to show that $R^i\varphi_*\mathcal{O}^*_{Y\times D}(1) = 0$ for $i \neq 0$. If $\bar{k}$ has characteristic 0 then $\mathcal{O}^*_{Y\times D}(1) = \mathcal{O}_{Y\times D}(1)$ (use exp. and log.). So the required result follows from (3.17). If the characteristic of $\bar{k}$ is $\neq 0$ then one can give a proof of $R^i\varphi_*\mathcal{O}^*_{Y\times D}(1) = 0$ for $i \neq 0$ along the lines of (3.17). This shows (1). The lemma (3.26) shows that for a connected $U \subset Y$ every invertible function on $U \times D$ can uniquely be written as $f(u)\cdot(z-a_1)^{n_1}\ldots(z-a_s)^{n_s}\cdot(1+h)$ with $f \in \mathcal{O}(U)^*$ and $h \in \mathcal{O}(1)(U \times D)$ such that $h(u,a) = 0$ for all $u \in U$. This yields the decomposition. The sheaf G consist of the elements $h \in \varphi_*\mathcal{O}^*_{Y\times D}(1)$ with $h(u,a) \equiv 1$. Using the Mittag–Leffler decomposition for functions on D one calculates easily that $G = G_0^{s+1}$. Part (3) of the proposition is now clear. Finally, if $\bar{k}$ has characteristic 0 then $\varphi_*\mathcal{O}^*_{Y\times E^1}(1)$ is isomorphic to $\varphi_*\mathcal{O}_{Y\times E^1}(1)$. Using this and the result

(3.20) one finds $H^i(Y, G_0) = H^i(Y, \mathcal{O}_Y(1)) \hat{\otimes}_{k^0} M$ with

$$M = \left\{ \sum_{n=1}^{\infty} a_n z^n \mid a_n \in k^0, \lim a_n = 0 \right\} \subset \mathcal{O}(E_k^1).$$

The statement (4) follows. We note that every element of $H^i(Y, \mathcal{O}_Y(1))$ is a torsion element. If some $\pi \in k^0$, $\pi \neq 0$, annihilates all of $H^i(Y, \mathcal{O}_Y(1))$ then $H^i(Y, G_0)$ is simply the direct sum of a countable number of copies of $H^i(Y, \mathcal{O}_Y(1))$.

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Mathematisch Instituut
Postbus 800
9700 AV Groningen
The Netherlands