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MULTIPLICITY ONE FOR THE GELFAND-GRAEV REPRESENTATION OF A LINEAR GROUP

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§1. Introduction

Let F be a local field and let $\mathbf{G}$ be a connected, reductive, quasi-split group over F. Then there exists a Borel subgroup $\mathbf{B}$ defined over F with Levi decomposition $\mathbf{B} = \mathbf{A} \ltimes \mathbf{N}$. Let $\bar{\mathbf{N}}$ be the group opposed to $\mathbf{N}$. Set $G = \mathbf{G}(F)$, $B = \mathbf{B}(F)$, $A = \mathbf{A}(F)$, $N = \mathbf{N}(F)$ and $\bar{N} = \bar{\mathbf{N}}(F)$, and regard them as locally compact groups. Further let $\mathcal{N}(A)$ and $\mathcal{Z}(A)$ denote respectively the normalizer and centralizer of A in G.

We say that a (unitary) character of N is generic if:

(1.1) its restriction to $N \cap w\bar{N}w^{-1}$ is non-trivial for each w in $\mathcal{N}(A)$ not in $\mathcal{Z}(A)$; and

(1.2) it extends to a character of $\mathbf{N}(E)$ where E is the smallest Galois extension of F that splits $\mathbf{G}$.

The second condition is always satisfied if F has characteristic zero or if G is split over F.

The Gelfand-Graev representation of G attached to a generic character η of N is the representation π_η unitarily induced by η. In this paper we show that just as in the case of finite field ([15]), this representation is without multiplicity.

In the non-archimedean case, this result follows rather easily from a characterization of certain relatively invariant distributions due to Shalika ([18]) and Gelfand-Kajdan ([6]). This case is included here for completeness, and because it does not make the proof longer.

The archimedean case turns out to be more delicate. The distributions which arise naturally here have the property of not being eigendistributions for the Casimir operator on G. This fact creates an obstruction to a straight-forward application of the results of Shalika

0010–437X/81010003–12$00.20/0

([18]). To circumvent this problem, we make essential use of the nuclear version of the spectral theorem due to Maurin ([12]). This forms the central part of the paper.

To every operator L in the commutant of the representation π_η, we construct in a natural way a dense nuclear subspace X_L which is stable under L and G. Then by use of Maurin's theorem, for any given continuous sum decomposition $\int_\Lambda V_\lambda \, d\mu_\lambda$ of π_η into factors, we obtain a continuous G-Linear map $M_\lambda : X_L \to V_\lambda$ for every λ outside a fixed negligible set Λ_0 in Λ. This allows us to decompose (over Λ) bilinear forms on X_L. This crucial fact then enables us to prove certain invariance property of the associated distributions on X_L by checking locally at every λ outside Λ_0.

In the case of $\mathbf{G} = \mathbf{GL_n}$, we may take B to be the subgroup of G consisting of upper triangular matrices. Then N is of the form $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$. Fixing a non-trivial additive character ψ of F, we see that every generic character of N is equivalent under the action of $A = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$ to the character $(n_{ij}) \xrightarrow{\eta} \psi(\Sigma_{i=1}^{n-1} n_{i,i+1})$. Thus up to equivalence there is a unique Gelfand-Graev representation of G.

We can give a simpler proof of the multiplicity one theorem for GL_n([16]) than the one given below. It does not work for other groups, and goes roughly as follows: of G ($= GL_n(F)$) generic if the restriction of σ to P is equivalent to τ. By Mackey's generalized Frobenius reciprocity theorem ([11]), we may decompose Ind(G, P; τ) (which is equivalent to π_η) as a continuous sum of irreducibles in the dual $\hat{G}$ of G such that the multiplicities of almost all of the irreducibles in the decomposition are equal to the multiplicities of τ in their restrictions to P. But the dual of G consists of generic representations outside a set negligible for the Plancherel measure ([8]). Hence the multiplicities of almost all of the irreducibles in the decomposition of Ind(G, P; τ) are one.

The L^2-theorem of this paper is much in the spirit of, but does not follow from, the question of uniqueness of Whittaker models for irreducible, admissible representations of G, treated by Jacquet-Langlands ([9]), Gelfand-Kajdan ([6]), Rodier ([17]), Bernshtein and Zelevinsky ([1]), Shalika ([18]), Pjateckii-Shapiro ([13]), ([14]) and Kostant ([10]). One can show, however, using Whittaker model arguments, that the discrete spectrum (modulo the center) of the Gelfand-Graev representation of G has multiplicity one.

It is a pleasure to thank my advisor, Professor H. Jacquet, for his constant encouragement and advice, without which this paper could not have been written. I would also like to thank Professor P. Green for many valuable discussions.

§2. A distribution

Fix a generic character η of N, and let $V = V(\eta)$ be the space of the representation π_η. Then V consists of (classes of) functions f on G such that

(i) $f(ng) = \eta(n)f(g), n \in N, g \in G$;

and

(ii) f is square-integrable mod N.

If f is any function on G, set:

$$\lambda_x f(g) = f(x^{-1}g), \rho_x f(g) = f(gx), x, g \in G.$$

Then the representation of G on V is defined by $\pi(x)f = \rho_x f$.

Let $\mathcal{D}$ denote the space of smooth functions on G with compact support, and let $\mathcal{D}'$ be the space of distributions on G. Then we can extend the right and left actions of G to $\mathcal{D}'$ by duality.

For every f in $\mathcal{D}$, set: $\sigma(f)(g) = \int_N f(ng)\eta(n^{-1})\,dn$, $g \in G$. This is well-defined and we get:

$$\sigma(\rho_x f) = \rho_x\sigma(f), \quad x \in G;$$

$$\sigma(\lambda_{n^{-1}} f) = \eta(n)\sigma(f), \quad n \in N.$$

Further, $\sigma(f)$ is smooth (on the right), and has compact support mod N. Thus σ is a (right) G-equivariant linear map from $\mathcal{D}$ into V. If L is in $\mathrm{Hom}_G(V, V)$, we have

$$L\sigma(\rho_g f) = L\rho_g\sigma(f) = \rho_g L\sigma(f), \text{for all } g \text{ in } G \text{ and } f \text{ in } \mathcal{D}.$$

Now define a sesquilinear form on $\mathcal{D}$ by:

$B_L(f, h) = (L\sigma(f), \sigma(h))$, the scalar product being the one on V.

Clearly, $B_L(\rho_x f, \rho_x h) = B_L(f, h)$, $x \in G$. Thus there exists a unique distribution T_L such that $B_L(f,h) = T_L(f * h^*)$, where h^* denotes the function $x \mapsto \overline{h(x^{-1})}$. Moreover (for n in N)

$$(\lambda_n T_L)(f * h^*) = T_L(\lambda_{n^{-1}}(f * h^*)) = T_L(\lambda_{n^{-1}} f * h^*) = (L\sigma(\lambda_{n^{-1}} f), \sigma(h))$$
$$= \eta(n)(L\sigma(f), \sigma(h)) = \eta(n) T_L(f * h^*).$$

Thus $\lambda_n T_L = \eta(n) T_L$, for all n in N. Similarly,

$$(\rho_{n^{-1}} T_L)(f * h^*) = T_L(f * \rho_{n^{-1}} h^*) = T_L(f * (\lambda_n) *) = (L\sigma(f), \sigma(\lambda_n h))$$
$$= (L\sigma(f), \eta(n^{-1})\sigma(h)) = \eta(n) T_L(f * h^*).$$

Hence

(2.1) $$\lambda_{n_1} \rho^{\rho}_{n_2^{-1}} T_L = \eta(n_1 n_2) T_L, \quad n_1, n_2 \in N.$$

§3. The main theorem

We may choose an element w_0 in $\mathcal{N}(A)$ such that $w_0 N w_0^{-1} = \bar{N}$. Now, as in [18], we can find an antiautomorphism θ of G satisfying:

(3.1) $$\theta^2 = 1;$$

(3.2) $$\theta(a) = w_0 a w_0^{-1}, \quad a \in A;$$

(3.3) $$\theta(N) = N;$$

(3.4) $$\eta(\theta(n)) = \eta(n), \quad n \in N;$$

and

(3.5) $$\text{for fixed } w \text{ in } \mathcal{N}(A),\ \eta(n) = \eta(wnw^{-1}),$$
$$\forall n \in N \cap w^{-1} N w \Rightarrow \theta(w) = w.$$

When $G = GL_r(F)$, $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$, and $\eta(n_{ij}) = \psi(\Sigma_{i=1}^{r-1} n_{i,i+1})$, we can take w_0 to be $\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ and $\theta(x) = w_0 {}^t x w_0^{-1}$.

Next we extend θ to $\mathcal{D}$ by ${}^{\theta}f(g) = f({}^{\theta}g)$, and by duality to $\mathcal{D}'$. And we define $\tau \in \mathrm{Aut}(G)$ by ${}^{\tau}g = {}^{\theta}g^{-1}$. Since $\tau^2 = 1$, τ leaves stable the Haar measures of G and N, and the invariant measure on $N \backslash G$.

THEOREM: *For every L in* $\mathrm{Hom}_G(V, V)$, *T_L is fixed by θ.*

Before proving this theorem, let us show how this implies our result.

COROLLARY: *The algebra* $\mathrm{Hom}_G(V, V)$ *is commutative.*

PROOF: For any function φ on G, define ${}^\nu\varphi$ by ${}^\nu\varphi(g) = {}^\theta\varphi^*(g)$. Since θ fixes η and $\overline{\eta(n^{-1})} = \eta(n)$ for $n \in N$, we have:

$${}^\nu\varphi(ng) = \eta(n){}^\nu\varphi(g), \quad \varphi \in V.$$

We also have

$${}^\nu(\rho_x\varphi) = \overline{\rho_{\tau(x)}\varphi}.$$

Thus we get an algebra automorphism of $\mathrm{Hom}_G(V, V)$ by $L \mapsto \nu^{-1}L\nu$. Futher, since τ fixes the measure of the quotient $N\backslash G$ and since ${}^\nu\varphi(g) = \overline{{}^\tau\varphi(g)}$, we have $(\varphi_1, \varphi_2) = ({}^\nu\varphi_2, {}^\nu\varphi_1)$, for all φ_1, φ_2 in V. Now ${}^\theta T_L(f * h^*) = T_L({}^\theta h^* * {}^\theta f) = T_L({}^\nu h * {}^\nu f) = (L\sigma({}^\nu h), \sigma({}^\nu f))$. Since τ fixes the Haar measure of N and sends η to $\bar{\eta}$, $\sigma({}^\nu f) = {}^\nu\sigma(f)$ for every f in $\mathcal{D}$. Thus $L\sigma({}^\nu h) = {}^\nu(\nu^{-1}L\nu)\sigma(h)$. So

$$\begin{aligned}{}^\theta T_L(f * h^*) = ({}^\nu\nu^{-1}L\nu\sigma(h), {}^\nu\sigma(f)) &= (\sigma(f), \nu^{-1}L\nu\sigma(h)) \\ &= ((\nu^{-1}L\nu)^*\sigma(f), \sigma(h)).\end{aligned}$$

Comparing this with $T_L(f * h^*) = (L\sigma(f), \sigma(h))$, and taking into account the fact that $\mathrm{Im}(\sigma)$ is dense in V, we get:

$$L = (\nu^{-1}L\nu)^*.$$

If M is another commuting operator,

$$LM = (\nu^{-1}LM\nu)^* = (\nu^{-1}L\nu \cdot \nu^{-1}M\nu)^* = (\nu^{-1}M\nu)^* \cdot (\nu^{-1}L_\nu)^*$$

Hence

$$LM = ML. \qquad \text{Q.E.D.}$$

It remains to prove the theorem.

If F is non-archimedean, this follows directly from the relation (2.1) and theorem 1.6 in [18] (see also [6]).

Now suppose that F is archimedean. $G(F)$ as a Lie group is separable and type I. The latter property is a consequence of Harish Chandra's subquotient theorem for reductive Lie groups.

Let $\mathcal{U}$ be the universal enveloping algebra of Lie(G). Then we may identify it with the algebra of right invariant differential operators on G, and regard Lie(G) as a subset. Then $\mathcal{U}$ acts on distributions in such a way that:

$$(-1)(DT)(f) = T(Df), D \in \mathrm{Lie}(G), T \in \mathcal{D}' \text{ and } f \in \mathcal{D}.$$

Let Δ denote the Casimir operator on G. Then Δ belongs to the center of $\mathcal{U}$.

We next denote by $\mathcal{A}$ the center of $\mathrm{Hom}_G(V, V)$. Then according to a theorem of Von Neumann ([3] chap. V, or [12] chap. V), there exists a Borel space Λ, and a continuous sum decomposition:

$$(3.6) \qquad (\pi, V) \cong \bigoplus \int_{\Lambda} (\pi_\lambda, V_\lambda)\, d\mu_\lambda$$

such that

(a) $\mathcal{A}$ gets transformed into the algebra of diagonal operators on the direct integral;
(b) each π_λ is a unitary representation of G on V_λ satisfying

$$(\pi(g)v)_\lambda = \pi_\lambda(g)v_\lambda, v \in V, g \in G;$$

and
(c) for almost all λ, π_λ is a multiple of an irreducible representation.

A consequence of (c) is that for all f in $\mathcal{D}$,

$$(3.7) \qquad \pi_\lambda(\Delta f) = a_\lambda \pi(f), \quad \text{where } a_\lambda \text{ is a scalar.}$$

Indeed this is clear if G is connected as a reductive (algebraic) Lie group, as then π_λ has an infinitesimal central character. This is the case when $F = \mathbb{C}$. When $F = \mathbb{R}$, the restriction of π_λ to the connected component G^0 splits up into a finite sum of factors π_λ^j with infinitesimal central characters ν_j. Each ν_j differs from another by a twisting by an outer automorphism of G^0 (which has finite order). But Δ is fixed by every automorphism, since G is reductive and algebraic. Hence (3.7).

We call a locally convex space W with a G-action π a nuclear G-module if:

(3.8) W is a nuclear space in the sense of Grothendieck ([7]);

and

(3.9) the map $G \times W \overset{\pi}{\rightarrow} W$ is continuous.

It is classical (cf. [7], for example) that $\mathcal{D}$ is a nuclear G-module with respect to the right (as well as the left) action of G. This yields in a natural way a nuclear G-module structure on $\mathcal{D} \oplus \mathcal{D}$. The action is, as usual, given by $\rho^{\oplus}$: $(g, (f, h)) \mapsto (\rho_g f, \rho_g h)$, for $g \in G$, f, $h \in \mathcal{D}$.

We next define a linear map $\mu_L : \mathcal{D} \oplus \mathcal{D} \rightarrow V$ by $(f, h) \mapsto L\sigma(f) + \sigma(h)$, and let $X_L = \operatorname{Im} \mu_L$. This map is continuous since σ and $L\sigma$ are continuous as maps from D into V. So we may give the quotient nuclear topology to $X = X_L$. Further,

$$\mu_L(\rho_g f, \rho_g h) = L\sigma(\rho_g f) + \sigma(\rho_g h) = \rho_g L\sigma(f) + \rho_g \sigma(h) = \rho_g \mu_L(f, h).$$

And we get a commutative diagram:

$$\begin{array}{ccc} G \times (\mathcal{D} \oplus \mathcal{D}) & \overset{\rho^{\oplus}}{\longrightarrow} & \mathcal{D} \oplus \mathcal{D} \\ \Big\downarrow (1, \mu_L) & & \Big\downarrow \mu_L \\ G \times X & \overset{\rho}{\longrightarrow} & X \end{array}$$

This shows that $\rho : G \times X \rightarrow X$ is continuous with respect to the induced nuclear topology on X. We thus get a nuclear G-Module X embedded continuously in V. We now claim the following:

There exists an orthonormal basis (v_j) of V, a family (f_j) of continuous linear forms on X, and a sequence (β_j) of positive real numbers such that:

(3.10)
(a) $(x, v_j) = b_j f_j(x)$;
(b) $\sum \beta_j^2 < \infty$, and
(c) $x \mapsto \left(\sum |f_j(x)|^2\right)^{1/2} \underset{\text{def}}{=} \|x\|_N$ defines a continuous semi-norm on X.

Indeed, according to a theorem of Pietsch ([15]), there exists a countable projective family (X_n) of pre-Hilbert spaces with Hilbert-Schmidt transition maps such that $X = \varprojlim X_n$. Consequently the continuous embedding $X \hookrightarrow V$ factors as: $X \xrightarrow{\pi_n} X_n \xrightarrow{T} V$ with π_n: continuous and T: Hilbert-Schmidt. Now T extends to the completion $\tilde{X}_n$ of X_n, and T^*T is a positive trace-class operator on $\tilde{X}_n$. So we can find an orthonormal basis (e_j) of $\tilde{X}_n$ consisting of eigenvectors for T^*T with eigenvalues β_j^2 such that $\beta_j > 0$ and $\Sigma \beta_j^2 < \infty$. Set $v_j = \beta_j^{-1} T e_j$. Then (v_j) is an orthonormal basis of $\overline{T(X_n)}$ $(= V$, since X is dense in $V)$:

$$(\beta_j^{-1} T e_j, \beta_k^{-1} T e_k) = \beta_j^{-1} \beta_k^{-1} (T^* T e_j, e_k) = \beta_j^{-1} \beta_k^{-1} \beta_j^2 \delta_{jk}.$$

Thus

$$\begin{aligned}(x, v_j) = (T\pi_n x, v_j) = (T\pi_n x, \beta_j^{-1} T e_j) &= \beta_j^{-1} \beta_j^2 (\pi_n x, e_j) \\ &= \beta_j^{-1} \beta_j^2 (\pi_n x, e_j) = \beta_j (\pi_n x, e_j).\end{aligned}$$

For every j, set $f_j(x) = (\pi_n x, e_j)$. Then f_j is a continuous linear form on X, and $(x, v_j) = \beta_j f_j(x)$. Further, we have $\|x\|_N = (\Sigma\, |f_j(x)|^2)^{1/2} = \|\pi_n x\|$, and so $x \mapsto \|x\|_N$ is a continuous semi-norm on X. In particular, $\|x\|_N$ is finite.

Now we utilize the decomposition (3.6). Let $\lambda \mapsto v_j(\lambda)$ be the vector field representing v_j. Since $x = \Sigma\, \beta_j f_j(x) v_j$, we know that for almost all λ, the series $\Sigma\, \beta_j f_j(x) v_j(\lambda)$ converges in V_λ and the almost everywhere defined vector field $\lambda \mapsto \Sigma\, \beta_j f_j(x) v_j(\lambda)$ represents x. On the other hand, following Maurin [12],

$$\begin{aligned}\sum \|\beta_j f_j(x) v_j(\lambda)\| = \sum \beta_j |f_j(x)| \cdot \|v_j(\lambda)\| &\le \Big(\sum \beta_j^2 \|v_j(\lambda)\|^2\Big)^{1/2} \cdot \Big(\sum |f_j(x)|^2\Big)^{1/2} \\ &= \Big(\sum \beta_j^2 \|v_j(\lambda)\|^2\Big)^{1/2} \cdot \|x\|_N.\end{aligned}$$

Now,

$$\int_\Lambda \sum \beta_j^2 \|v_j(\lambda)\|^2 \, d\mu_\lambda = \sum \beta_j^2 \int_\Lambda \|v_j(\lambda)\|^2 \, d\mu_\lambda = \sum \beta_j^2 \|v_j\|^2 = \sum \beta_j^2 < \infty.$$

Hence the function $\lambda \mapsto \Sigma\, \beta_j^2 \|v_j(\lambda)\|^2$ is integrable and, in particular, almost everywhere finite, say for $\lambda \in \Lambda - \Lambda_1$, $\mu(\Lambda_1) = 0$. Hence for λ

outside Λ_1, the series $\Sigma\, \beta_j v_j(\lambda) f_j(x)$ converges absolutely and hence converges in V_λ. Moreover, if we set $c(\lambda) = (\Sigma\, \beta_j^2 \|v_j(\lambda)\|^2)^{1/2}$, then $\|\Sigma\, \beta_j f_j(x) v_j(\lambda)\| \leq c(\lambda) \|x\|_N$. Thus we get a continuous linear map

$$(3.11) \qquad M_\lambda : X \to V_\lambda,$$

$$x \mapsto \sum \beta_j f_j(x) v_j(\lambda), \quad \text{for every } \lambda \notin \Lambda_1.$$

Given g in G and x in X, we have for almost all λ:

$$M_\lambda(\pi(g)x) = (\pi(g)x)(\lambda) = \pi_\lambda(g)x(\lambda) = \pi_\lambda(g) M_\lambda(x).$$

Since X and G are separable, we can find countable dense sets X_0 in X and G_0 in G, and also a negligible set $\Lambda_0 \supset \Lambda_1$ such that (for $x \in X_0$, $g \in G_0$):

$$(3.12) \qquad M_\lambda \pi(g) = \pi_\lambda(g) M_\lambda.$$

Since for every g, the map $\{g\} \times X \xrightarrow{\pi} X$ is continuous, this relation holds for all x. And since $g \mapsto \pi(g)x$ and $g \mapsto \pi_\lambda(g)v_\lambda$ are both continuous, (3.12) actually holds for all x and all g.

Now for λ outside Λ_0, define a sesquilinear form B_λ on $\mathscr{D}$ by $B_\lambda(f, h) = (M_\lambda L\sigma(f),\ M_\lambda\sigma(h))$. Then B_λ is separately continuous in each variable, and $B_L(f, h) = \int_\Lambda B_\lambda(f, h)\, d\mu_\lambda$. Further, for all g in G, we have $B_\lambda(\rho_g f,\ \rho_g h) = (M_\lambda L\sigma(\rho_g f),\ M_\lambda\sigma(\rho_g h)) = (M_\lambda \pi_g L\sigma(f),\ M_\lambda \pi_g \sigma(h)) = (\pi_\lambda M_\lambda L\sigma(f),\ \pi_\lambda M_\lambda \sigma(h)) = (M_\lambda L\sigma(f),\ M_\lambda\sigma(h)) = B_\lambda(f, h)$. Thus there exists a unique distribution T_λ on G such that $T_\lambda(f * h^*) = B_\lambda(f, h)$, and

$$(3.13) \qquad T_L(f * h^*) = \int_\Lambda T_\lambda(f * h^*)\, d\mu_\lambda.$$

Clearly,

$$(3.14) \qquad \lambda_{n_1}\rho_{n_2^{-1}} T_\lambda = \eta(n_1 n_2) T_\lambda, \quad \text{for} \quad n_1, n_2 \in N.$$

Next observe that if f_1, f_2 are in $\mathscr{D}$, $\Delta(f_1 * f_2) = \Delta f_1 * f_2 = f_1 * \Delta f_2$. This is so because Δ is a bi-invariant operator. Thus for f_1, f_2, h in $\mathscr{D}$, $\Delta(f_1 * f_2 * h^*) = \Delta(f_1 * f_2) * h^* = f_1 * \Delta f_2 * h^*$, and

$$\begin{aligned}(\Delta T_\lambda)(f_1 * f_2 * h^*) &= T_\lambda(\Delta(f_1 * f_2 * h^*)) = T_\lambda(f_1 * \Delta f_2 * h^*)\\ &= B_\lambda(f_1 * \Delta f_2, h) = (M_\lambda \sigma_L(f_1 * \Delta f_2), M_\lambda \sigma(h))\\ &= (M_\lambda \pi(\Delta f_2)\sigma_L(f_1), M_\lambda\sigma(h))\\ &= (\pi_\lambda(\Delta f_2) M_\lambda \sigma_L(f_1), M_\lambda \sigma(h))\\ &= a_\lambda(\pi_\lambda(f_2) M_\lambda \sigma_L(f_1), M_\lambda\sigma(h))\\ &= a_\lambda(M_\lambda \pi(f_2)\sigma_L(f_1), M_\lambda\sigma(h))\\ &= a_\lambda(M_\lambda\sigma_L(f_1 * f_2), M_\lambda\sigma(h))\\ &= a_\lambda T_\lambda(f_1 * f_2 * h^*).\end{aligned}$$

Next we note that such triple products $f_1 * f_2 * h$, with $f_1, f_2, h \in \mathscr{D}$, are dense $\mathscr{D}$. Actually every function in $\mathscr{D}$ can be written as a finite linear combination of them ([4]). In any case, we have

$$(\Delta T_\lambda)(\varphi) = a_\lambda \varphi, \text{ for every } \varphi \text{ in } \mathscr{D}. \tag{3.15}$$

The relations (3.14) and (3.15) together with Theorem 2.1 of Shalika [18] imply that:

$$^\theta T_\lambda = T_\lambda, \text{ for every } \lambda \notin \Lambda_0. \text{ Now,} \tag{3.16}$$

$$\begin{aligned}^\theta T_L(f * h^*) = T_L({}^\nu h * {}^\nu f^*) &= \int_\Lambda T_\lambda({}^\nu h * {}^\nu f^*)\,\mathrm{d}\mu_\lambda = \int_\Lambda {}^\theta T_\lambda(f * h^*)\,\mathrm{d}\mu_\lambda\\ &= \int_\Lambda T_\lambda(f * h^*)\,\mathrm{d}\mu_\lambda = T_L(f * h^*).\end{aligned}$$

Thus

$$^\theta T_L = T_L. \qquad \text{Q.E.D.}$$

§4. Decomposition over the center

Now let ω be any character of Z, and let $\pi_\eta(\omega) = \mathrm{Ind}(G, ZN; \omega \cdot \eta)$. Let $V(\omega)$ denote the corresponding space. Clearly we have a decomposition:

$$(\pi_\eta, V) \cong \bigoplus \int_{\omega \in \hat{Z}} (\pi_\eta(\omega), V(\omega))\,\mathrm{d}\omega.$$

Fix $\omega \in \hat{Z}$. We can define a (right) G-equivariant linear map σ^{ω} from $\mathscr{D}$ to $V(\omega)$ by: $f \mapsto \int_{ZN} f(zng)\eta(n^{-1})\omega(z^{-1})\,dn\,dz$. And for every L in $\mathrm{Hom}_G(V(\omega), V(\omega))$, we can get a distribution T_L^{ω} satisfying $T_L^{\omega}(f * h^*) = (L\sigma^{\omega}(f), \sigma^{\omega}(h))$, $f, h \in \mathscr{D}$. Proceeding as above we can show that T_L^{ω} is θ-invariant, and obtain:

THEOREM: *$\mathrm{Hom}_G(V(\omega), (V(\omega))$ is commutative.*

COROLLARY: *The discrete components of $V(\omega)$ appear with multiplicity one.*

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