COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 45, nº 1 (1982), p. 109-113
<http://www.numdam.org/item?id=CM_1982__45_1_109_0>

COMPOSITIO MATHEMATICA, Vol. 45, Fasc. 1, 1981, pag. 109–113

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Introduction

The purpose of this note is to show that the space of linear combinations of certain Eisenstein series on a rational tube domain is defined over an explicitly computable ground field in the following sense: the space has a basis of automorphic forms all of whose Fourier coefficients lie in the ground field. We also take this opportunity to correct an omission in an earlier version of this theorem, which appeared as 5.5.3 in [3].

To illustrate and motivate the theorem, consider for a positive square-free integer m the normalizer $\Gamma_0(m)^+$ *in* $PSL(2, \mathbb{R})^0$ of Hecke's congruence subgroup $\Gamma_0(m)$; see [1]. It is known that $\Gamma_0(m)^+$ is a maximal discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})^0$ and that it has just one cusp. In particular, it therefore has one Eisenstein series of each weight satisfying the condition that the first Fourier coefficient be 1. An immediate consequence of what we are going to prove is that all the Fourier coefficients of these Eisenstein series are rational numbers. The earlier version of our theorem is not applicable here because it required that the arithmetic group be a congruence subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ or one of its conjugates. This answers a question posed by John Thompson and kindly communicated to the author by W.L. Baily, Jr. The referee has remarked that, for m a prime number, the answer is contained in [2], formula 19, since $G(\sqrt{m})$ is conjugate to $\Gamma_0(m)^+$ here.

* Supported in part by a grant from the National Science Foundation.

0010-437X81010109-05$00.20/0

1.

We now adopt the language of [3] and examine Theorem 5.5.3. Let G be a linear algebraic group defined over $\mathbb{Q}$ with maximal $\mathbb{Q}$-split torus T contained in a maximal parabolic $\mathbb{Q}$-subgroup P and let $\rho: P \to \mathrm{GL}(1)$ be a rational character defined over $\mathbb{Q}$, all subject to assumptions (A0) to (A5) of ([3], 1.2). In particular, the space of maximal compact subgroups of $G(\mathbb{R})$ is a rational tube domain D (= Siegel domain of type I with a 0-dimensional rational boundary component) and $P(\mathbb{R})^0$ acts as affine transformations on D. Coordinates are chosen so that

$$G(\mathbb{Q}_p) = P(\mathbb{Q}_p)G(\mathbb{Z}_p)$$

for each rational prime p. We identify $\mathrm{GL}(1, \hat{\mathbb{Z}})$ with the Galois group $\mathfrak{G}$ of the maximal abelian extension $\mathbb{Q}_{ab}$ of $\mathbb{Q}$, and take an embedding defined over $\mathbb{Q}$

$$\mu: \mathrm{GL}(1) \to \mathrm{Int}_G(T)$$

whose image is a split component of $\mathrm{Int}_G(P)$ and which gives vector space structure to the (abelian) unipotent radical of P.

The Eisenstein series are automorphic forms for a congruence subgroup

$$\Gamma(K) = G(\mathbb{Q}) \cap G(\mathbb{R})^0 \cdot K$$

for some open subgroup K of the group $G(\mathbb{A}_{na})$ of "finite adèle" points of G, see ([3], 2.3). Then, for 5.5.3, one assumes that $K \subseteq G(\hat{\mathbb{Z}})$, and the statement should read as follows.

THEOREM: *Suppose that* $\mathfrak{G}$ *normalizes both* $G(\hat{\mathbb{Z}})$ *and* K. *Then each of the spaces* $\mathfrak{E}$ *and* ${}^0\mathfrak{E}$ *has a basis of Eisenstein series defined over* $\mathbb{Q}$.

In [3] we omitted the hypothesis that $\mathfrak{G}$ normalize K both in 5.5.2 and 5.5.3, where it is used implicitly when Lemma 5.2.1 is invoked.

We want to eliminate the requirement that K lie in $G(\hat{\mathbb{Z}})$. The idea is to show that the Galois group $\mathfrak{G}$, acting via its effect on Fourier coefficients, preserves the space of Eisenstein series for $\Gamma = \Gamma(G)$. This in turn comes from the following: the Eisenstein series in question lift to adèlic Eisenstein series ${}_{\mathfrak{A}}\Phi_s$ as in ([3], 1.2) with

$\mathfrak{A} = (G, T, P, \rho, K \cap G(\hat{\mathbb{Z}}))$ and with s varying throughout a space ${}^0V$ of functions on $G(\mathbb{A})$, which is stable under a certain action of $\mathfrak{G}$.

2.

Fix an open compact subgroup K of $G(\mathbb{A})$, let $\mathfrak{A} = (G, T, P, \rho, K \cap G(\hat{\mathbb{Z}}))$ be as above, and recall that the character ρ determines a holomorphic factor of automorphy j on $G(\mathbb{R})^0 \times D$, as in ([3], 1.5.3). To start with, we lift the Eisenstein series $\mathscr{E}_a$ attached to a point-cusp $P(\mathbb{Q})a\Gamma \subseteq G(\mathbb{Q})$ for $\Gamma = \Gamma(K)$. By definition, if Γ_a is the stability group of a in Γ, then

$$\mathscr{E}_a(z) = \sum_{\gamma} j(a\gamma, z), \quad (\gamma \in \Gamma_a/\Gamma).$$

For the lifting we introduce the operator Tr_K on the space $C^\infty(G(\mathbb{A}))$ of all complex-valued locally constant functions on $G(\mathbb{A})$ as follows: For $f \in C^\infty(G(\mathbb{A}))$ we define

$$\mathrm{Tr}_K(f) : g \mapsto \mathrm{meas}(K)^{-1} \int_K f(gx)\, \mathrm{d}x$$

for any Haar measure $\mathrm{d}x$ on $G(\mathbb{A})$ giving volume $\mathrm{meas}(K)$ to K. In ([3], 2.3) there are attached to the character ρ a norm N and a spherical function φ. Let

$$N_K = \mathrm{Tr}_K(N) \quad \text{and} \quad \varphi_K = \mathrm{Tr}_K(\varphi).$$

Define s_a to be $N_K(a)^{-1}$ times the characteristic function of $P(\mathbb{Q}) \cdot a \cdot G(\mathbb{R})^0 \cdot K$ in $G(\mathbb{A})$. Then a brief calculation shows that the function

$$\Phi : g \mapsto \sum_{\gamma} s_a(\gamma s)\varphi_K(\gamma g), \quad (\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})),$$

is a lift of $\mathscr{E}_a$ to $G(\mathbb{A})$.

Let ${}^0\mathfrak{E}$ denote the space of all linear combinations of the Eisenstein series $\mathscr{E}_a$, $a \in G(\mathbb{Q}) \cap G(\mathbb{R})^0$. We next write down a space ${}^0V$ such that each ${}_{\mathfrak{A}}\Phi_s$ with $s \in {}^0V$ is the lift of an element (Eisenstein series) in ${}^0\mathfrak{E}$, and conversely.

Given an open compact subgroup H of $G(\mathbb{A}_{na})$ let $H^+ = G(\mathbb{R})^0 \cdot H$ and $S(H)$ (*resp.* ${}^0S(H)$) be the space of all complex-valued functions on $P(\mathbb{Q})\backslash G(\mathbb{A})/H^+$ (*resp.* $P(\mathbb{Q})\backslash P(\mathbb{Q})H^+/H^+$). Let

$$V = S(K)\varphi_K\varphi^{-1} \quad \text{and} \quad {}^0V = {}^0S(K)\varphi_K\varphi^{-1}.$$

Both are clearly subspaces of $S(K \cap G(\hat{\mathbb{Z}}))$ and if $s \in {}^0V$, then

$${}_{\mathfrak{A}}\Phi_s : g \mapsto \sum_{\gamma} s(\gamma g)\varphi(\gamma g), \quad (\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q}))$$

is the lift of some $\mathscr{E} \in {}^0\mathfrak{E}$, (i.e., an Eisenstein series for Γ). Let $\mathfrak{E}$ be the space of Γ-automorphic forms that lift to functions ${}_{\mathfrak{A}}\Phi_s$ with $s \in V$.

3.

Suppose now that $\mathfrak{G}$ normalizes both $G(\hat{\mathbb{Z}})$ and K. For each $\tau \in \mathfrak{G}$ recall that there is an operator τ^* on $C^\infty(G(\mathbb{A}))$ defined by

$$\tau^* f : g \mapsto f(w \circ \mu(\tau)^{-1}(w^{-1}g))$$

for $f \in C^\infty(G(\mathbb{A}))$, where w is chosen as in 1.4.3 (it lies in $G(\mathbb{Q})$ and gives a Cartan involution of D). To show that $\mathfrak{E}$ and ${}^0\mathfrak{E}$ are defined over $\mathbb{Q}$ it suffices to show that V and ${}^0V$ are stable under all τ^* with $\tau \in \mathfrak{G}$; see Lemma 5.5.1 of [3]. Since $\mathfrak{G}$ normalizes $G(\hat{\mathbb{Z}})$ and K, Lemmas 5.2.1 and 5.5.2 of [3] show that $S(K)$ and ${}^0S(K)$ are invariant under all τ^*. Thus it suffices to show that $\tau^*\varphi = \varphi$, which follows at once from Lemma 5.2.2 of [3], and that $\tau^*\varphi_K = \varphi_K$. Since $\mathfrak{G}$ normalizes K, the operators τ^* all commute with Tr_K; hence, $\tau^*\varphi_K = \varphi_K$. This gives the desired extension of the theorem stated at the beginning to cover the case of arbitrary compact open K normalized by $\mathfrak{G}$.

4.

Now drop the assumption that $\mathfrak{G}$ normalize $G(\hat{\mathbb{Z}})$ and K. If one intersects the normalizers $\mathcal{N}_{\mathfrak{G}}(G(\hat{\mathbb{Z}}))$ and $\mathcal{N}_{\mathfrak{G}}(K)$, one obtains an open subgroup of $\mathfrak{G}$, necessarily of finite index. Call it $\mathfrak{G}_K$. Then it is an easy exercise to prove that $\mathfrak{E}$ and ${}^0\mathfrak{E}$ are both defined over the fixed field of $\mathfrak{G}_K$.

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(Oblatum 20-VI-1980 & 4-XII-1980)

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