COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 44, nº 1-3 (1981), p. 375-394

<http://www.numdam.org/item?id=CM_1981__44_1-3_375_0>

COMPOSITIO MATHEMATICA, Vol. 44, Fasc. 1-3, 1981, pag. 375–394

COMPLEX GEODESICS

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Let D be a domain in a complex, locally convex, Hausdorff vector space $\mathscr{E}$. The Carathéodory and Kobayashi invariant pseudo-distances have been introduced in D, together with the corresponding infinitesimal pseudo-metrics [14]. A holomorphic map of the unit disc Δ of $\mathbb{C}$ into D which is an isometry for the Poincaré distance of Δ and the Carathéodory or the Kobayashi pseudo-distances of D is called a complex geodesic. It is easily seen that complex geodesics do not always exist. However, their existence turns out to be a useful tool in the investigation of the group of all holomorphic automorphisms of D.

The primary purpose of this paper is that of establishing basic properties of complex geodesics, clarifying in particular the relationship between the Carathéodory pseudo-distance and the Carathéodory pseudo-metric along a complex geodesic. This latter result will lead to conditions whereby holomorphic maps of D which preserve complex geodesics are necessarily affine maps. As a consequence, some results obtained in [10] and in [13] will be improved.

All these facts will be established in nn.3 and 4. Some basic material will be collected in n.1, and in n.2 the relationship between the "size" of D and the behaviour of the Kobayashi pseudo-metric will be investigated; as a consequence, classical results of the theory of conformal mappings will be extended to the domain D in $\mathscr{E}$.

1. Let $\mathscr{E}$ and $\mathscr{E}_1$ be two complex locally convex Hausdorff vector spaces, and let D and D_1 be domains in $\mathscr{E}$ and in $\mathscr{E}_1$ respectively. By definition (cf. e.g. [8]), a holomorphic map $F: D \to D_1$ is a continuous, Gateaux-analytic map F of D into $\mathscr{E}_1$ such that $F(D) \subset D_1$. The

0010-437X/81/09375-20$00.20/0

symbol $\mathrm{Hol}(D, D_1)$ will denote the set of all holomorphic maps of D into D_1.

If D is an open convex neighborhood of 0 in $\mathscr{E}$, the support function m of D is a convex, continuous function in $\mathscr{E}$. Let Δ be the open unit disc in $\mathbb{C}$, and let $f \in \mathrm{Hol}(\Delta, D)$. For any $\zeta \in \Delta$ and any sufficiently small $r > 0$,

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta + \mathrm{r}e^{i\theta}) \mathrm{d}\theta.$$

Hence

$$m(f(\zeta)) = m\left(\frac{1}{2\pi} \int_0^{2\pi} f(\zeta + \mathrm{r}e^{i\theta}) \mathrm{d}\theta\right) \leq \frac{1}{2\pi} \int_0^{2\pi} m(f(\zeta + \mathrm{r}e^{i\theta})) \mathrm{d}\theta.$$

That proves

LEMMA 1.1: *For any $f \in \mathrm{Hol}(\Delta, D)$, the function $\zeta \mapsto m(f(\zeta))$ is subharmonic on Δ.*

REMARK: If the domain D is furthermore a balanced neighborhood of 0, then m is a semi-norm and the function $\zeta \mapsto \log m(f(\zeta))$ is subharmonic [12].

By lemma 1.1 the function $m \circ f$ satisfies a maximum principle, whereby, if $f(\Delta) \cap \partial D \neq \emptyset$, then $f(\Delta) \subset \partial D$. In the following a stronger form of the maximum principle will be needed, which was established by E. Thorp and R. Whitley in [11] for complex Banach spaces. As it was shown in [11] (cf. also [12]), there exists a constant c, $0 < c < 1$, such that, for every positive integer n, complex numbers $z_1, \ldots, z_n$ can be found satisfying the following conditions:

$$|z_\alpha| < \frac{1}{c} \ (\alpha = 1, \ldots, n); \sum_{\alpha=1}^{n} z_\alpha = n; \sum_{\alpha=1}^{n} z_\alpha^p = 0 \text{ for } p = 2, \ldots, n.$$

The proof of the following lemma—given originally in [11] in the case of a Banach space—can be adapted to locally compact spaces.

LEMMA 1.2: *For every non-constant holomorphic function $f: \Delta \to \mathscr{E}$, there is an element $a \in \mathscr{E} \backslash \{0\}$ such that the image of the open disc $\{\zeta \in \mathbb{C}: |\zeta| < c\}$ by the affine map $\zeta \mapsto f(0) + \zeta a$ belongs to the closure of the convex hull of $f(\Delta)$.*

PROOF: Assuming $f(0) = 0$, let

$$f(\zeta) = \zeta a_1 + \zeta^2 a_2 + \cdots$$

be the power series expansion of f in Δ. If p is any continuous semi-norm on $\mathscr{E}$, the series $\Sigma_1^{+\infty} p(a_\nu)t^\nu$ converges for $0 \leq t < 1$. Then, choosing n complex numbers $z_1, \ldots, z_n$ as before,

$$\sum_{\alpha=1}^{n} f(z_\alpha \zeta) = n\zeta a_1 + h_n(\zeta),$$

where

$$h_n(\zeta) = \left(\sum_{\alpha=1}^{n} z_\alpha^{n+1}\right)\zeta^{n+1} a_{n+1} + \left(\sum_{\alpha=1}^{n} z_\alpha^{n+2}\right)\zeta^{n+2} a_{n+2} + \cdots.$$

Since

$$p\left(\frac{1}{n} h_n(\zeta)\right) \leq \sum_{n+1}^{+\infty} p(a_\nu)\left(\frac{|\zeta|}{c}\right)^\nu,$$

then

$$\lim_{n\to\infty} p\left(\frac{1}{n} h_n(\zeta)\right) = 0,$$

for $|\zeta| < c$ and for any continuous semi-norm p. Therefore

$$\lim_{n\to\infty} \frac{1}{n} h_n(\zeta) = 0,$$

i.e.

$$\lim_{n\to\infty} \frac{1}{n} f(z_\alpha \zeta) = \zeta a_1 = \zeta f'(0) \quad \text{for} \quad |\zeta| < c.$$

That proves the theorem when $f'(0) \neq 0$. If $f'(0) = 0$, let

$$f(\zeta) = \zeta^n a_n + \zeta^{n+1} a_{n+1} + \cdots \qquad (a_n \neq 0)$$

be the power series expansion of f. Denoting by $\epsilon_1, \ldots, \epsilon_n$ the n-roots of 1, the holomorphic function $g: \Delta \to \mathscr{E}$ defined by

$$g(\zeta) = \frac{1}{n}(f(\epsilon_1 \zeta^{1/n}) + \cdots + f(\epsilon_n \zeta^{1/n}))$$

belongs to the convex envelope of $f(\Delta)$ and is such that

$$g'(0) = a_n. \qquad \text{QED}$$

Recall that a point $x_0 \in \partial D$ is a complex extreme point of $\bar{D}$ if $y = 0$ is the only vector such that $x_0 + \Delta y \subset \bar{D}$. Lemmas 1.1 and 1.2 yield

Proposition 1.3: *Let D be an open, convex neighborhood of 0 in $\mathscr{E}$. If $f \in \mathrm{Hol}(\Delta, D)$ is such that $f(\Delta) \cap \partial D \neq \emptyset$, then $f(\Delta) \subset \partial D$. If $f(\Delta) \cap \partial D$ contains a complex extreme point of $\bar{D}$, then f is constant.*

Remark: Proposition 1.3 extends a similar statement proved in [14] in the case in which D is balanced. The proof given in [14] is a straightforward consequence of a simplified version of Thorp and Whitley's original argument, which was established by L.A. Harris in [5] (cf. also [3]).

2. The Poincaré metric of Δ

$$ds^2 = (1 - |\zeta|^2)^{-2}|d\zeta|^2 \qquad (\zeta \in \Delta)$$

has Gaussian curvature -4. For $\zeta \in \Delta$, $\tau \in \mathbb{C}$, set

$$\langle \tau \rangle_\zeta = (1 - |\zeta|^2)^{-1}|\tau|.$$

For ζ_1, ζ_2 in Δ, the Poincaré distance is expressed by

$$\omega(\zeta_1, \zeta_2) = \frac{1}{2} \log \frac{1 + \left|\dfrac{\zeta_1 - \zeta_2}{1 - \bar{\zeta}_1\zeta_2}\right|}{1 - \left|\dfrac{\zeta_1 - \zeta_2}{1 - \bar{\zeta}_1\zeta_2}\right|}.$$

Given any two points x and y in the domain $D \subset \mathscr{E}$, an analytic chain joining x and y in D consists of $\nu + 1$ points $\zeta_0, \ldots, \zeta_\nu$ in Δ, and of ν functions $f_j \in \mathrm{Hol}(\Delta, D)(j = 1, \ldots, \nu)$ such that:

$$f_1(\zeta_0) = x,\ f_j(\zeta_j) = f_{j+1}(\zeta_j)(j = 1, \ldots, \nu - 1),\ f_\nu(\zeta_\nu) = y.$$

The open set D being connected, analytic chains joining x and y in D do exist. The Kobayashi pseudo-distance $k_D(x, y)$ is defined by

$$k_D(x, y) = \inf \sum_{j=1}^{\nu} \omega(\zeta_{j-1}, \zeta_j),$$

where the infimum is taken over all analytic chains joining x and y in D.

The Carathéodory pseudo-distance $c_D(x, y)$ is, by definition,

$$c_D(x, y) = \sup\{\omega(h(x), h(y)) \colon h \in \mathrm{Hol}(D, \Delta)\}$$

It turns out [14] that

$$c_D(x, y) \leq k_D(x, y),$$

and

$$c_\Delta = k_\Delta = \omega.$$

The infinitesimal versions of k_D and c_D are defined as follows: the Kobayashi pseudo-metric is given, for $x \in D$, $v \in \mathscr{E}$, by

$$\kappa_D(x; v) = \inf\{\langle\tau\rangle_\zeta \colon \zeta \in \Delta, \tau \in \mathbb{C}, f \in \mathrm{Hol}(\Delta, D), f(\zeta) = x, \tau f'(\zeta) = v\}, \tag{1}$$

and the Carathéodory pseudo-metric by

$$\gamma_D(x; v) = \sup\{\langle \mathrm{d}h(x)v\rangle_{h(x)} \colon h \in \mathrm{Hol}(D, \Delta)\}.$$

The domain D being open, the set appearing on the right-hand side of (1) is not empty. Furthermore [3]

$$\gamma_D(x; v) \leq \kappa_D(x; v)$$

and

$$\gamma_\Delta(\zeta; \tau) = \kappa_\Delta(\zeta; \tau) = \langle\tau\rangle_\zeta \quad (\zeta \in \Delta, \ \tau \in \mathbb{C}).$$

These pseudo-distances and pseudo-metrics are all contracted by holomorphic maps: for $F \in \mathrm{Hol}(D, D_1)$

$$c_{D_1}(F(x), F(y)) \leq c_D(x, y), \ k_{D_1}(F(x), F(y)) \leq k_D(x, y),$$
$$\gamma_{D_1}(F(x); \mathrm{d}F(x)v) \leq \gamma_D(x; v), \ \kappa_{D_1}(F(x); \mathrm{d}F(x)v) \leq \kappa_D(x; v)$$

for all x, $y \in D$, $v \in \mathscr{E}$.

For further details on the above defined pseudo-distances and pseudo-metrics see e.g. [3] in the case of domains in complex Banach spaces, and [14] for the general case.

For any continuous semi-norm p on $\mathscr{E}$, any $x \in \mathscr{E}$ and any $r \geq 0$, $B_p(x, r)$ denotes the open ball with center x and radius r for the pseudo-distance defined by p. Then ([3], [13], [14])

$$\gamma_{B_p(x,\, r)}(x;\, v) = \kappa_{B_p(x,\, r)}(x;\, v) = \frac{p(v)}{r},$$

$$c_{B_p(x,r)}(x,\, y) = k_{B_p(x,r)}(x,\, y) = \omega\left(0, \frac{p(y-x)}{r}\right) \tag{2}$$

for all $y \in B_p(x, r)$, $v \in \mathscr{E}$.

The following theorem links the behaviour of the Kobayashi pseudo-metric κ_D of a domain $D \subset \mathscr{E}$ to the "size" of D.

THEOREM 1: *Let $x_0 \in D$, $v_0 \in \mathscr{E}$ be such that $\kappa_D(x_0; v_0) > 0$. Then*

$$(\mathscr{E} \backslash D) \cap \left\{x_0 + \frac{\zeta}{\kappa_D(x_0;\, v_0)}\, v_0\colon \zeta \in \bar{\Delta}\right\} \neq \emptyset. \tag{3}$$

Furthermore, there is no continuous semi-norm p on $\mathscr{E}$ such that D is completely interior to $B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0;\, v_0)}\right)$; i.e. there is no continuous semi-norm p such that

$$D \subset B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0;\, v_0)}\right) \tag{4}$$

and

$$\inf\left\{p(x-y)\colon x \in D,\ y \notin B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0;\, v_0)}\right)\right\} > 0 \tag{5}$$

PROOF: If (3) does not hold, there is some ϵ such that $0 < \epsilon < \kappa_D(x_0; v_0)$ and

$$\left\{x_0 + \frac{\zeta}{\kappa_D(x_0;\, v_0) - \epsilon}\, v_0\colon \zeta \in \Delta\right\} \subset D.$$

Since the holomorphic function $f\colon \zeta \mapsto x_0 + \dfrac{\zeta}{\kappa_D(x_0;\, v_0) - \epsilon}\, v_0$ maps Δ into D, and moreover

$$\varphi(0) = x_0,\ (\kappa_D(x_0;\, v_0) - \epsilon) f'(0) = v_0,$$

then, by (1),

$$\kappa_D(x_0;\ v_0) \leq \kappa_D(x_0;\ v_0) - \epsilon,$$

which is a contradiction.

Let p be a continuous semi-norm on $\mathscr{E}$ for which (4) and (5) hold. Then, there is some $\epsilon > 0$ such that

$$D \subset B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0;\ v_0) + \epsilon}\right),$$

and therefore, setting $A = B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0;\ v_0) + \epsilon}\right)$,

$$\kappa_D(x_0;\ v_0) \geq \kappa_A\,(x_0;\ v_0) = \frac{p(v_0)}{\dfrac{p(v_0)}{\kappa_D(x_0;\ v_0) + \epsilon}} = \kappa_D(x_0;\ v_0) + \epsilon.$$

Contradiction. QED

Let D be a simply connected domain $D \subsetneq \mathbb{C}$. For any $x \in D$, let h be the conformal map of D onto Δ with $h(x) = 0$, $h'(x) > 0$. Then

$$\kappa_D(x;1) = \langle h'(x)\rangle_{h(x)} = h'(x),$$

and therefore $\kappa_D(x;1)^{-1}$ is the conformal radius of D at x. Theorem 1 becomes in this case a classical result of complex function theory (cf. [1], [4]).

The relative topology of D in $\mathscr{E}$ is finer than the topologies defined by the pseudo-distances c_D and k_D. If the relative topology coincides with the topology defined by k_D, D is called a *hyperbolic domain.* A bounded domain is hyperbolic [14]. Since $\mathscr{E}$ is a Hausdorff space, on a hyperbolic domain k_D is a distance. Moreover the pseudo-metric κ_D cannot degenerate, as is shown by the following

THEOREM 2: *Let D be a hyperbolic domain. For $x_0 \in D$ let $r > 0$ and let p be a continuous semi-norm on $\mathscr{E}$ such that*

$$B_p(x_0, r) \subset D.$$

There exists a positive constant c such that

$$\kappa_D(x_0;\ v) \geq cp(v)$$

for all $v \in \mathscr{E}$.

PROOF: For $s > 0$ let $B_k(x_0, s)$ be the open ball with center x_0 and radius s for the distance k_D. The domain D being hyperbolic, there is $s > 0$ such that

$$B_k(x_0, s) \subset B_p(x_0, r).$$

If the conclusion of the theorem is false, there is a sequence $\{v_\nu\}$ in $\mathscr{E}$ such that $p(v_\nu) = 1$ and $\lim_{\nu\to\infty} \kappa_D(x_0; v_\nu) = 0$. Assume $\kappa_D(x_0; v_\nu) < 1$ for all ν, and let ϵ_ν be such that

$$\kappa_D(x_0;\ v_\nu) < \epsilon_\nu < 1,$$

and $\lim_{\nu\to\infty} \epsilon_\nu = 0$. Let $f_\nu \in \mathrm{Hol}(\Delta, D)$ and $\tau_\nu \in \mathbb{C}$ be such that

$$f_\nu(0) = x_0, \quad \tau_\nu f'_\nu(0) = v_\nu, \quad |\tau_\nu| < \epsilon_\nu.$$

The latter condition implies that

$$p(f'_\nu(0)) > \frac{1}{\epsilon_\nu}. \tag{6}$$

Let $\delta_\nu = \epsilon_\nu^{1/2}$. Being, for $\nu \gg 0$,

$$f'_\nu(0) = \frac{1}{2\pi\delta_2}\int_0^{2\pi} e^{-i\theta} f_\nu(\delta_\nu e^{i\theta})d\theta,$$

then

$$p(f'_\nu(0)) \leq \frac{1}{\delta_\nu}\sup\{p(f_\nu(\zeta)) \colon |\zeta| < \delta_\nu\},$$

and (6) yields

$$\sup\{p(f_\nu(\zeta)) \colon |\zeta| < \delta_\nu\} > \frac{\delta_\nu}{\epsilon_\nu} = \epsilon_\nu^{-1/2}.$$

Since $\lim_{\nu\to\infty} \epsilon_\nu^{-1/2} = +\infty$, there is an index ν_0 and—for every $\nu > \nu_0$—

some $|\zeta_\nu| \le \delta_\nu$, such that $p(f_\nu(\zeta_\nu)) > r$. Therefore $k_D(x_0, f_\nu(\zeta_\nu)) > s$. But this is a contradiction, for

$$k_D(x_0, f_\nu(\zeta_\nu)) \le \omega(0, \zeta_\nu) = \frac{1}{2}\log\frac{1+|\zeta_\nu|}{1-|\zeta_\nu|} \le \frac{1}{2}\log\frac{1+\delta_\nu}{1-\delta_\nu} \to 0$$

as $\nu \to \infty$. QED

If $\mathscr{E}$ is locally bounded, then for any $v \in \mathscr{E}\backslash\{0\}$ the continuous semi-norm p can be chosen in such a way that $p(v) > 0$.

COROLLARY: *If D is a hyperbolic domain in a complex Banach space $\mathscr{E}$, for any $x_0 \in D$ there is a positive constant c such that*

$$\kappa_D(x_0;\ v) \ge c\|v\|$$

for all $v \in \mathscr{E}$.

The corollary improves Proposition V.1.9 of [3].

3. Let D be a domain in $\mathscr{E}$ and let $f \in \mathrm{Hol}(\Delta, D)$. For all ζ_0, $\zeta \in \Delta$,

$$c_D f(\xi_0), f(\zeta)) \le k_D(f(\zeta_0), f(\zeta)) \le \omega(\zeta_0, \zeta). \tag{7}$$

If there is $\zeta_0 \in \Delta$ such that $c_D(f(\zeta_0), f(\zeta)) = \omega(\zeta_0, \zeta)$ $(k_D(f(\zeta_0), f(\zeta)) = \omega(\zeta_0, \zeta))$ for all $\zeta \in \Delta$, f is called a *complex geodesic* for c_D (respectively for k_D) at $f(\zeta_0)$. Inequality (7) yields the first part of the following lemma (the second part is trivial).

LEMMA 3.1: *If f is a complex geodesic for c_D, then f is a complex geodesic for k_D. If $k_D(c_D)$ is a distance and if f is a complex geodesic for $k_D(c_D)$ at $f(\zeta_0)$, then $f(\Delta)$ is closed in D for the k_D-topology (c_D-topology).*

REMARK: Let D be a non simply connected bounded domain in $\mathbf{C}$. Since any nonconstant $f \in \mathrm{Hol}(\Delta, D)$ is an open map, the above lemma shows that there are no complex geodesics for either c_D or k_D.

PROPOSITION 3.2: *If $f \in \mathrm{Hol}(\Delta, D)$ and $\zeta_0 \in \Delta$ are such that*

$$\gamma_D(f(\zeta_0);\ f'(\zeta_0)) = \langle 1 \rangle_{\zeta_0},$$

then f is a complex geodesic for c_D at $f(\zeta_0)$ in D.

PROOF: By composing f on the right with the Moebius transformation

$$\zeta \mapsto \frac{\zeta + \zeta_0}{1 + \bar{\zeta}_0 \zeta},$$

there is no restriction in assuming $\zeta_0 = 0$, and therefore

$$\gamma_D(f(0); f'(0)) = 1.$$

By definition, there is a sequence $\{h_\nu\}(h_\nu \in \mathrm{Hol}(D, \Delta))$ such that $h_\nu(f(0)) = 0$, and

$$\lim_{\nu \to \infty} |\mathrm{d}h_\nu(f(0))f'(0)| = 1.$$

In view of Montel's theorem, the sequence $\{h_\nu \circ f\}$ of holomorphic maps $h_\nu \circ f \in \mathrm{Hol}(\Delta, \Delta)$ contains a subsequence $\{h_{\nu_j} \circ f\}$ normally convergent on compact sets of Δ to a map $g \in \mathrm{Hol}(\Delta, \Delta)$. Since $g(0) = \lim_{j \to \infty} h_{\nu_j}(f(0)) = 0$, $|g'(0)| = \lim_{j \to \infty} |dh_{\nu_j}(f(0))f'(0)| = 1$, then, by the Schwarz lemma, g is a holomorphic automorphism of Δ. The sequence of inequalities

$$\omega(0, \zeta) \geq c_D(f(0), f(\zeta)) \geq \omega(h_{\nu_j}(f(0)), h_{\nu_j}(f(\zeta)))$$

yields, as j tends to infinity,

$$\omega(0, \zeta) \geq c_D(f(0), f(\zeta)) \geq \omega(g(0), g(\zeta)) = \omega(0, \zeta),$$

for all $\zeta \in \Delta$. QED

A similar argument leads to the following

PROPOSITION 3.3: *If there are two distinct points ζ_0, ζ_1 in Δ, such that*

$$c_D(f(\zeta_0), f(\zeta_1)) = \omega(\zeta_0, \zeta_1),$$

then $f \colon \Delta \to D$ is a complex geodesic for c_D at $f(\zeta_0)$.

PROOF: By definition there is a sequence $h_\nu \in \mathrm{Hol}(D, \Delta)$ such that

$$\lim_{\nu \to \infty} \omega(h_\nu(f(\zeta_0)), h_\nu(f(\zeta_1))) = \omega(\zeta_0, \zeta_1).$$

By Montel's theorem, the sequence $\{h_\nu \circ f\}$ contains a subsequence $\{h_{\nu_j} \circ f\}$ normally convergent on compact sets of Δ to a map $g \in \mathrm{Hol}(\Delta, \Delta)$ for which

$$\omega(g(\zeta_0), g(\zeta_1)) = \omega(\zeta_0, \zeta_1).$$

Hence, by the Schwarz–Pick lemma g is a holomorphic automorphism of Δ, and therefore

$$\begin{aligned}\omega(\zeta_0, \zeta) \geq c_D(f(\zeta_0), f(\zeta)) &\geq \lim_{j\to\infty} \omega(h_{\nu_j}(f(\zeta_0)), h_{\nu_j}(f(\zeta))) = \\ &= \omega(g(\zeta_0), g(\zeta)) = \omega(\zeta_0, \zeta)\end{aligned}$$

for all $\zeta \in \Delta$. QED

COROLLARY: *If $f: \Delta \to D$ is a complex geodesic for c_D at some point $f(\zeta_0)(\zeta_0 \in \Delta)$, then f is a complex geodesic for c_D at any point $\zeta \in \Delta$.*

If $f \in \mathrm{Hol}(\Delta, D)$ is a complex geodesic for k_D or c_D, and if φ is any Moebius transformation of Δ, then $f \circ \varphi$ is a complex geodesic for k_D or c_D respectively. To discuss the converse to this statement, let f and g be two injective holomorphic maps of Δ into D, having the same range: $f(\Delta) = g(\Delta)$.This fact, together with injectivity, sets up a bi-jective map φ of Δ onto Δ, defined by $g = f \circ \varphi$. To prove that φ is holomorphic, let λ be a continuous linear form on $\mathscr{E}$ such that $\lambda \circ f$ is not constant. The maps $\zeta \mapsto \lambda(f(\zeta))$ and $\zeta \mapsto \lambda(g(\zeta))(\zeta \in \Delta)$ are holomorphic. If $(\lambda \cdot f)'(\tau_0) = \lambda(f'(\tau_0)) \neq 0$ at some point $\tau_0 \in \Delta$, then, by the inverse function theorem, φ is holomorphic in a neighborhood of $\varphi^{-1}(\tau_0)$. The set of points $\{\tau \in \Delta: (\lambda \circ f)'(\tau) = 0\}$ is discrete in Δ. Since $|\varphi|$ is bounded in Δ, the Riemann extension theorem implies that φ is holomorphic on Δ, and—being bijective—is therefore a holomorphic automorphism of Δ. The above argument yields

PROPOSITION 3.4: *Let f and g be two complex geodesics for k_D (or for c_D). Then f and g have the same range if, and only if, there is a Moebius transformation φ of Δ such that $g = f \circ \varphi$.*

Let p be a continuous semi-norm on $\mathscr{E}$, and let $B_p = B_p(0, 1)$ be the open unit ball for p. A direct application of (2), of the Hahn–Banach theorem and of Proposition 1.3 yields the following lemma [13], [14].

LEMMA 3.5: *For any $x \in B_p$ for which $p(x) > 0$ the map $f \in$ Hol(Δ,B) defined by $f(\zeta) = \dfrac{\zeta}{p(x)}x$ is a complex geodesic for c_{B_p}. If*

$f(\bar{\Delta}) \cap \partial B_p$ contains a complex extreme point of $\bar{B}_p$, then f is (up to a change of parameter expressed by a Moebius transformation in Δ) the unique complex geodesic for c_{B_p} (unique also for k_{B_p}) whose range contains 0 *and x.*

EXAMPLE: Let B be the open unit ball of a complex Hilbert space $\mathscr{H}$. Every boundary point of B is a real (hence complex) extreme point of $\bar{B}$. Lemma 3.5 implies that for any $x \in B\backslash\{0\}$ the map $\zeta \mapsto \frac{\zeta}{\|x\|} x$ of Δ into B is the unique complex geodesic for c_B whose range contains 0 and x. The group Aut(B) of all holomorphic automorphisms of B acts transitively on B (cf. e.g. [3], Proposition VI.1.5, pp. 148–149). Hence, for any $y \in B$, there is $F \in \text{Aut}(B)$ such that $F(y) = 0$. Thus, given $x \in B\backslash\{y\}$, the map

$$\zeta \mapsto F^{-1}\left(\frac{\zeta}{\|F(x)\|} F(x)\right) \tag{8}$$

of Δ into B is the unique complex geodesic for c_B whose range contains x and y. By Theorem VI.1.7 (p. 150) of [3], the range of the map (8) is the intersection of B with a complex affine line of $\mathscr{H}$. To describe F, let T be the continuous linear operator of $\mathscr{H}$ defined by

$$T(z) = \frac{(z, y)}{1 + \alpha(y)} y + \alpha(y) z \quad (z \in \mathscr{H}),$$

where (,) is the scalar product of $\mathscr{H}$ and $\alpha(y) = \sqrt{1 - \|y\|^2}$.

The map F is defined by

$$F(z) = \frac{1}{1 - (z, y)} (T(z) - y) \quad (z \in B)$$

and

$$F^{-1}(z) = \frac{1}{1 + (z, y)} (T(z) + y) \quad (z \in B).$$

Hence the complex geodesic (8) is given by

$$\zeta \mapsto \frac{1}{\|F(x)\| + \zeta(F(x), y)} (\zeta T(F(x)) + \|F(x)\| y).$$

Since

$$(T(x),\ y) = (x,\ y),$$

then

$$(F(x),\ y) = \frac{(x,\ y) - \|y\|^2}{1 - (x,\ y)},$$

and

$$T(F(x)) = \frac{(F(x),\ y)}{1 + \alpha(y)}\, y + \alpha(y) F(x) = \frac{1 - \|y\|^2}{1 - (x,\ y)}\, x - y.$$

Finally, the norm $\|F(x)\|$ is given by ([3], p. 156):

$$\|F(x)\|^2 = \frac{1}{|1 - (x,\ y)|^2}(|1 - (x,\ y)|^2 - (1 - \|x\|^2)(1 - \|y\|^2)).$$

PROPOSITION 3.5: *Let $x_0 \in D$, $v_0 \in \mathscr{E}\backslash\{0\}$ and $h \in \mathrm{Hol}(D, \Delta)$ be such that*

$$\langle dh(x_0)v_0\rangle_{h(x_0)} = \kappa_D(x_0;\ v_0). \tag{9}$$

If the domain D is bounded, convex and if $\bar{D}$ is complete and compact for the weak topology, then there is a complex geodesic $f \in \mathrm{Hol}(\Delta, D)$ for c_D in D such that $f(0) = x_0$ and $f'(0)$ is collinear to v_0.

PROOF: Let p be a continuous semi-norm on $\mathscr{E}$ such that $B_p(x_0, r) \subset D$ for some $r > 0$, and such that $p(v_0) > 0$. The domain D, being bounded, is hyperbolic. Thus, by Theorem 2, $\kappa_D(x_0; v_0) > 0$. By definition,

$$\kappa_D(x_0;\ v_0) = p(v_0)/\sup\{p(g'(0)) \colon g \in \mathrm{Hol}(\Delta, D), g(0) = x_0, g'(0)$$

collinear to $v_0\}$.

Let $\{g_\nu\}$ be a sequence of functions $g_\nu \in \mathrm{Hol}(\Delta, D)$ such that: $g_\nu(0) = x_0$, $g'_\nu(0) \neq 0$ is collinear to v_0, and

$$\lim_{\nu\to\infty} p(g'_\nu(0)) = \frac{p(v_0)}{\kappa_D(x_0;\ v_0)} \neq 0. \tag{10}$$

Since D is bounded and the ranges of the functions g_ν lie in a complete weakly compact subset of $\mathscr{E}$, then there is a subsequence $\{g_{\nu_j}\}$ weakly convergent on compact subsets of Δ to a function $f \in \mathrm{Hol}(\Delta, D)$([1]). The range of f belongs to the weak closure of D, which coincides with the closure $\bar{D}$. Thus $f(\Delta) \subset \bar{D}$. Suppose now that $f(\Delta) \cap \partial D \neq \emptyset$. Denoting by m the support function of D, lemma 1.1 implies that $m(f(\zeta)) = 1$ for all $\zeta \in \Delta$. On the other hand, for every continuous linear form λ on $\mathscr{E}$,

$$\lambda(f(0)) = \lim_{j\to\infty} \lambda(g_{\nu_j}(0)) = \lambda(x_0).$$

Therefore $f(0) = x_0$ and by consequence $m(f(0)) < 1$. This contradiction proves that $f(\Delta) \subset D$.

If $f'(0)$ is not collinear to v_0, there is a continuous linear form λ on $\mathscr{E}$ such that

$$\lambda(f'(0)) = 0 \qquad \lambda(v_0) \neq 0.$$

Since $g'_\nu(0) = c_\nu v_0$ with $c_\nu \in \mathbb{C}\backslash\{0\}$, then

$$\lambda(f'(0)) = \lim_{j\to\infty} \lambda(g'_{\nu_j}(0)) = \lambda(v_0) \lim_{j\to\infty} c_{\nu_j}. \tag{11}$$

Thus $\lim_{j\to\infty} c_{\nu_j} = 0$, contradicting (10). Then there exists $c \in \mathbb{C}\backslash\{0\}$ such that $f'(0) = cv_0$.

The same computation (11), for *every* continuous linear form λ on $\mathscr{E}$, yields

$$c = \lim_{j\to\infty} c_{\nu_j},$$

and therefore, by (10),

$$p(f'(0)) = |c|\,p(v_0) = \lim_{j\to\infty} p(c_{\nu_j} v_0) = \lim_{j\to\infty} p(g'_{\nu_j}(0)) = \frac{p(v_0)}{\kappa_D(x_0;\, v_0)},$$

[1] That is theorem 3.14.2 (pp. 105–106) of [7] in the case where $\mathscr{E}$ is a complex Banach space. The proof of the theorem as given in [7] carries over with no change to the case of a bounded, convex domain D in a locally convex complex space $\mathscr{E}$, such that $\bar{D}$ is complete. This latter condition (which is automatically satisfied in the case of Banach spaces) ensures the applicability—as in the original proof given in [7]—of the Eberlein–Smulian theorem [9, pp. 187–188].

i.e.

$$|c|\kappa_D(x_0;\ v_0) = 1.$$

Since $h \circ f \in \mathrm{Hol}(\Delta, \Delta)$, and

$$h \circ f(0) = h(x_0),$$

$$\langle (h \circ f)'(0)\rangle_{h(x_0)} = \langle \mathrm{d}h(x_0)f'(0)\rangle_{h(x_0)} = |c|\langle \mathrm{d}h(x_0)v_0\rangle_{h(x_0)} = |c|\kappa_D(x_0;\ v_0) = 1,$$

by the Schwarz–Pick lemma $h \circ f$ is a holomorphic automorphism of Δ. Thus

$$1 = \kappa_D(x_0; f'(0)) \geq \gamma_D(x_0; f'(0)) \geq \langle \mathrm{d}h(x_0)f'(0)\rangle_{h(x_0)} = 1,$$

and therefore $\gamma_D(x_0; f'(0)) = 1$. Proposition 3.2 yields the conclusion.

QED

COROLLARY: *Let $\mathscr{E}$ be a reflexive Banach space, and let D be a convex bounded domain in $\mathscr{E}$. If $x_0 \in D$, $v_0 \in \mathscr{E}\setminus\{0\}$, and $h \in \mathrm{Hol}(D, \Delta)$ satisfy* (9), *then there exists a complex geodesic f for c_D such that $f(0) = x_0$ and $f'(0)$ is collinear to v_0.*

4. LEMMA 4.1: *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be normed spaces over $\mathbb{C}$. Let B_1 and B_2 be the open unit balls for $\mathscr{E}_1$ and $\mathscr{E}_2$, and assume that every boundary point of B_2 is a complex extreme point of $\bar{B}_2$. If $F \in \mathrm{Hol}(B_1, B_2)$ is such that*

(12) $$\|F(x)\| = \|x\| \qquad \textit{for all } x \in B_1,$$

then F is a linear $\|\ \ \|$-isometry.

PROOF: Condition (12) is equivalent to

$$c_{B_2}(0,\ F(x)) = c_{B_1}(0,\ x) \quad \text{for all } x \in B_1.$$

For $u \in \mathscr{E}_1$, $\|u\| = 1$, the map $\zeta \mapsto \zeta u$ is a complex geodesic for c_{B_1}. Thus by (2) and (12) $\zeta \mapsto F(\zeta u)$ is a complex geodesic for c_{B_2} at 0. By lemma 3.4. $\zeta \mapsto F(\zeta u)$ is linear, i.e.

$$F(\zeta u) = \zeta \mathrm{d}F(0)u.$$

Let

$$F(x) = \mathrm{d}F(0)x + P_2(x) + P_3(X) + \cdots$$

be the power series expansion of F around 0, where P_2, $P_3, \ldots$ are continuous homogeneous polynomials of degrees $2, 3, \ldots$ from $\mathscr{E}_1$ to $\mathscr{E}_2$. Then $P_\nu(u) = 0$ for all $u \in \mathscr{E}_1$ and all $\nu = 2, 3, \ldots$. Therefore

$$F(x) = \mathrm{d}F(0)x \qquad \text{for all } x.$$

and the conclusion follows. QED

The hypothesis concerning the complex extreme points cannot be dropped as is shown by the map $\zeta \mapsto (\zeta, \zeta^2)$ of Δ into the bi-disc $\Delta \times \Delta$. However, according to a result of L.A. Harris [5][3], that hypothesis can be avoided if $\mathrm{d}F(0)$ is assumed to be a linear isometry of $\mathscr{E}_1$ onto $\mathscr{E}_2$.

The above lemma could be compared with the theorem of Mazur–Ulam [2, pp. 166–168]. Does a statement similar to Lemma 4.1 hold for real analytic mappings?

The following result is a direct consequence of Lemma 4.1.

THEOREM 3: *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two locally convex, locally bounded, complex vector spaces. Let D_1 and D_2 be two bounded, convex, balanced open neighborhoods of* 0 *in $\mathscr{E}_1$ and $\mathscr{E}_2$, and let $F \in \mathrm{Hol}(D_1, D_2)$ be such that $F(0) = 0$ and that either*

$$\gamma_{D_2}(0; \mathrm{d}F(0)u) = \gamma_{D_1}(0; u) \quad \textit{for every } u \in \mathscr{E}_1,$$

or any one of the following conditions

$$k_{D_2}(0, F(x)) = k_{D_1}(0, x),$$

$$c_{D_2}(0, F(x)) = c_{D_1}(0, x)$$

holds for all $x \in D_1$. If every point of ∂D_2 is a complex extreme point of $\bar{D}_2$, then F is (the restriction to D_1 of) a linear map of $\mathscr{E}_1$ into $\mathscr{E}_2$:

$$F(x) = \mathrm{d}F(0)x \qquad \textit{for all } x \in D_1.$$

COROLLARY: *Under the same hypotheses of Theorem* 3 *for D_1 and D_2, let $F \in \mathrm{Hol}(D_1, D_2)$ be such that any one of the following conditions holds:*

$$\gamma_{D_2}(F(0); \mathrm{d}F(0)u) = \gamma_{D_1}(0;u) \quad \textit{for all } u \in \mathscr{E}_1,$$

or

$$c_{D_2}(F(0), F(x) = c_{D_1}(0, x) \quad \textit{for all } x \in D_1. \tag{13}$$

If the semi-group S_c of all holomorphic c_{D_2}-isometries is transitive on D_2, then for any $g \in S_c$ such that $g(F(0)) = 0$, $g \circ F$ is linear and $g(F(D_1))$ is the intersection of D_2 with a linear subspace of $\mathscr{E}_2$.

A similar conclusion holds if condition (13) is replaced by

$$k_{D_2}(F(0),\ F(x)) = k_{D_1}(0,\ x) \quad \text{for all } x \in D_1,$$

and if the semi-group S_k of all holomorphic k_{D_2}-isometries is transitive on D_2.

The above conditions concerning S_c and S_k are fulfilled if the group $\mathrm{Aut}(D_2)$ of all bi-holomorphic automorphisms of D_2 acts transitively on D_2. That is the case if, e.g., D_2 is the open unit ball B of a complex Hilbert space $\mathscr{H}$. Since every boundary point of B is a real (hence complex) extreme point of $\bar{B}$, all the hypotheses of the above corollary are satisfied. Furthermore, for any linear subspace $\mathscr{P}$ of $\mathscr{H}$ and any $g \in \mathrm{Aut}(B)$ there is an affine subvariety $\mathscr{L}$ of $\mathscr{H}$ such that $g(\mathscr{P} \cap B) = \mathscr{L} \cap B$. Moreover, for all $y_1,\ y_2 \in \mathscr{L} \cap B$, $c_{B\cap\mathscr{L}}(y_1,\ y_2) = k_{B\cap\mathscr{L}}(y_1,\ y_2) = c_B(y_1,\ y_2) = k_B(y_1,\ y_2)$[3].

COROLLARY: *Let D_1 be a bounded, convex, balanced open neighborhood of 0 in $\mathscr{E}_1$, and let $F \in \mathrm{Hol}(D_1, B)$ be such that any one of the following conditions*

$$\gamma_B(F(0); \mathrm{d}F(0)x) = \gamma_{D_1}(0; x), \tag{14}$$

$$c_B(F(0), F(x)) = c_{D_1}(0, x), \tag{15}$$

$$k_B(F(0), F(x)) = k_{D_1}(0, x), \tag{16}$$

holds for all $x \in D_1$. Then there is an affine sub-variety $\mathscr{L}$ of $\mathscr{H}$ such that $F(D_1) = \mathscr{L} \cap B$ and,

$$c_{B\cap\mathscr{L}}(F(0),\ F(x)) = c_{D_1}(0,\ x) \qquad (x \in D_1)$$

if either (14) *or* (15) *holds, or*

$$k_{B\cap\mathscr{L}}(F(0),\ F(x)) = k_{D_1}(0,\ x) \qquad (x \in D_1)$$

if (16) *holds. Moreover for any $g \in \mathrm{Aut}(B)$ such that $g \circ F(0) = 0$, $g \circ F$ is a continuous linear map.*

5. The following result, concerning the non-homogeneous case, im-

proves previous statements of [13]. Let Ξ be a σ-algebra on a set M and let μ be a positive measure on Ξ. Let B be the open unit ball in the complex Banach space $L^1(M, \Xi, \mu)$ and let D be a convex hyperbolic domain in $L^1(M, \Xi, \mu)$.

PROPOSITION 5.1: *Let* $F \in \mathrm{Hol}(B, D)$ *be such that* $\mathrm{d}F(0)$ *has a continuous inverse and moreover*

(17) $\gamma_D(F(0); \mathrm{d}F(0)u) = \gamma_B(0; u)$ *for all* $u \in L^1(M, \Xi, \mu)$.

If $\dim_{\mathbb{C}} L^1(M, \Xi, \mu) > 1$, *and if, for every* $s > 0$, *there is* $0 < r < s$ *such that the open set* $D_r = \{y \in D: c_D(F(0), y) < r\}$ *is bounded and convex, then F is an affine map*

$$F(x) = F(0) + \mathrm{d}F(0)x \qquad (x \in B)$$

of B onto D.

PROOF: There is no restriction in assuming $F(0) = 0$. By the inverse mapping theorem (cf. e.g. [3]) there is an open neighborhood U of 0 in B such that $F(U)$ is an open neighborhood of 0 in D and the restriction $F_{|U}$ is a holomorphic diffeomorphism of U onto $F(U)$. The domain D being hyperbolic, in view of the hypothesis there is some $r > 0$ such that the set $D_r = \{y \in D: c_D(0, y) < r\}$ is bounded, convex and contained in $F(U)$. For any $y \in D_r \setminus \{0\}$, let $x \in U$ be the unique point of U such that $F(x) = y$. The map $\zeta \mapsto \frac{\zeta}{\|x\|} x$ $(\zeta \in \Delta)$ is a complex geodesic at 0 for c_B whose range contains x (2). By Proposition 3.2 $\zeta \mapsto F\left(\frac{\zeta}{\|x\|} x\right)$ is a complex geodesic at 0 for c_D, whose range contains $y = F(x)$. Hence

$$c_D(0, y) = c_D(0, F(x)) = \omega(0, \|x\|) = c_B(0, x),$$

and therefore $x \in B' := \{x \in L^1(M, \Xi, \mu): c_B(0, x) < r\} =$

$$= \left\{x \in L^1(M, \Xi, \mu): \|x\| < \frac{e^{2r} - 1}{e^{2r} + 1}\right\}.$$

[2] Since every boundary point of B is a complex extreme point of $\bar{B}$ [11], that map is actually the *unique* complex geodesic at 0 for c_B whose range contains 0 (Lemma 4.3 of [13]).

Since F contracts the Carathéodory distances, then

$$F(B') = D_r,$$

and $F_{|B'}$ is a holomorphic diffeomorphism of the ball B' onto the convex domain D_r. By a theorem of T.J. Suffridge [10, Theorem 7] $F_{|B_r}$ is linear. Hence F itself is linear.

For any $u \in L^1(M, \Xi, \mu)$ with $\|u\| = 1$, the map $\zeta \mapsto F(\zeta u) = \zeta \mathrm{d}F(0)u$ is a complex geodesic for c_D at 0 in the convex open set D. By Lemma 3.1, its range is closed in D. Since $\mathrm{d}F(0)$ has a continuous inverse, that implies that $F(B) = D$. QED

COROLLARY: *Let $F \in \mathrm{Hol}(B, D)$ be such that $\mathrm{d}F(0)$ has a continuous inverse and that (17) holds. If $\dim_{\mathbf{C}} L^1(M, \Xi, \mu) > 1$, and if D is an open, convex, balanced, hyperbolic neighborhood of $F(0)$, then F is an affine map of B onto D.*

The following result, which is a consequence of the above corollary, improves Theorem II of [13].

THEOREM 4: *Let $F \in \mathrm{Hol}(B, B)$ be such that $\mathrm{d}F(0)$ has a continuous inverse and that*

$$\gamma_B(F(0); \mathrm{d}F(0)x) = \gamma_B(0, x) \quad \textit{for all } x \in B.$$

If $\dim_{\mathbf{C}} L^1(M, \Xi, \mu) > 1$, F is the restriction to B of a linear isometry of $L^1(M, \Xi, \mu)$ onto itself.

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(Oblatum 3-IV-1981 & 29-VI-1981)

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