PAOLO DE BARTOLOMEIS
GIUSEPPE TOMASSINI

Traces of pluriharmonic functions

Compositio Mathematica, tome 44, nº 1-3 (1981), p. 29-39
<http://www.numdam.org/item?id=CM_1981__44_1-3_29_0>

COMPOSITIO MATHEMATICA, Vol. 44, Fasc. 1-3, 1981, pag. 29-39

TRACES OF PLURIHARMONIC FUNCTIONS

Paolo de Bartolomeis and Giuseppe Tomassini

Introduction

Let M be a real oriented hypersurface in a complex manifold X, which divides X into two open sets X^+ and X^-.

In this paper we characterize in terms of tangential linear differential operators on M the distributions T on M which are "jumps" or traces (in the sense of currents) of pluriharmonic functions in X^+ and X^-.

The starting point of our investigation is the non-tangential characterizing equation $\bar{\partial}_b \partial T = 0$, which can be deduced from the theory of boundary values of holomorphic forms. If M is not Levi-flat, we construct a second order tangential local linear differential operator ω_M such that if T is the trace on M of a pluriharmonic function h, then $\omega_M(T) = \partial h$. This enables us to prove that the tangential equation $\bar{\partial}_b \omega_M(T) = 0$ characterizes locally the traces on M of pluriharmonic functions on X^+ or X^- (local Cauchy-Dirichlet problem). From this local result we deduce directly the global solvability of the Cauchy-Dirichlet problem in the case M is either compact or its Levi form has everywhere at least one positive eigenvalue.

Finally, using standard cohomological arguments the global Riemann-Hilbert problem ("jumps" of pluriharmonic functions on $X \backslash M$) is solved if $H^2(X, \mathbb{C}) = 0$ or $H^1(M, \mathbb{R}) = 0$. Particular cases of our problem have been investigated in [1], [3] (cf. also [2], [4]).

The present paper contains an improved version of the results announced in [5].

1. Preliminaries and notations

In the present paper X will be a complex manifold of dimension $n \geq 2$, and $M \subset X$ a real oriented connected C^∞ hypersurface.

0010-437X/81/07029-11$00.20/0

We assume M is defined by $\rho = 0$ where ρ: $X \to \mathbb{R}$ is a C^∞ function such that $d\rho \neq 0$ on M.

We say that such a ρ is a *defining function* for M; M divides X into two open sets X^+ and X^-, defined respectively by $\rho > 0$ and $\rho < 0$; if U is an open subset of X, we will set $U^\pm = U \cap X^\pm$; we can also assume that there exists an $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$ and M_ϵ is the level hypersurface defined by $\rho = \epsilon$, there exists a diffeomorphism π_ϵ: $M_\epsilon \to M$.

We will use the standard notations for currents and distributions spaces (cf. e.g. [9]); in particular we fix the orientation on M in such a way that $d[X^+] = [M]$.

Furthermore we list the following definitions:

i) Let L: $\mathscr{E}^{(r)}(X) \to \mathscr{E}^{(r)}(M)$ be the restriction operator; we set: $\mathscr{E}^{(p,q)}(M) = L(\mathscr{E}^{(p,q)}(X))$.

ii) Let $K \in \mathscr{D}'^{(r)}(M)$: $K \wedge [M]$ will be the $(r+1)$-current on X defined by: $\langle K \wedge [M], \varphi \rangle = \langle K, L(\varphi) \rangle$, $\varphi \in \mathscr{D}^{(2n-r-1)}(X)$.

iii) We say that $K \in \mathscr{D}'^{(r)}(M)$ is a $(r, 0)$-current (resp. $(0, r)$-current) if $(K \wedge [M])^{p,q} = 0$ for $p \leq r$ (resp. $(K \wedge [M])^{q,p} = 0$ for $p \leq r$); we denote by $\mathscr{D}'^{(r,0)}(M)$ (resp. $\mathscr{D}'^{(0,r)}(M)$) the space of $(r, 0)$-currents (resp. $(0, r)$-currents).

iv) If $K \in \mathscr{D}'^{(r,0)}(M)$, then $K \wedge [M]^{1,0}$ is the $(r+1, 0)$-current defined by: $\langle K \wedge [M]^{1,0}, \varphi \rangle = \langle K, L(\varphi) \rangle$, $\varphi \in \mathscr{D}^{(n-r-1,n)}(X)$ and $K \wedge [M]^{0,1}$ is the $(r, 1)$-current defined by: $\langle K \wedge [M]^{0,1}, \varphi \rangle = \langle K, L(\varphi) \rangle$ $\varphi \in \mathscr{D}^{(n-r,\ n-1)}(X)$.

v) Let $\alpha \in \mathscr{E}^{(2)}(X^+)$; we say that α *admits trace* $K \in \mathscr{D}'^{(r)}(M)$ on M in the sense of currents if for every $\varphi \in \mathscr{D}^{(2n-r-1)}(M)$ we have:

$$\lim_{\epsilon \to 0^+} \int_{M_\epsilon} \alpha \wedge \pi_\epsilon^*(\varphi) = \lim_{\epsilon \to 0^+} \int_M \pi_{\epsilon *}(\alpha) \wedge \varphi =$$

$$= \lim_{\epsilon \to 0^+} \langle \pi_{\epsilon *}(\alpha), \varphi \rangle = \langle K, \varphi \rangle$$

(cf. [8]); we set $K = \gamma_+(\alpha)$; in the same manner we define $\gamma_-(\alpha)$ if $\alpha \in \mathscr{E}^{(r)}(X^-)$.

Let $a \in \mathscr{E}^{(r)}(X \backslash M)$ such that $\gamma_+(\alpha)$ and $\gamma_-(\alpha)$ exist: we refer to $\gamma_+(\alpha) - \gamma_-(\alpha)$ as the *jump* of α on M.

We denote by $\mathscr{E}_*^{(r,s)}(X^\pm)$ the space of forms $\alpha \in \mathscr{E}^{(r,s)}(X^\pm)$ such that $\gamma_\mp(\alpha)$ and $\gamma_\pm(d\alpha)$ exist.

Observe that if $\alpha \in \mathscr{E}_*^{(r,0)}(X^+)$, in particular we have

$$\partial(\gamma_+(\alpha) \wedge [M]^{1,0}) = (-1)^{r+1} \gamma_+(\partial\alpha) \wedge [M]^{1,0}.$$

2. Tangential operators on M

2a) *The smooth case*

A local linear operator Φ: $\mathscr{E}^{(p,q)}(X) \to \mathscr{E}^{(r,s)}(M)$ is said to be *tangential on M* if from $L(f)=0$ it follows $\Phi(f)=0$; a tangential operator induces a new operator: $\mathscr{E}^{(p,q)}(M) \to \mathscr{E}^{(r,s)}(M)$ which will be denoted again by Φ.

Let now (,) be a Hermitian structure on X; without loss of generality we can assume $(\partial\rho, \partial\rho) \equiv \frac{1}{2}$ on M. Define:

$$\mathscr{N}^{(p,q)}(M) = \{\alpha \in \mathscr{E}^{(p,q)}(M) \mid \alpha = \varphi \wedge L(\partial\rho)\} \quad p \geq 1$$

$$\mathfrak{z}^{(p,q)}(M) = \{\alpha \in \mathscr{E}^{(p,q)}(M) \mid (\alpha, \beta) = 0 \quad \forall \beta \in \mathscr{N}^{(p,q)}(M)\}$$

$$\bar{\mathscr{N}}^{(p,q)}(M) = \{\alpha \in \mathscr{E}^{(p,q)}(M) \mid \alpha = \varphi \wedge L(\bar\partial\rho)\} \quad q \geq 1$$

$$\bar{\mathfrak{z}}^{(p,q)}(M) = \{\alpha \in \mathscr{E}^{(p,q)}(M) \mid (\alpha, \beta) = 0 \quad \forall \beta \in \bar{\mathscr{N}}^{(p,q)}(M)\}$$

We have the decompositions:

$$\mathscr{E}^{(p,q)}(M) = \mathscr{N}^{(p,q)}(M) \oplus \mathfrak{z}^{(p,q)}(M)$$

$$\mathscr{E}^{(p,q)}(M) = \bar{\mathscr{N}}^{(p,q)}(M) \oplus \bar{\mathfrak{z}}^{(p,q)}(M)$$

and we denote by:

$$\tau\colon \mathscr{E}^{(p,q)}(M) \to \mathfrak{z}^{(p,q)}(M)$$

$$\bar\tau\colon \mathscr{E}^{(p,q)}(M) \to \bar{\mathfrak{z}}^{(p,q)}(M)$$

the natural projections; observe that $\tau(\alpha) = \tau(\beta)$ is equivalent to $\alpha \wedge [M]^{1,0} = \beta \wedge [M]^{1,0}$; we set $\partial_b = \tau \circ L \circ \partial$ and $\bar\partial_b = \bar\tau \circ L \circ \bar\partial$; by definition ∂_b and $\bar\partial_b$ are tangential operators on M (cf. [6]).

The induced operators

$$\partial_b\colon \mathscr{E}^{(p,q)}(M) \to \mathfrak{z}^{(p+1,q)}(M)$$

$$\bar\partial_b\colon \mathscr{E}^{(p,q)}(M) \to \mathfrak{z}^{(p,q+1)}(M)$$

are described by the formulas

$$\partial_b\alpha = \tau \circ L \circ \partial(\alpha \wedge [M]^{1,0})$$

$$\bar\partial_b\alpha = \bar\tau \circ L \circ \bar\partial(\alpha \wedge [M]^{0,1}).$$

Thus

$$\partial(\alpha \wedge [M]^{1,0}) = \partial_b\alpha \wedge [M]^{1,0}$$

$$\bar{\partial}(\alpha \wedge [M]^{0,1}) = \bar{\partial}_b\alpha \wedge [M]^{0,1}.$$

We consider in particular the following cases: $f \in \mathscr{E}^{(0,0)}(X)$ and $\beta \in \mathscr{E}^{(1,0)}(X)$.

a) if $f \in \mathscr{E}^{(0,0)}(X)$ set:

$$N(f) = L[(\partial f, \partial\rho)] \qquad \bar{N}(f) = L[(\bar{\partial} f, \bar{\partial}\rho)];$$

so we obtain on M:

$$L(\partial f) = \partial_b f + 2N(f)L(\partial\rho)$$

$$L(\bar{\partial} f) = \bar{\partial}_b f + 2\bar{N}(f)L(\bar{\partial}\rho);$$

furthermore it is easy to check that:

i) at every point of M, $N(f)$ represents the complex normal derivative of f and $N(f) + \bar{N}(f)$ is the real normal derivative of f.

ii) $f \mapsto N(f) - \bar{N}(f)$ is a tangential operator on M; thus we have also e.g.:

$$L(\partial f) = \partial_b f + [N(f) - \bar{N}(f)]L(\partial\rho) + [N(f) + \bar{N}(f)]L(\partial\rho)$$

b) if $\beta \in \mathscr{E}^{(1,0)}(X)$ define $N_1(\beta)$ and $\bar{N}_1(\beta)$ by the relations:

$$L(\partial\beta) = \partial_b\beta + 2N_1(\beta) \wedge L(\partial\rho)$$

$$L(\partial\bar{\beta} + \bar{\partial}\beta) - \tau(\partial\bar{\beta} + \bar{\partial}\beta) = \bar{N}_1(\beta) \wedge L(\partial\rho)$$

we observe that $\beta \mapsto N_1(\beta) - \bar{N}_1(\beta)$ is a tangential operator, and so we obtain on M the decomposition:

$$L(\partial\beta) = \partial_b\beta + [N_1(\beta) - \bar{N}_1(\beta)] \wedge L(\partial\rho) + [N_1(\beta) + \bar{N}_1(\beta)] \wedge L(\partial\rho)$$

which again enables us to isolate the genuine non-tangential component of $L(\partial\beta)$, namely $(N_1(\beta) + \bar{N}_1(\beta)) \wedge L(\partial\rho)$.

Let $U \subset X$ be an open set and let $\mathscr{P}(U)$ be the space of real pluriharmonic functions on U (i.e. $f \in \mathcal{P}(U)$ iff f is real valued and $\partial\bar{\partial} f = 0$).

We have the following:

PROPOSITION 2.1: *Assume $L(\partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho) \neq 0$; then there exists a local linear differential operator R: $\mathscr{E}^{(0,0)}(M) \to \mathscr{E}^{(0,0)}(M)$ such that if $h \in \mathcal{P}(X)$ then $R(L(h)) = L(N(h) + \bar{N}(h))$.*

PROOF: Let $h \in \mathcal{P}(X)$: define $\delta_b = \partial_b + \bar{\partial}_b$, $*\delta_b = \bar{\tau}\partial_b + \tau\bar{\partial}_b$, $\delta_b^c = i(\bar{\partial}_b - \partial_b)$; then omitting L to simplify our notations:

$$0 = \tau\bar{\tau}(\bar{\partial}\partial h) = \tau(\bar{\partial}_b \partial h) = \tau(\bar{\partial}_b[\partial_b h + 2N(h)\partial\rho]) =$$
$$= \tau(\bar{\partial}_b\partial_b h + 2\bar{\partial}_b N(h) \wedge \partial\rho + 2N(h)\bar{\partial}_b\partial\rho) = \tau(\partial_b\bar{\partial}_b h) + 2N(h)\tau(\bar{\partial}_b\partial\rho)$$

and also

$$0 = \bar{\tau}\tau(\partial\bar{\partial} h) = \tau(\partial_b\bar{\partial}_b h) + 2\bar{N}(h)\bar{\tau}(\partial_b\bar{\partial}\rho) = \bar{\tau}(\partial_b\bar{\partial}_b h) - 2\bar{N}(h)\tau(\bar{\partial}_b\partial\rho)$$

thus we obtain the relation:

$$(\#\#) \qquad *\delta_b\delta_b^c h = 2i[N(h) + \bar{N}(h)]\tau(\bar{\partial}_b\partial\rho);$$

taking the Hermitian product, we have:

$$(*\delta_b\delta_b^c h, i\tau(\bar{\partial}_b\partial\rho)) = 2[N(h) + \bar{N}(h)](\|i\tau(\bar{\partial}_b\partial\rho)\|^2).$$

Since $i\tau(\bar{\partial}_b\partial\rho)$, which is a real operator, represents the restriction to M of the Levi form of ρ, in our assumption $\|i\tau(\bar{\partial}_b\partial\rho)\|^2 > 0$ everywhere and so the R we are looking for is given by:

$$R(f) = \tfrac{1}{2}[(*\delta_b\delta_b^c f,\ i\tau(\bar{\partial}_b\partial\rho))]\|i\tau(\bar{\partial}_b\partial\rho)\|^{-2}.$$

We observe that in [10] a similar formula is proved in a more laborious way.

For example in the case $X = \mathbb{B}^2$, the unit ball in $\mathbb{C}^2$, and $M = b\mathbb{B}^2$, we obtain the formula:

$$R(f) = \frac{1}{\sqrt{2}}\Bigg[\frac{\partial f}{\partial z_1} z_1 + \frac{\partial f}{\partial z_2} z_2 + \frac{\partial f}{\partial \bar{z}_1}\bar{z}_1 + \frac{\partial f}{\partial \bar{z}_2}\bar{z}_2$$
$$+ 2\left(\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_2} z_1\bar{z}_2 + \frac{\partial^2 f}{\partial \bar{z}_1 \partial z_2} z_1\bar{z}_2 - \frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1} z_2\bar{z}_2 - \frac{\partial^2 f}{\partial z_2 \partial \bar{z}_2} z_1\bar{z}_1\right)\Bigg]$$

(cf. also [1]).

Proposition 2.1 shows that if h is a pluriharmonic function on X and M is not Levi-flat anywhere, then the real normal derivative of h on M can be expressed by means of a real tangential operator R.

Thus we have on M:

$$(*)\quad L(\partial h)=\partial_b h+[N(h)-\bar{N}(h)]L(\partial\rho)+R(h)L(\partial\rho)$$

let $\omega_M:\mathscr{E}^{(0,0)}(M)\to\mathscr{E}^{(1,0)}(M)$ be defined by the right member of (*) we have the following:

REMARKS 2.2: a) if $f\in\mathscr{E}^{(0,0)}(M)$, then $\partial_b\omega_M(f)=0$.

(b) if f is the restriction of a pluriharmonic function F then $\bar{\partial}_b\omega_M(f)=0$.

PROOF: a) Let $\tilde{f}\in\mathscr{E}^{(0,0)}(X^+)$ be an extension of f; by Stokes' theorem, $L(\partial\rho)\wedge[M]^{1,0}=0$, so $\omega_M(f)\wedge[M]^{1,0}=\partial_b f\wedge[M]^{1,0}=\partial(\tilde{f}[M]^{1,0})$ and therefore $\partial\omega_M(f)\wedge[M]^{1,0}=0$, so $\partial_b\omega_M(f)=0$.

b) By (*), $\omega_M(f)=L(\partial F)$ so $\omega_M(f)\wedge[M]^{0,1}=\partial F\wedge[M]^{0,1}$. Since F is pluriharmonic, $\bar{\partial}(\omega_M(f)\wedge[M]^{0,1})=0$ and therefore $\bar{\partial}_b\omega_M(f)=0$.

2b) *Extension to the general case.*

Consider the dual decompositions:

$$\mathscr{D}'^{(p,q)}(M)=\mathscr{N}'^{(p,q)}(M)\oplus\mathfrak{z}'^{(p,q)}(M)$$

$$\mathscr{D}'^{(p,q)}(M)=\bar{\mathscr{N}}'^{(p,q)}(M)\oplus\bar{\mathfrak{z}}'^{(p,q)}(M)$$

where the projections τ and $\bar{\tau}$ are defined in an obvious way, so the operators ∂_b and $\bar{\partial}_b$ extend naturally to currents on M. Furthermore it turns out that N_1 and $\bar{N}_1$ are defined as N_1, $\bar{N}_1$: $\mathscr{E}_*^{1,0}(X^{\pm})\to\mathscr{D}'^{(1,0)}(M)$ and they induce a continuous operator $N_1-\bar{N}_1$: $\mathscr{D}'^{(1,0)}(M)\to\mathscr{D}'^{(1,0)}(M)$; so if e.g. $\beta\in\mathscr{E}_*^{(1,0)}(X^+)$ we have:

$$\gamma_+(\partial\beta)=\partial_b\beta+(N_1-\bar{N}_1)(\gamma_+(\beta))\wedge L(\partial\rho)+(N_1+\bar{N}_1)(\beta)\wedge L(\partial\rho).$$

Of course $\partial\beta=0$ implies $N_1(\beta)=0$; we note also that if β is holomorphic or $\beta=\partial f$ for a real valued f, then $\bar{N}_1(\beta)=0$; in particular if $\gamma_+(\beta)=\gamma_+(f)\partial\rho$ for a real valued function f, then:

$$(^\circ)\quad (N_1-\bar{N}_1)(\beta)=(N_1-\bar{N}_1)(\partial f\rho-\rho\partial f)=(N_1-\bar{N}_1)(\partial f\rho)=0.$$

Finally we have that ω_M can be extended as an operator $\omega_M:\mathscr{D}'^{(0,0)}(M)\to\mathscr{D}'^{(1,0)}(M)$ and remarks 2.2 a), b) hold.

3. Traces of pluriharmonic functions

We are able now to give the following local solution to the trace problem (Cauchy–Dirichlet problem) for the $\bar{\partial}\partial$ operator.

THEOREM 3.1: *Let $p \in M$ and U be a neighbourhood of p; assume $L(\partial\rho \wedge \bar{\partial}\rho \wedge \bar{\partial}\partial\rho) \neq 0$ on $M \cap U$ and let T be a real distribution defined on $U \cap M$; then the following statements are equivalent:*

i) $\bar{\partial}_b\omega_M(T) = 0$ *on* $U \cap M$

ii) *there exists a neighbourhood V of p and there exists $F \in \mathscr{P}(V \backslash M)$ such that $\gamma_+(F) - \gamma_-(F) = T$ in $U \cap V \cap M$; more precisely if the Levi form of ρ has a positive (resp. negative) eigenvalue at p we can choose $F|_{V^-} = 0$ (resp. $F|_{V^+} = 0$) and so T is actually the trace of a pluriharmonic function.*

PROOF: We have already observed that ii) implies i); conversely assume $\bar{\partial}_b\omega_M(T) = 0$ on $U \cap M$; let $W \subset U$ be a Stein neighbourhood of p: by assumption we have: $\bar{\partial}[\omega_M(T) \wedge [M \cap W]^{0,1}] = 0$ in W and so there exists $\tilde{K} \in \mathscr{D}'^{(1,0)}(W)$ such that $\bar{\partial}\tilde{K} = \omega_M(T) \wedge [M \cap W]^{0,1}$.

$\tilde{K}|_{W^+}$ and $\tilde{K}|_{W^-}$ are holomorphic $(1,0)$-forms in W^+ and W^- respectively, $\tilde{K}|_{W^\pm} \in \mathscr{E}_*^{(1,0)}(W^\pm)$ and $\gamma_+(\tilde{K}|_{W^+}) - \gamma_-(\tilde{K}|_{W^-}) = \omega_M(T)$ on $M \cap W$ (cf. [8] th.II 1.3). Now assume the Levi form of ρ has e.g. a positive eigenvalue at p; then $\tilde{K}|_{W^-}$ extends across M as a holomorphic (1,0)-form β to a Stein neighbourhood V of p: if $K_+ = \tilde{K}|_{V^+}$ of course we have: $\gamma_+(K_+ - \beta) = \omega_M(T)$ on $M \cap W$.

Furthermore we have:

$$(^{**}) \quad \gamma_+[\partial(K_+ - \beta)] = 0$$

in fact $\bar{N}_1(K_+ - \beta) = 0$ and so:

$$\begin{aligned} \gamma_+[\partial(K_+ - \beta)] &= \partial_b[\gamma_+(K_+ - \beta)] + 2(N_1 - \bar{N}_1)(\gamma_+(K_+ - \beta)) \\ &= \partial_b\omega_M(T) + 2(N_1 - \bar{N}_1)(\omega_M(T)) = 2(N_1 - \bar{N}_1)(\partial_b T) \end{aligned}$$

Let now $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathscr{E}^{(0,0)}(V)$ such that $L(f_n) \to T$ in $\mathscr{D}'^{(0,0)}(M \cap V)$; from (°) it follows

$$2(N_1 - \bar{N}_1)(\partial_b T) = \lim_n 2(N_1 - \bar{N}_1)(\partial_b f_n) = \lim_n 2(N_1 - \bar{N}_1)(\partial f_n) = 0$$

which proves (**).

Now in V we have:

$$\bar{\partial}[\partial(K_+ - \beta) \wedge [V^+]] = \gamma_+(\partial K_+ - \beta) \wedge [V \cap M]^{0,1} = 0$$

and thus $\partial(K_+ - \beta) \wedge [V^+]$ is holomorphic in V and so $\partial(K_+ - \beta) \equiv 0$ on V^+.

It follows that on V: $\partial[(K_+ - \beta) \wedge [V^+] - T \wedge [M \cap V]^{1,0}] = 0$ and so it is possible to find $\tilde{G} \in \mathscr{D}'^{(0,0)}(V)$ such that:

$$\partial\tilde{G} = (K_+ - \beta) \wedge [V^+] - T \wedge [M \cap V]^{1,0}.$$

Then if we set $G = \tilde{G}|_{V \setminus M}$ we obtain $\bar{\partial}\partial G = 0$.

Thus we have the following:

a) G is a pluriharmonic function in $V \setminus M$
b) Since G can be extended as a distribution across M and satisfies $\bar{\partial}\partial G = 0$, then (cf. again [8] corollaire I, 2.6.) $\gamma_+(G)$ and $\gamma_-(G)$ exist.
c) On V^+ one has $\partial G = K_+ - \beta$.

We have also that $\gamma_+(G) + T$ is ∂_b-closed: in fact:

$$\partial[(\gamma_+(G) + T) \wedge [M \cap V]^{1,0}] = \omega_M(T) \wedge [M \cap V]^{1,0} +$$
$$- \gamma_+(\partial G) \wedge [M]^{1,0} = \omega_M(T) \wedge [M \cap V]^{1,0} - \gamma_+(K_+ - \beta) \wedge [M \cap V]^{1,0}.$$

Thus there exists an antiholomorphic function H on V^+ such that $\gamma_+(H) = \gamma_+(G) + T$.

It follows that $F = H - G$ is a pluriharmonic function on V^+ such that $\gamma_+(F) = T$ and since T is real we actually have $\gamma_+(\mathrm{Re}\, F) = T$ and the proof of Theorem 3.1. is complete.

From Theorem 3.1. we can deduce first the following global solutions of the Cauchy–Dirichlet problem:

PROPOSITION 3.2: *Suppose the Levi form of ρ has least one positive eigenvalue at every point $p \in M$. Then there exists a neighbourhood U of M such that the equation $\bar{\partial}_b \omega_M(T) = 0$ characterizes those distributions on M which are traces of pluriharmonic functions in U^+.*

PROOF: Theorem 3.1. assures that there exists a covering $\mathscr{U} = (U_n)_{n \in \mathbb{N}}$ of M by open set of X such that for every $n \in \mathbb{N}$ there exists $f_n \in \mathscr{P}(U_n^+)$ for which $\gamma_+(f_n) = T$ on $M \cap U_n$; furthermore, if

$U_n \cap U_m \cap M \neq \emptyset$, one has $\gamma_+(f_m) = \gamma_+(f_n)$ on $U_m \cap U_n \cap M$ and thus, since the trace on M characterizes a pluriharmonic function, we have $f_m = f_n$ on $U_n^+ \cap U_m^+$ etc ...

PROPOSITION 3.3: *Suppose X is a Stein manifold, X^+ is relatively compact and $L(\partial\rho \wedge \bar\partial\rho \wedge \bar\partial\partial\rho) \neq 0$. If T is a real distribution on M, then the following statements are equivalent*:

i) $\bar\partial_b \omega_M(T) = 0$
ii) *there exists $F \in \mathcal{P}(X^+)$ such that $\gamma_+(F) = T$*

PROOF: i) follows immediately from ii); assume now i) holds: in order to prove ii), we argue in the same way as in Theorem 3.1, setting $W = X$ and using Hartog's theorem to extend $\tilde K|_{X^-}$ to the whole X.

Using standard cohomological arguments we can investigate the global Riemann–Hilbert problem.

Let $\mathcal{S}$ be the sheaf of germs of real distributions T on M such that $\bar\partial_b \omega_M(T) = 0$ and let $\hat{\mathcal{S}}$ be its trivial extension to X. Let $\mathcal{A}$ be the sheaf of germs of distributions T on M such that $\bar\partial_b T = 0$. Furthermore, let $\mathcal{P}_X$ be the sheaf of germs of real pluriharmonic functions on X and ${}_*\mathcal{P}_M$ the sheaf on X associated to the canonical presheaf:

$$ {}_*\mathcal{P}(U) = \begin{cases} \mathcal{P}(U) & \text{if } \ U \cap M = \emptyset \\ \{f \in \mathcal{P}(U) \mid \gamma_+(f),\ \gamma_-(f) \text{ exist on } M\} & \text{if } \ U \cap M \neq \emptyset. \end{cases} $$

Assume $L(\partial\rho \wedge \bar\partial\rho \wedge \bar\partial\partial\rho) \neq 0$ on M.

Let Re: $\mathcal{A} \to \mathcal{S}$ be the sheaf homomorphism defined by $T \mapsto$ real part of T, and let α be the sheaf homomorphism defined by: $\alpha : {}_*\mathcal{P}_M \to \hat{\mathcal{S}}$

$$ \alpha_x(f) = \begin{cases} [\gamma_+(f) - \gamma_-(f)]_x & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases}. $$

COROLLARY 3.4: *The sequences*

(1) $0 \to \mathcal{P}_X \to {}_*\mathcal{P}_M \overset{\alpha}{\to} \hat{\mathcal{S}} \to 0$

(2) $0 \to \mathbb{R} \overset{i}{\to} \mathcal{A} \overset{\text{Re}}{\to} \mathcal{S} \to 0$

are exact.

PROOF: We need the following

LEMMA 3.5: *Suppose M is not Levi-flat at p and let U be a neighbourhood of p; let $f_\pm \in \mathcal{P}(U^\pm)$ admitting traces $\gamma_+(f_+)$ and $\gamma_-(f_-)$ on $U \cap M$. If furthermore $\gamma_+(f_+) = \gamma_-(f_-)$, then there exists $f \in \mathcal{P}(U)$ such that $f|_{U^\pm} = f_\pm$.*

PROOF OF LEMMA 3.5: Since the problem is local, we can assume U is a domain of $\mathbb{C}^n$: then $\frac{\partial f_\pm}{\partial z_j}$, $1 \le j \le n$, are holomorphic in $U^\pm$, $\gamma_+\left(\frac{\partial f_+}{\partial z_j}\right)$ and $\gamma_-\left(\frac{\partial f_-}{\partial z_j}\right)$ exist and Proposition 2.1. assures that $\gamma_+\left(\frac{\partial f_+}{\partial z_j}\right) = \gamma_-\left(\frac{\partial f_-}{\partial z_j}\right)$, $1 \le j \le n$; hence, we are essentially reduced to the case $f_\pm$ holomorphic which follows from [8], Corollaire II 1.2.

PROOF OF THE COROLLARY 3.4: (1) In virtue of Theorem 3.1. α is surjective and the previous lemma concludes the proof of the exactness of (1).

(2) As a consequence of Lemma 3.5, we deduce easily that $\ker \mathrm{Re} = \mathrm{Im}(i)$ and so we have to check that if $p \in M$ and $T \in \mathcal{S}_p$ then there exists $\tilde{T} \in \mathcal{A}_p$ such that $\mathrm{Re}\, \tilde{T} = T$.

Now in virtue of Theorem 3.1, there exist a neighbourhood U of p in X and $F \in \mathcal{P}(U \backslash M)$ such that: $T = \gamma_+(F) - \gamma_-(F)$; from [8] we deduce that there exists $G \in \mathcal{O}(U \backslash M)$ such that $\mathrm{Re}\, G = F$ and $\gamma_+(G)$, $\gamma_-(G)$ exist; (more in detail the argument runs as follows: holomorphic and pluriharmonic functions with traces in the sense of currents are characterized by finite order of growth with respect to ρ ([8] Corollaire I 2.6.) so F has finite order of growth with respect to ρ and so does G, which can be expressed locally as $G = F + iH$, where H satisfies $dH = d^cF$ etc ...); it follows that Re: $\mathcal{A} \to \mathcal{S}$ is surjective and (2) is exact: so the proof of Corollary 3.4 is complete.

THEOREM 3.6: *(Global solution of the Riemann–Hilbert problem for $\bar{\partial}\partial$) Suppose X is a Stein manifold and $L(\partial\rho \wedge \bar{\partial}\rho \wedge \bar{\partial}\partial\rho) \neq 0$; assume furthermore $H^2(X, \mathbb{C}) = 0$ or $H^1(M, \mathbb{R}) = 0$; then if T is a real distribution on M, the following statements are equivalent:*

i) $\bar{\partial}_b \omega_M(T) = 0$

ii) *there exists $F \in \mathcal{P}(X \backslash M)$ such that $\gamma_+(F) - \gamma_-(F) = T$.*

PROOF: We observe that for a Stein manifold X we have the isomorphism $H^r(X, \mathcal{P}_X) \approx H^{r+1}(X, \mathbb{C})$ for $r \ge 1$; moreover we have: $H^0(X, \hat{\mathcal{S}}) \approx H^0(M, \mathcal{S})$.

If $H^2(X, \mathbb{C}) = 0$ we obtain the exact sequence:

$$0 \to H^0(X, \mathscr{P}_X) \to H^0(X, {}_*\mathscr{P}_M) \to H^0(M, \mathscr{S}) \to 0$$

If $H^1(M, \mathbb{R}) = 0$ we obtain the exact sequence:

$$0 \to \mathbb{R} \to H^0(M, \mathscr{A}) \to H^0(M, \mathscr{S}) \to 0$$

This concludes the proof.

REFERENCES

[1] T. AUDIBERT: *Caractérisation locale par des opérateurs differentiels des restrictions à la sphère de* $\mathbb{C}^n$ *des fonctions pluriharmoniques* C.R.A.S. (1977) A284 1029–1031.
[2] E. BEDFORD: $(\partial\bar{\partial})_b$ *and the real part of CR functions* Indiana Univ. Math. J. (1980) 29 3 333–340.
[3] E. BEDFORD, P. FEDERBUSH: *Pluriharmonic boundary values* Tohoku Math. (1974) 26 505–511.
[4] V. BELOSAPKA: *Functions pluriharmonic on manifolds* Math. U.S.S.R. Isvestija (1978) 12 439–447.
[5] P. de BARTOLOMEIS, G. TOMASSINI: *Traces of pluriharmonic functions* Analytic Functions Kozubnik 1979 Lecture Notes in Math. 798 Springer (1980) 10–17.
[6] J.J. KOHN, U. ROSSI: *On the extension of holomorphic functions from the boundary of a complex manifold* Ann. of Math. (1965) 81 451–473.
[7] G. LAVILLE: *Fonctions pluriharmoniques et solution fondamentale d'un opérateur du* 4^e *ordre* Bull. Sc. Math. 2s. (1977) 101 305–317.
[8] S. LOJASIEWICZ, G. TOMASSINI: *Valeurs au bord des formes holomorphes* Several Complex Variables, Proceedings of International Conferences, Cortona Italy 1976–1977 222–245 Scuola Normale Superiore, Pisa (1978).
[9] J. POLKING, R. O. WELLS: *Boundary values of Dolbeault Cohomology classes and a generalized Bochner–Hartogs Theorem* Abh. Math. Sem. Univ. Hamburg (1978) 47 3–24.
[10] G.B. RIZZA: *Dirichlet problem for n-harmonic functions and related geometrical properties* Math. Ann. Bd. (1955) 130 s.202–218.

(Oblatum 6-III-1981 & 5-VI-1981)

Giuseppe Tomassini
Universitá delqe Studi
Istituto Matematico Ulisse
Dine
Viale Morgagni 67-A
Firenze 50134
Italy