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MAURO NACINOVICH

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ON THE ABSENCE OF POINCARÉ LEMMA FOR SOME SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

Mauro Nacinovich

dedicated to Aldo Andreotti

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Introduction

In this paper we study necessary conditions for the validity of the Poincaré lemma for complexes of differential operators. We extend to systems the technique of short waves asymptotic solutions that Hörmander used in the case of a single operator in his generalization of the example of Hans Lewy (cf. [7]). Introducing asymptotic operators and asymptotic series at a characteristic direction, we reduce to the study of complexes of differential operators on spaces of rapidly decreasing functions. Some of these are similar to a complex defined in the book [11] of Guillemin and Sternberg. By this method we are able to prove non local solvability for systems of differential equations in some cases that - to our knowledge - were not previously considered in the literature.

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There is a special reason to me to dedicate this paper to the memory of Aldo Andreotti, as an ideal continuation of an investigation we had started together.

§1. The Poincaré and the special Poincaré lemma

(a) Let $A(x,D):\mathscr{E}^p(\Omega)\to\mathscr{E}^q(\Omega)$ and $B(x,D):\mathscr{E}^q(\Omega)\to\mathscr{E}^r(\Omega)$ be differential operators with smooth coefficients on an open set $\Omega\subset\mathbb{R}^n$, such that

$$(1)\qquad \mathscr{E}^p(\Omega)\xrightarrow{A(x,D)}\mathscr{E}^q(\Omega)\xrightarrow{B(x,D)}\mathscr{E}^r(\Omega)$$

is a complex, i.e. $B(x,D)\circ A(x,D)=0$.

Let us denote by $\mathscr{E}_{x_0}$ the ring of germs of (complex valued) C^∞ functions at x_0; for every $x_0\in\Omega$ by restriction to germs we obtain from the complex (1) a new complex:

$$(2)\qquad \mathscr{E}^p_{x_0}\xrightarrow{A(x,D)}\mathscr{E}^q_{x_0}\xrightarrow{B(x,D)}\mathscr{E}^r_{x_0}.$$

When (2) is an exact sequence we say that (1) admits the *Poincaré Lemma* at $\mathscr{E}^q_{x_0}$. We say that (1) admits the *Special Poincaré Lemma* at $\mathscr{E}^q_{x_0}$ when it admits the Poincaré lemma at $\mathscr{E}^q_{x_0}$ and moreover the map $B(x,D):E^q_{x_0}\to\mathscr{E}^r_{x_0}$ has a sequentially closed image in the following sense:

if ω is a fixed neighborhood of x_0 in Ω and $\{f_n\}$ a sequence of functions in $\mathscr{E}^q(\omega)$ such that $\{B(x,D)f_n\}$ converges to a function $g\in\mathscr{E}^r(\omega)$ uniformly with all derivatives on compact subsets of ω, then the germ of g at x_0 belongs to the image of $B(x,D):\mathscr{E}^q_{x_0}\to\mathscr{E}^r_{x_0}$.

We note that when $B(x,D)=0$ (determined or under-determined systems) there is no distinction between the Poincaré and the special Poincaré lemma.

When complex (1) is part of a longer complex:

$$(1')\qquad \mathscr{E}^p(\Omega)\xrightarrow{A(x,D)}\mathscr{E}^q(\Omega)\xrightarrow{B(x,D)}\mathscr{E}^r(\Omega)\xrightarrow{C(x,D)}\mathscr{E}^s(\Omega)\to\cdots$$

then the failure of the special Poincaré lemma at $\mathscr{E}^q_{x_0}$ implies the failure of the Poincaré lemma at either the place $\mathscr{E}^q_{x_0}$ or at the place $\mathscr{E}^r_{x_0}$; indeed, when the Poincaré lemma holds at $\mathscr{E}^r_{x_0}$, the map $B(x,D):\mathscr{E}^q_{x_0}\to\mathscr{E}^r_{x_0}$ has a sequentially closed image in the sense precised above.

(b) LEMMA 1: *If the image of* $B(x, D)\colon \mathscr{E}^q_{x_0} \to \mathscr{E}^r_{x_0}$ *is sequentially closed in the sense precised above, then, given any open neighborhood* ω *of* x_0 *in* Ω *there is an open neighborhood* ω' *of* x_0 *in* ω *such that for every* $f \in \mathscr{E}^r(\omega)$ *belonging to the closure in* $\mathscr{E}^r(\omega)$ *of* $B(x, D)\mathscr{E}^q(\omega)$ *there is a function* $w \in E^q(\omega')$ *with* $f|_{\omega'} = B(x, D)w$.

PROOF: Let $\mathscr{G}(\omega)$ denote the closure of $B(x, D)\mathscr{E}^q(\omega)$ in $\mathscr{E}^r(\omega)$. It is a linear space and is a Fréchet space with the topology induced by $\mathscr{E}^r(\omega)$. Let $\{\omega_\nu\}$ be a fundamental system of open neighborhoods of x_0 in ω and set for every $\nu = 1, 2, \ldots$; $\mathscr{G}_\nu = \{f \in \mathscr{G}(\omega) \mid f|_{\omega_\nu} \in B(x, D)\, \mathscr{E}^q(\omega_\nu)\}$. By assumption we have $\mathscr{G}(\omega) = \cup \mathscr{G}_\nu$ and thus, by Baire's category argument, one of the sets in the reunion, say $\mathscr{G}_{\nu_0}$, is of the second category in $\mathscr{G}(\omega)$. But $\mathscr{G}_{\nu_0}$ is the image of the Fréchet space $F = \{(f, w) \in \mathscr{G}(\omega) \times \mathscr{E}^q(\omega_{\nu_0}) \mid f|_{\omega_{\nu_0}} = B(x, D)w\}$ under the projection into the first component. Then it follows from a theorem of Banach that $\mathscr{G}_{\nu_0} = \mathscr{G}(\omega)$ and the statement follows with $\omega' = \omega_{\nu_0}$.

Given a compact set K and an integer $m \geq 0$, for every function f which is C^∞ on a neighborhood of K we set

$$\|f\|_{K,m} = \sup_K \sup_{|\alpha| \leq m} |D^\alpha f|.$$

COROLLARY: *Let the image of* $B(x, D)\colon \mathscr{E}^q_{x_0} \to \mathscr{E}^r_{x_0}$ *be sequentially closed. Then for every open neighborhood* ω *of* x_0 *in* Ω *we can find an open neighborhood* ω' *of* x_0 *in* ω *with the property*:
given any compact set $K' \subset \omega'$ *and any integer* $m' \geq 0$ *there are*: *a constant* $c > 0$, *a compact* $K \subset \omega$, *an integer* $m \geq 0$ *such that for every* $v \in \mathscr{E}^q(\omega)$ *there is* $w \in \mathscr{E}^q(\omega')$ *with*

$$B(x, D)v|_{\omega'} = B(x, D)w \text{ and}$$

$$\|w\|_{K',m'} \leq c\|B(x, D)v\|_{K,m}.$$

PROOF: With the same notations of the above lemma, we choose ω' to satisfy the statement of the lemma. Moreover we can assume that ω' is relatively compact in ω. By a theorem of Banach, the projection $F \to \mathscr{G}(\omega)$ is an open map and then the image of the open set $\{(f, w) \in F \mid \|w\|_{K,m} < 1\}$ contains a neighborhood of 0 in $\mathscr{G}(\omega)$ of the form $G = \{f \in \mathscr{G}(\omega) \mid \|f\|_{K,m} \leq \epsilon\}$, for K compact $\subset \omega$, m integer ≥ 0 and $\epsilon > 0$. We can take $K \supset \omega'$. Then the statement of the corollary follows with $c = \epsilon^{-1}$. Indeed, for $v \in \mathscr{E}^q(\omega)$, if $B(x, D)v|_{\omega'} = 0$ we can choose $w = 0$; if $B(x, D)v$ is not identically 0 on ω' then $L = \|B(x, D)v\|_{K,m} \neq 0$ and $L^{-1}\epsilon B(x, D)v \in G$. Hence we have $L^{-1}\epsilon B(x, D)v|_{\omega'} = B(x, D)u$ for some $u \in \mathscr{E}^q(\omega')$ with $|u|_{K',m'} < 1$ and then $B(x, D)v|_{\omega'} = B(x, D)w$ with $w = L\epsilon^{-1}u$ and $\|w\|_{K',m'} \leq \epsilon^{-1}L$.

(c) We derive now some a priori estimates that are consequences of the Poincaré and of the special Poincaré lemma.

We denote by dx the Lebesgue measure on $\mathbb{R}^n$ and by

$$(u, v)_\Omega = \int {}^t u\bar{v}\, dx$$

the scalar product in $L^2(\Omega)^s$, for Ω open in $\mathbb{R}^n$ and any integer $s \geq 1$.

Given a differential operator with smooth coefficients on Ω $A(x, D): \mathscr{E}^p(\Omega) \to \mathscr{E}^p(\Omega)$, with $A(x, D) = \Sigma A_\alpha(x)D^\alpha$ (the $A_\alpha(x)$ are $q \times p$ matrices with entries in $\mathscr{E}(\Omega)$), the formal adjoint of $A(x, D)$ is the differential operator with smooth coefficients in Ω: $A^*(x, D): \mathscr{E}^q(\Omega) \to \mathscr{E}^p(\Omega)$ defined by the formula:

$$A^*(x, D)v = \sum (-1)^{|\alpha|} D^\alpha({}^t\bar{A}_\alpha(x)v).$$

It is characterized by the identity:

$$(A(x, D)u, v)_\Omega = (u, A^*(x, D)v)_\Omega \quad \text{for every } u \in C_0^\infty(\Omega)^p \text{ and } v \in C_0^\infty(\Omega)^q.$$

PROPOSITION 1: *Assume that the complex* (1) *admits the Poincaré lemma at* $\mathscr{E}^q_{x_0}$. *Then, given any open neighborhood* ω *of* x_0 *in* Ω *we can find an open neighborhood* $\omega_1 \Subset \omega$ *of* x_0 *such that for a compact set* $F \subset \omega$, *an integer* $l \geq 0$ *and a constant* $c > 0$ *we have*:

$$|(f, v)\omega_1| \leq c\|A^*(x, D)v\|_{\omega_1,0}\|f\|_{F,l}$$

$$\text{for every } f \in \mathscr{E}^q(\omega) \text{ with } B(x, D)f = 0 \text{ and every } v \in C_0^\infty(\omega_1)^q. \tag{3}$$

The proof of this statement can be found in [3].

Let us now assume that the complex (1) admits the special Poincaré lemma at $\mathscr{E}^q_{x_0}$. Given an open neighborhood ω of x_0 in Ω, let $\omega' \Subset \omega$ be an open neighborhood of x_0 for which the statement of the corollary to lemma 1 holds. We choose a neighborhood $\omega_1 \Subset \omega'$ of x_0 in such a way that the statement of the proposition above holds with ω'

replacing ω. Then we choose a compact K in ω, m and $c>0$ in such a way that the statement of the corollary to lemma 1 holds with $K'=\bar{\omega}_1$ and $m'=l$, and moreover we require that $m-l$ is an upper bound for the order of differential operators that are entries of the matrix $A(x,D)$ and $K\supset\bar{\omega}'$. Let now $v\in C_0^\infty(\omega_1)^q$ and $f\in\mathscr{E}^q(\omega)$. We take $w\in\mathscr{E}^q(\omega')$ such that $B(x,D)f\mid\omega'=B(x,D)w$ and $\|w\|_{\bar{\omega}_1,l}\le c\|B(x,D)f\|_{K,m}$. Then $B(x,D)(f-w)=0$ on ω' and by the proposition above we obtain:

$$\begin{aligned}|(f,w)_{\omega_1}| &\le |(f-w,v)_{\omega_1}|+|(w,v)_{\omega_1}|\\ &\le c\|A^*(x,D)v\|_{\omega_1,0}\|f-w\|_{F,l}+\text{meas }\omega_1\|w\|_{\omega_1,0}\|v\|_{\omega_1,0}\\ &\le c(\|A^*(x,D)v\|_{\omega_1,0}\|f\|_{K,m}+\|B(x,D)f\|_{K,m}\|v\|_{\omega_1,0}).\end{aligned}$$

We have obtained the following:

PROPOSITION 2: *Assume that the complex* (1) *admits the special Poincaré Lemma at* $\mathscr{E}^q_{x_0}$. *Then, given any open neighborhood* ω *of* x_0 *in* Ω *we can find an open neighborhood* $\omega_1\Subset\omega$ *of* x_0, *a compact* $K\subset\omega$, *an integer* $m\ge 0$ *and a constant* $c>0$ *such that*:

$$(4)\qquad |(f,v)_{\omega_1}|\le c(\|A^*(x,D)v\|_{\omega_1,0}\|f\|_{K,m}+\|v\|_{\omega_1,0}\|B(x,D)f\|_{K,m})$$

for every $f\in\mathscr{E}^q(\omega)$ and every $v\in C_0^\infty(\omega_1)^q$.

§2. The radiation principle

(a) *The space* $\mathscr{S}$ *and the Weyl algebra* W

We denote by $\mathscr{S}=\mathscr{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. For each integer $m\ge 0$ we introduce the scalar product:

$$(u,v)_m=\sum_{|\alpha+\beta|\le m}\int x^\beta D^\alpha u\overline{x^\beta D^\alpha v}\,dx\quad\text{for } u,v\in\mathscr{S},$$

and the norm

$$\|u\|_m=(u,u)_m^{1/2}$$

associated to it. We denote by S_m the Hilbert space obtained by completing $\mathscr{S}$ with this norm. Then $\mathscr{S}$ is the inverse limit of this family of Hilbert spaces:

$$\mathscr{S} = \xleftarrow{\lim} S_m.$$

Let us denote by W the Weyl algebra of differential operators with polynomial coefficients in $\mathbb{R}^n$. The elements of W define linear automorphisms of $\mathscr{S}$, that in this way can be considered as a left W-module. We define the canonical ordering of W as the ascending chain of linear subspaces of W:

$$W_0 \subset W_1 \subset W_2 \subset W_3 \subset \cdots$$

where for each m we have denoted by W_m the linear span of the operators $x^\beta D^\alpha$ for $|\alpha+\beta| \le m$. It is obvious that $P \in W_m$ defines for every $k \ge m$ a linear and continuous map from S_k into S_{k-m}.

(b) *The symplectic structure*

Let $E = \mathbb{R}^n \oplus \mathbb{R}^n$ and on E let us consider the symplectic structure defined by the bilinear form

$$\sigma(u, v) = \langle \xi, y\rangle - \langle \eta, x\rangle \quad \text{for } u = (\xi, x),\ v = (\eta, y) \in E.$$

Let J denote the matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ where I is the $n \times n$ identity matrix. Then

$$\sigma(u, v) = {}^t vJu \quad \text{for } u, v \in E \text{ (written as column-vectors).}$$

We denote by $Sp(n, \mathbb{R})$ the group of linear symplectic transformations of E. This is the group of linear maps $S : E \to E$ such that $\sigma(Su, Sv) = \sigma(u, v)$ for $u, v \in E$; still denoting by S the $2n \times 2n$ matrix representing the transformation S, we have $S \in Sp(n, \mathbb{R})$ iff ${}^t SJS = J$.

Then we denote by $Sp^1(n, \mathbb{R})$ the group of affine symplectic transformations of E. This group is isomorphic to the group of $(2n+1) \times (2n+1)$ matrices of the form

$$\tilde{S} = \begin{pmatrix} S & u \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad S \in Sp(n, \mathbb{R}) \text{ and } u \in E.$$

It is well known that the group $Sp^1(n, \mathbb{R})$ has a representation into the group of automorphisms of $\mathcal{S}$ that extend for each $m \geq 0$ to automorphisms of S_m which for $m = 0$ are unitary. This representation is defined on a set of generators of $Sp^1(n, \mathbb{R})$ in the following way:

(i) The translation $(\xi, x) \to (\xi + \xi^0, x + x_0)$ corresponds to the map

$$f(x) \to f(x - x_0) \exp(i\langle \xi^0, x - x_0 \rangle).$$

(ii) to the map changing x_j into ξ_j and ξ_j into $-x_j$ while leaving all other coordinates fixed corresponds the partial Fourier transform:

$$f(x) \to (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) \exp(-itx_j)\, dt.$$

(iii) to $\tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$ with $S(\xi, x) = (\xi - Tx, x)$ for a real symmetric matrix T corresponds the map $f(x) \to f(x) \exp(-i\langle Tx, x \rangle)$.

(iv) to $\tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$ with $S(\xi, x) = ({}^tT^{-1}\xi, Tx)$ for $T \in Gl(n, \mathbb{R})$ corresponds the map $f(x) \to |\det T|^{-1/2} f(T^{-1}x)$.

We denote by $T_{\tilde{S}} : \mathcal{S} \to \mathcal{S}$ the map corresponding to $\tilde{S} \in Sp^1(n, \mathbb{R})$. These maps induce automorphisms of W preserving the canonical ordering: indeed $T_{\tilde{S}} \circ P \circ T_{\tilde{S}}^{-1} \in W_m$ if $P \in W_m$.

To describe these automorphisms we introduce "complete symbols": for $P(x, D) = \sum_{|\alpha+\beta| \leq m} a_{\alpha\beta} x^\beta D^\alpha \in W$ we set

$$\begin{aligned} P^0(x, \xi) &= \exp\Big(-1/2i \sum_{j=1}^n \partial^2/\partial x_j\, \partial \xi_j\Big)\Big(\sum_{|\alpha+\beta| \leq m} a_{\alpha\beta} x^\beta (i\xi)^\alpha\Big) \\ &= \sum_{|\alpha+\beta| \leq m} a_{\alpha\beta} x^\beta (i\xi)^\alpha \\ &\quad + \sum_{h=1}^m (-1/2i)^h 1/h! \Big(\sum_{j=1}^n \partial^2/\partial x_j\, \partial \xi_j\Big)^h \Big(\sum_{|\alpha+\beta| \leq m} a_{\alpha\beta} x^\beta (i\xi)^\alpha\Big). \end{aligned}$$

This polynomial P^0 of the $2n$ indeterminates x, ξ is called the *complete symbol* of $P(x, D)$. We note that any given polynomial of $2n$ indeterminates $Q(x, \xi)$ is the complete symbol of a differential operator in W, that is given by the formula:

$$P(x, D) = \exp\Big(1/2i \sum_{j=1}^n \partial^2/\partial x_j\, \partial \xi_j\Big) Q(x, \xi) \mid \xi = (1/i)D.$$

If $\tilde{S} \in \mathrm{Sp}^1(n, \mathbb{R})$, $P(x, D) \in W$ and $Q(x, D) = T_{\tilde{S}} \circ P(x, D) \circ T_{\tilde{S}}^{-1}$, then the complete symbols P^0 and Q^0 of the two operators are related by

$$Q^0(x, \xi) = P^0 \circ S^{-1}(x, \xi).$$

(cf. for instance Leray [12], Chap I, §1).

From the statements above one easily obtains the following

LEMMA 2: *For every integer $m \geq 0$ there is a constant $c(m)$ such that for every $\tilde{S} = \begin{pmatrix} S & v \\ 0 & 1 \end{pmatrix} \in Sp^1(n, \mathbb{R})$ and every $u \in S_m$ we have*

$$\|T_{\tilde{S}} u\|_m \leq c(m) \sum_{h=0}^{m} |v|^{m-h}(\|S\| + \|S^{-1}\|)^h \|u\|_h$$

where $\|S\| = \sup_{|w|=1} |Sw|$.

(c) *Asymptotic functions and fading waves*

We denote by $\mathbb{R}^+$ the set of strictly positive real numbers. For fixed $\epsilon \geq 0$ we denote by $F_\epsilon(\mathbb{R}^+, \mathcal{S})$ the space of all functions $a: \mathbb{R}^+ \to \mathcal{S}$ such that for every integer $m \geq 0$ the norms $t^{\epsilon m}\|a(t, x)\|_m$ are bounded on $\mathbb{R}^+$. We say that two functions $a(t, x)$ and $b(t, x)$ in $F_\epsilon(\mathbb{R}^+, \mathcal{S})$ are asymptotically equivalent if

$$\limsup_{t \to 0} t^{-m}\|a(t, x) - b(t, x)\|_m = 0 \text{ for all } m.$$

In this case we write $a(t, x) \sim b(t, x)$. This is an equivalence relation and the quotient space $A_\epsilon \mathcal{S}$ is called the space of ϵ-asymptotics of $\mathcal{S}$.

LEMMA 3: *Let $0 \leq \epsilon < 1$ and let φ be a C^∞ function with compact support in $\mathbb{R}^n$ that is 1 on a neighborhood of 0. Then $a(t, x) \sim \varphi(tx)a(t, x)$ for all $a \in F_\epsilon(\mathbb{R}^+, \mathcal{S})$.*

PROOF: Let $R > 0$ be such that $\varphi = 1$ on $|x| \leq R$. Then

$$R^{-s}t^s|x|^s \geq 1 \text{ on support of } 1 - \varphi(tx)$$

and therefore we have

$$\|a(t, x) - \varphi(tx)a(t, x)\|_m \leq c(m, s)R^{-s}t^s\|a(t, x)\|_{m+s}$$

for a constant $c(m, s)$ independent of t. Because $\epsilon < 1$, we can choose s so large that $s(1 - \epsilon) \geq (1 + \epsilon)m + 1$ and then the statement follows.

DEFINITION: A fading wave at $(x_0, \xi^0) \in \mathbb{R}^n \times (\mathbb{R}^n - \{0\})$ is a function of the form $u(t, x) = a(t, (x - x_0)/t) \exp(i\langle \xi^0, x\rangle/t^2)$, with a function $a(t, x)$, called the amplitude, belonging to $F_\epsilon(\mathbb{R}^+, \mathcal{S})$ for some $0 \le \epsilon < 1$.

LEMMA 4: *For every integer $m \ge 0$ there is a constant $c(m)$ such that*

$$\|u(t, x)\|_{R^n,m} \le c(m) \sum_{h=0}^{m} t^{-m-h} \|a(t, x)\|_{m-h-n} \quad \forall t > 0$$

for every fading wave u with amplitude a.

Let $\varphi \in C_0(\mathbb{R}^n)$ be 1 on a neighborhood of 0. Then for every integer $m \ge 0$ there is a constant $k(m)$ such that

$$\|\varphi(tx)a(t, x)\|_m \le k(m) \sum_{h=0}^{m+n} t^{-m-n-2h} \|u(t, x)\|_{\mathbb{R}^n,h} \quad \forall t > 0$$

for every fading wave u with amplitude a.

PROOF: By Sobolev's lemma we have

$$\|v\|^2_{\mathbb{R}^n,0} \le \text{const} \sum_{|\alpha| \le n} \int |D^\alpha v|^2 \, dx \le \text{const.} \|v\|^2_n \quad \forall v \in \mathcal{S},$$

and, on the other hand,

$$\|v\|_0^2 \le \text{const} \|(1 + |x|^2)^{n/2} v\|^2_{\mathbb{R}^n,0}.$$

Thus, being

$$|D^\alpha u(t, x)| = \left| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (i\xi^0)^\beta t^{-|\alpha+\beta|} (D^{\alpha-\beta} a)(t, (x - x_0)/t) \right|$$

we obtain

$$\begin{aligned} \|u(t, x)\|_{\mathbb{R}^n,m} &\le \text{const} \sum_{h=0}^{m} t^{-m-h} \|a(t, x)\|_{\mathbb{R}^n,m-h} \\ &\le \text{const} \sum_{h=0}^{m} t^{-m-h} \|a(t, x)\|_{m-h+n}. \end{aligned}$$

To prove the second estimate, we note that

$$\varphi(tx)a(t,x) = \varphi(tx)u(t, x_0 + tx)\exp(-i\langle \xi^0, x_0 + tx\rangle/t^2)$$

and then

$$|D^\alpha(\varphi(tx)a(t,x))| = \left| \sum_{\mu\le\nu\le\alpha} \binom{\alpha}{\nu}\binom{\nu}{\mu} t^{-|\alpha|-2|\nu|}(D^{\alpha-\nu}\varphi)(tx) \times \right.$$
$$\left. \times (-i\xi^0)^{\nu-\mu}(D^\mu u)(t, x_0 + tx) \right|.$$

If $R = \sup|x|$ on support of φ, we obtain

$$\|x^\beta D^\alpha \varphi(tx)a(t,x)\|_{R^n,0} \le R^{|\beta|}t^{-|\beta|} \text{ const} \sum_{h\le|\alpha|} t^{2h-|\alpha|}\|u(t,x)\|_{\mathbb{R}^n,h}.$$

Therefore

$$\|\varphi(tx)a(t,x)\|_m \le \text{const} \sum_{|\alpha+\beta|\le m} \|x^\beta D^\alpha(\varphi(tx)a(t,x))\|_0$$
$$\le \text{const} \sum_{|\alpha+\beta|\le m} \|(1+|x|^2)^{n/2}x^\beta D^\alpha(\varphi(tx)a(t,x))\|_{\mathbb{R}^n,0}$$
$$\le k(m) \sum_{h\le m+n} t^{2h-m-n}\|u(t,x)\|_{\mathbb{R}^n,h}.$$

From this lemma we obtain the following

COROLLARY: *Two fading waves u and v at (x_0, ξ^0) satisfy*

$$\limsup_{t\to 0} t^{-m}\|u(t,x) - v(t,x)\|_{\mathbb{R}^n,m} = 0 \quad \forall m$$

if and only if their amplitudes are asymptotically equivalent.

Let $A(x, D): \mathscr{E}^p(\Omega) \to \mathscr{E}^q(\Omega)$ be a differential operator with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$ and let $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$.

DEFINITION: A fading wave solution for $A(x, D)$ at (x_0, ξ^0) is a p-vector valued function $u(t, x)$, with fading wave components, such that

$$\limsup_{t\to 0} t^{-m}\|A(x,D)u(t,x)\|_{K,m} = 0$$

for all compact subsets K of Ω and all $m \ge 0$.

REMARK 1: Every fading wave with amplitude asymptotic to 0 is trivially a fading wave solution for any operator $A(x, D)$. In particular, if $\varphi \in C_0^\infty(\mathbb{R}^n)$ is 1 on a neighborhood of x_0, and u is a fading wave solution for $A(x, D)$ at (x_0, ξ^0), then also $\varphi(x)u(t, x)$ is a fading wave solution for $A(x, D)$ at (x_0, ξ^0).

REMARK 2: Let φ be a real valued C^∞ function bounded with all derivatives on $\mathbb{R}^n$, vanishing with its first derivatives at 0. Then multiplication times $\exp(i\varphi(tx)/t^2)$ defines a linear automorphism of the space $F_\epsilon(\mathbb{R}^+, \mathcal{S})$.

Indeed we have $D^\alpha(t^{-2}\varphi(tx)) = t^{|\alpha|-2}(D^\alpha\varphi)(tx)$. For $|\alpha| = 1$ we apply the Lagrange formula to obtain:

$$|(D^\alpha\varphi)(tx)| \leq \Big(\sup_{\mathbb{R}^n} \sum_{|\beta|=2} |D^\beta\varphi|\Big) t|x|.$$

Therefore we have: $|D^\alpha(t^{-2}\varphi(tx))| \leq c_\alpha$ on $\mathbb{R}^n$ for $|\alpha| \geq 2$

$$|D^\alpha(t^{-2}\varphi(tx))| \leq c|x| \quad \text{for} \quad |\alpha| = 1.$$

Thus for $a \in F_\epsilon(\mathbb{R}^+, \mathcal{S})$ we obtain:

$$\begin{aligned} |x^\beta D^\alpha(a(t, x)\exp(i\varphi(tx)/t^2)| &= \Big| x^\beta \sum_{\nu\leq\alpha} (D^{\alpha-\nu}a(t, x)) \sum_{h=1}^{|\nu|} \sum_{\mu_1+\cdots+\mu_h=\nu} i^h \times \\ &\times c(\nu, \mu_1, \ldots, \mu_h) D^{\mu_1}(t^{-2}\varphi(tx)) \ldots D^{\mu_h}(t^{-2}\varphi(tx)) \\ &\leq \text{const.}\, |x|^\beta \sum_{h=1}^{|\alpha|} (1 + |x|^2)^{h/2} \sum_{|\nu|=|\alpha|-h} |D^\nu a(t, x)|. \end{aligned}$$

Therefore we have:

$$\|a(t, x)\exp(i\varphi(tx)/t^2)\|_m \leq c(m)\|a(t, x\|_m \forall a \in F_\epsilon(\mathbb{R}^+, \mathcal{S}), \quad \forall t > 0.$$

And it is obvious that also the opposite inequality holds for a new constant $k(m)$ independent of a and t.

This remark explains the reason to consider only elementary phase functions $x \to \langle\xi^0, x\rangle$: terms in the phase function not of the first order can be absorbed in the complex fading amplitude.

(d) *First formulation of the radiation principle*

Let us consider a complex of differential operators with smooth coefficients on an open subset Ω of $\mathbb{R}^n$:

$$(1) \qquad \mathcal{E}^p(\Omega) \xrightarrow{A(x, D)} \mathcal{E}^q(\Omega) \xrightarrow{B(x, D)} \mathcal{E}^r(\Omega).$$

Let $A^*(x, D)$ denote the formal adjoint of $A(x, D)$ for the L^2 scalar product. Let $v(t, x) = a(t, (x - x_0)/t) \exp(i\langle \xi^0, x\rangle/t^2)$ and

$$f(t, x) = b(t, (x - x_0)/t) \exp(i\langle \xi^0, x\rangle/t^0)$$

be fading wave solutions at $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$ for $A^*(x, D)$ and $B(x, D)$ respectively, with $a, b \in F_\epsilon(\mathbb{R}^+, \mathcal{S})$ for some $0 \le \epsilon < 1$.

We choose a C^∞ function φ with compact support in Ω that is 1 on a neighborhood of x_0. By the remarks at the end of the previous section,

$$\|A^*(x, D)(\varphi(x)v(t, x))\|_{\omega,0}\|\varphi(x)f(t, x)\|_{\omega,m} \\ + \|\varphi(x)v(t, x)\|_{\omega,0}\|B(x, D)(\varphi(x)f(t, x))\|_{\omega,m}$$

where ω is a neighborhood of x_0 containing the support of φ, is $0(t^l)$ for every l and m. We consider now the integral

$$\int \varphi(x)v(t, x)\overline{\varphi(x)f(t, x)}\, dx = t^n \int |\varphi(tx)|^2 a(t, x)\overline{b(t, x)}\, dx = \psi(t)$$

We note that $\psi(t) \sim \int a(t, x)\overline{b(t, x)}\, dx$ as asymptotic numbers (*).

(*) Two functions $\psi_1(t)$ and $\psi_2(t)$ defined for $t > 0$ are said to define the same asymptotic number, and we write $\psi_1 \sim \psi_2$, if $\limsup_{t\to 0} t^{-m}|\psi_1(t) - \psi_2(t)| = 0\ \forall m$. We denote by $\tilde{\mathbb{C}}$ the complex asymptotic numbers, i.e. the quotient of all functions defined for $t > 0$ by the equivalence relation $\sim$.

Thus, if we want that estimate (4) be valid for all functions v and f with sufficiently small support about x_0, then we must have $\psi(t) \sim 0$. We have obtained the following:

PROPOSITION 3: (*Radiation Principle*).

A necessary condition in order that the complex (1) *admits the special Poincaré lemma at* $x_0 \in \Omega$ *is that for every* $\xi^0 \in \mathbb{R}^n - \{0\}$ *and every fading wave solutions* v *for* $A^*(x, D)$ *with amplitude* $a \in F_\epsilon(\mathbb{R}^+, \mathcal{S}^q)$ *and* f *of* $B(x, D)$ *with amplitude* $b \in F_\epsilon(\mathbb{R}^+, \mathcal{S}^q)$ *at* (x_0, ξ^0) (with $0 \le \epsilon < 1$) one has:

$$\int a(t, x)b(t, x)\, dx \sim 0$$

In particular, for determined or under-determined operators one obtains the following

COROLLARY: *A necessary condition in order that the equation*

$$A(x, D)u = f$$

has a solution $u \in \mathscr{E}^p_{x_0}$ *for every* $f \in \mathscr{E}^q_{x_0}$ *is that the operator* $A^*(x, D)$ *has no fading wave solution* v *with amplitude* $a \in F_\epsilon(\mathbb{R}^+, \mathscr{S}^q)$ *for* $0 \le \epsilon < 1$ *at* $(x_0, \xi^0) \in \Omega \times \mathbb{R}^n - \{0\}$ *with*

$$\int |a(t, x)|^2 \, dx \quad \textit{not} \sim 0.$$

(e) *Asymptotic operators*

Let $A(x, D) = (A_{ij}(x, D))_{i=1,\dots,q;\, j=1,\dots,p}$ be a $q \times p$ matrix of differential operators with smooth coefficients on Ω. We say that $A(x, D)$ is of type (a_j, b_i) for some integers $a_1, \dots, a_p$, $b_1, \dots, b_q$ if for every i, j the operator A_{ij} has order $\le a_j - b_i$ (is 0 if $a_j - b_i < 0$). Then, having fixed $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$, we set:

$$A(t, x, D) = \operatorname{diag}(t^{-2b_1}, \dots, t^{-2b_q})\, e^{-i\langle \xi^0, x\rangle/t} A\left(x_0 + tx, \frac{1}{t} D\right)$$
$$\times\, e^{i\langle \xi^0, x\rangle/t} \operatorname{diag}(t^{2a_1}, \dots, t^{2a_p})$$

This is a differential operator with coefficients that depend smoothly on t. We can consider the formal Taylor series of $A(t, x, D)$ at $t = 0$:

$$A(t, x, D) = \sum_{h=0}^{\infty} t^h A_h(x, D)$$

and we note that $A_h(x, D)$ has all components in W_h. We call $A(t, x, D)$ the (canonical) asymptotic operator of $A(x, D)$ at (x_0, ξ^0). We denote by $\tilde{W}$ the ring of all formal power series of the form

$$\sum_{h=0}^{\infty} t^h p_h(x, D) \quad \text{with } p_h(x, D) \in W_h \, \forall h.$$

This is called the ring of (canonical) asymptotic operators.

The asymptotic operators have a natural action on the space of ϵ – asymptotics of $\mathscr{S}$ for $0 \le \epsilon < 1$, that is described by the following

Proposition 4: *Let* $p(t, x, D) = \Sigma\, t^h p_h(x, D)(p_h \in W_h\ \forall h)$ *be an asymptotic operator. Let* $0 \le \epsilon < 1$ *be fixed. Then for every* $a = a(t, x) \in F_\epsilon(\mathbb{R}^+, \mathcal{S})$ *there is* $b = b(t, x) \in F_\epsilon(\mathbb{R}^+, \mathcal{S})$, *unique modulo* $\sim$, *such that*

$$(*)\quad t^{\epsilon k}\left\| b(t, x) - \sum_{s=0}^{l} t^s p_s(x, D) a(t, x)\right\|_k \le c(k, l) t^{(l+1)(1-\epsilon)} \quad \text{for} \quad t > 0.$$

Proof: Because $p_s \in W_s$, there are constants $C(s, k)$ such that

$$\|p_s v\|_k \le C(s, k)\|v\|_{s+k} \quad \forall v \in \mathcal{S}.$$

We choose an integer m so large that $2/m < 1 - \epsilon$ and then we define for each integer $s \ge 0$ $h(s) = j$ if $mj \le s < (m+1)j$.

Having done this, we choose a sequence of positive real numbers $\{t_s\}$ with $0 < t_s < 2^{-1-s}$, $t_s > t_{s+1}\ \forall s$ and such that

$$t_s^{s/m} t^{\epsilon(s+h(s))}\|p_s(x, D) a(t, x)\|_{h(s)} < 2^{-s} \quad \forall t > 0.$$

This is possible because it is sufficient to choose t_s in such a way that

$$t_s^{-s/m} > \sup_{t>0} C(s, h(s)) t^{\epsilon(s+h(s))}\|a(t, x)\|_{s+h(s)},$$

the right hand side being bounded because of the assumption that $a \in F_\epsilon(\mathbb{R}^+, \mathcal{S})$. Now we choose a sequence of smooth functions $\{\psi_s\}$ on $\mathbb{R}^+$ with $0 \le \psi_s \le 1$, $\psi_s(t) = 1$ if $0 < t \le t_s/2$, $\psi_s(t) = 0$ for $t \ge t_s$. The series

$$\sum_{s=0}^{\infty} \psi_s(t) t^s p_s(x, D) a(t, x)$$

contains for each fixed $t > 0$ only finitely many terms different from 0 and thus pointwise defines a function $b = b(t, x) : \mathbb{R}^+ \to \mathcal{S}$. We have the estimate:

$$\begin{aligned} t^{\epsilon k}\|b(t, x)\|_k &\le \sum_{0}^{\infty} \psi_s(t) t^{s+\epsilon k}\|p_s(x, D) a(t, x)\|_k \\ &\le \sum_{0}^{mk} C(s, k)\psi_s(t) t^{s(1-\epsilon)}(t^{\epsilon(s+k)}\|a(t, x)\|_k) \\ &\quad + \sum_{mk+1}^{\infty} \psi_s(t) t^{s+\epsilon k}\|p_s(x, D) a(t, x)\|_{h(s)} \end{aligned}$$

$$\leq \sum_{0}^{mk} C(s,k) \sup_{t>0} t^{\epsilon(s+k)} \|a(t,x)\|_{s+k} + \sum_{mk+1} \psi_s(t) t^{s+\epsilon k-\epsilon(s+h(s))} t_s^{-s/m} 2^{-s}.$$

We have

$$s+\epsilon(k-s-h(s)) > s(1-\epsilon)+\epsilon k - \frac{s}{m} > \frac{s}{m} + \epsilon k > 0.$$

Because $t \leq t_s$ on the support of ψ_s, we obtain

$$\psi_s(t) t^{s+\epsilon(k-s-h(s))} \leq t_s^{(s/m)+\epsilon k} \leq t_s^{s/m} \quad \forall t > 0.$$

Therefore $t^{\epsilon k}\|b(t,x)\|_k$ is majorized by the convergent numerical series

$$\sum_{0}^{mk} C(s,k) \sup_{t>0} t^{\epsilon(s+k)} \|a(t,x)\|_{s+k} + \sum_{mk+1}^{\infty} 2^{-s}.$$

Thus $t^{\epsilon k} b(t,x)$ is a bounded function from $\mathbb{R}^+$ to S_k. This shows that $b(t,x) \in F_\epsilon(\mathbb{R}^+, \mathcal{S})$. Let us prove that (*) holds. For $0 < t < t_l/2$ we have:

$$t^{\epsilon k}\left\| b(t,x) - \sum_{0}^{l} t^s p_s(x,D) a(t,x) \right\|_k = t^{\epsilon k} \left\| \sum_{l+1}^{\infty} t^s \psi_s p_s(x,D) a(t,x) \right\|_k$$

$$\leq t^{\epsilon k} \sum_{l+1}^{m(k+l+1)} \psi_s(t) t^s C(s,k) \|a(t,x)\|_{s+k} + t^{\epsilon k} \sum_{m(k+l+1)+1}^{\infty} \psi_s(t) t^{s+\epsilon(k-s-h(s))} t_s^{-s/m} 2^{-s}.$$

We have

$$s-\epsilon(s+h(s)) \geq s-\epsilon\left(s+\frac{s}{m}\right) = s\left(1-\epsilon-\frac{\epsilon}{m}\right) > s\frac{2-\epsilon}{m} = \frac{s}{m} + s\,\frac{1-\epsilon}{m}.$$

Therefore

$$\psi_s(t) t^{s+\epsilon(k-s-h(s))} \leq t_s^{s/m} t^{s(1-\epsilon)/m+\epsilon k}.$$

For $s > m(k+l+1)$ one has

$$\epsilon k + s(1-\epsilon)/m > m(k+l+1)(1-\epsilon)/m$$

$$= (1-\epsilon)(k+l+1) \geq (l+1)(1-\epsilon).$$

As $t<1$ on support of ψ_s, we obtain:

$$t^{\epsilon k}\left\| b(t,x)-\sum_0^l t^s p_s(x,D)a(t,x)\right\|_k \le t^{(1-\epsilon)(l+1)}\left\{\sum_{l+1}^{m(k+l+1)} C(s,k)\times\right.$$

$$\left.\times \sup_{t>0} t^{\epsilon(s+k)}\|a(t,x)\|_{s+k}+\sum_{m(k+l+1)+1}^{\infty} 2^{-s}\right\}$$

$$\le c(k,l)t^{(1-\epsilon)(l+1)}.$$

Uniqueness modulo asymptotic equivalence is obvious from (*).

REMARK: With obvious notations, a $q\times p$ matrix with entries in W can be applied to elements of $F_\epsilon(\mathbb{R}^+,\mathcal{S}^p)$ to obtain an element of $A_\epsilon\mathcal{S}^q$. Because the image of an element asymptotic to 0 is asymptotic to 0, we obtain:

PROPOSITION 5: *A $q\times p$ matrix $\mathfrak{A}(t,x,D)$ of asymptotic operators defines for all $0\le\epsilon<1$ a linear map from $A_\epsilon\mathcal{S}^p$ into $A_\epsilon\mathcal{S}^q$ (that we will denote by the same symbol $\mathfrak{A}(t,x,D)$).*

Let us consider again the operator $A(x,D)$. It is not restrictive to assume that $a_j\le 0$ for all j. By lemma 4 in section (c) we have the following

PROPOSITION 6: *Let $a(t,x)\in F_\epsilon(\mathbb{R}^+,\mathcal{S}^p)$ (for some $0\le\epsilon<1$) and let*

$$u(t,x)=\operatorname{diag}(t^{-2a_1},\ldots,t^{-2a_p})a(t,(x-x_0)/t)\,e^{i\langle\xi,x-x_0\rangle/t^2}$$

Then u is a fading wave solution at (x_0,ξ^0) for $A(x,D)$ if and only if

$$A(t,x,D)a(t,x)\sim 0.$$

(f) *A remark on homological algebra*

Let A be a (non commutative) unitary ring and let R and L be respectively a right and a left A-module. Given any left A-module M, we define a map

$$R\otimes M\times \operatorname{Hom}(M,L)\to R\otimes L$$

by associating to $\lambda\in\operatorname{Hom}(M,L)$ and $f\in R\otimes M$ the element

$$(1\otimes\lambda)f \quad\text{in}\quad R\otimes L.$$

Let us consider now a projective resolution of the left A-module M:

$$\cdots \to P_{k+1} \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Then for each k the map

$$R \otimes P_k \times \mathrm{Hom}(P_k, L) \to R \otimes L$$

passes to the quotient, defining a homomorphism of abelian groups:

$$\omega_k : \mathrm{Tor}_k(R, M) \times \mathrm{Ext}^k(M, L) \to R \otimes L.$$

We also note that, if Q_2 is projective and we have a complex

$$Q_2 \to P_1 \to P_0 \to M \to 0$$

and we denote by H the cohomology of the complex

$$\mathrm{Hom}(P_0, L) \to \mathrm{Hom}(P_1, L) \to \mathrm{Hom}(Q_2, L)$$

there is an injective map $i : \mathrm{Ext}^1(M, L) \to H$.
Analogously, if H^* denotes the cohomology of the complex

$$R \otimes Q_2 \to R \otimes P_1 \to R \otimes P_0,$$

there is a surjective map $s : H^* \to \mathrm{Tor}_1(R, M)$.

The map $R \otimes P_1 \times \mathrm{Hom}(P_1, L) \to R \otimes L$ induces also a homomorphism

$$h : H^* \times H \to R \otimes L.$$

Let G be an abelian group and $\lambda : R \otimes L \to G$ a homomorphism of abelian groups. If the map $H^* \times H \to R \otimes L \to G$ has zero image, then also the map $\mathrm{Tor}_1(R, M) \times \mathrm{Ext}^1(M, L) \to R \otimes L \to G$ has zero image.

Indeed the maps ω_1 and h are obtained by passing to the quotients from the maps

$$\ker(R \otimes P_1 \to R \otimes P_0) \times \ker(\mathrm{Hom}(P_1, L) \to \mathrm{Hom}(P_2, L)) \to R \otimes L$$

and

$$\ker(R \otimes P_q \to R \otimes P_0) \times \ker(\mathrm{Hom}(P_1, L) \to \mathrm{Hom}(Q_2, L)) \to R \otimes L$$

and hence the statement is obtained by factoring through the inclusion

$$\ker(\mathrm{Hom}(P_1, L) \to \mathrm{Hom}(P_2, L)) \to \ker(\mathrm{Hom}(P_1, L) \to \mathrm{Hom}(Q_2, L)).$$

(g) *Invariant Formulation of the Radiation Principle*

For an asymptotic operator $\mathfrak{A}(t, x, D) = \Sigma\, t^s \mathfrak{A}_s(x, D)$ the formal adjoint is defined by $\mathfrak{A}^*(t, x, D) = \Sigma\, t^2 \mathfrak{A}_s^*(x, D)$. We have (cf. sect. (e)):

$$(\tilde{A}(t, x, D))^* = \tilde{A}^*(t, x, D).$$

Defining a right action of $\tilde{W}$ on $\overline{A_\epsilon \mathcal{S}}$ by

$$\bar{w} \cdot p = \overline{p^* w} \quad \text{for} \quad \bar{w} \in \overline{A_\epsilon \mathcal{S}} \quad \text{and} \quad p \in W,$$

the space $\overline{A_\epsilon \mathcal{S}}$ turns into a right-$\tilde{W}$-module and the map

$$(\bar{u}, v) \to \int \bar{u} v\, dx \in \tilde{\mathbb{C}} \quad \text{from} \quad \overline{A_\epsilon \mathcal{S}} \times A_\epsilon \mathcal{S} \quad \text{into} \quad \tilde{\mathbb{C}}$$

induces a homomorphism of abelian groups from the corresponding tensor product:

$$\lambda : \overline{A_\epsilon \mathcal{S}} \otimes A_\epsilon \mathcal{S} \to \tilde{\mathbb{C}}.$$

Let $\mathfrak{M} = \mathfrak{M}(x_0, \xi^0) = \mathrm{coker}(\tilde{A}(t, x, D) : \tilde{W}^q — \tilde{W}^p)$. By the remarks made in the previous section we obtain

PROPOSITION 7: *If the complex* (1) *admits the special Poincaré lemma at* x_0*, then for all* $\xi^0 \in \mathbb{R}^n - \{0\}$ *the map*

$$\lambda_0 \omega_1 : \mathrm{Tor}_1(\overline{A_\epsilon \mathcal{S}}, \mathfrak{M}(x_0, \xi^0)) \times \mathrm{Ext}^1(\mathfrak{M}(x_0, \xi^0), A_\epsilon \mathcal{S}) \to \tilde{\mathbb{C}}$$

is identically 0.

§3. Rings of asymptotic operators

(a) Let us denote by W the ring of differential operators with polynomial coefficients on $\mathbb{R}^n$. An ordering of W is the datum of an ascending chain of $\mathbb{C}$-linear subspaces of W:

$$W_0 \subset W_1 \subset W_2 \subset W_3 \subset \cdots$$

satisfying the following conditions:

(i) $W_0 = \mathbb{C}$

(ii) $\cup W_j = W$

(iii) $W_i W_j \subset W_{i+j}$ for $i, j = 0, 1, 2, \ldots$

(iv) $[W_i, W_j] = \{[P, Q] = PQ - QP \mid P \in W_i, Q \in W_j\} \subset W_{i+j-1}$ for $i, j = 0, 1, 2, \ldots$ (with the convention that $W_i = 0$ if $i < 0$).

We set $H = \bigoplus_{j=0}^{\infty} W_j / W_{j-1} = \bigoplus_{j=0}^{\infty} H_j$. By (iii) we can define on H a ring structure by taking as product of the equivalence class of P_1 in W_{j_1}/W_{j_1-1} and of the equivalence class of P_2 in W_{j_2}/W_{j_2-1} the equivalence class of P_1P_2 in $W_{j_1+j_2}/W_{j_1+j_2-1}$. By (iv) the ring H is commutative.

EXAMPLES: (1) Take $W_0 = \mathbb{C}$, $W_j = W$ for $j > 0$. Note that in this case H is not Noetherian.

(2) If $W_j = \{\Sigma_{|\alpha+\beta| \le j}\, a_{\alpha\beta} x^\beta D^\alpha\}$, then H is the ring of polynomials in $2n$ indeterminates with usual gradings.

(3) More in general, we can fix n-uples h, k of strictly positive integers and set $W_j = \{\Sigma\, a_{\alpha\beta} x^\beta D^\alpha \mid \langle \beta, h\rangle + \langle \alpha, k\rangle \le j\}$.
Then H is again the ring of polynomials in $2n$ indeterminates $x_1, \ldots, x_n$, $\xi_1, \ldots, \xi_n$, with the grading obtained by considering x_j homogeneous of degree h_j and ξ_j homogeneous of degree k_j ($j = 1, \ldots, n$, h_j and k_j are the j-th component of h and k respectively).

(b) Having fixed an ordering for W, we define the ring $\mathfrak{W}$ of asymptotic differential operators as the ring of formal power series of the indeterminate t of the form:

$$\sum_{s=0}^{\infty} t^s P_s(x, D) \quad \text{with} \quad P_s \in W_s\ \forall s.$$

We denote by $\mathfrak{m}$ the ideal of $\mathfrak{W}$ of elements of this form with $P_0 = 0$. We have the following

PROPOSITION 8: *The ring W is a local ring with maximal ideal $\mathfrak{m}$, separated and complete for the $\mathfrak{m}$-adic topology.*

The verification of this statement is straightforward and is omitted.

(c) On $\mathfrak{W}$ we consider the $\mathfrak{m}$-adic filtration

$$\mathfrak{W} = \mathfrak{m}^0 \supset \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \ldots$$

Let $\mathfrak{G}_j = \mathfrak{m}^j/\mathfrak{m}^{j+1}$ and let $\mathfrak{G} = \bigoplus \mathfrak{G}_j$ be the graded ring associated to this filtration. We note that $\mathfrak{G}_j$ is isomorphic to $t^j W_j$ and therefore $\mathfrak{G}$ can be identified to the subring of elements of $\mathfrak{W}$ that are polynomials in t. We denote by $\mathfrak{W} \xrightarrow{\pi} \mathfrak{G}$ the natural projection.

Let us consider a free $\mathfrak{W}$-module $\mathfrak{W}^p$, of finite type. Given $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{Z}^p$ we define canonically a filtration on $\mathfrak{W}^p$ by settting

$$(\mathfrak{W}^p)_k^\alpha = \{(p_1, \dots, p_p) \in \mathfrak{W}^p \mid p_j \in \mathfrak{m}^{k-\alpha_j} \text{ for } j = 1, \dots, p\}.$$

To this filtration corresponds a canonical graduation of the module $\mathfrak{G}^p$ given by $\mathfrak{G}_k^\alpha = \{(t^{k-\alpha_1}P_1, \dots, t^{k-\alpha_p}P_p) \mid P_j \in W_{k-\alpha_j} \text{ for } j = 1, \dots, p\}$.
We denote by π^α the natural map $\mathfrak{W}^p \to \mathfrak{G}^p$. We will write $\mathfrak{W}^\alpha$ and $\mathfrak{G}^\alpha$ instead of $\mathfrak{W}^p$ and $\mathfrak{G}^p$ to emphasize the choice of the filtration and of the graduation.

Let $\mathfrak{M}$ be a left $\mathfrak{W}$-module of finite type. Then we can define on $\mathfrak{M}$ a filtration compatible with the $\mathfrak{m}$-adic filtration of $\mathfrak{W}$ and then consider the associated graded module that we denote by $\mathfrak{M}^0$.

(d) Note that under the assumptions made up to now, the rings $\mathfrak{G}$ and $\mathfrak{W}$ can be non Noetherian. This happens for instance with the filtration of $\mathfrak{W}$ given by:

$$W_j = \left\{\sum a_{\alpha\beta} x^\beta D^\alpha \mid |a + \beta| \le j^2\right\}.$$

Then the maximal ideal $\mathfrak{m}$ itself is not finitely generated and the same happens for $\mathfrak{m} \cap \mathfrak{G}$. To avoid this unpleasant fact, we make the following: *assumption*: *H is Noetherian.*

This is the case of examples (2) and (3) of section (a), that are the ones relevant for our applications. We note that in the case of example (1) $\mathfrak{G}$ and $\mathfrak{W}$ are Noetherian, while H is non Noetherian.

(e) Let (t) denote the ideal of $\mathfrak{G}$ generated by t. This is a central ideal as t commutes with all elements of $\mathfrak{G}$. We consider then the (t)-adic filtration of G:

$$\mathfrak{G} = (t)^0 \supset (t) \supset (t)^2 \supset (t)^3 \supset \cdots$$

The graded ring associated to this filtration is (isomorphic to) the ring $H[s]$ of polynomials in one indeterminate with coefficients in H. Because $H[s]$ is Noetherian in view of the assumption above, we obtain (cf. [15], Corollary 2 at p. 414):

LEMMA 5: *$\mathfrak{G}$ is Noetherian.*

From this lemma it follows at once:

PROPOSITION 9: *$\mathfrak{W}$ is a Noetherian ring.*

(f) Let us go back to the situation of section (c). Let $\mathfrak{M}$ be a filtered left $\mathfrak{W}$-module of finite type, with a filtration compatible with the $\mathfrak{m}$-adic filtration. Let $\mathfrak{M}^0$ denote the associated graded $\mathfrak{G}$-module.

PROPOSITION 10: *There is a free resolution*

$$(5)\qquad \cdots \to \mathfrak{G}^{\alpha^{(k)}} \xrightarrow{\mathfrak{a}^{(k)}} \mathfrak{G}^{\alpha^{(k-1)}} \xrightarrow{\mathfrak{a}^{(k-1)}} \cdots \to \mathfrak{G}^{\alpha^{(1)}} \xrightarrow{\mathfrak{a}^{(1)}} \mathfrak{G}^{\alpha^{(0)}} \to \mathfrak{M}^0 \to 0$$

of $\mathfrak{M}^0$ by homogeneous free modules of finite type $\mathfrak{G}^{\alpha^{(k)}} = \mathfrak{G}^{p_k}$ for $\alpha^{(k)} = (\alpha_1^{(k)}, \ldots, \alpha_{p_k}^{(k)}) \in \mathbb{Z}^{p_k}$ and homogeneous left $\mathfrak{G}$-homomorphisms. To any such resolution there correspond left $\mathfrak{W}$-homomorphisms $\mathfrak{A}^{(k)}: \mathfrak{W}^{p_{k+1}} \to \mathfrak{W}^{p_k}$ $(k = 0, 1, \ldots)$ such that $\mathfrak{a}^{(k)} \circ \pi^{\alpha^{(k+1)}} = \pi^{\alpha^{(k)}} \circ \mathfrak{A}^{(k)}$ and

$$(6)\qquad \cdots \to \mathfrak{W}^{p_{k+1}} \xrightarrow{\mathfrak{A}^{(k)}} \mathfrak{W}^{p_k} \to \cdots \to \mathfrak{W}^{p_1} \xrightarrow{\mathfrak{A}^{(0)}} \mathfrak{W}^{p_0} \to \mathfrak{M} \to 0$$

is a free resolution of $\mathfrak{M}$ as a left $\mathfrak{W}$-module.

A resolution (6) with the property of the proposition above is called *correct.*

(g) Let us denote by $(\tilde{t})$ the ideal of $\mathfrak{W}$ generated by t. This is a central ideal of $\mathfrak{W}$ and we consider the $(\tilde{t})$-adic filtration of $\mathfrak{W}$:

$$\mathfrak{W} = (\tilde{t})^0 \supset (\tilde{t}) \supset (\tilde{t})^2 \supset (\tilde{t})^3 \supset \cdots$$

The associated graded ring is (isomorphic to) the ring $\tilde{H}[s]$ of polynomials in one indeterminate s with coefficients in the ring $\tilde{H}$ of formal series $\Sigma_j h_j$ with $h_j \in H_j$ $\forall j$.

Because $\tilde{H}$ is obtained from H by I-adic completion with respect to the ideal $I = \bigoplus_{j \geq 1} H_j$, the ring $\tilde{H}$ is a flat ring extension of H and it follows in an obvious way that $\tilde{H}[s]$ is a flat ring extension of $H[s]$.

By taking "correct resolutions" with respect to the (t)-adic (resp. the $(\tilde{t})$-adic) filtration one can easily prove ("correct resolutions" exist by the generalization of Artin-Rees lemma to central ideals in Noetherian non commutative rings; cf. [15], Theorem 1, p. 292):

PROPOSITION 11: *The ring $\mathfrak{W}$ is a flat ring extension of the ring $\mathfrak{G}$.*

Let Λ denote the subring of $\mathfrak{W}$ of elements that have a finite order with respect to $\partial/\partial x_1, \ldots, \partial/\partial x_n : \Lambda$ is the ring of all formal power series of t:

$$\sum_{h=0}^{\infty} t^h P_h(x, D) \text{ with } P_h \in W_h \text{ and } \sup_h \text{ order of } P_h$$

$$\text{with respect to } \partial/\partial x_1, \ldots, \partial/\partial x_n < +\infty.$$

If the ordering of W is the one given in example (2) or (3) in sect. (a), then $\tilde{H}$ is the ring of formal power series in $2n$ indeterminates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$. If we take on Λ the $(\hat{t})$-adic filtration, for $(\hat{t})$ denoting the central ideal of Λ generated by t, then the associated graded ring is the ring $\hat{H}[s]$ of polynomials in s with coefficients in the ring $\hat{H}$ of polynomials of $\xi_1, \ldots, \xi_n$ with coefficients formal power series of $x_1, \ldots, x_n$. Then Λ is Noetherian and again by the generalization of Artin-Rees lemma there are "correct resolutions" of finitely generated left Λ-modules and we can conclude as above by:

PROPOSITION 12: *If the ordering of W is as in examples* (2) *or* (3) *in sect.* (*a*), *then $\mathfrak{W}$ is a flat ring extension of Λ.*

(g) We consider now a complex of differential operators with smooth coefficients on an open set $\Omega < \mathbb{R}^n$:

$$\mathscr{E}^p(\Omega) \xrightarrow{A(x, D)} \mathscr{E}^q(\Omega) \xrightarrow{B(x, D)} \mathscr{E}^r(\Omega)$$

Let $a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r$ be fixed integers such that

$$A(x, D) = (A_{ij}(x, D)) \text{ with } A_{ij}(x, D) = \sum_{|\alpha| \le a_j - b_i} A_{ij}^{\alpha}(x) D^{\alpha} \text{ of order } \le a_j - b_i$$

$$B(x, D) = (B_{hi}(x, D)) \text{ with } B_{hi}(x, D)$$
$$= \sum_{|\alpha| \le b_i - c_h} B_{hi}^{\alpha}(x) D^{\alpha} \text{ of order } \le b_i - c_h$$

We can say $A(x, D)$ is of type (a_j, b_i) and $B(x, D)$ is of type (b_i, c_h). Let us define:

$$\hat{A}(x, \xi) = (\hat{A}_{ij}(x, \xi)) \quad \text{with} \quad \hat{A}_{ij}(x, \xi) = \sum_{|\alpha| = a_j - b_i} A_{ij}^{\alpha}(x) \xi^{\alpha}$$

$$\hat{B}(x, \xi) = (\hat{B}_{hi}(x, \xi)) \quad \text{with} \quad \hat{B}_{hi}(x, \xi) = \sum_{|\alpha| = b_i - c_h} B_{hi}^{\alpha}(x) \xi^{\alpha}$$

Let ϕ_{x_0} denote the ring of formal power series centered at x_0 and let us write still $\hat{A}(x, \xi)$ and $\hat{B}(x, \xi)$ for the matrices obtained by substituting to the coefficients of $\hat{A}(x, \xi)$ and $\hat{B}(x, \xi)$ their formal Taylor power series centered at x_0. We say that the complex (1) is formally correct at $x_0 \in \Omega$ if the sequence

$$\phi_{x_0}[\xi]^r \xrightarrow{\hat{B}(x,\xi)} \phi_{x_0}[\xi]^q \xrightarrow{\hat{A}(x,\xi)} \Phi_{x_0}[\xi]^p \tag{7}$$

is an exact sequence of $\phi_{x_0}[\xi]$-modules.

We define $\tilde{A}(t, x, D)$ and $\tilde{B}(t, x, D)$ as the Taylor series with respect to t of the operators

$$\operatorname{diag}(t^{-2b_1}, \ldots, t^{-2b_q})\, e^{-i\langle \xi^0, x\rangle/t} A\left(x_0 + tx, \frac{1}{t} D\right) e^{i\langle \xi^0, x\rangle/t} \operatorname{diag}(t^{2a_1}, \ldots, t^{2a_p})$$

and

$$\operatorname{diag}(t^{-2c_1}, \ldots, t^{-2c_r})\, e^{-i\langle \xi^0, x\rangle/t} B\left(x_0 + tx, \frac{1}{t} D\right) e^{i\langle \xi^0, x\rangle/t} \operatorname{diag}(t^{2b_1}, \ldots, t^{2b_q}).$$

Then we obtain from proposition 12:

PROPOSITION 13: *If the complex* (1) *is formally correct at* $x_0 \in \Omega$, *then*

$$\tilde{W}^r \xrightarrow{B(t,x,D)} \tilde{W}^q \xrightarrow{A(t,x,D)} \tilde{W}^p \text{ is exact.} \tag{8}$$

When the statement of Proposition 13 holds, the resolution (5) of the graded module $\mathfrak{M}^0(x_0, \xi^0)$ associated to $\mathfrak{M}(x_0, \xi^0) = \operatorname{coker}(\tilde{A}(t, x, D) : \tilde{W}^q \to \tilde{W}^p)$ is called *the first subprincipal complex at* (x_0, ξ^0) *of the complex* (1).

Sometimes it is more convenient to consider localizations of the complex (1) by rings of asymptotic operators obtained from different orderings of W.

Let $m = (m_1, \ldots, m_n)$ be a n-uple of integers and let $m_0 > |m_i|\ \forall i$. Then we consider the ordering of W given by

$$W_{(k)} = \left\{ \sum_{\langle \beta, m\rangle - \langle \alpha, m\rangle + m_0|\alpha+\beta| \le k} a_{\alpha\beta} x^\beta D^\alpha \right\}.$$

We denote by $\mathfrak{W}$ the associated ring of asymptotic operators. To avoid confusion while comparing the rings $\mathfrak{W}$ and $\tilde{W}$, we denote by $y_1, \dots, y_n$, $\partial_1, \dots, \partial_n, h$ the generators of $\mathfrak{W}$ and by $x_1, \dots, x_n$, $D_1, \dots, D_n, t$ the generators of $\tilde{W}$. Then by the change of variables

$$(9) \qquad y_j = h^{-m_j}x_j, \quad \partial_j = h^{m_j}D_j, \quad t = h^{m_0}$$

we obtain an injective ring homomorphism $\tilde{W} \hookrightarrow \mathfrak{W}$.

It is easy to check the following:

PROPOSITION 14: *The ring $\mathfrak{W}$ is a flat ring extension of $\tilde{W}$.*

Denoting by $\tilde{\tilde{A}}(h, y, \partial)$ the image of $\tilde{A}(t, x, D)$ under the above homomorphism, we can consider the module $\mathfrak{N}(x_0, \xi^0) = \operatorname{coker}(\tilde{\tilde{A}}(h, y, \partial): \mathfrak{W}^q \to \mathfrak{W}^p)$. Having chosen a filtration of this module, the resolution (5) of the associated graded module $\mathfrak{N}^0(x_0, \xi^0)$ is called *the second subprincipal complex of* (1) *at* (x_0, ξ^0).

PROPOSITION 15: *Let $\mu = \sup|m_j| < m_0$. Then for $0 \le \epsilon < 1$ the map*

$$a(h, y) \to a(t^{1/m_0}, t^{-m_1/m_0}x_1, \dots, t^{-m_n/m_0}x_n)$$

is linear and continuous from $F_\epsilon(\mathbb{R}^+, \mathcal{S})$ into $F_\delta(\mathbb{R}^+, \mathcal{S})$ where $\delta = (\mu + \epsilon)/m_0$, and in particular defines a map from $A_\epsilon\mathcal{S}$ into $A_\delta\mathcal{S}$.

PROOF: The relations (9) describe for each fixed h a symplectic change of coordinates in $E = \mathbb{R}^n \oplus \mathbb{R}^n$, linear and of norm $h^{-\mu}$ for $h \le 1$. Then the statement follows by lemma 2 in §2, (b).

As a corollary, we obtain the following

PROPOSITION 16: *If the complex* (1) *admits the special Poincaré lemma at x_0 then for every $\xi^0 \in \mathbb{R}^n - \{0\}$ the map*

$$\lambda \circ \omega_1 : \operatorname{Tor}_1(\overline{A_\epsilon\mathcal{S}}, \mathfrak{N}(x_0, \xi^0)) \times \operatorname{Ext}^1(\mathfrak{N}(x_0, \xi^0), A_\epsilon\mathcal{S}) \to \tilde{\mathbb{C}}$$

is identically 0.

(h) *Asymptotic series.*

We denote by $A\mathcal{S}$ the ring of formal power series of an indeterminate h with coefficients in $\mathcal{S}$: an element of $A\mathcal{S}$ has the form:

$$u(h, y) = \sum_{s=0}^{\infty} h^s u_s(y) \quad \text{with} \quad u_s \in \mathcal{S} \quad \forall s.$$

The following proposition is well known:

PROPOSITION 17: *$A\mathcal{S}$ is isomorphic to the quotient of the space $C^\infty(\bar{\mathbb{R}}^+, \mathcal{S})$ by asymptotic equivalence (i.e. two elements $u(h, y)$, $v(h, y)$ of $C^\infty(\bar{\mathbb{R}}^+, \mathcal{S})$ are equivalent if*

$$\|u(h, y) - v(h, y)\|_m = o(h^m)\ \forall \text{ integer } m \geq 0.)$$

In particular we have an inclusion $A\mathcal{S} \hookrightarrow A_0\mathcal{S}$.

COROLLARY: *If the complex* (1) *admits the special Poincaré lemma at x_0 then for every $\xi^0 \in \mathbb{R}^n - \{0\}$ the map*

$$\lambda \circ \omega_1 : \mathrm{Tor}_1(\overline{A\mathcal{S}}, \mathfrak{N}(x_0, \xi^0)) \times \mathrm{Ext}^1(\mathfrak{N}(x_0, \xi^0), A\mathcal{S}) \to \tilde{\mathbb{C}}$$

is the 0 *map.*

(i) Let $\mathfrak{N}$ be a left-$\mathfrak{W}$-module of finite type and let

$$\cdots \to \mathfrak{W}^{p_{k+1}} \xrightarrow{\mathfrak{A}^{(k)}} \mathfrak{W}^{p_k} \xrightarrow{\mathfrak{A}^{(k-1)}} \cdots \to \mathfrak{W}^{p_1} \xrightarrow{\mathfrak{A}^{(0)}} \mathfrak{W}^{p_0} \to \mathfrak{N} \to 0 \tag{10}$$

be a resolution of $\mathfrak{N}$ by free $\mathfrak{W}$-modules of finite type, correct with respect to gradings $\alpha^{(k)} = (\alpha_1^{(k)}, \ldots, \alpha_{p_k}^{(k)}) \in \mathbb{Z}^{p_k} (k = 0, 1, 2, \ldots)$. For $h \neq 0$ and $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}^p$ we write $h^\alpha = \mathrm{diag}(h^{\alpha_1}, \ldots, h^{\alpha_p})$. Then we have:

$$\mathfrak{A}^{(k)} = h^{\alpha^{(k+1)}} \mathfrak{B}^{(k)} h^{-\alpha^{(k)}} = h^{\alpha^{(k+1)}} \left(\sum h^s \mathfrak{B}_s^{(k)} \right) h^{-\alpha^{(k)}} \quad (k = 0, 1, 2, \ldots).$$

It follows by the correctness assumption that

$$\cdots \to \mathfrak{G}^{p_{k+1}} \xrightarrow{h^{\alpha^{(k+1)}} \mathfrak{B}_0^{(k)} h^{-\alpha^{(k)}}} \mathfrak{G}^{p_k} \to \cdots \to \mathfrak{G}^{p_0} \to \mathfrak{N}^0 \to 0 \tag{11}$$

is an exact sequence. But this implies that the sequence

$$\cdots \to W^{p_{k+1}} \xrightarrow{\mathfrak{B}_0^{(k)}} W^{p_k} \to \cdots \to W^{p_1} \xrightarrow{\mathfrak{B}_0^{(0)}} W^{p_0} \tag{12}$$

is exact and therefore, denoting by $\{W\}$ the ring of formal power series of h with coefficients in W, also the sequence

$$\cdots \to \{W\}^{p_{k+1}} \xrightarrow{\mathfrak{B}^{(k)}} \{W\}^{p_k} \to \cdots \to \{W\}^{p_1} \xrightarrow{\mathfrak{B}^{(0)}} \{W\}^{p_0} \tag{13}$$

is exact. We call the sequence (13) *reduced asymptotic sequence* and the sequence (12) *reduced subprincipal sequence.* We set

$$\mathfrak{X} = \operatorname{coker}(\mathfrak{B}^{(0)} : \{W\}^{p_1} \to \{W\}^{p_0})$$

$$\mathfrak{X}^0 = \operatorname{coker}(\mathfrak{B}_0^{(0)} : W^{p_1} \to W^{p_0}).$$

Let $\mathscr{L}$ be a left W-module. Then the set $A\mathscr{L}$ of formal power series of h with coefficients in $\mathscr{L}$ turns in a natural way into a left $\{W\}$-module and $\mathscr{L}$ is isomorphic to the quotient $A\mathscr{L}/hA\mathscr{L}$. By this identification the functors $\operatorname{Hom}_W(-, \mathscr{L})$ applied to (12) and $\operatorname{Hom}_{\{W\}}(-, A\mathscr{L}/hA\mathscr{L})$ applied to (13) give the same complex:

$$\mathscr{L}^{p_0} \xrightarrow{\mathfrak{B}_0^{(0)}} \mathscr{L}^{p_1} \xrightarrow{\mathfrak{B}_0^{(1)}} \mathscr{L}^{p_2} \to \cdots \tag{14}$$

and thus we obtain an isomorphism:

$$\operatorname{Ext}^j_W(\mathfrak{X}^0, \mathscr{L}) = \operatorname{Ext}^j_W(\mathfrak{X}, A\mathscr{L}/hA\mathscr{L}). \quad (j = 0, 1, 2, \ldots).$$

Taking into account this isomorphism, from the exact sequence

$$0 \to A\mathscr{L} \xrightarrow{h} A\mathscr{L} \to A\mathscr{L}/hA\mathscr{L} \to 0$$

we deduce the long exact sequence:

$$0 \to \operatorname{Hom}_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \xrightarrow{h} \operatorname{Hom}_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \to \operatorname{Hom}_W(\mathfrak{X}^0, \mathscr{L}) \to$$

$$\to \operatorname{Ext}^1_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \xrightarrow{h} \operatorname{Ext}^1_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \to \operatorname{Ext}^1_W(\mathfrak{X}^0, \mathscr{L}) \to$$

$$\cdots \to \operatorname{Ext}^j_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \xrightarrow{h} \operatorname{Ext}^j_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \to \operatorname{Ext}^j_W(\mathfrak{X}^0, \mathscr{L}) \to$$

$$\to \operatorname{Ext}^{j+1}_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \to \cdots$$

We note that the groups $\operatorname{Ext}^j_W(\mathfrak{X}^0, \mathscr{L})$ measure the cohomology of the complex (14), while the groups $\operatorname{Ext}^j_{\{W\}}(\mathfrak{X}, A\mathscr{L})$ measure the cohomology of the complex:

$$A\mathscr{L}^{p_0} \xrightarrow{\mathfrak{B}^{(0)}} A\mathscr{L}^{p_1} \xrightarrow{\mathfrak{B}^{(1)}} A\mathscr{L}^{p_2} \to \cdots. \tag{15}$$

PROPOSITION 18: *If* $\operatorname{Ext}^j_W(\mathfrak{X}^0, \mathscr{L}) = 0$, *then also* $\operatorname{Ext}^j_{\{W\}}(\mathfrak{X}, A\mathscr{L}) = 0$. *If* $\operatorname{Ext}^{j+1}_{\{W\}}(\mathfrak{X}, A\mathscr{L}) = 0$, *then the map* $\operatorname{Ext}^j_{\{W\}}(\mathfrak{X}, A\mathscr{L}) \to \operatorname{Ext}^j_W(\mathfrak{X}^0, \mathscr{L})$ *is onto.*

PROOF: The last statement is a trivial consequence of the long exact sequence established above. Let us prove the first. We note that the case $j=0$ is trivial: if $f=\Sigma\, h^s f_s \in A\mathscr{L}^{p_0}-\{0\}$ satisfies $\mathfrak{B}^{(0)}f=0$, then the first term of the series defining f which is different from 0, say f_m, satisfies $\mathfrak{B}_0^{(0)}f_m=0$. But, if $\mathrm{Ext}^0_W(\mathfrak{X}^0,\mathscr{L})=0$, this implies that $f_m=0$ and brings a contradiction, that shows that $\mathrm{Ext}^0_{\{W\}}(\mathfrak{X},A\mathscr{L})=0$.

Let us consider then the case $j\geq 1$. Let $f\in A\mathscr{L}^{p_j}$ satisfy $\mathfrak{B}^{(j)}f=0$. Let $f=\Sigma\, h^s f$. Then $f_0\in\mathscr{L}^{p_j}$ satisfies $\mathfrak{B}_0^{(j)}f_0=0$ and then by the assumption that $\mathrm{Ext}^j_W(\mathfrak{X}^0,\mathscr{L})=0$ there is $u_0\in\mathscr{L}^{p_{j-1}}$ such that $\mathfrak{B}_0^{(j-1)}u_0=f_0$. Then we consider $f^{(1)}=f-\mathfrak{B}^{(j-1)}u_0=\Sigma_{s\geq 1} f_s^{(1)}h^s$. We have $\mathscr{B}_0^{(j-1)}f_1^{(1)}=0$ and then we can find $u_1\in\mathscr{L}^{p_{j-1}}$ such that $\mathfrak{B}_0^{(j-1)}u_1=f_1^{(1)}$. By iteration we construct a sequence $\{u_k\}\subset\mathscr{L}^{p_{j-1}}$ such that

$$f^{(k+1)}=f-\mathfrak{B}^{(j-1)}(u_0+hu_1+h^2u_2+\cdots+h^ku_k)\in h^{k+1}A\mathscr{L}^{p_j}.$$

Then $u=\Sigma\, h^s u_s\in A\mathscr{L}^{p_{j-1}}$ solves $f=\mathfrak{B}^{(j-1)}u$.

We note now that, taking any right W-module $\mathscr{R}$, the set of formal power series of h with coefficients in $\mathscr{R}$ forms a right $\{W\}$-module, that we denote by $A\mathscr{R}$. We have $A\mathscr{R}/hA\mathscr{R}\cong\mathscr{R}$ and, by this identification, the functor $\mathscr{R}\otimes_W$-applied to (12) and the functor $A\mathscr{R}/hA\mathscr{R}\otimes_{\{W\}}$-applied to the resolution (13) yield the same complex:

$$\cdots\to\mathscr{R}^{p_k}\xrightarrow{\mathfrak{B}_0^{(k-1)}}\mathscr{R}^{p_{k-1}}\to\cdots\to\mathscr{R}^{p_1}\xrightarrow{\mathfrak{B}_0^{(0)}}\mathscr{R}^{p_0}. \tag{16}$$

Therefore we obtain the isomorphism

$$\mathrm{Tor}_j^W(\mathscr{R},\mathfrak{X}^0)=\mathrm{Tor}_j^{\{W\}}(A\mathscr{R}/hA\mathscr{R},\mathfrak{X})\quad (j=0,1,2,\ldots).$$

Taking into account this isomorphism and the exact sequence

$$0\to A\mathscr{R}\xrightarrow{h}A\mathscr{R}\to A\mathscr{R}/hA\mathscr{R}\to 0$$

we deduce the long exact sequence:

$$\cdots\to\mathrm{Tor}_j^{\{W\}}(A\mathscr{R},\mathfrak{X})\xrightarrow{h}\mathrm{Tor}_{j-1}^{\{W\}}(A\mathscr{R},\mathfrak{X})\to\mathrm{Tor}_{j-1}^{W}(\mathscr{R},\mathfrak{X}^0)\to$$

$$\to\mathrm{Tor}_{j-1}^{\{W\}}(A\mathscr{R},\mathfrak{X})\xrightarrow{h}\mathrm{Tor}_{j-1}^{\{W\}}(A\mathscr{R},\mathfrak{X})\to\mathrm{Tor}_{j-1}^{\{W\}}(\mathscr{R},\mathfrak{X}^0)\to$$

$$\cdots\to A\mathscr{R}\otimes_{\{W\}}\mathfrak{X}\xrightarrow{h}A\mathscr{R}\otimes_{\{W\}}\mathfrak{X}\to R\otimes_W\mathfrak{X}^0\to 0.$$

Arguing as in the previous proposition one obtains

PROPOSITION 19: *If* $\mathrm{Tor}_j^W(\mathscr{R}, \mathfrak{X}^0) = 0$, *then* $\mathrm{Tor}_j^{\{W\}}(A\mathscr{R}, \mathfrak{X}) = 0$. *If* $\mathrm{Tor}_{j-1}^{\{W\}}(A\mathscr{R}, \mathfrak{X}) = 0$, *then the map* $\mathrm{Tor}_j^{\{W\}}(A\mathscr{R}, \mathfrak{X}) \to \mathrm{Tor}_j^W(\mathscr{R}, \mathfrak{X}^0)$ *is surjective.*

(j) Let us consider a complex of differential operators with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$:

$$\mathscr{E}^{q_0} \xrightarrow{A_0(x,D)} \mathscr{E}^{q_1} \xrightarrow{A_1(x,D)} \mathscr{E}^{q_2} \xrightarrow{A_2(x,D)} \mathscr{E}^{q_3} \to \cdots \tag{17}$$

We assume that this complex is formally correct at $x_0 \in \Omega$ for suitable multigradings. For a fixed $\xi^0 \in \mathbb{R}^n - \{0\}$ we consider the module $\mathfrak{N}(x_0, \xi^0) = \mathfrak{N}$ defined in sect. (g), so that (10) is an asymptotic sequence for (17) at (x_0, ξ^0). Let $\mathscr{L}$ be any left W-submodule of $\mathscr{S}$ and let $\mathscr{R} = \bar{\mathscr{L}}$. Then the map

$$(\bar{u}, v) \to \int \bar{u} \cdot v\, dx \text{ induces a natural map}$$

$$\mathrm{Tor}_j(\bar{\mathscr{L}}, \mathfrak{X}^0) \times \mathrm{Ext}^j(\mathfrak{X}^0, \mathscr{L}) \to \mathbb{C}.$$

If $\mathrm{Tor}_{j-1}(\bar{\mathscr{L}}, \mathfrak{X}^0) = 0$ and $\mathrm{Ext}^{j+1}(\mathfrak{X}^0, \mathscr{L}) = 0$, then we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Tor}_j^{\{W\}}(A\bar{\mathscr{L}}, \mathfrak{X}) & \times & \mathrm{Ext}^j_{\{W\}}(\mathfrak{X}, A\mathscr{L}) & \to & \mathbb{C}\{\{h\}\} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Tor}_j^W(\bar{\mathscr{L}}, \mathfrak{X}^0) & \times & \mathrm{Ext}^j_W(X^0, \mathscr{L}) & \to & \mathbb{C}
\end{array}$$

(where the vertical arrow to the right is evaluation at 0). For $m \geq \sup_i |\alpha_i^{(j)}|$ we obtain commutative diagrams:

$$\begin{tikzcd}
\mathrm{Tor}_j^W(A\bar{\mathscr{L}}, \mathfrak{N}) \times \mathrm{Ext}^j_W(\mathfrak{N}, A\mathscr{L}) \arrow[rr, "h^{m+\alpha^{(j)}}"] \arrow[dr, "h^m"'] & & \mathrm{Tor}_j^W(A\bar{\mathscr{L}}, \mathfrak{X}) \times \mathrm{Ext}^j_W(\mathfrak{X}, A\mathscr{L}) \arrow[dl] \\
& A\bar{\mathscr{L}} \otimes A\mathscr{L} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Tor}_j^W(A\bar{\mathscr{L}}, \mathfrak{X}) \times \mathrm{Ext}^j_W(\mathfrak{X}, A\mathscr{L}) \arrow[rr, "h^{m-\alpha^{(j)}}"] \arrow[dr, "h^m"'] & & \mathrm{Tor}_j^W(A\bar{\mathscr{L}}, \mathfrak{N}) \times \mathrm{Ext}^j_W(\mathfrak{N}, A\mathscr{L}) \arrow[dl] \\
& A\mathscr{L} \otimes A\mathscr{L} &
\end{tikzcd}$$

Therefore we obtain the following statement:

PROPOSITION 20: *Let* $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$ *be fixed. Assume that* $\mathrm{Tor}_{j-1}^{W}(\bar{\mathscr{L}}, \mathfrak{X}^0) = 0$ *and* $\mathrm{Ext}_W^{j+1}(\mathfrak{X}^0, \mathscr{L}) = 0$. *Then, if the complex* (17) *admits the special Poincaré lemma at* $\mathscr{C}_{x_0}^{q_j}$, *the map*:

$$\mathrm{Tor}_j^W(\bar{\mathscr{L}}, \mathfrak{X}^0) \times \mathrm{Ext}_W^j(\mathfrak{X}^0, \mathscr{L}) \to \mathbb{C}$$

is identically 0.

§4. Subelliptic estimates for reduced subprincipal complexes

(a) *Preliminaries.* In §2 we defined the Hilbert spaces S_m as completion of $\mathscr{S}$ for the norm

$$\|u\|_m = \Big(\sum_{|\alpha+\beta|\le m} \int |x^\beta D^\alpha u|^2\, dx\Big)^{1/2}$$

associated to the scalar product

$$(u, v)_m = \sum_{|\alpha+\beta|\le m} \int \overline{x^\beta D^\alpha v} \cdot x^\beta D^\alpha u\, dx$$

To $u \in S_m$ we let correspond the linear functional on $\mathscr{S}$:

$$v \to \int u \cdot v\, dx.$$

In this way we identify S_m to a subspace of the space $\mathscr{S}'$ of tempered distributions. We have continuous and dense inclusions:

$$\mathscr{S} \subset S_m \subset \mathscr{S}'.$$

By Riesz representation theorem, for every $f \in S_m'$ (linear and continuous functional $f: S_m \to \mathbb{C}$) there is a unique $u \in S_m$ such that

$$(u, \bar{v})_m = f(v)\ \forall v \in S_m.$$

Identifying u and f to the corresponding elements of $\mathscr{S}'$, we obtain

$$E_m u = \sum_{|\alpha+\beta|\le m} (-1)^\alpha D^\alpha(x^{2\beta} D^\alpha u) = f \text{ in } \mathscr{S}'.$$

Note that the operator E_m belongs to W_{2m} (canonical ordering). We identify S'_m to its image in $\mathscr{S}'$, that we will denote by S_{-m}. Then $E_m : S_m \to S_{-m}$ is an isometry. Denoting by $E_{-m} : S_{-m} \to S_m$ its inverse, the natural scalar product of S_{-m} is given by:

$$(u, v)_{-m} = (E_{-m}u, E_{-m}v)_m \text{ for all } u, v \in S_{-m}.$$

We set also $\|u\|_{-m} = (u, u)^{1/2}_{-m}$, for $u \in S_{-m}$.

For $P = \sum_{|\alpha+\beta|\leq m} a_{\alpha\beta}x^\beta D^\alpha \in W_m$ we denote by ${}^tP$ the transposed operator:

$$tPu = \sum_{|\alpha+\beta|\leq m} (-1)^{|\alpha|}D^\alpha(x^\beta u), \text{ so that } {}^tP \text{ also belongs to } W_m.$$

If $k \leq 0$, one realizes that $P : S_k \to S_{k-m}$ is the transposed map of the linear map ${}^tP : S_{m-k} \to S_{-k}$ and thus is linear and continuous. Because every $P \in W_m$ is obtained as sum of compositions of operators in W_1 and W_0, one obtains:

Lemma 6: *Every $P \in W_m$ defines for each $k \in \mathbb{Z}$ a linear and continuous map from S_k to S_{k-m}.*

(b) *Notations.* Let N be a positive integer. For $\mu \in \mathbb{Z}^N$, $\mu = (\mu_1, \ldots, \mu_N)$ we set $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_N}$. For $u, v \in S_\mu$, $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N)$ we set

$$(u, v)_\mu = \sum_{i=1}^{N} (u_i, v_i)_{\mu_i}$$

$$\|u\|_\mu = (u, u)^{1/2}_\mu.$$

If $k \in \mathbb{Z}$ we set $\mu + k = (\mu_1 + k, \ldots, \mu_N + k)$.

(c) We make the following remark:

Proposition 21: *Let $m_1 < m_2 \in \mathbb{Z}$. Then we have a compact inclusion $S_{m_2} \hookrightarrow S_{m_1}$.*

Proof: The general statement follows if we can prove it when $m_1 \geq 0$ and $m_2 - m_1 = 1$. Indeed the inclusion $S_{m_2} \hookrightarrow S_{m_1}$ is the composition of the inclusions $S_{m_2} \hookrightarrow S_{m_1+1} \hookrightarrow S_{m_1}$ and when $m_1 < 0$ the inclusion $S_{m_1+1} \hookrightarrow S_{m_1}$ is the dual of the inclusion $S_{-m_1} \hookrightarrow S_{-m_1-1}$, where $-m_1 - 1 \geq 0$.

Let us assume therefore that $m_1 = m \geq 0$ and $m_2 = m + 1$. We recall the following theorem, due to Fréchet and Kolmogorov:

"A subset C of $L^2(\mathbb{R}^n)$ has compact closure in $L^2(\mathbb{R}^n)$ if and only if the following conditions are satisfied:

(i) C is bounded in $L^2(\mathbb{R}^n)$;

(ii) For every vector $\xi \in \mathbb{R}^n$ we have

$$\lim_{t\to 0} \int |u(x+t\xi) - u(x)|^2\, dx = 0 \text{ uniformly for } u \in C;$$

(iii) We have

$$\lim_{t\to\infty} \int_{|x|>t} |u(x)|^2\, dx = 0 \text{ uniformly for } u \in C."$$

Let us prove first that the ball $B = \{\|u\|_1 \leq 1\}$ of S_1 is relatively compact in $L^2(\mathbb{R}^n) = S_0$. Because $\|u\|_0 \leq \|u\|_1\ \forall u \in S_1$, B is bounded in S_0. For $t > 0$ we have:

$$\int_{|x|>t} |u(x)|^2\, dx \leq t^{-2} \int_{|x|>t} |x|^2 |u|^2\, dx \leq t^{-2}\|u\|_1 \leq t^{-2} \quad \forall u \in B.$$

Thus condition (iii) of the theorem of Fréchet-Kolmogorov is satisfied. Moreover we have:

$$\begin{aligned}\int |u(x+t\xi) - u(x)|^2\, dx &= \int \left| \int_0^t \sum_j \partial u/\partial x_j \circ (x+s\xi)\xi_j\, ds \right|^2 dx \\ &\leq |t| \int \int_0^{|t|} |\xi|^2 |\text{gradu}(x+s\xi)|^2\, ds\, dx \\ &\leq t^2|\xi|^2 \int |\text{grad}\, u|^2\, dx \quad \text{for every } u \in \mathcal{S}.\end{aligned}$$

Because $\mathcal{S}$ is dense in S_1 we deduce that

$$\int |u(x+t\xi) - u(x)|^2\, dx \leq t^2|\xi|^2\|u\|_1 \leq t^2|\xi|^2 \quad \forall u \in B.$$

Hence also condition (ii) is satisfied and by the above cited theorem it follows that B is relatively compact in S_0, i.e. that the inclusion $S_1 \hookrightarrow S_0$ is compact.

When $m \geq 1$, we note that if $\{u_k\}$ is a bounded sequence in S_{m+1}, then for all $|\alpha+\beta|\le m\{x^\beta D^\alpha u_k\}$ is a bounded sequence in S_1. Because the inclusion $S_1 \hookrightarrow S_0$ is compact, we can extract a subsequence $\{u_j\}$ such that $\{x^\beta D^\alpha u_j\}$ converges in S_0 for every $|\alpha+\beta|\le m$, but this implies that $\{u_j\}$ converges in S_m and then the proof is complete.

(d) *The density theorem.*

Let $L(x, D): \mathcal{S}^q \to \mathcal{S}^N$ be a $N\times q$ matrix with entries in W. Let $L=(L_{ij})$ with $L_{ij}\in W_{\mu_j}$ for $i=1,\dots,N$ and $j=1,\dots,q$.

Let us denote by $\mathcal{H}_k$ (for any fixed $k\in\mathbb{Z}$) the subspace of $\mathcal{S}'^q$ of generalized functions $u\in\mathcal{S}'^q$ such that $u\in S_{k+\mu-1}$ and $Lu\in S_k^N$. We have:

PROPOSITION 22: *$\mathcal{S}^q$ is dense in $\mathcal{H}_k$ for the norm:*

$$\|u\|_{\mathcal{H}_k}=(\|u\|^2_{k+\mu-1}+\|Lu\|^2_k)^{1/2}.$$

PROOF: Let φ be a C^∞ function with compact support in $\mathbb{R}^n$ that is 1 on a neighborhood of 0. We have, for $u\in\mathcal{H}_k$ and $\varphi_\nu=\varphi(x/\nu)$:

$$\begin{aligned}L(\varphi_\nu u)&=\varphi_\nu Lu+\sum_{|\alpha|\ge1}1/\alpha!(D^\alpha\varphi_\nu)L^{(\alpha)}(x,D)u\\&=\varphi(x/\nu)Lu+\sum_{|\alpha|\ge1}\nu^{-|\alpha|}/\alpha!(D^\alpha\varphi)(x/\nu)L^{(\alpha)}(x,D)u\end{aligned}$$

where, for $L(x,D)=\Sigma\, L_\alpha(x)D^\alpha$ we have set

$$L^{(\alpha)}(x,D)=\sum_{\beta\ge\alpha}L_\beta(x)\frac{\beta!}{(\beta-\alpha)!}D^{\beta-\alpha}.$$

Therefore $\varphi_\nu u\in\mathcal{H}_k$ if $u\in\mathcal{H}_k$ and one easily checks that

$$\|\varphi_\nu u-u\|_{k+\mu-1}\to0.$$
$$\|L(\varphi_\nu u)-Lu\|_k\to0.$$

Hence one proves that $\mathcal{H}_k\cap\mathcal{E}'$ is dense in $\mathcal{H}_k$, and thus it is sufficient to approximate elements with compact support. Let $\mathcal{F}:\mathcal{S}'\to\mathcal{S}'$ denote the Fourier transform. Then $\mathcal{F}:\mathcal{H}_k\to\hat{\mathcal{H}}_k$ is an isomorphism of $\mathcal{H}_k$ with the space $\hat{H}_k$ of generalized functions u such that $u\in S_{k+\mu-1}$ and $L_1u=\mathcal{F}L\mathcal{F}^{-1}u\in S_k^N$. We note that $\mathcal{F}(\mathcal{H}_k\cap\mathcal{E}')\subset\hat{\mathcal{H}}_k\cap\mathcal{E}$ is dense in $\hat{\mathcal{H}}_k$. If $u\in\hat{\mathcal{H}}_k\cap\mathcal{E}$, then $\varphi_\nu u\in\mathcal{S}^q$ $\forall\nu\ge1$ and approximates u in $\hat{\mathcal{H}}_k$ by the argument given at the beginning of the proof. Therefore, for $u\in\mathcal{H}_k\cap\mathcal{E}'$, the sequence $\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}(u))$ is in $\mathcal{S}^q$ and approximates u in $\mathcal{H}_k$. This completes the proof.

(e) *Regularization for coercive operators.*

We consider a $N \times q$ matrix L as in point (d). We say that the form

$$q(u, v) = (Lu, Lv)_0 \quad u, v \in S_\mu$$

is *coercive* if there exists a constant c such that

$$\|u\|_\mu^2 \le c\|Lu\|_0^2 \quad \forall u \in \mathcal{S}^q.$$

We have the following:

PROPOSITION 23: *Assume that the form $q(u, v)$ is coercive. Then for every integer k, the map $L^*L: S_{k+\mu} \to S_{k-\mu}$ is an isomorphism.*

PROOF: The assumption that $q(u, v)$ is coercive implies that for every $S \in Sp(n, \mathbb{R})$ the operator $T_S L^* L T_S^{-1}$ is elliptic (in the sense of Douglis-Nirenberg). Because, denoting by H^m the classical Sobolev spaces, the spaces S_m are characterized as the subspaces of $\mathcal{S}'$ of distributions u for which $T_S u \in H^m$ for every $S \in Sp(n, \mathbb{R})$, the statement of the proposition follows from the classical Hilbertian theory of elliptic partial differential operators in Sobolev spaces.

REMARK: In particular for every positive integer m, $E_m: S_k \to S_{k-2m}$ is an isomorphism for every integer k. We denote by E_{-m} its inverse.

(f) *Sub-elliptic estimates.*

Let $L: \mathcal{S}^q \to \mathcal{S}^N$ be as in section (d). We say that the form $q(u, v) = (Lu, Lv)_0$ satisfies a sub-elliptic estimate if there is a constant c such that

$$\|u\|_{\mu-1}^2 \le c\|Lu\|_0^2 \quad \forall u \in \mathcal{S}^q. \tag{18}$$

Let us set $\mathcal{H} = \mathcal{H}_0$ (cf. sect. (d) for the notation). This is a Hilbert space and $\mathcal{S}^q$ is a dense subspace of $\mathcal{H}$ by Proposition 19. If (18) holds, then $q(u, v)$ is an equivalent scalar product on $\mathcal{H}$. Then, by Riesz representation theorem, given any $f \in S'^q$ such that

$$|f(v)| \le C(f)\{\|u\|_{\mu-1}^2 + \|Lu\|_0^2\}^{1/2}$$

there is a unique $u \in \mathcal{H}$ such that

$$(Lu, Lv)_0 = f(\bar{v}) \quad \forall v \in \mathcal{S}^q,$$

and

$$\|u\|_{\mu-1}^2 + \|Lu\|_0^2 \leq (1+c)^2 C(f)^2.$$

Let N_1 be the number of $2q$-uples of multiindices α^j, $\beta^j, \in \mathbb{N}^n$ with $|\alpha^j + \beta^j| \leq \mu_j$. Then we define $F: \mathcal{S}^q \to \mathcal{S}^{N_1}$ by $Fu = F(u_1, \ldots, u_q) = (x^{\beta^j} D^{\alpha^j} u_j)_{j=1,\ldots,q,|\alpha^j+\beta^j|\leq\mu_j}$, so that $(u, v)_\mu = (Fu, Fv)_0 \, \forall u, v \in S_\mu$. Then for every $\epsilon > 0$ we consider the operator $L_\epsilon u = Lu \oplus \epsilon Fu$; $L_\epsilon : \mathcal{S}^q \to \mathcal{S}^{N+N_1}$, and the sesquilinear form

$$q_\epsilon(u, v) = (L_\epsilon u, L_\epsilon v)_0 = (Lu, Lv)_0 + \epsilon(Fu, Fv)_0, \quad u, v \in \mathcal{S}^q.$$

If (18) holds we have:

$$\|\mu\|_{\mu-1}^2 \leq c q_\epsilon(u, u) \quad \forall u \in \mathcal{S}^q$$

$$\|u\|_\mu^2 \leq \epsilon^{-1} q_\epsilon(u, u) \quad \forall u \in \mathcal{S}^q.$$

PROPOSITION 24: *Assume that* (18) *holds. If* $f \in S_{k-\mu+1} (k \geq 0)$ *and* $u \in \mathcal{H}$ *solves*

$$(Lu, Lv)_0 = f(\bar{v}) \quad \forall v \in \mathcal{S}^q,$$

then $u \in S_{k+\mu-1}$ *and* $Lu \in S_k^N$. *In particular* $u \in \mathcal{S}^q$ *if* $f \in \mathcal{S}^q$.

PROOF: For every $\epsilon > 0$ we denote by u_ϵ the solution $u_\epsilon \in S_\mu$ of

$$q_\epsilon(u_\epsilon, v) = f(\bar{v}) \quad \forall v \in \mathcal{S}^q.$$

This equation is equivalent to

$$(L^*L + \epsilon F^*F)u_\epsilon = f$$

and thus by the result of sect. (e) $u \in S_{k+\mu+1}$ if $f \in S_{k-\mu+1}$. Let $P \in W_h$ for $h \leq k$. Then

$$q_\epsilon(Pu_\epsilon, v) = ([L, P]u_\epsilon, Lv)_0 + \epsilon([F, P]u_\epsilon, Fv)_0 + (Lu_\epsilon, [P^*, L]v)_0$$
$$+ \epsilon(Fu, [P^*, L]v)_0 + (Pf)(\bar{v})$$

Then:

$$|q_\epsilon(Pu_\epsilon, v)| \leq \text{const}(1+\epsilon)\{\|u_\epsilon\|_{h+\mu-2}^2 + \|Lu_\epsilon\|_{h-1}^2 + \epsilon\|Fu_\epsilon\|_{h-1}^2 + \|f\|_{h-\mu+1}^2\}^{1/2} \times$$
$$\times \{\|v\|_{\mu-1}^2 + \|Lv\|_0^2 + \epsilon\|Fv\|_{h-1}^2\}^{1/2}.$$

By the remarks made at the beginning of the section these estimates imply that

$$\|Pu_\epsilon\|_{\mu-1}^2 + \|LPu_\epsilon\|_0^2 + \epsilon\|FPu_\epsilon\|_0^2 \leq \text{const}(1+\epsilon)^2\{\|u\|_{h+\mu-2}^2 + \|Lu_\epsilon\|_{h-1}^2 + \epsilon\|Fu_\epsilon\|_{h-1}^2 + \|f\|_{h-\mu+1}^2\},$$

with a constant independent of ϵ. Noticing that:

$$\|PLu_\epsilon\|_0 \leq \text{const}\{\|LPu_\epsilon\|_0 + \|u_\epsilon\|_{h+\mu-2}\} \quad \text{and}$$

$$\|PFu_\epsilon\|_0 \leq \text{const}\{\|FPu_\epsilon\|_0 + \|u_\epsilon\|_{h+\mu-2}\}$$

one deduces, again with a constant independent of $\epsilon > 0$, that:

$$\|Pu_\epsilon\|_{\mu-1}^2 + \|PLu_\epsilon\|_0^2 + \epsilon\|PFu_\epsilon\|_0^2 \leq \text{const}(1+\epsilon)^2\{\|u_\epsilon\|_{h+\mu-2}^2 + \|Lu_\epsilon\|_{h-1}^2 + \epsilon\|Fu_\epsilon\|_{h-1}^2 + \|f\|_{h-\mu+1}^2\}.$$

Summing over all P of the form $P = x^\beta D^\alpha$ with $|\alpha+\beta| \leq h$, using again commutation estimates for the first term, one deduces, with a constant C independent of $\epsilon > 0$:

$$\|u_\epsilon\|_{h+\mu-1}^2 + \|Lu_\epsilon\|_h^2 + \epsilon\|Fu_\epsilon\|_h^2 \leq C(1+\epsilon)^2\{\|u_\epsilon\|_{h+\mu-2}^2 + \|Lu_\epsilon\|_{h-1}^2 + \epsilon\|Fu_\epsilon\|_{h-1}^2 + \|f\|_{h-\mu+1}^2\}$$

for $1 \leq h \leq k$. Because

$$\|u_\epsilon\|_{\mu-1}^2 + \|Lu_\epsilon\|_0^2 + \epsilon\|Fu_\epsilon\|_0^2 \leq \text{const}\|f\|_{1-\mu}^2$$

one obtains that

$$\|u_\epsilon\|_{k+\mu-1}^2 + \|Lu_\epsilon\|_k^2 + \epsilon\|Fu_\epsilon\|_k^2 \leq \text{const}(1+\epsilon)^{2k}\|f\|_{k-\mu+1}^2 \quad \text{for all } \epsilon > 0.$$

Then we can find $w \in S_{k+\mu-1}$ such that $Lw \in S_k^N$ and a sequence $\{\epsilon_\nu\}$ of real positive numbers converging to 0 such that:

$$u_{\epsilon_\nu} \rightharpoonup w \text{ weakly in } S_{k+\mu-1}$$

$$Lu_{\epsilon_\nu} \rightharpoonup Lw \text{ weakly in } S_k^N.$$

Assuming that $k \geq 1$ (otherwise there is nothing to prove) we have

$$u_{\epsilon_\nu} \to w \text{ strongly in } S_\mu \text{ and therefore}$$

$$q(w, v) = f(\bar{v}) \quad \forall v \in \mathcal{S}^q$$

and then because this equation has a unique solution in $\mathscr{H}$, we must have $u = w$ and therefore the proposition is proved.

COROLLARY: *Assume that with some constant $c > 0$ we have the estimate:*

$$\|Lu\|_0^2 + \|u\|_{\mu-2}^2 \geq c\|u\|_{\mu-1}^2 \quad \forall u \in \mathscr{S}^q. \tag{19}$$

Then, if $f \in S_{k-\mu+1}$ for some $k \geq 1$ and $u \in \mathscr{H}$ solves

$$(Lu, Lv)_0 = f(\bar{v}) \quad \forall v \in \mathscr{S}^q \tag{*}$$

then $u \in S_{k+\mu-1}$ and $Lu \in S_k^N$. In particular $u \in \mathscr{S}^q$ for $f \in \mathscr{S}^q$.

PROOF: Let us set $(L_1u, L_1v)_0 = (Lu, Lv)_0 + (u, v)_{\mu-2}$. Then the sesquilinear form $(L_1u, L_1v)_0$ satisfies all assumptions of the previous proposition. If u solves (*), then u solves also the equation

$$(L_1u, L_1v) = (E_{\mu-2}u + f)(\bar{v}) \quad \forall v \in \mathscr{S}^q,$$

where $E_{\mu-2}$ is the diagonal operator having the entry E_{μ_j-2} at the j-th place on the diagonal. We have $E_{\mu-2}u \in S_{-\mu+3}$ because $u \in S_{\mu-1}$. Noticing that the space $\mathscr{H}$ is the same if computed from the original form $q(u, v)$ or from the one we have introduced now, we can apply the previous proposition. We have

$$f + E_{\mu-2}u \in S_{\inf(2, k)-\mu+1}$$

and hence

$$u \in S_{\inf(2, k)+\mu-1} \quad \text{and} \quad Lu \in S^N_{\inf(2, k)}.$$

But then

$$E_{\mu-2} \in S_{\inf(2, k)-\mu+1}$$

and thus the statement follows by iteration.

(g) *Complexes.*

We consider a complex

$$\mathscr{S}^p \xrightarrow{A(x, D)} \mathscr{S}^q \xrightarrow{B(x, D)} \mathscr{S}^r \tag{20}$$

where $A(x, D)$ and $B(x, D)$ are matrices of differential operators with polynomial coefficients and $B(x, D)A(x, D) = 0$.
Let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}^p$, $\beta = (\beta_1, \ldots, \beta_q) \in \mathbb{Z}^q$, $\gamma = (\gamma_1, \ldots, \gamma_r) \in \mathbb{Z}^r$ be such that $A = (A_{ij})$ with $A_{ij} \in W_{\beta_i - \alpha_j}$ and $B = (B_{hi})$ with $B_{hi} \in W_{\gamma_h - \beta_i}$.

We can assume that $\gamma_h \leq 0\ \forall h$. Let $m \in \mathbb{Z}$ be fixed in such a way that $2m + \alpha_j \geq 0\ \forall j$ and $m + \beta_i \geq 0\ \forall i$. We choose $F : \mathcal{S}^q \to \mathcal{S}^N$ in such a way that

$$\|u\|^2_{m+\beta} \leq \text{const}\|Fu\|^2_0 \quad \forall u \in \mathcal{S}^q$$

(we can take for instance $(Fu, Fv)_0 = (u, v)_{m+\beta}\ \forall u, v \in \mathcal{S}^q$).
Then we define the operator $E = F^*F$ and we consider the quadratic form:

$$Q(u) = q(u, u) = \|A^*u\|^2_{2m+\alpha} + \|BEu\|^2_{-\gamma}, \quad u \in \mathcal{S}^q,$$

associated to the sesquilinear form:

$$q(u, v) = (A^*u, A^*v)_{2m+\alpha} + (BEu, BEv)_{-\gamma} \quad \text{for} \quad u, v \in \mathcal{S}^q.$$

We have $q(u, v) = (Lu, Lv)_0$ for a suitable $L : \mathcal{S}^q \to \mathcal{S}^M$ (for some integer $M > 0$) with $L = (L_{ij})$ and $L_{ij} \in W_{2m+\beta_j}$ for every $i = 1, \ldots, M$ and every $j = 1, \ldots, q$. Then we obtain the following:

PROPOSITION 25: *Assume that the form $q(u, v)$ satisfies the sub-elliptic estimate*:

$$c\|u\|^2_{2m+\beta-1} \leq \|A^*u\|^2_{2m+\alpha} + \|BEu\|^2_{-\gamma} \tag{21}$$

$$\forall u \in \mathcal{S}^q \textit{ with a constant } c > 0.$$

Then, if $f \in S_{k-\mu+1}$ for some $k \geq 0$ (resp $f \in \mathcal{S}^q$) satisfies $Bf = 0$, then there exists $v \in S_{k-\alpha}$ (resp $v \in \mathcal{S}^q$) such that $Av = f$.

PROOF: By proposition 21 we can find $u \in S_{4m+\mu+k-1}$ such that

$$(*)\ E_{2m+\alpha}A^*u \in S_{k-\alpha},\ BEu \in S_{-\gamma+k} \quad \text{and} \quad q(u, v) = f(\bar{v}) \quad \forall v \in \mathcal{S}^q.$$

Let $v_0 = B^*E_{-\gamma}BEu$. Then $v_0 \in S_{k+\beta+2m}$ and we obtain, as $v_0 \in \mathcal{H}$:

$$q(u, v_0) = (BEu, BEB^*E_{-\gamma}BEu)_{-\gamma} = \|FB^*E_{-\gamma}BEu\|^2_0$$
$$= \overline{f(B^*E_{-\gamma}BEu)} = (Bf)\overline{(E_{-\gamma}BEu)} = 0.$$

Therefore $B^*E_{-\gamma}BEu = 0$ and then we obtain from (*):

$$A(E_{2m+\alpha}A^*u) + EB^*E_{-\gamma}BEu = A(E_{2m+\alpha}A^*u) = f$$

and thus the statement follows taking $v = E_{2m+\alpha}A^*u$.

COROLLARY: *Under the same assumptions of the previous proposition: let $k \geq 0$ and let $g \in S_{2m+\beta+k+1}$ satisfy $A^*g = 0$. Then there exists $w \in S_{2m+\gamma+k}$ such that $B^*w = g$. If $g \in \mathcal{S}^q$, then we can take $w \in \mathcal{S}^r$.*

PROOF: We take $f = Eg$ and then we solve, with $u \in \mathcal{H}$, the equation:

$$q(u, v) = f(\bar{v}) \quad \forall v \in \mathcal{S}^q.$$

Because $f \in S_{k+1-\beta}$ we have $u \in S_{4m+\beta-k-1}$, $A^*u \in S_{4m+\alpha+k}$, $BEu \in S_{-\gamma+k}$. Let $Ev_0 = AE_{2m+\alpha}A^*u \in S_{k-\beta}$. Then $v_0 \in S_{2m+\beta+k}$ and we have

$$\begin{aligned} q(u, v_0) &= (A^*u, A^*E^{-1}AE_{2m+\alpha}A^*u)_{2m+\alpha} + (BEu, BAE_{2m+\alpha}A^*u)_{-\gamma} \\ &= (AE_{2m+\alpha}A^*u, E^{-1}AE_{2m+\alpha}A^*u)_0(Eg)\overline{(E^{-1}AE_{2m+\alpha}A^*u)} \\ &= (A^*g)(E_{2m+\alpha}A^*u) = 0, \end{aligned}$$

from which we deduce that $AE_{2m+\alpha}A^*u = 0$. Therefore we have

$$f = Eg = AE_{2m+\alpha}A^*u + EB^*E_{-\gamma}BEu = EB^*E_{-\gamma}BEu.$$

The statement follows taking $w = E_{-\gamma}BEu$. If $g \in \mathcal{S}^q$ this procedure yields $w \in \mathcal{S}^r$.

PROPOSITION 26: *Under the assumption of Proposition 25: the linear closed densely defined operators:*

$$A^*: S_{2m+\beta+k-1} \cdots \to S_{2m+\alpha+k} \quad \textit{for} \quad k \geq 2m+1$$

and

$$B: S_{-\beta+k-1} \cdots \to S_{-\gamma+k} \quad \textit{for} \quad k \geq 2m+1$$

have a closed image.

PROOF: Let $u \in S_{2m+\beta+k-1}$ with $A^*u \in S_{2m+\alpha+k}$. Then $f = AE_{2m+\alpha}A^*u \in S_{-2m-\beta+k}$. We solve the equation

$$q(g, v) = (A^*u, A^*v)_{2m+\alpha} \quad \forall v \in \mathscr{S}^q \quad \text{with} \quad g \in \mathscr{H}.$$

Then by the regularity theorem (Proposition 24)

$$g \in S_{2m+\beta+k-1},\ A^*g \in S_{2m+\alpha+k},\ BEg \in S_{-\gamma+k} \text{ and we have}$$

$$(*) \qquad \|g\|_{2m+\beta+k-1} \leq \text{const}\|A^*u\|_{2m+\alpha+k}$$

with a constant independent of u. Because $k > 2m$ we obtain by the previous proposition that $B^*E_{-\gamma}BEg = 0$, since $Bf = 0$. Hence

$$\begin{aligned} 0 &= (A^*(u-g), A^*(u-g))_{2m+\alpha} + (BEg, BE(g-u))_{-\gamma} \\ &= (A^*(g-u), A^*(g-u))_{2m+\alpha} \\ &\quad + (B^*E_{-\gamma}BEg, E(g-u))_{-\gamma} \\ &= \|A^*(u-g)\|^2_{2m+\alpha}. \end{aligned}$$

Therefore $A^*g = A^*u$ and the statement follows then from estimate (*). The proof for B is analogous.

By this proposition and the abstract dualtity theorem for Hilbert spaces and linear closed densely defined operators we obtain the following:

PROPOSITION 27: *For every $k \geq 0$ and every $f \in S_{-4m-\beta-k-2}$ such that $Bf = 0$ we can find $u \in S_{-4m-\alpha-k-3}$ such that $Au = f$.*

For every $k \geq 0$ and every $f \in S_{\beta-2m-k-2}$ such that $A^*f = 0$ we can find $u \in S_{\gamma-2m-k-3}$ such that $B^*u = f$.

Moreover the linear closed densely defined operators:

$$B: S_{-4m-\beta-k-2} \cdots \to S_{-4m-\gamma-k-1}$$

and

$$A^*: S_{\beta-2m-k-2} \cdots \to S_{\alpha-2m-k-1}$$

have a closed image.

Then we obtain the following:

PROPOSITION 28: *If the assumptions of Proposition 25 are satisfied,* *then all the sequences*:

$$\mathcal{S}^p \xrightarrow{A(x,D)} \mathcal{S}^q \xrightarrow{B(x,D)} \mathcal{S}^r$$

$$\mathcal{S}'^p \xrightarrow{A(x,D)} \mathcal{S}'^q \xrightarrow{B(x,D)} \mathcal{S}'^r$$

$$\mathcal{S}^r \xrightarrow{A^*(x,D)} \mathcal{S}^q \xrightarrow{B^*(x,D)} \mathcal{S}^p$$

$$\mathcal{S}'^r \xrightarrow{A^*(x,D)} \mathcal{S}'^q \xrightarrow{B^*(x,D)} \mathcal{S}'^p$$

are exact and the maps $B:\mathcal{S}^q \to \mathcal{S}^r$, $B:\mathcal{S}'^q \to \mathcal{S}'^r$, $A^*:\mathcal{S}^q \to \mathcal{S}^p$, $A^*: S'^q \to \mathcal{S}'^p$ *have closed images.*

PROOF: Indeed by proposition 27 it follows that the second and the fourth sequences are exact with the last map having a closed image. Then the exactness of the first and the third sequences and the fact that their last map has a closed image follows from the duality theorem for Fréchet-Schwartz and dual of Fréchet-Schwartz.

We have also the following proposition:

PROPOSITION 29: *Under the assumptions of Proposition* 25, *for every integer k we have exact sequences of Hilbert spaces and linear closed densely defined operators*:

$$S_{k-\alpha} \cdots \xrightarrow{A(x,D)} S_{k-\beta+1} \cdots \xrightarrow{B(x,D)} S_{k-\gamma+2} \tag{22}$$

$$S_{k+\gamma} \cdots \xrightarrow{B^*(x,D)} S_{k+\beta+1} \cdots \xrightarrow{A^*(x,D)} S_{k+\alpha+2}. \tag{23}$$

PROOF: By Propositions 25, 26 and 27 the statement is true when $|k|$ is large. The intermediary results can be obtained in a standard way by interpolation (cf. for instance [16]).

Vice versa, let us assume that for some fixed integer k, the sequence (22) is acyclic and that $B: S_{k-\beta+1} \cdots \to S_{k-\gamma+2}$ has a closed image. Then (cf. [0], Theorem B, pp. 750–751) there is a constant $C > 0$ such that

$$\|A^{\#}v\|^2_{k-\alpha} + \|Bv\|^2_{k-\beta+1} \geq C\|v\|^2_{k-\beta+1} \quad \forall v \in \mathcal{S}^q,$$

where $A^{\#}$ denotes the adjoint of $A: S_{k-\alpha} \cdots \to S_{k-\beta+1}$ in the sense of linear closed densely defined operators on Hilbert spaces. Obviously the actual form of $A^{\#}$ depends on the choice of the scalar products on the two spaces, i.e. on the duality operators $S_{k-\beta+1} \to S_{\beta-k-1}$ and $S_{k-\alpha} \to S_{\alpha-k}$.

Let m and E be chosen as in section (g). By subtracting from α, β, γ either 0 or 1 according k is even or odd, we can assume that $k = 2s$ with $s \in \mathbb{Z}$. Then we obtain, for a suitable choice of a (non standard) scalar product on $S_{k-\beta+1}$:

$$\|A^* E_{m+s} E^{-1} E_{1+s} v\|^2_{\alpha-2s} + \|Bv\|^2_{2s-\gamma+2} \geq C\|v\|^2_{2s-\beta+1} \quad \forall v \in \mathcal{S}^q$$

from which we derive

$$\|A^* E^{-1} E_{1+s} v\|^2_{2m+\alpha} + \|E_{1+s} Bv\|^2_{-\gamma} + \text{const}\|v\|^2_{2s-\beta} \geq C'\|v\|^2_{2s-\beta+1} \quad \forall v \in \mathcal{S}^q,$$

with strictly positive constants, from which we derive

$$\|A^* E^{-1}(E_{1+s} v)\|^2_{2m+\alpha} + \|B(E_{1+s} v)\|^2_{-\gamma} + \text{const}'\|v\|^2_{2s-\beta} \geq C''\|v\|^2_{2s-\beta+1} \qquad \forall v \in \mathcal{S}^q$$

with strictly positive constants.

Let us set $u = E^{-1} E_{1+s} v \in \mathcal{S}^q$. Then we obtain:

$$(*)\ \|A^* u\|^2_{2m+\alpha} + \|BEu\|^2_{-\gamma} + \text{const}''\|u\|^2_{2m+\beta-2} \geq c'''\|u\|^2_{2m+\beta-1} \quad \forall u \in \mathcal{S}^q$$

with a constant $c''' > 0$. Then the subelliptic estimate holds if we prove that, if $u \in S_{2m+\beta-1}$ and $A^* u = 0$, $BEu = 0$, then also $u = 0$. By the corollary to the Proposition 24 (regularity theorem), it follows that $u \in \mathcal{S}^q$. By the exactness of (22) we have $Eu = Aw$ for some $w \in S_{k-\alpha}$ and we obtain

$$\|Fu\|^2_0 = (Eu)(\bar{u}) = (Aw)(\bar{u}) = \overline{w(A^* u)} = 0$$

and hence $u = 0$. Thus we have proved the following

PROPOSITION 30: *The following statements are equivalent*:

(1) *There is an integer k such that the complex* (22) *is acyclic and the map $B : S_{k-\beta+1} \cdots \to S_{k-\gamma+2}$ has a closed image.*

(2) *For every integer k the complex* (22) *is acyclic and the map $B : S_{k-\beta+1} \cdots \to S_{k-\gamma+2}$ has a closed image.*

(3) *There is an integer k such that the complex* (23) *is acyclic and the map $A^* : S_{k+\beta+1} \cdots \to S_{k+\alpha+2}$ has a closed image.*

(4) *For every integer k the complex* (23) *is acyclic and the map* $A^*: S_{k+\beta+1} \cdots \to S_{k+\alpha+2}$ *has a closed image.*

(5) *There exists an integer m such that* $2m + \alpha_j \geq 0\ \forall j$, $m + \beta_i \geq 0\ \forall i$ *and a differential operator* $F = (F_{si}): \mathcal{S}^q \to \mathcal{S}^N$ *with* $F_{si} \in W_{\beta_i+m}\ \forall s, \forall i$, *such that* $\|u\|^2_{\beta+m} \leq \text{const}\|Fu\|^2_0\ \forall u \in \mathcal{S}^q$, *with the property that, setting* $E = F^*F$ *the subelliptic estimate holds, with a constant* $c > 0$:

$$\|A^*(x, D)u\|^2_{2m+\alpha} + \|B(x, D)Eu\|^2_{-\gamma} \geq c\|u\|^2_{2m+\beta-1} \quad \forall u \in \mathcal{S}^q.$$

(6) *The subelliptic estimate* (21) *holds for any choice of m and E satisfying the conditions in point 5 above, with a positive constant* $c = c(m, E)$ *that depends on m and E.*

Definition: If any one of the conditions of the proposition above is satisfied, we say that the complex (20) is subelliptic at $\mathcal{S}^q$.

(*h*) *The approximation theorem.*

Proposition 31: *If the complex* (20) *is subelliptic at* $\mathcal{S}^q$, *then every solution* $f \in \mathcal{S}'^p$ (*resp* $f \in \mathcal{S}'^r$) *of* $A(x, D)f = 0$(*resp* $B^*(x, D)f = 0$) *can be approximated in* $\mathcal{S}'^p$(*resp in* $\mathcal{S}'^r$) *by a sequence of solutions* $u \in \mathcal{S}^p$(*resp* $u \in \mathcal{S}^r$) *of* $A(x, D)u = 0$(*resp* $B^*(x, D)u = 0$).

Proof: Let $f \in \mathcal{S}'^p$ solve $Af = 0$. We have $f \in S_{-2m-\alpha}$ for a sufficiently large integer m and then by the density theorem we can find a sequence $\{f_\nu\}$ in $\mathcal{S}^p$ such that

$$f_\nu \to f \quad \text{in} \quad S_{-2m-\alpha} \quad \text{and} \quad Af_n \to 0 \quad \text{in} \quad S_{-2m-\beta+1}.$$

We can find $u_\nu \in \mathcal{S}^q$ such that

$$\begin{aligned}(A^*u_\nu, A^*v)_{2m+\alpha} &= (A^*u_\nu, A^*v)_{2m+\alpha} + (BE_{m+\beta}u_\nu, BE_{m+\beta}v)_{-\gamma} \\ &= (Af_\nu, v)_0 \quad \forall v \in \mathcal{S}^q.\end{aligned}$$

Therefore $Af = AE_{2m+\beta}A^*u$ and

$$\|u_\nu\|^2_{2m+\beta-1} \leq \text{const}\|A^*u_\nu\|^2_{2m+\alpha} \leq \text{const}\|Af_\nu\|_{-2m-\beta+1}\|u_\nu\|_{2m+\beta-1}.$$

Thus, for $v_\nu = E_{2m+\beta}A^*u_\nu$ we have

$$Av_\nu = Af_\nu \quad \text{and} \quad \|v_\nu\|_{-2m-\alpha} \leq \text{const}\|Af_\nu\|_{-2m-\beta-1},$$

with a constant independent of ν. Therefore $v_\nu \to 0$ in $S_{-2m-\alpha}$ and hence $f_\nu - v_\nu = w_\nu \in \mathcal{S}^p$, $Aw_\nu = 0$ and $w_\nu \to f$ in $S_{-2m-\alpha}$. The proof for the case $B^*f = 0$, $f \in \mathcal{S}'^r$, is analogous and is omitted.

(i) Let us consider now a longer complex:

$$\{\mathcal{S}^{p_0} \xrightarrow{\mathfrak{A}_0} \mathcal{S}^{p_1} \xrightarrow{\mathfrak{A}_1} \mathcal{S}^{p_2} \xrightarrow{\mathfrak{A}_2} \mathcal{S}^{p_3} \to \cdots\} = (\mathcal{S}^*) \tag{24}$$

with $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \ldots$ matrices of differential operators with polynomial coefficients. For each j let be given a p_j-uple $\alpha^j = (\alpha_1^j, \ldots, \alpha_{p_j}^j)$ of integers such that for every j we have $\mathfrak{A}_j = (\mathfrak{A}_j^{rs})$ with $\mathfrak{A}_j^{rs} \in W_{\alpha_r^{j+1} - \alpha_s^j}$. We consider subellipticity with respect to this choice of multigradings. We have:

Proposition 32: *Assume that the complex $(\mathcal{S}^*)$ is subelliptic at $\mathcal{S}^{p_j}$. Then the cohomology group $H^{j+1}(\mathcal{S}^*) = \dfrac{\operatorname{Ker}(\mathfrak{A}_{j+1} : \mathcal{S}^{p_{j+1}} \to \mathcal{S}^{p_{j+2}})}{\operatorname{Image}(\mathfrak{A}_j : \mathcal{S}^{p_j} \to \mathcal{S}^{p_{j+1}})}$ is separated.*

Proof: Indeed we proved that $\mathfrak{A}_j : \mathcal{S}^{p_j} \to \mathcal{S}^{p_{j+1}}$ has a closed image (Proposition 28).

Proposition 33: *If the complex $(\mathcal{S}^*)$ is subelliptic at $\mathcal{S}^{p_j}$ and at $\mathcal{S}^{p_{j+2}}$, then $H^{j+1}(\mathcal{S}^*) = 0$ if and only if $(\mathcal{S}^*)$ is subelliptic at $\mathcal{S}^{p_{j+1}}$.*

Proof: Indeed the map $\mathfrak{A}_j^\# \oplus \mathfrak{A}_{j+1} : S_{-\alpha^{j+1}-1} \cdots \to S_{-\alpha^j-2} \oplus S_{-\alpha^{j+2}}$ where we denoted by $\mathfrak{A}_j^\#$ the adjoint of $\mathfrak{A}_j : S_{-\alpha^j-2} \cdots \to S_{-\alpha^{j+1}-1}$ in the sense of linear closed densely defined operators on Hilbert spaces, has a closed image by Proposition 30. Indeed, if $\{u_\nu\}$ is a sequence in $S_{-\alpha^{j+1}-1}$ such that $\mathfrak{A}_j^\# u_\nu \to f$ in $S_{-\alpha^j-2}$ and $\mathfrak{A}_{j+1} u_\nu \to g$ in $S_{-\alpha^{j+2}}$, then we can find $v, u \in S_{-\alpha^{j+1}-1}$ such that $f = \mathfrak{A}_j v$ and $g = \mathfrak{A}_{j+1} u$. But v can be chosen in $(\ker \mathfrak{A}_j^\#)^\perp \subset (\operatorname{Image} \mathfrak{A}_{j+1}^\#)^\perp$ and hence in such a way that $\mathfrak{A}_{j+1} v = 0$ and u can be chosen in $(\ker \mathfrak{A}_{j+1})^\perp \subset (\operatorname{Image} \mathfrak{A}_j)^\perp$ and hence satisfying $\mathfrak{A}_j^\# u = 0$. Therefore $f = \mathfrak{A}_j^\#(u + v)$ and $g = \mathfrak{A}_{j+1}(u + v)$, proving that $f \oplus g$ belongs to the image of $\mathfrak{A}_j^\# \oplus \mathfrak{A}_{j+1}$.

Then we obtain (cf. [0], Theorem A, p. 749), with some constant $c > 0$:

$$\|\mathfrak{A}_j^\# u\|^2_{-\alpha^j-2} + \|\mathfrak{A}_{j+1} u\|^2_{-\alpha^{j+2}} \geq c\|u\|^2_{-\alpha^{j+1}-1}$$

for every u in domain $\mathfrak{A}_j^\# \cap$ domain $\mathfrak{A}_{j+1} \cap (\ker \mathfrak{A}_j^\# \cap \ker \mathfrak{A}_{j+1})^\perp$.

If $\mathcal{S}^{p_{j+1}} \cap \ker \mathfrak{A}_j \cap \ker \mathfrak{A}_{j+1} = 0$, then the inequality above holds for esery $u \in \mathcal{S}^{p_{j+1}}$ and the argument given in section (g) proves that the complex $(\mathcal{S}^*)$ is subelliptic at $\mathcal{S}^{p_{j+1}}$. We note that that intersection is zero when $H^{j+1}(\mathcal{S}^*)$ is 0. Because the viceversa is also true by proposition 30, the proof is complete.

(j) *Necessary conditions for the validity of the special Poincaré Lemma.*

Let us assume that the complex (17) is formally correct at x_0, and that the complex (24) is a reduced subprincipal complex for (17) at (x_0, ξ^0) for some $\xi^0 \in \mathbb{R}^n - \{0\}$. We consider the extension of complex (24):

$$(24)' \qquad (\mathcal{S}^*) = \{0 \to \mathcal{S}^{p_0} \xrightarrow{\mathfrak{A}_0} \mathcal{S}^{p_1} \xrightarrow{\mathfrak{A}_1} \mathcal{S}^{p_2} \xrightarrow{\mathfrak{A}_2} \mathcal{S}^{p_3} \to \cdots\}$$

PROPOSITION 34: *If* $(\mathcal{S}^*)$ *is subelliptic at* $\mathcal{S}^{p_{j-1}}$ *and at* $\mathcal{S}^{p_{j+1}} (j \geq 1)$ *but* $H^j(\mathcal{S}^*) \neq 0$, *then* (17) *does not admit the special Poincaré lemma at* $\mathscr{E}^{q_j}_{x_0}$.

§5. Principal type presentation

(a) Let us consider a complex (17) of differential operators with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$:

$$(\mathscr{E}^*(\Omega)) = \{\mathscr{E}^{p_0} \xrightarrow{A_0(x, D)} \mathscr{E}^{p_1} \xrightarrow{A_1(x, D)} \mathscr{E}^{p_2} \xrightarrow{A_2(x, D)} \cdots\}$$

and let $a^j = (a^j_1, \ldots, a^j_{p_j}) \in \mathbb{Z}^{p_j}$ (for $j = 0, 1, 2, \ldots$) be multigradings for which the given complex is formally correct at $x_0 \in \Omega$. If $\hat{A}_j(x, \xi)$ denotes the principal symbol of $A_j(x, D)$ at x (*cf.* §3, sect. (g)) and

$$\tilde{A}_j(t, x, D) = \sum_{h=0}^{\infty} t^h \tilde{A}_{jh}(x, D)$$

is the asymptotic operator of $A_j(x, D)$ at (x_0, ξ^0), then we have

$$\tilde{A}_{j0}(x, D) = \tilde{A}_{j0} = \hat{A}_j(x_0, i\xi^0)$$

$$\tilde{A}_{j1}(x, D) = \sum_{s=1}^{n} \frac{\partial \hat{A}_j}{\partial \xi_j}(x_0, i\xi^0) \frac{\partial}{\partial x_s} + x_s \frac{\partial \hat{A}_j}{\partial x_j}(x_0, i\xi^0)$$

We set $K_j = \ker \tilde{A}_{j0} : \mathbb{C}^{p_j} \to \mathbb{C}^{p_{j+1}}$

$$I_j = \text{Image } \tilde{A}_{j-1,0} : \mathbb{C}^{p_{j-1}} \to \mathbb{C}^{p_j} \quad (= 0 \text{ if } j = 0).$$

Then $I_j \subset K_j\ \forall j$. Moreover, for $u \in \mathcal{S} \otimes K_j$ we have

$$\tilde{A}_{j+1,0}\tilde{A}_{j1}u = -\tilde{A}_{j+1,1}\tilde{A}_{j0} = 0,$$

i.e. $\tilde{A}_{j1}: \mathcal{S} \otimes K_j \to \mathcal{S} \otimes K_{j+1}$.
If $u \in \mathcal{S} \otimes I_j$ we have $u = \tilde{A}_{j-1,0}w$ for some $w \in \mathcal{S}^{p_{j-1}}$ and hence

$$\tilde{A}_{j1}u = \tilde{A}_{j1}\tilde{A}_{j-1,0}w = -\tilde{A}_{j0}\tilde{A}_{j-1,1}w \in I_{j+1} \otimes \mathcal{S}.$$

i.e. $\tilde{A}_{j1}: \mathcal{S} \otimes I_j \to \mathcal{S} \otimes I_{j+1}$.
Therefore, if we denote by L_j the quotient space K_j/I_j the operators $\tilde{A}_{j1}$ define differential operators

$$\mathfrak{a}_j = \mathfrak{a}_j(x, D): \mathcal{S} \otimes L_j \to \mathcal{S} \otimes L_{j+1}.$$

If $u \in \mathcal{S} \otimes K_j$, then

$$\tilde{A}_{j+1,1}\tilde{A}_{j1}u = -\tilde{A}_{j+1,0}\tilde{A}_{j2}u - \tilde{A}_{j+1,2}\tilde{A}_{j0}u = -\tilde{A}_{j+1,0}\tilde{A}_{j2}u \in I_{j+2} \otimes \mathcal{S}.$$

Therefore

$$(25) \qquad 0 \to \mathcal{S} \otimes L_0 \xrightarrow{\mathfrak{a}_0} \mathcal{S} \otimes L_1 \xrightarrow{\mathfrak{a}_1} \mathcal{S} \otimes L_2 \to \cdots$$

is a complex. From the remarks made in the preceding sections it follows that, when the sequence

$$(26) \qquad \cdots \to {}^tL_2 \otimes W \xrightarrow{\mathfrak{a}_1} {}^tL_1 \otimes W \xrightarrow{\mathfrak{a}_0} {}^tL_0 \otimes W$$

is exact (${}^tL_i$ denoting the dual spaces of the spaces L_i), then it is the reduced subprincipal sequence for trivial gradings. When this happens, we say that the complex (17) is in principal type presentation at (x_0, ξ^0).

(b) Let $s_i = \dim_{\mathbb{C}} L_i$ (for $i = 0, 1, \ldots$). After performing linear changes of coordinates in the spaces $\mathbb{C}^{p_i}$, we can assume that

$$\hat{A}_i(x_0, i\xi^0) = \begin{pmatrix} * & 0 \\ I & 0 \end{pmatrix}$$

where I is the $(p_i - s_i) \times (p_i - s_i)$ identity matrix.

By a linear change of coordinates in $\mathbb{R}^n$ we can assume that

$\xi^0 = (0, \ldots, 0, 1)$. Then we have:

$$\frac{\partial \hat{A}_i}{\partial \xi_n}(x_0, i\xi^0) = \sum_{r=1}^{n} \frac{\partial \hat{A}_i}{\partial \xi_r}(x_0, i\xi^0)\xi_r^0 = ((-a_h^{i+1} + a_r^i)\hat{A}_i^{hr}(x_0, i\xi^0)) = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$$

and hence $\frac{\partial \hat{A}_i}{\partial \xi_n}(x_0, i\xi^0)|_{Ki} = 0$.

From this we deduce that the operators $\mathfrak{a}_j(x, D)$ are independent of $\partial/\partial x_n$. Let us consider the subspace N of $E = \mathbb{R}^n \oplus \mathbb{R}^n$ where $\mathfrak{a}_0(x, i\xi) = 0$. Let $N^\sigma = \{u \in E \mid \sigma(u, v) = 0 \ \forall v \in N\}$ and let λ be a maximal isotropic space contained in N (the fact that λ is isotropic means that $\sigma(u, v) = 0 \ \forall u, v \in \lambda$). After a symplectic change of coordinates we can assume that

$$\lambda = \{\xi_1 = 0, \ldots, \xi_m = 0, x_1 = 0, \ldots, x_n = 0\}.$$

In these new symplectic coordinates $\mathfrak{a}_0$ is written as a differential operator independent of $\partial/\partial x_{m+1}, \ldots, \partial/\partial x_n$. Then we can assume that all operators $\mathfrak{a}_j$ have the same property, as this can be obtained by passing to an equivalent asymptotic sequence.
We notice that the space $\{x_{m+1} = 0, \ldots, x_n = 0, \xi_{m+1} = 0, \ldots, \xi_n = 0\}$ is then isomorphic to $N^\sigma/N \cap N^\sigma$ and thus has an invariant meaning (cf. [10], p. 355).

Let us set

$$y_j = x_j \quad \text{for} \quad j = 1, \ldots, m \qquad \partial/\partial y_j = \partial/\partial x_j \quad \text{for} \quad j = 1, \ldots, m$$
$$t = h^3 y_j = hx_j + \eta_j \quad \text{for} \quad j = m+1, \ldots, n,$$
$$\partial/\partial y_j = (1/h)\, \partial/\partial x_j \quad \text{for } j = m+1, \ldots, n.$$

Then we form the second asymptotic sequence. We obtain operators:

$$\tilde{\tilde{A}}_i(h, y, \partial) = \sum_0^\infty b_{ik}(y, \partial)h^k$$

with $b_{i0}(y, \partial) = \hat{A}_i(x_0, i\xi^0)$, $b_{i1} = b_{i2}$ on K_i,

$$b_{i3} = \sum_{r=1}^{m} \frac{\partial \hat{A}_i}{\partial \xi_r}(x_0, i\xi^0)\frac{\partial}{\partial y_r} + \frac{\partial \hat{A}_i}{\partial x_r}(x_0, i\xi^0)y_r + \sum_{r=m+1}^{n} \frac{\partial \hat{A}_i}{\partial x_r}(x_0, i\xi^0)\eta_r.$$

Because W is flat over the ring of polynomials of $x_{m+1}, \ldots, x_n$ with coefficients in the Weyl algebra ${}_mW$ of differential operators with polynomial coefficients in $\mathbb{R}^m$, we obtain an exact sequence:

$$\cdots \to {}^tL_2 \otimes {}_mW[x_{m+1}, \ldots, x_n] \xrightarrow{\mathfrak{a}_1} {}^tL_1 \otimes {}_mW[x_{m+1}, \ldots, x_n] \xrightarrow{\mathfrak{a}_0} {}^tL_0 \otimes {}_mW[x_{m+1}, \ldots, x_n].$$

The exactness of this sequence is equivalent to the exactness of the sequences

$$\cdots \to {}^tL_2 \otimes {}_mW_k \otimes \mathcal{P}_{n-m} \xrightarrow{\mathfrak{a}_1} {}^tL_1 \otimes {}_mW_{k+1} \otimes \mathcal{P}_{n-m} \xrightarrow{\mathfrak{a}_0} {}^tL_0 \otimes {}_mW_{k+2} \otimes \mathcal{P}_{n-m}$$

for every integer k (we have denoted by $\mathcal{P}_{n-m}$ the ring of polynomials in $n-m$ indeterminates $x_{m+1}, \ldots, x_n$). For every fixed k this sequence can be considered as an exact sequence of finitely generated free $\mathcal{P}_{n-m}$-modules. Thus for every k the set of $\eta \in \mathbb{C}^{n-m}$ such that the complex obtained by tensoring the exact sequence by $\mathcal{P}_{n-m}/I_\eta$ where I_η is the ideal generated by $x_{m+1} - \eta_1, \ldots, x_n - \eta_{n-m}$ is not exact is a nowhere dense algebraic subvariety V_k of $\mathbb{C}^{n-m}$ (cf. [2]). Then $\mathbb{R}^{n-m} - \cup V_k$ is non empty and, for η in this set, denoting by $\mathfrak{b}_j$ the operators obtained from b_{j3} by passing to the quotient, we obtain an exact sequence:

$$\cdots \to {}^tL_2 \otimes {}_mW \xrightarrow{\mathfrak{b}_1} {}^tL_1 \otimes {}_mW \xrightarrow{\mathfrak{b}_0} {}^tL_0 \otimes {}_mW.$$

(c) Let $\mathcal{L}$ be the subspace of $\mathcal{S}$ of functions of the form

$$f = g(y_1, \ldots, y_m)p(y_{m+1}, \ldots, y_n)\exp(-y_{m+1}^2 - \cdots - y_n^2)$$

where $g \in \mathcal{S}(\mathbb{R}^m)$ and p is a polynomial in $n-m$ indeterminates.

If $A(y, \partial)$ and $B(y, \partial)$ are respectively $q \times p$ and $r \times q$ matrices with entries in ${}_mW$, then the sequence

$$\mathcal{L}^p \xrightarrow{A(y,\partial)} \mathcal{L}^q \xrightarrow{B(y,\partial)} \mathcal{L}^r$$

is exact if and only if the sequence

$$\mathcal{S}(\mathbb{R}^m)^p \xrightarrow{A(y,\partial)} \mathcal{S}(\mathbb{R}^m)^q \xrightarrow{B(y,\partial)} \mathcal{S}(\mathbb{R}^m)^r$$

is exact. Therefore, as an easy corollary of the results obtained in §3 and §4 we obtain the following statement:

PROPOSITION 32: *If the complex*:

$$C^* = \{0 \to L_0 \otimes \mathcal{S}(\mathbb{R}^m) \xrightarrow{\mathfrak{b}_0} L_1 \otimes \mathcal{S}(\mathbb{R}^m) \xrightarrow{\mathfrak{b}_1} L_2 \otimes \mathcal{S}(\mathbb{R}^m) \to \cdots\} \tag{27}$$

is subelliptic at $L_{j-1} \otimes \mathcal{S}(\mathbb{R}^m)$ *and at* $L_{j+1} \otimes \mathcal{S}(\mathbb{R}^m)$ (*here* $j \geq 1$), *but* $H^j(C^*) \neq 0$, *then the complex* (17) *does not admit the special Poincaré lemma at* $\mathscr{E}^{q_j}_{x_0}$.

(d) Let us first study the case in which (17) is defined on an open set Ω of $\mathbb{R}^2$. Then we are reduced to consider ordinary differential operators and we are interested in estimates of the form

$$\|Au' + yBu + Cu\|_0^2 \geq c\|u\|_0^2 \quad \forall u \in \mathcal{S}^p, \tag{$*$}$$

where $c > 0$ and A, B, C are $q \times p$ matrices with complex entries. Let us discuss the case where $C = 0$.

Let us set $X_1 = \mathbb{C}^p$ and $Y_0 = \mathbb{C}^q$. Then we define recursively:

$$Y_j = A(X_j) \quad \text{and} \quad X_{j+1} = \ker(X_j \xrightarrow{B} Y_{j-1} \to Y_{j-1}/Y_j)(j \geq 1).$$

In this way we obtain two decreasing chains:

$$X_1 \supset X_2 \supset X_3 \supset \cdots \quad \text{and} \quad Y_0 \supset Y_1 \supset Y_2 \supset Y_3 \supset \cdots$$

and it is obvious that these two sequences become stationary together, for a first integer $m \geq 1$. Note that, if $m > 1$ and $1 \leq j < m$, then B induces an injective map $X_j/X_{j+1} \to Y_{j-1}/Y_j$.

Therefore, by a suitable choice of coordinates in the linear spaces $\mathbb{C}^p$ and $\mathbb{C}^q$ we can assume that A and B have the form:

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & I \end{pmatrix}$$

$$B=\begin{pmatrix} B_0 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & B_1 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & B_2 & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & & B_{m-2} & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & B_{m-1} & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & H \end{pmatrix}$$

where I are identity matrices of suitable size, $B_0, \dots, B_{m-1}$ are injective and H is a square matrix in Jordan form.

Decomposing in an obvious way $u \in \mathcal{S}^p$ as $u = (u_1, \dots, u_m)$ we obtain:

$$\|Au' + yBu\|_0^2 = \|yB_0u_1\|_0^2 + \|u_1' + yB_1u_2\|_0^2 + \cdots + \|u_j' + yB_ju_{j+1}\|_0^2 + \cdots \\ + \|u'_{m-1}\|_0^2 + \|u'_m + yHu_m\|_0^2.$$

The operator obtained by deleting from $Ad/dy + yB$ the block of the last rows and columns is elliptic in the sense precised by Hörmander in [10] (i.e. its principal symbol is injective for all (y, η) in $\mathbb{R}^2 - \{0\}$) and because solutions $u = (u_1, \dots, u_{m-1}, 0) \in \mathcal{S}^p$ of $Au' + yBu = 0$ must be zero, it follows that with some constant $c_1 > 0$ we have

$$\|yB_0u_1\|_0^2 + \cdots + \|u'_{m-1}\|_0^2 \geq c_1 \sum_{j=1}^{m-1} \|u_j\|_1^2.$$

Then we are reduced to the study of the estimate

$$(*) \qquad \|v' + yHv\|_0^2 \geq c\|v\|_0^2 \quad \forall v \in \mathcal{S}^k \quad \text{where } k \text{ is the size of } H.$$

We have proved that when estimate $(*)$ holds with a positive constant c, then there are no non-trivial solutions v of $v' + yHv = 0$ in $\mathcal{S}'^k$. Hence we obtain:

a necessary and sufficient in order that estimate $(*)$ *holds is that all eigenvalues of H have negative real part.*

Note that, when this is the case, we end up with an estimate

$$\|Au' + yBu\|_0^2 \geq c'\|u\|_1^2$$

for some positive constant c'.

(e) Let us consider now the general form of a complex (27) when $m = 1$. For simplicity we assume that $\mathfrak{b}_0$ is of the form $A_0d/dy +$

$yB_0 : L_0 \otimes \mathcal{S}(\mathbb{R}) \to L_1 \otimes \mathcal{S}(\mathbb{R})$. By a suitable choice of symplectic coordinates in $\mathbb{R}^2$ and of linear coordinates in L_0 and L_1, we can assume that A_0 and B_0 have the form:

$$A_0 = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & I & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 & I \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} Q_0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & Q_1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & Q_{m-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & Q_{m-1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & H \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the I's denote identity matrices of suitable size and $Q_0, \dots, Q_{m-1}$ are matrices representing surjective linear maps. Then we note the only integrability conditions of order ≤ 1 for $\mathfrak{b}_0$ are those expressing the fact that the image of $\mathfrak{b}_0$ is contained in (Image of $A_0) \otimes \mathcal{S}(\mathbb{R})$. Erasing the zero rows from A_0 and B_0 we are therefore reduced to an equivalent complex:

$$0 \to L_0 \otimes \mathcal{S}(\mathbb{R}) \xrightarrow{\mathfrak{b}_0} L_{1,0} \otimes \mathcal{S}(\mathbb{R}) \to 0$$

where $L_{1,0} = \mathrm{Image}(A_0 : L_0 \to L_1)$. We have also to assume that Q_0 is injective.

Therefore, when we reduce to a subprincipal complex of ordinary differential operators of the form specified above, we can conclude that:

if H is a matrix of positive size with all eigenvalues with a strictly negative real part, then the complex (17) *does not admit the strict Poincaré Lemma at $\mathscr{E}^{q_1}_{x_0}$.*

Indeed when this condition is satisfied the subelliptic estimate holds at $L_0 \otimes \mathcal{S}(\mathbb{R})$ and does not hold at $L_{1,0} \otimes \mathcal{S}(R)$ (while at 0 the subelliptic estimate is trivial).

(f) *Koszul type complexes.*

An important example in which we obtain a complex in "principal type presentation" is offered by the Koszul type complexes that we are going to discuss now.

Let $P_1(x, D), \ldots, P_s(x, D)$ be differential operators with smooth coefficients in an open set $\Omega \subset \mathbb{R}^n$, of orders $m_1, \ldots, m_s$ respectively. Let

$$A_0(x, D) = {}^t(P_1(x, D), \ldots, P_s(x, D))$$

and let us assume that there is a complex of the form (17) in which $q_j = \binom{s}{j}$ and the space $E^{q_j}(\Omega)$ being identified to the space of alternated forms of degree j in the indeterminates $dt^1, \ldots, dt^s$ with coefficients in $E(\Omega)$, the principal symbol of $A_j(x, D)$ corresponds to multiplication by the 1-form

$$\hat{P}_1(x, \xi)\, dt^1 + \cdots + \hat{P}_s(x, \xi)\, dt^s.$$

The existence of such a complex imposes algebraic conditions on the operators $P_1, \ldots, P_s$, that we will not specify here. We only note that we must have

$$\sum \frac{\partial \hat{P}_j}{\partial x_k}(x, \xi) \frac{\partial \hat{P}_i}{\partial \xi_k}(x, \xi) = \sum \frac{\partial \hat{P}_i}{\partial x_k}(x, \xi) \frac{\partial \hat{P}_j}{\partial \xi_k}(x, \xi) \quad \forall i, j = 1, \ldots, s$$
$$\text{if } \hat{P}_j(x, \xi) = 0\ \forall j.$$

(Here $\hat{P}_j$ denotes the principal symbol of degree m_j of P_j).

Let $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$ be such that $\hat{P}_j(x_0, \xi^0) = 0\ \forall j$. We assume for simplicity that $\xi^0 = (0, \ldots, 0, 1)$. Then, with

$$d = p_1(y, \partial)\, dt^1 + \cdots + p_s(y, \partial)\, dt^s$$

for

$$p_j(y, \partial) = \sum_{h=1}^{n-1} \frac{\partial \hat{P}_j}{\partial \xi_h}(x_0, i\xi^0) \frac{\partial}{\partial y_h} + \frac{\partial \hat{P}_j}{\partial x_h}(x_0, i\xi^0) y_h$$

we obtain the reduced subprincipal complex:

$$\begin{aligned} 0 \to \Lambda_0^s \otimes \mathcal{S}(\mathbb{R}^{n-1}) &\xrightarrow{d} \Lambda_1^s \otimes \mathcal{S}(\mathbb{R}^{n-1}) \\ &\xrightarrow{d} \cdots \to \Lambda_s^s \otimes \mathcal{S}(\mathbb{R}^{n-1}) \to 0 \end{aligned} \tag{28}$$

where Λ_k^s is the linear space of k-alternated forms in $\mathbb{C}^s$. In [8] Hörmander proved that a necessary and sufficient condition in order that this complex be subelliptic at $\Lambda_p^s \otimes \mathscr{S}(\mathbb{R}^{n-1})$ is that the quadratic form (Levi form)

$$(*)\qquad \operatorname{Re}\Big(\sum_{k,i,j} \frac{\partial \hat{P}_i}{\partial x_k}(x_0, i\xi^0)\,\overline{\frac{\partial \hat{P}_j}{\partial \xi_k}(x_0, i\xi^0)}\lambda_i\bar{\lambda}_j\Big) \quad \forall \lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s,$$

has at least $p+1$ or $s-p+1$ positive eigenvalues.

Assume that the quadratic form above has exactly q positive and $s-q$ negative eigenvalues. Then the complex (28) is subelliptic ad $\Lambda_{q-1}^s \otimes \mathscr{S}(\mathbb{R}^{n-1})$ and at $\Lambda_{q+1}^s \otimes \mathscr{S}(\mathbb{R}^{n-1})$, but is not subelliptic at $\Lambda_q^s \otimes \mathscr{S}(\mathbb{R}^{n-1})$. Thus we conclude by the statement:

if the quadratic form (∗) *has q negative and $s-q$ positive eigenvalues, then the complex* (27) *does not admit the special Poincaré Lemma at $\mathscr{E}_{x_0}^{q_q}$.*

Of course this criterion applies also to general complexes which have a reduced subprincipal complex of the form (28) at some point $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$.

REMARK: The statement above applies in particular to the Cauchy-Riemann complex induced on a generic real submanifold of an open set Ω of $\mathbb{C}^n$. Thus we recover (in a less precise form) the results obtained in [3].

(g) *Determined systems in principal type presentation.*

Let us consider a differential operator of the form $L(y, \partial) = \Sigma_1^m (H_j(\partial/\partial y_j) + K_j y_j) + M$ with H_j, K_j, M matrices of size $q \times p$ with entries in $\mathbb{C}$. If we exclude the case $p = 1$ and the Koszul type complexes, in general we cannot expect to find a basis for the syzygies of $L(y, \partial)$ by row-vectors of differential operators in ${}_mW_1$. Therefore, when we restrict to principal type presentation, it is reasonable to restrict the study to Koszul type complexes and to determined complexes of the form

$$(29)\qquad \mathscr{E}^p(\Omega) \xrightarrow{A(x,D)} \mathscr{E}^p(\Omega) \to 0$$

where $A(x, D)$ has a principal symbol of type (a_j, b_i), say $\hat{A}(x, \xi)$ that is injective on $\mathscr{P}^p = \mathbb{C}[\xi_1, \ldots, \xi_n]^p$ for all $x \in \Omega$.

By the construction of section (a), if the complex (28) is in principal type presentation at $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$ we have to study a reduced subprincipal complex of the form $(m < n)$:

$$0 \to L_0 \otimes \mathcal{S}(\mathbb{R}^m) \xrightarrow{L(y,\partial)} L_1 \otimes \mathcal{S}(\mathbb{R}^m) \to 0$$

where $\dim_{\mathbb{C}} L_0 = \dim_{\mathbb{C}} L_1$ and the operator L has the form described at the beginning of the section, with $H_j, K_j, M \in \operatorname{Hom}_{\mathbb{C}}(L_0, L_1)$. We set for simplicity $N = 2m$, $M = H_0$ and $K_j = H_{m+j}(j = 1, \ldots, m)$. Then we denote by Λ the linear subspace of $\operatorname{Hom}_{\mathbb{C}}(L_0, L_1)$ generated by $H_0, \ldots, H_N$. By the construction of $L(y, \partial)$, certainly Λ contains an invertible element S and then we consider the subspace $\Lambda_1 = S^{-1}\Lambda$ of $\operatorname{Hom}_{\mathbb{C}}(L_0, L_0)$. Then we can consider the series of Jordan-Hölder associated to Λ_1:

$$(*) \qquad 0 = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} V_2 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} V_k = L_0.$$

The V_j's are distinct linear subspaces of L_0 with $AV_j \subset V_j \ \forall A \in \Lambda_1$ and such that, considering the families of maps induced by Λ_1 on V_{j+1}/V_j, this space does not contain any non trivial proper subspace invariant for it.

Two series of Jordan-Hölder associated to Λ_1 are equivalent, in the sense that, if

$$0 = W_0 \underset{\neq}{\subset} W_1 \underset{\neq}{\subset} W_2 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} W_r = L_0$$

is another ascending chain with the same properties of $(*)$, then $r = k$ and there is a permutation g of $\{1, \ldots, k\}$ and linear isomorphisms $V_j/V_{j-1} \to W_{g_j}/W_{g_j-1}$ such that for all $H \in \Lambda_1$ we obtain a commutative diagram:

$$\begin{array}{ccc} V_j/V_{j-1} & \xrightarrow{H} & V_j/V_{j-1} \\ \downarrow & & \downarrow \\ W_{g_j}/W_{g_j-1} & \xrightarrow{H} & W_{g_j}/W_{g_j-1} \end{array} \qquad (j = 1, \ldots, k).$$

Let N be another invertible element of $\operatorname{Hom}_{\mathbb{C}}(L_0, L_1)$ and let us consider $\Lambda_2 = N^{-1}\Lambda$. If $V \subset L_0$ is an invariant subspace for Λ_2, then V is invariant for the operator $N^{-1}S$. Because $N^{-1}S$ is invertible, then V will also be invariant for $S^{-1}N$. Therefore V is invariant for $S^{-1}N\Lambda_2 =$

Λ_1. This shows that the Jordan-Hölder series is also invariant with respect to the choice of the invertible element S in Λ.

The considerations above show that by a suitable choice of coordinates on the spaces L_0 and L_1 we can assume that $H_0, \ldots, H_N$ are in the semi-triangular form:

$$H_j = \begin{pmatrix} H_j^{11} & 0 & 0 & \ldots & 0 \\ H_j^{21} & H_j^{22} & 0 & \ldots & 0 \\ H_j^{31} & H_j^{32} & H_j^{33} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ H_j^{k1} & H_j^{k2} & H_j^{k3} & \ldots & H_j^{kk} \end{pmatrix}$$

where the H_j^{hh} are square matrices with size equal to $\dim_{\mathbb{C}} V_h/V_{h-1}$.

We note that, if $k \neq 1$, we cannot expect in general to have an estimate of the form

$$\|L(y, \partial)u\|_0^2 \geq \text{const}\|u\|_0^2 \quad \forall u \in L_0 \otimes \mathcal{S}(\mathbb{R}^m)$$

or

$$\|L^*(y, \partial)u\|_0^2 \geq \text{const}\|u\|_0^2 \quad \forall u \in {}^tL_1 \otimes \mathcal{S}(\mathbb{R}^m)$$

with some const > 0.

Let $h_j = \dim_{\mathbb{C}} V_j/V_{j-1}$. Then we write $L_0 \otimes \mathcal{S}(\mathbb{R}^m) = \mathcal{S}(\mathbb{R}^m)^{h_1} \oplus \cdots \oplus \mathcal{S}(\mathbb{R}^m)^{h_k} = \mathcal{S}(\mathbb{R}^m)^h$. We define $\alpha_i = 2(k-j)$ if $h_j + 1 \leq i \leq h_{j+1}$ and $\beta_i = \alpha_i + 1$ $\forall i$ (we have set $h_0 = 0$).

Then, if for each $1 \leq h \leq k$ we have an estimate of the form:

$$(*h) \quad \left\| \sum_1^m H_j^{hh}\, \partial u/\partial y_j + H_{m+j}^{hh} y_j u + H_0^{hh} u \right\| \geq C_h \|u\|_0^2 \quad \forall u \in \mathcal{S}(\mathbb{R}^m)^{h_h}$$

with a constant $C_h > 0$, then we have with some constant $c > 0$:

$$\|L(y, \partial)u\|_\alpha^2 \geq c\|u\|_{\beta-1}^2 \quad \forall u \in \mathcal{S}(\mathbb{R}^m)^h,$$

and we can apply the results of §4 to discuss the reduced subprincipal complex. Let us set $L_h(y, \partial) = \Sigma_{j=1}^m H_j^{hh}\, \partial/\partial y_j + y_j H_{m+j}^{hh} + H_0^{hh}$ then we have the following criterion:

If for every $1 \leq h \leq k$ *the complex*

$$(30_h) \qquad 0 \to \mathcal{S}(\mathbb{R}^m)^{h_h} \xrightarrow{L_h} \mathcal{S}(\mathbb{R}^m)^{h_h} \to 0$$

is subelliptic at the first $\mathcal{S}(\mathbb{R}^m)^{h_h}$*, but there exists an index h for which* *this complex is not subelliptic at the second* $\mathscr{S}(\mathbb{R}^m)^{h_h}$, *then the complex* (29) *does not admit the Poincaré lemma at the point* x_0.

As an example, let us apply the criterion above to an operator $A(x, D)$ of type (a_j, b_i), represented by a triangular matrix:

$$A(x, D) = \begin{pmatrix} P_1(x, D) & 0 & \dots & 0 \\ * & P_2(x, D) & & 0 \\ \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots \\ * & * & \dots & P_p(x, D) \end{pmatrix}$$

We find that, denoting by $\hat{P}_h(x, \xi)$ the principal symbol of the operator $P_h(x, D)$ on the diagonal, the condition given by the criterion above is that for some direction $\xi^0 \neq 0$ in $\mathbb{R}^n$ we have:

$$(\ddagger h) \qquad \operatorname{Re}\left(\sum \frac{\partial \hat{P}_h}{\partial \xi_j}(x_0, i\xi^0)\overline{\frac{\partial \hat{P}_h}{\partial x_j}(x_0, i\xi^0)}\right) < 0$$

for $h = 1, 2, \dots, p$. However, for the case of a triangular matrix $A(x, D)$, it is obvious that the validity of $(\ddagger h)$ for a single h is sufficient to conclude that the equation $A(x, D)u = f$ has no solution $u \in \mathscr{E}^p_{x_0}$ for general $f \in \mathscr{E}^p_{x_0}$.

But when $(\ddagger h)$ holds for every h we can say more: namely, if we add to $A(x, D)$ any matrix $B(x, D)$ of differential operators with smooth coefficients in Ω, of type $(a_j - 1, b_i)$, the equation

$$A(x, D)u + B(x, D)u = f$$

has no solution $u \in \mathscr{E}^p_{x_0}$ for many $f \in \mathscr{E}^p_{x_0}$: the criterion we have obtained is invariant under perturbations by lower order operators. If we want to maintain this invariance, we are actually compelled to ask that $(\ddagger h)$ holds for every h, as the following example, essentially due to Grušin, shows:

Let us consider on $\mathbb{R}^2$ the differential operator:

$$\begin{pmatrix} \partial/\partial x_1 + ix_1\,\partial/\partial x_2 & 1 \\ x_1^2\,\partial/\partial x_2 & \partial/\partial x_1 - ix_1\,\partial/\partial x_2 \end{pmatrix}$$

with trivial gradings $a_1 = a_2 = 1$, $b_1 = b_2 = 0$. The reduced subprincipal operator at $(0, \xi^0)$, for $\xi^0 = (0, 1)$ is diagonal and $(\ddagger h)$ is satisfied for $h = 1$, but not for $h = 2$. The system

$$\begin{cases} \partial u/\partial x_1 + ix_1\,\partial u/\partial x_2 + v = f \\ x_1^2\,\partial u/\partial x_2 + \partial v/\partial x_1 - i\,\partial v/\partial x_2 = g \end{cases}$$

for $u, v, f, g \in \mathscr{E}_0$ is equivalent to the scalar equation:

$$u, f, g \in \mathscr{E}_0 \quad \text{and} \quad P(x, D)u = (\partial^2/\partial x_1^2 + x_1^2\, \partial^2/\partial x_2^2 - i\, \partial/\partial x_2 - x_1^2\, \partial/\partial x_2)u$$
$$= \partial f/\partial x_1 - i x_1\, \partial f/\partial x_2 - g.$$

In [6] Grušin proved that both the operator $P(x, D)$ and the operator $P^*(x, D)$ are hypoelliptic at 0 and this implies that the above equation is locally solvable at x_0.

REMARK: Criterions for the subellipticity at different places of the complexes $(30_h)(h = 1, \ldots, k)$ are described in the paper of Hörmander [10].

§6. Some remarks on double characteristics

(a) Let $P(x, D): \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ be a scalar differential operator of order m with smooth coefficients on an open set Ω of $\mathbb{R}^n$. We say that P is doubly characteristic at $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n - \{0\})$ if, denoting by $\hat{P}(x, \xi)$ the principal symbol of P we have:

$$\hat{P}(x_0, i\xi^0) = \frac{\partial \hat{P}}{\partial \xi_j}(x_0, i\xi^0) = \frac{\partial \hat{P}}{\partial x_j}(x_0, i\xi^0) = 0 \quad \text{for} \quad j = 1, \ldots, n.$$

If P is doubly characteristic at (x_0, ξ^0), then its asymptotic operator at (x_0, ξ^0) takes the form:

$$\tilde{P}(t, x, D) = \sum_{s=2}^{\infty} t^s \tilde{P}_s(x, D)$$

with $P_0 = P_1 = 0$ and P_2 of the form:

$$\tilde{P}_2(x, D) = (1/2) \sum_{i,j=1}^{m} \left(\frac{\partial^2 \hat{P}}{\partial \xi_i\, \partial \xi_j}(x_0, i\xi^0) \frac{\partial^2}{\partial x_i\, \partial x_j} + 2 \frac{\partial^2 \hat{P}}{\partial \xi_i\, \partial x_j}(x_0, i\xi^0) x_j \frac{\partial}{\partial x_i} \right.$$
$$\left. + \frac{\partial^2 \hat{P}}{\partial x_i\, \partial x_j}(x_0, i\xi^0) x_i x_j \right) + c$$

where the constant c also depends on the lower order terms in $P(x, D)$.

Let $L_1 \subset E = \mathbb{R}^n \oplus \mathbb{R}^n$ be a maximal isotropic subspace of E of which the total symbol of $\tilde{P}_2$ is independent. By homogeneity reasons, we can assume that $\xi^0 \in L_1$, and thus $L_1 \neq 0$. By a suitable choice of symplectic coordinates in E we reduce to the case in which

$$L_1 = \{x_1 = \cdots = x_n = 0, \xi_1 = \cdots = \xi_m = 0\} \quad (m < n).$$

Then we set

$$y_i = x_i \quad \text{for} \quad i = 1, \ldots, m, \quad hy_i = x_i \quad \text{for} \quad i = m+1, \ldots, n$$
$$\partial/\partial y_i = \partial/\partial x_i \quad \text{for} \quad i = 1, \ldots, m$$
$$\partial/\partial y_i = h\, \partial/\partial x_i \quad \text{for} \quad i = m+1, \ldots, n \quad t = h^4,$$

obtaining the asymptotic operator

$$\tilde{\tilde{P}}(h, y, \partial) = \sum_{s=8}^{\infty} h^s \tilde{\tilde{P}}_s(y, \partial)$$

where $\tilde{\tilde{P}}_8(y, \partial)$, that by simplicity we will denote by $L(y, \partial)$, has the form:

$$L(y, \partial) = \tilde{\tilde{P}}_8(y, \partial) = \sum_1^m a_{ij}\, \partial^2/\partial y_i\, \partial y_j + 2b_{ij} y_j\, \partial/\partial y_i + c_{ij} y_i y_j + d.$$

By the discussion of the preceding sections, we have the following criterion:

If the complex

$$0 \to \mathscr{S}(\mathbb{R}^m) \xrightarrow{L(y, \partial)} \mathscr{S}(\mathbb{R}^m) \to 0 \tag{31}$$

is subelliptic at the first $\mathscr{S}(\mathbb{R}^m)$ but not at the second, then the complex

$$\mathscr{E}(\Omega) \xrightarrow{P(x, D)} \mathscr{E}(\Omega) \to 0$$

does not admit the Poincaré lemma at x_0.

(b) Let us consider the quadratic form

$$q(y, \eta) = \sum_{i,j=1}^{m} -a_{ij}\eta_i\eta_j + i2b_{ij}y_j\eta_i + c_{ij}y_iy_j.$$

The case in which $q(y, \eta)$ has no non-trivial zeros $(y, \eta) \in$

$R^m \oplus R^m = E_m$ and $m \geq 2$ has been considered by many authors (cf. [5], [6]). But they show that in this case the problem

$$u \in \mathscr{S}(\mathbb{R}^m), \qquad L(y, \partial)u = f \in \mathscr{S}(\mathbb{R}^m)$$

has index 0, i.e. the cohomology groups of the complex (31) are finite dimensional and the alternated sum of their dimensions is 0. So the method we outlined does not apply to this case: this accounts for the unstability under lower order perturbations exhibited by the example of Grušin [6].

(c) Let us discuss first the case $m = 1$. This is always the case if $n = 2$. In this case the quadratic form q decomposes into a product of linear factors:

$$q(y, \eta) = (ia_1\eta + b_1y)(ia_2\eta + b_2y)$$

and then we can write:

$$L(y, \partial) = (a_1 d/dy + yb_1)(a_2 d/dy + yb_2) \pm d' \quad \text{with} \quad d' \in \mathbb{C}.$$

Let $a_1a_2b_1b_2 \neq 0$. By reducing to Hermite's degenerate hypergeometric equation we obtain:

if $\mathrm{Re}(b_1/a_1) < 0$ and $\mathrm{Re}(b_2/a_2) < 0$, then:

$$L^*(y, \partial): \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}) \text{ is onto}$$

$$L^*(y, \partial): \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}) \text{ is not injective.}$$

Therefore we obtain the following statement:

Let $P(x, D)$ be doubly characteristic at $(x_0, \xi^0) \in \Omega \times \mathbb{R}^n - \{0\}$ and assume that the total symbol $\tilde{P}_2^0(x, \xi)$ of P_2 is independent of the variables of a maximal isotropic subspace of the symplectic space $\{x_1 = \xi_1 = 0\} \subset E$. If all roots of the quadratic equation:

$$\lambda^2 \frac{\partial^2 \hat{P}}{\partial \xi_1^2}(x_0, i\xi^0) + 2\lambda \frac{\partial^2 \hat{P}}{\partial \xi_1 \partial x}(x_0, i\xi^0) + \frac{\partial^2 \hat{P}}{\partial x_1^2}(x_0, i\xi^0) = 0$$

have negative real part, then the complex $\mathscr{E}(\Omega) \xrightarrow{P(x,D)} \mathscr{E}(\Omega) \to 0$ *does not admit the Poincaré lemma at* x_0.

(d) *A priori estimates.*

Let us set $\nabla = {}^t(-i\partial/\partial y_1, \ldots, -i\partial/\partial y_m, y_1, \ldots, y_m) = {}^t(\nabla_1, \ldots, \nabla_{2m})$. We have $\nabla^* = {}^t\nabla$ and, setting

$$J = (J_{hk}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

with I the $m \times m$ identity matrix, we have the following commutation relations:

$$[\nabla_h, \nabla_k] = \nabla_h \nabla_k - \nabla_k \nabla_h = iJ_{hk}.$$

Then, for a uniquely determined symmetric matrix A with complex entries we can write:

$$L(y, \partial) = \nabla^* A \nabla + d \quad (d \in \mathbb{C}).$$

We have

$$L^*(y, \partial) = \nabla^* \bar{A} \nabla + \bar{d}.$$

then for $u \in \mathscr{S}(\mathbb{R}^m)$ we obtain:

$$\begin{aligned} \|L(y, \partial)u\|_0^2 - \|L^*(y, \partial)u\|_0^2 &= ((\nabla^* \bar{A} \nabla \nabla^* A \nabla - \nabla^* A \nabla \nabla^* \bar{A} \nabla)u, u)_0 \\ &= \sum_{h,k,r,s} \bar{a}_{hk} a_{rs} ((\nabla_h \nabla_k \nabla_r \nabla_s - \nabla_r \nabla_s \nabla_h \nabla_k)u, u)_0 \end{aligned}$$

where $A = (a_{rs})$ are the coefficients of the matrix A.

Using the commutation formulas established above, we obtain

$$\begin{aligned} \|L(y, \partial)u\|_0^2 - \|L^*(y, \partial)u\|_0^2 &= 2i((\bar{A}JA - AJ\bar{A})\nabla u, \nabla u)_0 \\ &= 4(\mathrm{Im}(AJ\bar{A})\nabla u, \nabla u)_0. \end{aligned} \tag{32}$$

If we assume that the matrix $\mathrm{Im}(AJ\bar{A})$ is positive definite, then we obtain with a constant $c > 0$:

$$\|L(y, \partial)u\|_0^2 \geq c\|u\|_1^2 \quad \forall u \in \mathscr{S}(\mathbb{R}^m).$$

The condition $\mathrm{Im}(AJ\bar{A}) > 0$ is symplectic invariant. It implies that the function

$$g(x, \xi) = -\mathrm{Re}\left(\sum_{j=1}^{m} \frac{\partial \hat{P}}{\partial \xi_j}(x, i\xi) \overline{\frac{\partial \hat{P}}{\partial x_j}(x, i\xi)}\right)$$

as a function of $x_1, \ldots, x_m, \xi_1, \ldots, \xi_m$ for $x_{m+1} = x_{0_{m+1}}, \ldots, x_n = x_{0_n}$ and $\xi_{m+1} = \xi^0_{m+1}, \ldots, \xi_n = \xi^0_n$ has an isolated maximum at $x_1 = x_{0_1}, \ldots, x_m = x_{0_m}, \xi_1 = \xi^0_1, \ldots, \xi_m = \xi^0_m$.

A straightforward computation shows that the condition $\operatorname{Im}(AJ\bar{A}) > 0$ is equivalent to the condition that $\operatorname{Re}(b_i/a_i) < 0$ for $i = 1, 2$ when $m = 1$.

(e) Let us set $H = -JA$. Then the matrix H is complex-symplectic, i.e. we have:

$$JH + {}^tHJ = 0$$

and hence if λ is an eigenvalue of H also $-\lambda$ is an eigenvalue of H. Let us set $\tilde{E} = E_m \otimes_{\mathbb{R}} \mathbb{C}$. Extending σ to a bilinear form on $\tilde{E}$, we define on $\tilde{E}$ a structure of complex symplectic space. For $\lambda \in \mathbb{C}$, we set

$$\tilde{E}(\lambda) = \bigcup_k \ker(H - \lambda I)^k.$$

Then we have

$$\sigma(u, v) = 0 \quad \text{if} \quad u \in E(\lambda), v \in E(\mu) \quad \text{and} \quad \lambda + \mu \neq 0.$$

We can choose a complex symplectic base of $\tilde{E}$ of vectors $u_1, \ldots, u_m, v_1, \ldots, v_m \in \tilde{E}$ such that $u_j \in \tilde{E}(\lambda_j)$ and $v_j \in \tilde{E}(-\lambda_j)$ for some eigenvalue λ_j of H $(j = 1, \ldots, m)$. Then we have

$$u = \sum f_j(u)u_j + g_j(u)v_j$$

for some linear functionals $f_j, g_j : \tilde{E} \to \mathbb{C}$. Then, with $\lambda_{jk} = \sigma(v_j, Hu_k)$ we obtain

$${}^tuAu = 2 \sum \lambda_{jk} g_j(u) f_k(u)$$

where we notice that $\lambda_{jk} = 0$ if $\lambda_j \neq \lambda_k$. In particular, when all eigenvalues of H are simple, we obtain the expression:

$${}^tuAu = 2 \sum \lambda_j g_j(u) f_j(u).$$

(f) Let Λ be a linear subspace of $\tilde{E}$. We say that Λ is Lagrangean if $\sigma(u, v) = 0\ \forall u, v \in \Lambda$ (i.e. Λ is isotropic) and $\dim_{\mathbb{C}} \Lambda = m$.

A Lagrangean subspace of $\tilde{E}$ is non-negative (positive/non-positive/negative) if the hermitean form $-i\sigma(u, \bar{v})$ restricted to Λ is positive semi-definite (positive definite/negative semi-definite/negative definite).

Any Lagrangean subspace Λ of $\tilde{E}$ can be brought by the complexification of a real symplectic change of coordinates of E_m to the form

$$\Lambda = \{(z, i\,\mathrm{grad}_z\, Q(z) \mid z \in \mathbb{C}^m\}$$

where $Q(z) = \Sigma\, a_j z_j^2$ with real a_j's. The condition that Λ is non-negative translates into the fact that $a_j \geq 0\, \forall j$ and analogously the other conditions can be translated in an obvious way into conditions on the signs of the a_j's. (Cf [10]).

We note now that the vectors $u_1, \ldots, u_m$ introduced in the previous section span a Lagrangean space. Assume that this Lagrangean space Λ is non-negative, and written in canonical form. Then the function $\exp(iQ(y)) \in \mathcal{S}'(\mathbb{R}^m)$ and we have, for some constants α_{ij}, β_{ij}, $\gamma \in \mathbb{C}$:

$$\exp(-iQ(y))L(y, \partial)(g(y)\exp(iQ(y)) = \sum_{i,j=1}^{m} \alpha_{ij} \frac{\partial^2 g}{\partial y_i\, \partial y_j} + y_j\beta_{ij}\frac{\partial g}{\partial y_i} + \gamma g.$$

It follows that $(L(y, \partial) - \gamma)\exp(iQ(y)) = 0$. Now we notice that, if $\mathrm{Im}(AJ\bar{A}) > 0$ then the estimate implied by (32) hold also for the operator $L(y, \partial) - \gamma$. Thus by the results of §4 if $\mathrm{Im}(AJ\bar{A}) > 0$ then the equation $(L(y, \partial) - \gamma)w = 0$ has no non-trivial solution in $\mathcal{S}'(\mathbb{R}^m)$. We have obtained the following:

if $\mathrm{Im}(AJ\bar{A}) > 0$, *then there exists no non-negative Lagrangean subspace* Λ *of* $\tilde{E}$ *invariant for* H.

(g) Assume that the condition $\mathrm{Im}(AJ\bar{A}) > 0$ holds. If the equation $L^*u = 0$ has no solution in $\mathcal{S}(\mathbb{R}^m) - \{0\}$, then, by Proposition 33 there is a constant $c > 0$ such that

$$\|L^*u\|_0^2 \geq c\|u\|_1^2 \quad \forall u \in \mathcal{S}(\mathbb{R}^m).$$

This implies that for each $f \in S_0$ there is a unique $u \in S_1$ such that $Lu = f$ (Proposition 26). Let us denote by $T: S_0 \to S_0$ the map that associates to $f \in S_0$ the solution $u = Tf \in S_1$ of the equation $Lu = f$. Then T is a compact operator by Proposition 21. Let $\lambda \neq 0$ and $u \in S_0$ be such that $Tu = \lambda u$. We have $Lu \in S_0$, $u \in S_1$ and $u = \lambda Lu$, i.e. $(L - (1/\lambda))u = 0$. But this implies that $u = 0$ because $(L - (1/\lambda))$ is injective on $\mathcal{S}'(\mathbb{R}^m)$ as we remarked above. Thus the spectrum of the operator T reduces to $\{0\}$, i.e. the compact operator T is quasi-nilpotent. This implies that T^* is also quasi-nilpotent: we have obtained the following statement:

if $L^*u = 0$ *has no solution* $u \in \mathscr{S}(\mathbb{R}^m) - \{0\}$, *then* $(L^* - \lambda)u = 0$ *has no solution* $u \in \mathscr{S}'(\mathbb{R}^m) - 0$ *for any* $\lambda \in \mathbb{C}$.

(h) The discussion of the previous sections yields the following criterion:

If the following conditions are satisfied:

(1) $\mathrm{Im}(AJ\bar{A}) > 0$,

(2) *There exists a non-positive Lagrangean subspace* Λ *of* E *invariant for* $H = -JA$, *then the complex*

$$\mathscr{E}(\Omega) \xrightarrow{P} \mathscr{E}(\Omega) \to 0$$

does not admit the Poincaré Lemma at x_0.

REMARK: Actually I think that condition (2) should be dropped and that (1) is sufficient to obtain the conclusion on the non validity of Poincaré lemma. Indeed it seems plausible that, if $L \in W_2$ has a compact inverse $T: S_0 \to S_0$ it never occurs that the spectrum of T reduces to 0. On the other hand I also suspect that an implication $(1) \Rightarrow (2)$ holds and moreover that (1) is the necessary and sufficient condition in order that all invariant Lagrangean subspaces for H are strictly negative.

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Università di Pisa
Istituto di Matematica "Leonide Tonelli"
I 56100 Pisa (Italy)