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ON THE LOGARITHMIC PLURIGENERA OF ALGEBRAIC SURFACES

Yoshiyuki Kuramoto

Introduction

Let S be a nonsingular algebraic surfaces over the complex number field $\mathbb{C}$. We study the logarithmic plurigenera $\bar{P}_m(S)$ under the assumption that there is a surjective morphism $f: S \to \Delta$ to a nonsingular curve Δ whose general fiber C_w is an irreducible curve such that the logarithmic Kodaira dimension $\bar{\kappa}(\Delta)$ is non-negative. Such a situation can arise from the quasi-Albanese mapping α_S of S. If $\bar{\kappa}(C_w) = -\infty$, i.e. $C_w \cong \mathbb{P}^1$ or $\mathbb{A}^1$, then $\bar{P}_m(S) = 0$ for all $m > 0$. Thus we assume that $\bar{\kappa}(C_w) \geqq 0$. Then by the addition formula for logarithmic Kodaira dimension ([6]), we have some $m > 0$ such that $\bar{P}_m(S) \geqq 1$. In this paper we shall look for integers m such that $\bar{P}_m(S) \geqq 1$.

We use the following notations:

- $\bar{S}$: a nonsingular complete algebraic surface which contains S as a Zariski open set,
- $D = \bar{S} - S$: the complement of S in $\bar{S}$. We assume that D has only normal crossings and every irreducible component of D is nonsingular.
- $\bar{\Delta}$: a nonsingular complete algebraic curve which contains Δ as a Zariski open set,
- $\bar{f}: \bar{S} \to \bar{\Delta}$: a rational mapping defined by f. We can assume that $\bar{f}$ is a morphism by taking a suitable $\bar{S}$.
- $\bar{C}_w$: a general fiber of $\bar{f}$,
- $K_{\bar{S}}$: the canonical divisor of $\bar{S}$,
- $p_g(\bar{S})$: the geometric genus of $\bar{S}$,

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$q(\bar{S})$: the irregularity of $\bar{S}$,
$P_m(\bar{S})$: the m-genus of $\bar{S}$,
$\bar{p}_g(S)$: the logarithmic geometric genus of S,
$\bar{q}(S)$: the logarithmic irregularity of S,
$\bar{P}_m(S)$: the logarithmic m-genus of S,
$\kappa(\bar{V})$: the Kodaira dimension of a nonsingular complete algebraic variety $\bar{V}$,
$\bar{\kappa}(V)$: the logarithmic Kodaira dimension of a nonsingular algebraic variety V,
$g(C)$: the genus of a nonsingular complete algebraic curve C, $\pi(C) = (1/2)(K_{\bar{S}} + C, C) + 1$, if C is a divisor on $\bar{S}$,
$\sim$: the linear equivalence of divisors,
$D_1 \geqq D_2$: means $D_1 - D_2$ is effective or 0 if D_1 and D_2 are divisors.

The following are our results.

THEOREM 1: *Under the above notations if $\bar{\kappa}(\Delta) \geqq 0$ and $\bar{\kappa}(C_w) \geqq 0$, then $\bar{P}_4(S) \geqq 1$ or $\bar{P}_6(S) \geqq 1$.*

COROLLARY: *Let $S = \mathbb{A}^2 - V(\varphi)$ where $\varphi \in \mathbb{C}[x, y]$ and φ is irreducible. If $\bar{P}_4(S) = \bar{P}_6 = 0$, then $S \cong \mathbb{A}^1 \times G_m$.*

THEOREM 2: *Let S be a nonsingular algebraic surface over $\mathbb{C}$. If $\bar{\kappa}(S) \geqq 0$ and $\bar{q}(S) \geqq 1$, then $\bar{P}_4(S) \geqq 1$ or $\bar{P}_6(S) \geqq 1$.*

If $\bar{S}$ is neither ruled nor rational, then $P_4(\bar{S}) \geqq 1$ or $P_6(\bar{S}) \geqq 1$ from the theory of complete algebraic surfaces. Thus to prove Theorem 1, it suffices to treat the case that $\bar{S}$ is ruled or rational. In §1, we give preliminary results. In §2–§7, we prove Theorem 1. The following theorem due to Tsunoda plays an essential role in §7.

THEOREM 3: (Tsunoda) *Let S, $\bar{S}$ and D be as above. If $\bar{\kappa}(S) = 2$ and the intersection matrix of D is not negative definite, then $\bar{P}_4(S) \geqq 1$ or $\bar{P}_6(S) \geqq 1$.*

In §8 we prove Theorem 3. In §9 we prove Corollary and Theorem 2.

The author would like to express his hearty thanks to Professor S. Iitaka for his kind advices and encouragement, and to Mr. S. Tsunoda who proved Theorem 3.

§1. Preliminaries

LEMMA 1: *Let S, $\bar{S}$ and D be as in the introduction. Suppose that there is an exceptional curve of the first kind E on $\bar{S}$ such that $(E, K_{\bar{S}} + D) \leqq 0$. Let $\mu : \bar{S} \to \bar{S}^b$ be the contraction of E. Then $\mu(D)$ is a divisor with only normal crossings and $\bar{P}_m(S) = \bar{P}_m(\bar{S}^b - \mu(D))$.*

PROOF: Easy and omitted. Especially if $D \not\supset E$ and $(D, E) = 1$, then $\bar{S}^b - \mu(D)$ is called a half point detachment from S, and if $D \supset E$ and $(D - E, E) = 2$, then μ is called a canonical blowing down. (c.f. [5])

LEMMA 2: *Let S, $\bar{S}$ and D be as in the introduction. Let $D = \Sigma D_j$ be the irreducible decomposition of D. Let $h(\Gamma(D))$ be the cyclotomic number of the dual graph $\Gamma(D)$ of D,* i.e.

$$\begin{aligned} h(\Gamma(D)) = &(\textit{number of connected components of } \Gamma(D)) \\ &-(\textit{number of vertices of } \Gamma(D)) \\ &+(\textit{number of 1-simplices of } \Gamma(D)). \end{aligned}$$

Then we have

$$\bar{p}_g(S) = \sum \pi(D_j) + p_g(\bar{S}) - q(\bar{S}) + h(\Gamma(D)) + t.$$

Where t is the dimension of the kernel of the canonical homomorphism $H^1(\bar{S}, \mathcal{O}_{\bar{S}}) \to H^1(D, \mathcal{O}_D)$. (c.f. [9; Theorem 2.2]).

LEMMA 3: *Let $\pi : V \to W$ be an r-sheeted Galois covering, where V and W are complete normal algebraic varieties over $\mathbb{C}$. Let D be a Cartier divisor on W. Then we have*

a) $\dim H^0(V, \mathcal{O}_V(\pi^*D)) \geqq 1 \Rightarrow \dim H^0(W, \mathcal{O}_W(rD)) \geqq 1$,

b) $\dim H^0(V, \mathcal{O}_V(\pi^*D)) \geqq 2 \Rightarrow \dim H^0(W, \mathcal{O}_W(rD)) \geqq 2$.

PROOF: Using the same argument as [2; §6] or [4; §10.11], we can prove the above lemma. Details are omitted.

In that follows we tacitly use the notation in the introduction.

§2. The case $\kappa(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = -\infty$

PROPOSITION 1: *If* $\bar{\kappa}(C_w) \geqq 0$, $\bar{\kappa}(\Delta) \geqq 0$, $\kappa(\bar{C}_w) = -\infty$ *and* $\kappa(\bar{\Delta}) = -\infty$, *then* $\bar{P}_2(S) \geqq 1$.

PROOF: By the assumption, $\bar{S}$ is rational and $\bar{\Delta} \cong \mathbb{P}^1$. Hence there exists a composition of a finite sequence of quadratic transformations $\mu : \bar{S} \to \hat{S}$ where $\hat{f} : \hat{S} \to \bar{\Delta}$ is a $\mathbb{P}^1$-bundle over $\bar{\Delta}$ and $\bar{f} = \hat{f} \circ \mu$. Since $\bar{\kappa}(\Delta) \geqq 0$, $\Delta \subset \mathbb{C}^*$ and so D contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2. If D contains two horizontal components with respect to $\bar{f}$, then by Lemma 2 $\bar{p}_g(S) \geqq 1$. Thus we assume that D contains only one horizontal component which we denote by H. If $(H, F_1) \geqq 2$, then by Lemma 2 $\bar{p}_g(S) \geqq 1$. Hence we assume $(H, F_1) = 1$. Let $\bar{C}_w$ be a general fiber of $\bar{f}$ and $\tilde{F}_1$ be the fiber such that $F_1 = \mathrm{supp}(\tilde{F}_1)$. Since $\bar{\kappa}(C_w) \geqq 0$, it follows that $(\bar{C}_w, H) = (F_1, H) \geqq 2$. Hence $F_1 \neq \tilde{F}_1$ and F_1 contains an exceptional curve E_1. If $E_1 \cap H = \emptyset$, we can contract E_1. Thus we may assume $(E_1, H) = 1$. If E_1 is an edge component of F_1 (i.e. $(E_1, F_1 - E_1) = 1$), then $(E_1, D - E_1) = 2$ and so by Lemma 1 we can contract E_1. Thus we may assume that F_1 contains two components C_1 and C_2 such that $(C_1, E_1) = (C_2, E_1) = 1$. Let $\mu' : \bar{S} \to \bar{S}'$ be the contraction of E_1. Put $\mu'_*(C_1) = C'_1$, $\mu'_*(C_2) = C'_2$ and $\mu'_*(H) = H'$. Since $2F_1 \geqq C_1 + C_2 + 2E_1$, we have

$$K_{\bar{S}} + H + 2F_1 \geqq K_{\bar{S}} - E_1 + H + C_1 + C_2 + 3E_1 = \mu'^*(K_{\bar{S}'} + H' + C'_1 + C'_2).$$

On the other hand by the Riemann-Roch theorem we have

$$\dim H^0(\bar{S}', \mathcal{O}_{\bar{S}'}(K_{\bar{S}'} + H' + C'_1 + C'_2))$$
$$\geqq (1/2)(K_{\bar{S}'} + H' + C'_1 + C'_2, H' + C'_1 + C'_2) + 1 = 1.$$

Thus we have

$$\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + H + 2F_1)) \geqq 1.$$

Similarly, $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + H + 2F_2)) \geqq 1$.

Hence, $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(2K_{\bar{S}} + 2H + 2F_1 + 2F_2)) \geqq 1$.

Therefore, $\bar{P}_2(S) = \dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(2(K_{\bar{S}} + D))) \geqq 1$. Q.E.D.

§3. The case $\kappa a(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = 0$

PROPOSITION 2: *If* $\bar{\kappa}(C_w) \geqq 0$, $\bar{\kappa}(\Delta) \geqq 0$, $\kappa(\bar{C}_w) = -\infty$ *and* $\kappa(\bar{\Delta}) = 0$, *then* $\bar{P}_2(S) \geqq 1$.

PROOF: By the assumption we have the following commutative diagram:

$$\begin{array}{ccccc} S & \hookrightarrow & \bar{S} & \xrightarrow{\mu} & \hat{S} \\ & {}_f\searrow & & {}_{\bar{f}}\searrow & \downarrow \hat{f} \\ & & \Delta & \hookrightarrow & \bar{\Delta} \end{array}$$

Where $\bar{\Delta}$ is a nonsingular elliptic curve, $\hat{f}: \hat{S} \to \bar{\Delta}$ is a $\mathbb{P}^1$-bundle, $\mu: \bar{S} \to \hat{S}$ is a composition of a finite number of quadratic transformations and $\bar{S} - S = D$ is a divisor with normal crossings. We may assume that all irreducible components of D are horizontal with respect to $\bar{f}$. Put $D = \Sigma H_j$, where the H_j are irreducible components. Since the H_j are horizontal $g(H_j) \geqq g(\bar{\Delta}) = 1$. Hence if $\Sigma g(H_j) \geqq 2$, then by Lemma 2 we have $\bar{p}_g(S) \geqq 1$. Thus we assume that D is an irreducible horizontal curve such that $g(D) = 1$. We put $\mu = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_r$, where $\mu_i : S_i \to \bar{S}_{i-1}$ is a quadratic transformation with center p_i, $\bar{S}_0 = \hat{S}$ and $\bar{S}_r = \bar{S}$. Let $E_i = \mu_i^{-1}(p_i)$. For the sake of simplicity we use the same letter E_i for $(\mu_{i+1} \circ \cdots \circ \mu_r)^* E_i$ also. Put $\mu(D) = \hat{D}$. Let ν_i be the multiplicity of the proper transform of $\hat{D}$ to $\bar{S}_{i-1}$ at p_i. Then we have

$$\mu^* \hat{D} = D + \sum_{i=1}^{r} \nu_i E_i, \quad \mu^* K_{\hat{S}} \sim K_{\bar{S}} - \sum_{i=1}^{r} E_i.$$

Let $\hat{F}$ be a fiber of $\hat{f}: \hat{S} \to \bar{\Delta}$. We put $(\hat{F}, \hat{D}) = d$. Since every exceptional curve on $\bar{S}$ is contained in some fiber of $\bar{f}$, it follows that $(\hat{F}, \hat{D}) = (\bar{C}_w, D)$. By the assumption that $\bar{\kappa}(C_w) \geqq 0$, we have $(\bar{C}_w, D) \geqq 2$. Hence $d \geqq 2$. Since $\hat{S}$ is a ruled surface of genus 1, there is a non-negative integer b such that

$$\hat{D} \equiv -(d/2) K_{\hat{S}} + b\hat{F},$$

where $\equiv$ means numerical equivalence. Since $g(D) = 1$, we have

$$0 = (D, D + K_{\bar{S}}) = \left(\hat{D} - \sum_{i=1}^{r} \nu_i E_i, \hat{D} + K_{\hat{S}} - \sum_{i=1}^{r} (\nu_i - 1) E_i\right)$$

$$= (\hat{D}, \hat{D} + K_{\hat{S}}) - \sum_{i=1}^{r} \nu_i(\nu_i - 1)$$

$$= (-(d/2)K_{\hat{S}} + b\hat{F}, (1 - (d/2))K_{\hat{S}} + b\hat{F}) - \sum_{i=1}^{r} \nu_i(\nu_i - 1)$$

$$= 2b(d-1) - \sum_{i=1}^{r} \nu_i(\nu_i - 1).$$

Thus we have

(1) $$\sum_{i=1}^{r} (1/2)\nu_i(\nu_i - 1) = b(d-1).$$

First we treat the case that $\nu_i = 1$ for all i. In this case $b = 0$ and $\hat{D} + (d/2)K_{\hat{S}} \equiv 0$. We may assume $\bar{S} = \hat{S}$. Then we have

$$(\hat{D} + K_{\hat{S}}, \hat{F}) = ((1 - (d/2))K_{\hat{S}}, \hat{F}) = d - 2 \geqq 0.$$

If $d > 2$, then $H^0(\hat{S}, \mathcal{O}_{\hat{S}}(-(m-1)\hat{D} - (m-1)K_{\hat{S}})) = 0$ for $m \geqq 2$, because $\hat{F}^2 = 0$. Thus from the exact sequence

$$\mathcal{O}_{\hat{S}}(-m\hat{D} - (m-1)K_{\hat{S}}) \to \mathcal{O}_{\hat{S}}(-(m-1)\hat{D} - (m-1)K_{\hat{S}})$$

$$\to \mathcal{O}_{\hat{D}}(-(m-1)(\hat{D} + K_{\hat{S}})|_{\hat{D}}) \cong \mathcal{O}_{\hat{D}},$$

we get the exact sequence

$$0 \to H^0(\hat{D}, \mathcal{O}_{\hat{D}}) \to H^1(\hat{S}, \mathcal{O}_{\hat{S}}(-m\hat{D} - (m-1)K_{\hat{S}})).$$

Hence, $\dim H^1(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) = \dim H^1(\hat{S}, \mathcal{O}_{\hat{S}}(-m\hat{D} - (m-1)K_{\hat{S}})) \geqq 1$.
Since $H^2(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) \cong H^0(\hat{S}, \mathcal{O}_{\hat{S}}(-m\hat{D} - (m-1)K_{\hat{S}})) = 0$, applying the Riemann-Roch theorem, we obtain

$$\dim H^0(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) \geqq 1 \text{ for } m \geqq 2.$$

Especially, we know $\bar{P}_2(S) \geqq 1$ in this case.

Now we assume that $d = 2$. Put $D = \tilde{\Delta}$. Corresponding to the morphism $\bar{f}|_D : D \to \bar{\Delta}$, we get the homomorphism $\psi : \tilde{\Delta} \to \bar{\Delta}$ of 1-dimensional Abelian varieties. We denote the kernel of ψ by G. Let $\tilde{S} = \bar{S} \times_{\bar{\Delta}} \tilde{\Delta}$ be the fiber product. Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \Psi\ } & \bar{S} \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \tilde{\Delta} & \xrightarrow{\ \psi\ } & \bar{\Delta} \end{array}$$

Where $\Psi : \tilde{S} \to \bar{S}$ and $\tilde{f} : \tilde{S} \to \tilde{\Delta}$ are projections. Since $\psi : \tilde{\Delta} \to \bar{\Delta}$ is a 2-sheeted unramified Galois covering, so is $\Psi : \tilde{S} \to \bar{S}$. And we know that $\tilde{f} : \tilde{S} \to \tilde{\Delta}$ is also a ruled surface of genus 1. Let $\tilde{D}$ be a cross-section of $\tilde{f}$ defined by $\tilde{D} = \{(a, a) | a \in D\}$. Then we have

$$\Psi^* D = \Psi^{-1}(D) = \{(a, a + \sigma) | \sigma \in G, a \in D\} = \sum_{\sigma \in G} \sigma(\tilde{D}),$$

and $\sigma(D) \cap \sigma'(D) = \phi$ if $\sigma \neq \sigma'$. Hence $\Psi^{-1}(D)$ consists of 2 connected components and each component is a nonsingular elliptic curve. Thus applying Lemma 2, we obtain

$$\dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\Psi^* D + K_{\tilde{S}})) = 1.$$

Since $\Psi^* D + K_{\tilde{S}} = \Psi^*(D + K_{\bar{S}})$, we can apply Lemma 3a) and obtain

$$\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(2(D + K_{\bar{S}}))) \geqq 1.$$

Thus we complete the case that $\nu_i = 1$ for all i.

Now we assume that $\Sigma_{i=1}^{r} (\nu_i - 1) > 0$. It is obvious that $\nu_i \leqq d$ for all i.

Suppose that there is ν_i such that $\nu_i = d$. Then $\hat{D}$ has a d-ple point p_0. Let $\hat{F}_0$ be the fiber of $\hat{f}$ which contains p_0. Since $(\hat{D}, \hat{F}_0) = d$, we know $\hat{D} \cap \hat{F}_0 = \{p_0\}$, and $\hat{D}$ and $\hat{F}_0$ have no common tangent line at p_0. The quadratic transformation with center p_0 appears in $\mu = \mu_1 \circ \cdots \circ \mu_r$. We denote it by μ_0. Let F'_0 and D' be proper transforms of $\hat{F}_0$ and $\hat{D}$ by μ_0, respectively. Then we have $F'_0 \cap D' = \phi$ and $(F'_0)^2 = -1$. And the proper transform of F'_0 to $\bar{S}$ is also an exceptional curve which does not intersect with D. By Lemma 1, we may assume that there is not such a curve on $\bar{S}$. Thus we assume that $\nu_i < d$ for all i.

If $d = 2$, then we have $\nu_i = 1$ for all i. But this contradicts the assumption. Therefore we assume that $d \geqq 3$.

Suppose that there is ν_i such that $\nu_i = d - 1$. Then $\hat{D}$ has a $(d-1)$-ple point p_0. Let $\hat{F}_0$ be the fiber of $\hat{f}$ which contains p_0. Let μ_0 be the quadratic transformation with center p_0, F'_0 and D' be the proper transforms of $\hat{F}_0$ and $\hat{D}$ by μ_0, respectively. If $\hat{D} \cap \hat{F}_0 - \{p_0\} = \phi$, then $\hat{D}$ and $\hat{F}_0$ have no common tangent line at p_0 and intersect simply at some other point. If $\hat{D} \cap \hat{F}_0 = \{p_0\}$, then D' and F'_0 intersect simply at one point of $\mu_0^{-1}(p_0)$. In each case, we have $(F'_0, D') = 1$ and $(F'_0)^2 = -1$. Thus, there is an exceptional curve E on $\bar{S}$ such that $(E, D) = 1$. By Lemma 1, we may assume that there is no such curves on $\bar{S}$. Hence we reduce the problem to the case that $d \geqq 3$ and $\nu_i \leqq d - 2$ for all i.

Since $D + K_{\bar{S}} \sim \mu^*(\hat{D} + K_{\hat{S}}) - \sum_{i=1}^{r} (\nu_i - 1)E_i$, we have

$$\text{(2)} \qquad \dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(m(D + K_{\bar{S}})))$$

$$\geqq \dim H^0(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) - \sum_{i=1}^{r} (1/2)m(\nu_i - 1)(m(\nu_i - 1) + 1).$$

On the other hand, by the Riemann-Roch theorem we have

$$\text{(3)} \qquad \dim H^0(\hat{S}, \mathcal{O}_{\hat{S}}(m(D + K_{\hat{S}})))$$
$$\geqq (1/2)(m(\hat{D} + K_{\hat{S}}), m\hat{D} + (m-1)K_{\hat{S}}) = m^2(d-1)b - m(m-1)b.$$

Combining (2) and (3), we calculate

$$\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(m(D + K_{\bar{S}})))$$

$$\begin{aligned}
&\geqq m^2(d-1)b - m(m-1)b - m^2 \sum_{i=1}^{r} (1/2)\nu_i(\nu_i - 1) \\
&\quad + m(m-1) \sum_{i=1}^{r} (1/2) \times (\nu_i - 1) \\
&= m(m-1)\left(\sum_{i=1}^{r} (1/2)(\nu_i - 1) - b\right) \\
&= (1/2)m(m-1)\left(\sum_{i=1}^{r} (\nu_i - 1) - (d-1)^{-1} \sum_{i=1}^{r} \nu_i(\nu_i - 1)\right) \\
&= (1/2)m(m-1) \sum_{i=1}^{r} (1 - (\nu_i/(d-1)))(\nu_i - 1) \\
&\geqq (1/2)m(m-1)(d-1)^{-1} \sum_{i=i}^{r} (\nu_i - 1) \\
&\geqq (1/2)m(m-1)(d-1)^{-1}.
\end{aligned}$$

Especially we get $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(2(D + K_{\bar{S}}))) \geqq (d-1)^{-1} > 0$. Therefore we have $\bar{P}_2(S) \geqq 1$. Q.E.D.

§4. The case $\kappa(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = 1$

PROPOSITION 3: *If* $\bar{\kappa}(C_w) \geqq 0$, $\bar{\kappa}(\bar{\Delta}) \geqq 0$, $\kappa(\bar{C}_w) - \infty$ *and* $\kappa(\bar{\Delta}) = 1$, *then* $\bar{p}_g(S) \geqq 1$ *and* $\bar{P}_2 \geqq 2$.

PROOF: Let $D = \Sigma_{i=1}^{r} C_i$ be the irreducible decomposition. From the assumption we know $(\bar{C}_w, D) \geqq 2$. We may assume that all C_i are horizontal with respect to $\bar{f}$. Since $g(C_i) \geqq g(\bar{\Delta}) = q(\bar{S}) \geqq 2$, by virtue of Lemma 2 we have $\bar{p}_g(S) \geqq \Sigma_{i=1}^{r} g(C_i) - q(\bar{S}) \geqq (r-1)q(\bar{S})$. Thus if $r \geqq 2$, we have $\bar{p}_g(S) \geqq 2$. Hence we assume that $r = 1$. Put $\bar{f}|_{C_1} = \psi$. Applying Hurwitz' formula to $\psi: C_1 \to \bar{\Delta}$, we get

$$g(C_1) = (\deg \psi)(q(\bar{S}) - 1) + 1 + (1/2) \sum_{p \in C_1} (e(p) - 1),$$

where $e(p)$ is the ramification index of ψ at p. Since $\deg \psi = (C_1, \bar{C}_w) = (D, \bar{C}_w) \geqq 2$, we see

$$g(C_1) \geqq 2(q(\bar{S}) - 1) + 1 \geqq q(\bar{S}) + 1.$$

Hence $\bar{p}_g(S) \geqq g(C_1) - q(\bar{S}) \geqq 1$, and if $\bar{p}_g(S) = 1$, then we have $q(\bar{S}) = 2$, $\deg \psi = 2$, $\Sigma_{p \in C_1} (e(p) - 1) = 0$ and $g(C_1) = 3$. Thus if $\bar{p}_g(S) = 1$, then $\psi: C_1 \to \bar{\Delta}$ is a 2-sheeted unramified covering. Let $\tilde{S} = \bar{S} \times_{\bar{\Delta}} C_1$ be the fiber product. Let $\Psi: \tilde{S} \to \bar{S}$ and $\tilde{f}: \tilde{S} \to C_1$ denote the first and the second projections respectively. Then Ψ is a 2-sheeted unramified covering and $q(\tilde{S}) = g(C_1) = 3$. Put $\Psi^{-1}(C_1) = \Sigma_{i=1}^{s} C_i'$ where the C_i' are irreducible components. Applying Hurwitz' formula to $\Psi|_{C_i'}: C_i' \to C_1$, we get

$$g(C_i') \geqq (\deg(\Psi|_{C_i'}))(3-1) + 1 = 2(\deg(\Psi|_{C_i'})) + 1.$$

Thus by Lemma 2 we have

$$\dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\Psi^{-1}(C_1) + K_{\tilde{S}})) \geqq \sum_{i=1}^{s} g(C_i') - q(\tilde{S})$$
$$\geqq 2 \sum_{i=1}^{s} \deg(\Psi|_{C_i'}) + s - 3 = 2 \deg \psi + s - 3 = s + 1 \geqq 2.$$

Since $\Psi^*(D + K_{\bar{S}}) \geqq \Psi^{-1}(C_1) + \Psi^* K_{\bar{S}} = \Psi^{-1} C_1 + K_{\tilde{S}}$, applying Lemma 3b), we infer that $\bar{P}_2(S) \geqq 2$. Q.E.D.

REMARK: $\bar{p}_g(S) \geqq 1$ was proved by Miyanishi-Sugie in [12; §2].

§5. The case $\kappa(\bar{C}_w) \geqq 0$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is an irrational ruled surface

PROPOSITION 4: *If $\bar{\kappa}(C_w) \geqq 0$, $\bar{\kappa}(\Delta) \geqq 0$, $\kappa(\bar{C}_w) \geqq 0$, $\kappa(\bar{\Delta}) = -\infty$, $\bar{S}$ is ruled and $q(\bar{S}) \geqq 1$, then $\bar{p}_g(S) \geqq 1$.*

PROOF: Let $\alpha : \bar{S} \to B := \alpha(\bar{S}) \hookrightarrow \mathrm{Alb}(\bar{S})$ be the Albanese mapping of $\bar{S}$, then B is a nonsingular curve of genus $q(\bar{S})$, and the ruling of $\bar{S}$ is given by α. By the assumption, we know that $D = \bar{S} - S$ contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2. For a general point p of $\bar{S}$, the fiber of $\bar{f}$ which contains p is a nonsingular curve of genus greater than 0 which we denote by C_1, and the fiber of α which contains p is a nonsingular rational curve which we denote by C_2. Then $C_1 \neq C_2$ and $C_1 \cap C_2 \neq \phi$, hence $(C_1, C_2) \geqq 1$. Therefore for any $u \in B$ and $w \in \bar{\Delta}$, we have $(\bar{f}^*(w), \alpha^*(u)) \geqq 1$. Especially, F_1 and F_2 contain horizontal components with respect to α which we denote by H_1 and H_2 respectively. Then from Lemma 2 we infer that $\bar{p}_g(S) \geqq g(H_1) + g(H_2) - q(\bar{S}) \geqq q(\bar{S}) \geqq 1$. Q.E.D.

§6. The case $\kappa(\bar{C}_w) = 0$, $\kappa(\bar{\Delta}) = -\infty$, and $\bar{S}$ is rational

PROPOSITION 5: *If $\bar{\kappa}(C_w) \geqq 0$, $\bar{\kappa}(\Delta) \geqq 0$, $\kappa(\bar{C}_w) = 0$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is rational, then $\bar{P}_3(S) \geqq 1$ or $\bar{P}_4(S) \geqq 1$.*

PROOF: By assumption, D contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2. We may assume $D = F_1 + F_2$. We contract all exceptional curves contained in fibers of $\bar{f}$ and then we have the following commutative diagram:

$$\begin{array}{ccc} \bar{S} & \xrightarrow{\ \mu\ } & \hat{S} \\ & {\scriptstyle \bar{f}}\searrow & \downarrow {\scriptstyle \hat{f}} \\ & & \bar{\Delta} \end{array}$$

where $\hat{f} : \hat{S} \to \bar{\Delta}$ is a rational elliptic surface which has no exceptional curves contained in fibers and $\mu : \bar{S} \to \hat{S}$ is composed of a finite number of quadratic transformations. Let $\hat{F}_1$ and $\hat{F}_2$ be fibers of $\hat{f}$ such that $F_1 = \mathrm{supp}\ \mu^*\hat{F}_1$ and $F_2 = \mathrm{supp}\ \mu^*\hat{F}_2$. Then by the canonical bundle formula [10], we know

$$K_{\hat{S}} \sim -\hat{F} + \sum_{\nu} (m_\nu - 1)P_\nu,$$

where the $m_\nu P_\nu$'s are all multiple fibers of $\hat{f}$ and $\hat{F}$ is a general fiber of $\hat{f}$. Since $\hat{S}$ has an exceptional curve E which is horizontal with respect to $\hat{f}$, we have

$$-1 = (E, K_{\hat{S}}) = -(E, \hat{F}) + \sum_\nu (m_\nu - 1)(E, P_\nu)$$
$$= \left(-1 + \sum_\nu (1 - m_\nu^{-1})\right)(E, \hat{F}).$$

Hence $\hat{f}: \hat{S} \to \bar{\Delta}$ has at most one multiple fiber which we denote by $m_0 P_0$. Putting $m_0 = 1$ if $\hat{f}$ is free from multiple fibers, we have

$$K_{\hat{S}} \sim -P_0.$$

Now we use the classification of singular fibers of elliptic surfaces by Kodaira [11]. Since $p_g(\hat{S}) = q(\hat{S}) = 0$, we have

$$(4) \qquad 12 = \sum_b b\nu(\mathrm{I}_b) + \sum_b (6+b)\nu(\mathrm{I}_b^*) + 2\nu(\mathrm{II}) + 10\nu(\mathrm{II}^*)$$
$$+ 9\nu(\mathrm{III}^*) + 4\nu(\mathrm{IV}) + 8\nu(\mathrm{IV}^*),$$

where $\nu(T)$ is the number of the singular fibers of $\hat{S}$ of type T.

LEMMA 4: If $\hat{F}_i$ is a singular fiber of type I_b or I_b^* or II or III or IV, then $|K_{\bar{S}} + 2F_i| \neq \phi$.

PROOF: If $\hat{F}_i$ is of type I_b, by Lemma 2 we have $|K_{\bar{S}} + F_i| \neq \phi$. If $\hat{F}_i$ is of type I_b^*, then 2 supp $\hat{F}_i \geqq \hat{F}_i$ and we may assume $\mu^*\hat{F}_i$ contains no exceptional curves. Hence we have

$$K_{\bar{S}} + 2F_i \geqq \mu^* K_{\hat{S}} + \mu^*\hat{F}_i \geqq 0.$$

If $\hat{F}_i$ is of type II i.e. a rational curve with one cusp, then F_i contains an exceptional curve E and nonsingular rational curves C_1, C_2 and C_3 such that $(C_1, E) = (C_2, E) = (C_3, E) = 1$. Let $\mu': \bar{S} \to \bar{S}'$ be the contraction of E and denote $\mu'_*(C_j)$ by C'_j for $j = 1, 2, 3$. Then we infer that

$$K_{\bar{S}} + 2F_i \geqq K_{\bar{S}} - E + C_1 + C_2 + C_3 + 3E = \mu'^*(K_{\bar{S}'} + C'_1 + C'_2 + C'_3).$$

And by the Riemann-Roch theorem we have

$$\dim H^0(\bar{S}', \mathcal{O}_{\bar{S}'}(K_{\bar{S}'} + C_1' + C_2' + C_3'))$$
$$\geqq (1/2)(K_{\bar{S}'} + C_1' + C_2' + C_3', C_1' + C_2' + C_3') + 1 = 1.$$

Thus we have $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + 2F_i)) \geqq 1$. If $\hat{F}_i$ is of type III i.e. two nonsingular rational curves intersecting at one point with multiplicity two, or type IV i.e. three nonsingular rational curves intersecting one point, then F_i contains also an exceptional curve E and nonsingular rational curves C_1, C_2 and C_3 such that $(C_1, E) = (C_2, E) = (C_3, E) = 1$. Hence similarly we have

$$\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + 2F_i)) \geqq 1. \qquad \text{Q.E.D.}$$

By Lemma 4 we infer that if both $\hat{F}_1$ and $\hat{F}_2$ are of type I_b or I_b^* or II or IV, then $|2(K_{\bar{S}} + F_1 + F_2)| \supset |K_{\bar{S}} + 2F_1| + |K_{\bar{S}} + 2F_2| \neq \phi$, and hence $\bar{P}_2(S) \geqq 1$.

Now we assume that one of $\hat{F}_i$ is of type II* or III* or IV*. By the assumption, the functional invariant of $\hat{S}$ (if $\hat{S}$ has a multiple fiber, the functional invariant of the corresponding elliptic surface free from multiple fibers) is not constant. Therefore we know that $\Sigma_b\ (\nu(\mathrm{I}_b) + \nu(\mathrm{I}_b^*)) \geqq 1$. Hence if $\nu(\mathrm{II}^*) \geqq 1$, then we infer from (4) that $\Sigma_b\ b\nu(\mathrm{I}_b) = 2$ and remaining $\nu(T)$ are 0. Thus one of $\hat{F}_i$ is of type I_b and therefore by Lemma 2 we have $\bar{p}_g(S) \geqq 1$. Thus we assume that one of $\hat{F}_i$ is of type III* or IV*.

Case 1. First we consider the case that one of $\hat{F}_i$ is of type III*. We assume that $\hat{F}_1$ is of type III*. We may assume that $\hat{F}_2$ is not of type I_b by Lemma 2. Then from (4) we know that F_2 is of type II. We put

$$\hat{F}_1 = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + 2\Theta_6 + \Theta_7, (\Theta_0, \Theta_1) = (\Theta_1, \Theta_2)$$
$$= (\Theta_2, \Theta_3) = (\Theta_3, \Theta_5) = (\Theta_3, \Theta_4) = (\Theta_4, \Theta_6) = (\Theta_6, \Theta_7) = 1,$$
$$\hat{F}_2 = C,$$

where the Θ_i are nonsingular rational curves with self-intersection number -2 and C is a rational curve with one cusp. Since $\mu: \bar{S} \to \hat{S}$ gives rise to an isomorphism on a neighborhood of F_1, we have

$$F_1 = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 + \Theta_6 + \Theta_7,$$

where we denote $\mu^*\Theta_i$ by the same letter Θ_i. On the other hand we have

$$\mu^*\hat{F}_2 = C''' + 2E_1'' + 3E_2' + 6E_3,$$

where $\mu = \mu_1 \circ \mu_2 \circ \mu_3$ and C''' is the proper transform of C by μ and E_i are the exceptional curves of the quadratic transformations μ_i and E_1'' and E_2' are the proper transforms of E_1 and E_2 to $\bar{S}$, respectively. We denote the total transform of E_i by the same letter E_i. Then we have

$$F_2 = C''' + E_1'' + E_2' + E_3.$$

Since $m_0P_0 \sim C$, we infer that

$$K_{\bar{S}} + F_2 + E_3 \sim \mu^*K_{\hat{S}} + E_1 + E_2 + E_3 + F_2 + E_3 \sim \mu^*(K_{\hat{S}} + C) \geqq 0.$$

On the other hand,

$$\begin{aligned} 4F_1 &= \mu^*F_1 + 3\Theta_0 + 2\Theta_1 + \Theta_2 + \Theta_4 + 2\Theta_5 + 2\Theta_6 + 3\Theta_7 \\ &\sim \mu^*C + 3\Theta_0 + 2\Theta_1 + \Theta_2 + \Theta_4 + 2\Theta_5 + 2\Theta_6 + 3\Theta_7. \end{aligned}$$

Hence we have

$$\begin{aligned} 4(K_{\bar{S}} + F_1 + F_2) = 4(K_{\bar{S}} + F_2) + 4F_1 &\geqq -4E_3 + \mu^*C \\ &= C''' + 2E_1'' + 3E_2' + 2E_3 \geqq 0. \end{aligned}$$

Therefore we get $\bar{P}_4(S) \geqq 1$.

Case 2. Now we consider the case that one of F_i is of type IV*. We assume F_1 is of type IV*. We may assume F_2 is not of type I_b. Then from (4) we know that F_2 is of type II or III. We put

$$\begin{aligned} \hat{F}_1 &= \Theta_0 + 2\Theta_1 + 3\Theta_3 + 2\Theta_4 + \Theta_5 + \Theta_6, \\ (\Theta_0, \Theta_1) &= (\Theta_1, \Theta_2) = (\Theta_2, \Theta_3) \\ &= (\Theta_2, \Theta_4) = (\Theta_3, \Theta_5) = (\Theta_4, \Theta_6) = 1, \end{aligned}$$

where the Θ_i are nonsingular rational curves with self-intersection number -2. Since μ gives rise to an isomorphism on a neighborhood of F_1, we denote $\mu^*\Theta_i$ by the same letter Θ_i.

Case 2.1: First we consider the case that $\hat{F}_2$ is of type II. Then $\hat{F}_2$ and F_2 are the same as in the case 1. Quite similarly we have

$K_{\bar{S}} + F_2 + E_3 \geqq 0$. On the other hand,

$$\begin{aligned} 3F_1 &= \mu^* \hat{F}_1 + 2\Theta_0 + \Theta_1 + \Theta_3 + \Theta_4 + 2\Theta_5 + 2\Theta_6 \\ &\sim \mu^* C + 2\Theta_0 + \Theta_1 + \Theta_3 + \Theta_4 + 2\Theta_5 + 2\Theta_6. \end{aligned}$$

Hence we have

$$\begin{aligned} 3(K_{\bar{S}} + F_1 + F_2) = 3(K_{\bar{S}} + F_2) + 3F_1 &\geqq -3E_3 + \mu^* C \\ &= C''' + 2E_1'' + 3E_2' + 3E_3 \geqq 0. \end{aligned}$$

Therefore we get $\bar{P}_3(S) \geqq 1$.

Case 2.2: Now we consider the case where $\hat{F}_2$ is of type III. Then we have $\hat{F}_2 = C_1 + C_2$, where C_1 and C_2 are nonsingular rational curves, $C_1 \cap C_2$ is one point and $(C_1, C_2) = 2$. We put $C_1 \cap C_2 = \{p\}$. Let $\mu_1 : \bar{S}_1 \to \hat{S}$ be the quadratic transformation with center p. We denote the exceptional curve of μ_1 by E_1 and the proper transforms of C_1 and C_2 by C_1' and C_2' respectively. Let $\mu_2 : \bar{S}_2 \to \bar{S}_1$ be the quadratic transformation with center $C_1' \cap C_2'$. We denote the exceptional curve of μ_2 by E_2 and the proper transforms of C_1', C_2' and E_1 by C_1'', C_2'' and E_1' respectively. Then we may assume that $\bar{S} = \bar{S}_2$ and $\mu = \mu_1 \circ \mu_2$. And we have

$$\mu^* \hat{F}_2 = C_1'' + C_2'' + 2E_1' + 4E_2.$$

Hence

$$F_2 = C_1'' + C_2'' + E_1' + E_2.$$

Thus we infer that

$$\begin{aligned} K_{\bar{S}} + F_2 &= \mu^* K_{\hat{S}} + E_1 + E_2 + C_1'' + C_2'' + E_1' + E_2 \\ &= \mu^* K_{\hat{S}} + \mu^* \hat{F}_2 - E_2 \geqq -E_2. \end{aligned}$$

Hence we have

$$\begin{aligned} 3(K_{\bar{S}} + F_1 + F_2) = 3(K_{\bar{S}} + F_2) + 3F_1 &\geqq -3E_2 + C_1'' + C_2'' + 2E_1' + 4E_2 \\ &= C_1'' + C_2'' + 2E_1' + E_2 \geqq 0. \end{aligned}$$

Therefore we obtain $\bar{P}_3(S) \geqq 1$. Q.E.D.

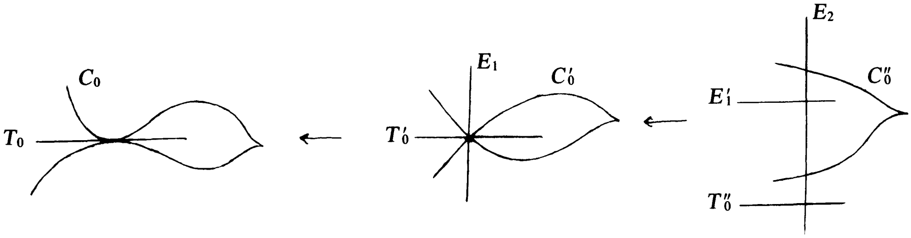

Figure 1.

REMARK: We can construct rational elliptic surfaces free from multiple fiber of case 1, case 2.1 and case 2.2 as follows.

Construction of case 1

Let C_0 be a quartic curve in $\mathbb{P}^2$ defined by

$$Z^2Y^2 - X^4 = X^3Y,$$

where $(X : Y : Z)$ is a homogenuous coodinate in $\mathbb{P}^2$. Then C_0 has a tacnode at $(0:0:1)$ and a cusp at $(0:1:0)$. Let T_0 be the tangent line of C_0 at $(0:0:1)$. We can resolve the singularity of C_0 at $(0:0:1)$ by the quadratic transformations as Figure 1.

Then we know T_0'' is an exceptional curve and $T_0'' \cap C_0'' = \phi$. We contract T_0'' and denote the resulting surface by $\hat{S}_0$. Let $\hat{E}_2$ be the direct image of E_2. Then we have $(\hat{E}_2)^2 = 0$, $(E_1')^2 = -2$ and $C_0'' \sim 4\hat{E}_2 + 2E_1'$. Hence $\hat{S}_0 \cong \Sigma_2$ and $C_0'' \in |2M + 4l|$, where M is the minimal section and l is a fiber. We put $\hat{E}_2 = l_0$ and $C_0'' \cap l_0 = \{p, q\}$. We perform quadratic transformations at p and q as Figure 2.

Let $\mu : \hat{S} \to \hat{S}_0$ be the composition of these quadratic transformations. Let C and Θ_3 be the proper transform of C_0'' and l_0 by μ, respectively. Put $E_{13} - E_{14} = \Theta_0$, $E_{23} - E_{24} = \Theta_7$, $E_{12} - E_{13} = \Theta_1$, $E_{22} - E_{23} = \Theta_6$, $E_{11} - E_{12} = \Theta_2$, $E_{21} - E_{22} = \Theta_4$ and $M = \Theta_5$, then the Θ_i are nonsingular rational curves with self-intersection number -2. Put

$$\hat{F}_1 = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + 2\Theta_6 + \Theta_7, \hat{F}_2 = C.$$

Then $\hat{F}_1 \sim \hat{F}_2$ and $\Phi_{|C|} : \hat{S} \to \mathbb{P}^1$ is a rational elliptic surface with singular fibers of type III* and of type II. Since E_{14} and E_{24} are cross-sections of $\Phi_{|C|}$, $\hat{S}$ is free from multiple fibers. We put $S = \hat{S} - \text{supp } \hat{F}_1 - \hat{F}_2$. Then we infer that $\bar{P}_2(S) = \bar{P}_3(S) = 0$, $\bar{P}_4(S) = 1$ and $\bar{P}_{12}(S) = 2$.

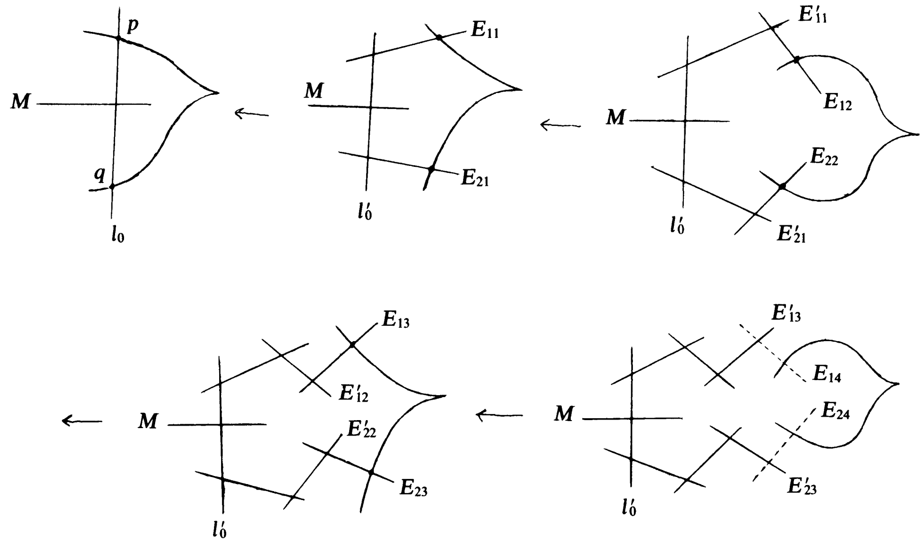

Figure 2.

Construction of case 2.2

Let C be a conic and l_1 a line in $\mathbb{P}^2$ such that C and l_1 intersect at one point q with multiplicity 2. Let l_0 be a line which does not contain q and intersects with C simply at two point p_2 and p_3. Put $l_0 \cap l_1 = p_1$. We perform quadratic transformations at p_1, p_2 and p_3 as follows:

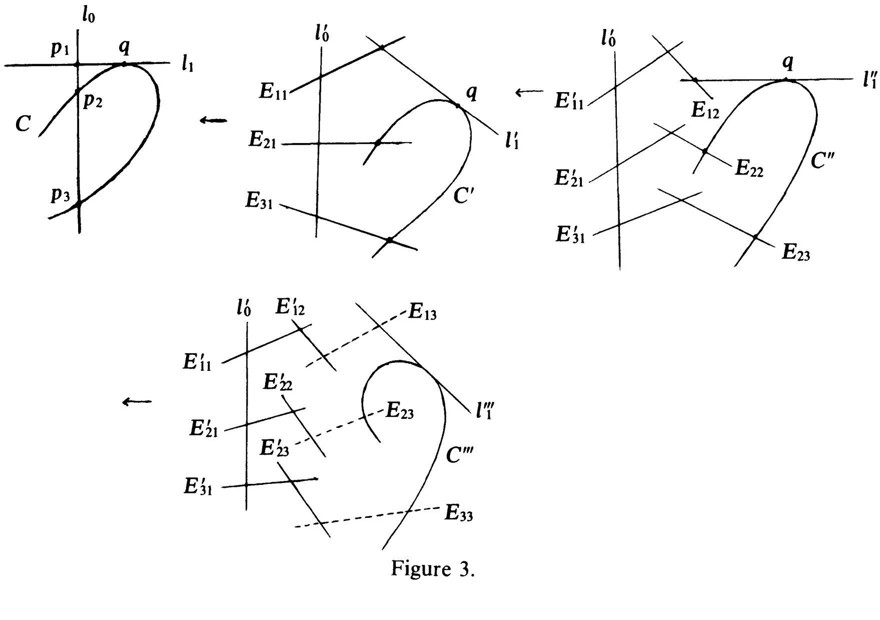

Figure 3.

Let $\mu : \hat{S} \to \mathbb{P}^2$ be the composition of these quadratic transformations. We denote the proper transforms of l_0, l_1 and C by l_0', l_1''' and C'''. We put $l_0' = \Theta_2$, $E_{12} - E_{13} = \Theta_0$, $E_{11} - E_{12} = \Theta_1$, $E_{22} - E_{23} = \Theta_5$, $E_{21} - E_{22} = \Theta_3$, $E_{32} - E_{33} = \Theta_6$ and $E_{31} - E_{32} = \Theta_4$. And we put

$$\hat{F}_1 = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 2\Theta_3 + 2\Theta_4 + \Theta_5 + \Theta_6,\ \hat{F}_2 = l_1''' + C'''.$$

Then $\hat{F}_1 \sim \hat{F}_2$ and $\Phi_{|F_1|} : \hat{S} \to \mathbb{P}^1$ is a rational elliptic surface free from multiple fibers and with singular fibers of type IV* and of type III. Put $S = \hat{S} - \operatorname{supp} \hat{F}_1 - \hat{F}_2$. Then we infer that $\bar{P}_2(S) = 0$, $\bar{P}_3(S) = 1$ and $\bar{P}_{12}(S) = 2$.

Construction of case 2.1

In the above construction of case 2.2, we replace $C + l_1$ by a cubic curve with one cusp and the remaining part goes quite similarly. Then $\bar{P}_2(S) = 0$, $\bar{P}_3(S) = 1$ and $\bar{P}_6(S) = 2$.

§7. The case $\kappa(\bar{C}_w) = 1$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is rational

PROPOSITION 6: If $\kappa(\bar{C}_w) = 1$, $\bar{\kappa}(\Delta) \geqq 0$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is rational, then $\bar{P}_4(S) \geqq 1$ or $\bar{P}_6(S) \geqq 1$.

PROOF: From the addition formula ([6]), we know $\bar{\kappa}(S) \geqq 1$. By assumption D contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2. Hence the intersection matrix of D is not negative definite. Therefore if $\bar{\kappa}(S) = 2$, then by Theorem 3 $\bar{P}_4(S) \geqq 1$ or $\bar{P}_6(S) \geqq 1$. Thus we assume that $\bar{\kappa}(S) = 1$. Then by [7; (2.3)–(2.8)] we have a fibering $g : \bar{S} \to \mathbb{P}^1$ such that a general fiber of $g|_S$ is a $\mathbb{C}^*$ or an elliptic curve. Since the genus of $\bar{C}_w$ is greater than 1, F_1 and F_2 contain horizontal components with respect to g. Therefore a general fiber of $g|_S$ is a $\mathbb{C}^*$. Hence using [7; (2.6) and (2.8)], we reduce the problem to the following:

> Under the condition that $-2 + \Sigma_i\ (1 - m_i^{-1}) > 0$, look for the integer m such that $-2m + \Sigma_i\ [m(1 - m_i^{-1})] \geqq 0$, where m_i are integers or ∞ and [ ] is the integral part.

We infer that for any such m_i, $m = 4$ or 6 is sufficient. Q.E.D.

§8. Proof of Theorem 3

First we note that when we contract an exceptional curve E on $\bar{S}$ such that $(E, K_{\bar{S}}+D) \leqq 0$, the condition on the intersection matrix of D is also preserved. Hence by Lemma 1 we may assume that there is no such exceptional curve on $\bar{S}$. By Proposition 2, 3, 4 we may assume that $\bar{S}$ is rational and by Lemma 2 we may assume that each connected component of D is a tree of nonsingular rational curves.

Let $K_{\bar{S}}+D=(K_{\bar{S}}+D)^{+}+(K_{\bar{S}}+D)^{-}$ be the Zariski decomposition of $\mathbb{Q}$-divisor (c.f. [7]). We put $(K_{\bar{S}}+D)^{-}=\sum_{i=1}^{r} a_iC_i$ where the a_i are rational numbers such that $0<a_i \leqq 1$ and the C_i are irreducible curves. Suppose that $C_i \not\subset D$ for $1 \leqq i \leqq t$ and $C_i \subset D$ for $t+1 \leqq i \leqq r$. If $(K_{\bar{S}}, C_i) \geqq 0$ for $1 \leqq i \leqq t$, then

$$\left(\sum_{i=1}^{t} a_iC_i, \sum_{i=1}^{t} a_iC_i\right)=\left(K_{\bar{S}}+D-(K_{\bar{S}}+D)^{+}-\sum_{i=t+1}^{r} a_iC_i, \sum_{i=1}^{t} a_iC_i\right)$$

$$=\left(K_{\bar{S}}+D-\sum_{i=t+1}^{r} a_iC_i, \sum_{i=1}^{t} a_iC_i\right) \geqq \left(K_{\bar{S}}, \sum_{i=1}^{t} a_iC_i\right) \geqq 0.$$

This contradicts to the negative definiteness of $(K_{\bar{S}}+D)^{-}$. Hence $(K_{\bar{S}}, C_{i_0})<0$ for some i_0 such that $1 \leqq i_0 \leqq t$. Since $(C_{i_0}, C_{i_0})<0$, C_{i_0} is an exceptional curve such that $(K_{\bar{S}}+D, C_{i_0}) \leqq 0$, which is a contradiction. Therefore $C_i \subset D$ for $1 \leqq i \leqq r$. Thus we have $(K_{\bar{S}}+D)^{+}=K_{\bar{S}}+D_m$, where $D_m=D-(K_{\bar{S}}+D)^{-}$ is an effective $\mathbb{Q}$-divisor. Since the intersection matrix of D is not negative definite, there are some irreducible components of D which don't occur in $(K_{\bar{S}}+D)^{-}$. Hence the part of D_m with coefficient 1 is an effective integral divisor which we denote by D_0. Then we have

$$D_m=D_0+\sum_{i=1}^{s} d_iC_i,$$

where the C_i are irreducible curves and the d_i are rational number such that $0<d_i<1$. It is easy to see that if C_i intersects with D_0, $d_i=1-m_i^{-1}$ where m_i is a positive integer. Since $n(K_{\bar{S}}+D) \geqq nK_{\bar{S}}-[-(n-1)D_m]+D_0 \geqq nK_{\bar{S}}+[nD_m]$, we have $\bar{P}_n(S)=\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(nK_{\bar{S}}-[-(n-1)D_m]+D_0))$ for $n \geqq 2$, where $[\quad]$ is the integral part of a $\mathbb{Q}$-divisor. Since $\dim H^2(\bar{S}, \mathcal{O}_{\bar{S}}(nK_{\bar{S}}-[-(n-1)D_m]+D_0))=0$, by the Riemann-Roch theorem we have

$$\bar{P}_n(S) \geqq (1/2)(nK_{\bar{S}}-[-(n-1)D_m], (n-1)K_{\bar{S}}-[-(n-1)D_m])+1$$

$$+(1/2)(D_0, (2n-1)K_{\bar{S}}-2[-(n-1)D_m]+D_0) \text{ for } n \geqq 2.$$

By Kawamata's vanishing theorem [8] we have

$$H^1(\bar{S}, \mathcal{O}_{\bar{S}}(nK_{\bar{S}} - [-(n-1)D_m]))$$
$$\cong H^1(\bar{S}, \mathcal{O}_{\bar{S}}(-(n-1)K_{\bar{S}} + [-(n-1)D_m]) \cong 0$$

for $n \geqq 2$. Hence we have

(5) $\bar{P}_n(S) \geqq (1/2)(D_0, (2n-1)K_{\bar{S}} - 2[-(n-1)D_m] + D_0)$ for $n \geqq 2$.

Suppose that $\bar{P}_2(S) = 0$. Then we have $(D_0, 3K_{\bar{S}} - 2[-D_m] + D_0) \leqq 0$ Hence $3(D_0, K_{\bar{S}} + D_0) + 2(D_0, \Sigma_{i=1}^s C_i) \leqq 0$. Let $D_{0_j} j = 1, \ldots, u$ be connected components of D_0. Then we have $(D_0, K_{\bar{S}} + D_0) = -2u$. Thus we have

$$\left(D_0, \sum_{i=1}^{s} C_i\right) \leqq 3u. \tag{6}$$

On the other hand, from $(K_{\bar{S}} + D_m, D_{0_j}) \geqq 0$ we know

$$(K_{\bar{S}} + D_{0_j}, D_{0_j}) + \left(\sum_{i=1}^{s} d_i C_i, D_{0_j}\right) + \left(\sum_{i \neq j} D_{0_i}, D_{0_j}\right) \geqq 0.$$

Therefore we have

$$\left(\sum_{i=1}^{s} d_i C_i, D_{0_j}\right) \geqq 2 \text{ for } 1 \leqq j \leqq u. \tag{7}$$

Since $0 < d_i < 1$, we obtain

$$\left(\sum_{i=1}^{s} C_i, D_{0_j}\right) \geqq 3 \text{ for } 1 \leqq j \leqq u. \tag{8}$$

From (6) and (8) we have

$$\left(\sum_{i=1}^{s} C_i, D_{0_j}\right) = 3 \text{ for } 1 \leqq j \leqq u. \tag{9}$$

Since every connected components of D is a tree of nonsingular rational curves, we know from (9) that 3 irreducible components of $\Sigma_{i=1}^s C_i$ intersect with D_{0_j} for $1 \leqq j \leqq u$. We denote the coefficients of these 3 components in D_m by $1 - a_j^{-1}$, $1 - b_j^{-1}$ and $1 - c_j^{-1}$, where a_j, b_j and c_j are positive integers such that $a_j \leqq b_j \leqq c_j$.

Suppose moreover that $\bar{P}_3(S)=0$. Then from (5) we infer that

$$-\sum_{j=1}^{u}([-2(1-a_j^{-1})]+[-2(1-b_j^{-1})]+[-2(1-c_j^{-1})])\leqq 5u.$$

Hence for some j_0 we have

$$-[-2(1-a_{j_0}^{-1})]-[-2(1-b_{j_0}^{-1})]-[-2(1-c_{j_0}^{-1})])\leqq 5$$

Since we have $a_j^{-1}+b_j^{-1}+c_j^{-1}\leqq 1$ from (7), we get $a_{j_0}=2$, $b_{j_0}\geqq 3$ and $c_{j_0}\geqq 3$, and therefore we have

$$-[-2(1-a_{j_0}^{-1})]-[-2(1-b_{j_0}^{-1})]-[-2(1-c_{j_0}^{-1})]=5.$$

Hence we have

$$-\sum_{j\neq j_0}([-2(1-a_j^{-1})]+[-2(1-b_j^{-1})]+[-2(1-c_j^{-1})])\leqq 5(u-1).$$

Thus we deduce that

$$a_j=2,\, b_j\geqq 3 \text{ and } c_j\geqq 3 \text{ for } 1\leqq j\leqq u.$$

Suppose moreover that $\bar{P}_4(S)=0$. Then we can deduce by similar way that $b_j=3$ and $c_j\geqq 6$ for $1\leqq j\leqq u$. Then we have

$$\begin{aligned}\bar{P}_6(S)&\geqq(1/2)(D_0,\,11K_{\bar{S}}-2[-D_m]+D_0)\\&=(11/2)(D_0,\,K_{\bar{S}}+D_0)+\sum_{j=1}^{u}(3+4-[-5(1-c_j^{-1})])\\&=-11u+\sum_{j=1}^{u}(3+4+5)=u\geqq 1.\end{aligned}$$

Q.E.D.

§9. Conclusion

By Proposition $1\sim 6$, we complete the proof of Theorem 1.

Proof of Theorem 2: Let $\alpha_S: S\to B\subset\mathscr{A}_S$ be the quasi-Albanese mapping where B is the closure of $\alpha_S(S)$ in the quasi-Albanese variety $\mathscr{A}_S$. Then by the assumption we may assume that dim $B=1$. Then by the property of the quasi-Albanese mapping ([3]), $\bar{\kappa}(B)\geqq 0$

and a general fiber of α_S is irreducible. Since $\bar{\kappa}(S) \geqq 0$, we have $\bar{\kappa}$ (a general fiber of $\alpha_S) \geqq 0$. Thus we can apply Theorem 1 to $\alpha_S : S \to B$.

Q.E.D.

PROOF OF COROLLARY TO THEOREM 1: We have a fibering φ: $S \to \mathbb{C}^*$. Let C_u be the affine curve defined by $\varphi = u$. Then by Theorem 1 we get $\bar{\kappa}(C_u) = -\infty$ for a general $u \in \mathbb{C}^*$. Since C_u is affine, we have $C_u \cong \mathbb{A}^1$. Thus by [4; Theorem 9.7] we complete the proof. Q.E.D.

REMARK: We can prove more sharpened result apart from Theorem 1 as follows. But to do this the deeper result [1] is necessary.

PROPOSITION 7: Let $S = \mathbb{A}^2 - V(\varphi)$ where $\varphi \in \mathbb{C}[x, y]$ and φ is irreducible. If $\bar{P}_2(S) = 0$, then $S \cong \mathbb{A}^1 \times \mathbb{C}^*$.

PROOF: Let C be the closure of $V(\varphi)$ in $\mathbb{P}^2$ and $H_\infty = \mathbb{P}^2 - \mathbb{A}^2$. Then $S = \mathbb{P}^2 - (C \cup H_\infty)$. We put $C \cup H_\infty = D$. If there are at least two points at which D is not normal crossing, it is easy to see that $\bar{P}_2(S) \geqq 1$. Hence C has at most one cusp. We may assume deg $C \geqq 2$. Since $\bar{p}_g(S) = 0$, $C \cap H_\infty$ is one point, C is rational and C has only cusp singularity. Put $C \cap H_\infty = \{p\}$. If p is not the cusp of C, then p and the cusp of C are two points at which D is not normal crossing. Thus C has one cusp at p and so $V(\varphi) = C - \{p\} \cong \mathbb{A}^1$. Therefore we can apply [1] and complete the proof. Q.E.D.

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