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Compositio Mathematica, tome 43, nº 3 (1981), p. 331-341
<http://www.numdam.org/item?id=CM_1981__43_3_331_0>

COMPOSITIO MATHEMATICA, Vol. 43, Fasc. 3, 1981, pag. 331–341

THE DOUBLE BAR AND COBAR CONSTRUCTIONS

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For the homology of a connected loop space $\Omega|X|$, Adams and Eilenberg-Moore found natural isomorphisms

$$H_*(\underline{\Omega} C_* X) \cong H_*(\Omega|X|),$$

$$H^*(\underline{B} C^* X) \cong H^*(\Omega|X|).$$

C_*X and C^*X are the normalized chains and cochains of the simplicial set X respectively. The functor $\underline{\Omega}$ is the cobar and $\underline{B}$ is the bar construction. We describe in this paper explicit mappings

$$\underline{\Delta} : \underline{\Omega} C_* X \longrightarrow \underline{\Omega} C_* X \otimes \underline{\Omega} C_* X$$

$$\underline{\mu} : \underline{B} C^* X \otimes \underline{B} C^* X \longrightarrow \underline{B} C^* X$$

which induce the diagonal on $H_*(\Omega|X|)$ and the cup product on $H^*(\Omega|X|)$. $\underline{\Delta}$ is a homomorphism of algebras and an associative diagonal for $\underline{\Omega} C_*$, $\underline{\mu}$ is a homomorphism of coalgebras and an associative multiplication for $\underline{B} C^*$. To construct such mappings is an old problem brought up for example in [1], [6] or [13]. By use of $\underline{\Delta}$ and $\underline{\mu}$ we can form the double constructions, which yield natural isomorphisms

$$H_*(\underline{\Omega}\,\underline{\Omega} C_* X) \cong H_*(\Omega\Omega|X|)$$

$$H^*(\underline{B}\,\underline{B} C^* X) \cong H^*(\Omega\Omega|X|)$$

for a connected double loop space. However there is no appropriate

0010-437X/81/060331-11$00.20/0

diagonal on $\underline{\Omega\Omega} C_* X$ and therefore further iteration is not possible. This answers the first of the two points made by Adams in the introduction of [1].

The diagonal on $\underline{\Omega} C_* X$ is also of special interest since it permits us to calculate the primitive elements in $H_*(\Omega|X|)$ over the integers. Over the rationals we thus can combine Adams' isomorphism and the Milnor-Moore theorem [10] to derive a combinatorial formula for the rational homotopy groups $\pi_*(|X|) \otimes \mathbb{Q}$. This is totally different from Quillen's and Sullivan's approach [4] and seems to be the easiest. For a finite simplicial set X the formula is of finite type and thus available for computations with computers. Such a finite combinatorial description is a classical aim of algebraic topology. The geometric background of the algebraic constructions here is discussed in [3].

§0. The algebraic bar and cobar constructions

We shall use the following notions of algebra and coalgebra. Let R be a fixed principal ideal domain of coefficients. A coalgebra C is a differential graded R-module C with an associative diagonal $\Delta : C \to C \otimes C$, a counit $1 : C \to R$ and a coaugmentation $\eta : R \to C$. An algebra A is a differential graded R-module A with an associative multiplication $\mu : A \otimes A \to A$ a unit $1 : R \to A$ and an augmentation $\epsilon : A \to R$, see [7]. All differentials have degree -1. An R-module V is positive if $V_n = 0$ for $n < 0$ and negative if $V_n = 0$ for $n > 0$. V is connected if $V_0 = 0$. For a connected V, which is positive or negative,

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} \tag{0.1}$$

has a canonical multiplication μ and a canonical diagonal Δ with

$$\mu(a_1 \otimes \cdots \otimes a_r, a_{r+1} \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_n$$

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{r=0}^{n} (a_1 \otimes \cdots \otimes a_r) \otimes (a_{r+1} \otimes \cdots \otimes a_n).$$

(0.2) Let C be a coalgebra, which is positive and connected, and let $\tilde{\Delta} : \tilde{C} \to \tilde{C} \otimes \tilde{C}$ be its reduced diagonal, $\tilde{C} = C/C_0$. The *cobar construction* $\underline{\Omega} C$ is the free algebra $(T(s^{-1}\tilde{C}), d_\Omega)$ with the differential d_Ω determined by the restriction

$$d_\Omega i^1 = -i^1(s^{-1} ds) + i^2(s^{-1} \otimes s^{-1})\tilde{\Delta} s$$

where $i^n : V^{\otimes n} \to T(V)$ is the inclusion. s denotes the suspension of graded modules, $(sV)_n = V_{n-1}$, and a switch with s involves an appropriate sign.

(0.3) Let A be an algebra, which is positive, or negative and connected, and let $\bar{\mu} : \bar{A} \otimes \bar{A} \to \bar{A}$ be the restriction of its multiplication μ to the augmentational ideal $\bar{A} = \ker \epsilon$. The (normalized) *bar construction* $\underline{B}A$ is the free coalgebra $(T(s\bar{A}), d_B)$ with the differential d_B determined by its component

$$p^1 d_B = -(sds^{-1})p^1 + s\bar{\mu}(s^{-1} \otimes s^{-1})p^2$$

where $p^n : T(V) \to V^{\otimes n}$ is the projection. see [12].

§1. A diagonal for the cobar construction

In this section coalgebras and algebras are positive and coalgebras are also connected. Let Δ^* be the simplicial category with objects $\Delta(n) = \{0, \ldots, n\}$. For a simplicial set $X : \Delta^* \to \mathrm{Ens}$ write

$$\sigma(a_0, \ldots, a_m) = i_a^*(\sigma) \text{ for } \sigma \in X_n = X(\Delta(n))$$

where $i_a : \Delta(m) \to \Delta(n)$ is the injective monotone function with image $a = \{a_0 < \cdots < a_m\}$. If $X_0 = *$ is a point, the normalised chain complex C_*X is a coalgebra by virtue of the Alexander-Whitney diagonal

$$\Delta : C_*X \longrightarrow C_*X \otimes C_*X \tag{1.1}$$

$$\Delta(\sigma) = \sum_{i=0}^{n} \sigma(0, \ldots, i) \otimes \sigma(i, \ldots, n).$$

A degenerate σ represents the zero element in C_*X. C_*X is also the cellular chain complex of the realisation $|X|$ which is a CW-complex. Now we assume that $|X|$ has trivial 1-skeleton $*$. Adams' result in [1] is a natural isomorphism

$$\phi_* : H_*(\underline{\Omega} C_*X) \cong H_*(\Omega|X|) \tag{1.2}$$

of Pontryagin algebras. $\underline{\Omega} C_*X$ is the cobar construction on the coalgebra (C_*X, Δ). In fact it can be regarded as the computation of the Adams-Hilton construction [2] for the CW-complex $|X|$. The algebra $\underline{\Omega} C_*X$ is the tensor algebra $T(s^{-1}\tilde{C}_*X)$ on the desuspension of $\tilde{C}_*(X) = C_*(X)/C_*(*)$. We introduce a diagonal

$$(1.3) \qquad \underline{\Delta} : \underline{\Omega} C_* X \longrightarrow \underline{\Omega} C_* X \otimes \underline{\Omega} C_* X$$

as follows. For a subset $b = \{b_1 < \cdots < b_r\}$ of $\overline{n-1} = \{1, \ldots, n-1\}$ let $\epsilon_{a,b}$ be the shuffle sign of the partition (a, b) with $a = \overline{n-1} - b$. We use the notation

$$[\sigma_1 | \ldots | \sigma_r] = s^{-1}\sigma_{i_1} \otimes \cdots \otimes s^{-1}\sigma_{i_k} \in T(s^{-1}\tilde{C}_* X)$$

where $\sigma_i \in X_{n_i}$ and where $i_1, \ldots, i_k$ are exactly those indices $i \in \{1, \ldots, r\}$ with $n_i > 1$. Now for $\sigma \in X_n$ we define

$$\underline{\Delta}[\sigma] = \sum_{b \subset \overline{n-1}} \epsilon_{a,b} [\sigma(0, \ldots, b_1) | \sigma(b_1, \ldots, b_2) | \ldots | \sigma(b_r, \ldots, n)] \otimes [\sigma(0, b, n)]$$

where $\sigma(0, b, n) = \sigma(0, b_1, b_2, \ldots, b_r, n)$. The sum is taken over all subsets $b = \{b_1, \ldots, b_r\}$, $r \geq 0$, of $\overline{n-1}$. Thus the indices $b = \emptyset$ and $b = \overline{n-1}$ yield the summands $[\sigma] \otimes 1$ or $1 \otimes [\sigma]$ respectively. The formula determines $\underline{\Delta}$ on all generators of the algebra $\underline{\Omega} C_* X$. Extend $\underline{\Delta}$ as an algebra homomorphism. (We make use of the convention that the tensor product $A \otimes A'$ of algebras is an algebra by means of the multiplication $(\mu \otimes \mu')(1_A \otimes T \otimes 1_{A'})$ where T is the switching homomorphism with $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$). One can check that $\underline{\Delta}$ provides $\underline{\Omega} C_* X$ with a coalgebra structure. $\underline{\Delta}$ is in fact a geometric diagonal, that is compare [3]:

(1.4) THEOREM: *Using Adams' isomorphism* (1.2) *the diagram*

$$\begin{array}{ccc} H_*(\underline{\Omega} C_* X) & \xrightarrow{\underline{\Delta}_*} & H_*(\underline{\Omega} C_* X \otimes \underline{\Omega} C_* X) \\ \cong \big\downarrow \phi_* & & \cong \big\downarrow (\phi \otimes \phi)_* \\ H_*(\Omega |X|) & \xrightarrow[D_*]{} & H_*(\Omega |X| \times \Omega |X|) \end{array}$$

commutes. D is the diagonal for the loop space $\Omega |X|$, *that is* $D(y) = (y, y)$.

By the Milnor-Moore theorem [10] we now obtain a purely algebraic description of the rational homotopy groups of a simplicial set X with trivial 1-skeleton $|X|^1 = *$.

(1.5) COROLLARY: *The Hurewicz map determines a natural* *isomorphism*

$$\pi_*(\Omega|X|)\otimes\mathbb{Q} = P(H_*(\underline{\Omega}C_*X)\otimes\mathbb{Q}, \underline{\Delta}_*)$$

of graded Lie algebras.

P denotes the primitive elements with respect to $\underline{\Delta}_*$, that is the kernel of the reduced diagonal $\underline{\Delta}_*:\tilde{H}\to\tilde{H}\otimes\tilde{H}$ with $\tilde{H}=\tilde{H}_*(\underline{\Omega}C_*X)\otimes\mathbb{Q}$.

If X is a finite simplicial set $\underline{\Omega}C_*X$ is finite dimensional in each degree and thus $\underline{\Delta}_*$ can be effectively calculated, see the examples below. A further advantage of (1.5) over Sullivan's approach [4], (which makes use of the rational simplicial de Rham algebra and is not of finite type), is that (1.4) allows us to compare the primitive elements over $\mathbb{Z}$ with rational homotopy groups. Moreover applying the cobar construction on $(\underline{\Omega}C_*X, \underline{\Delta})$ we obtain

(1.6) THEOREM: *If $|X|$ has trivial 2-skeleton, there is a natural isomorphism of algebras*

$$H_*(\underline{\Omega}\,\underline{\Omega}C_*X)\cong H_*(\Omega\Omega|X|).$$

We remark here that the method cannot be extended to iterated loop spaces $\Omega^r|X|$ for $r>2$, since it is impossible to construct a 'nice' diagonal on $\underline{\Omega}\,\underline{\Omega}C\,X$, compare [3].

(1.7) EXAMPLE: Let X be a simplicial complex and let $Y\subset X$ be a subcomplex containing the 1-skeleton of X. The quotient space X/Y is the realization of a simplicial set and (1.5) yields a formula for the rational homotopy groups $\pi_*(X/Y)\otimes\mathbb{Q}$. By (1.6) we have a formula for $H_*(\Omega(X/Y))$. An especially easy example is the computation for a wedge of spheres

$$W = \vee\,\Delta^{n_i}/\partial\Delta^{n_i},\ n_i\geq 2.$$

In this case the cobar construction $\underline{\Omega}C_*W = T(s^{-1}\tilde{H}_*W)$ has trivial differential and the diagonal $\underline{\Delta}$ in (1.3) has primitive values on generators $s^{-1}x$, $x\in\tilde{H}_*W$, that is

$$\underline{\Delta}s^{-1}x = 1\otimes(s^{-1}x)+(s^{-1}x)\otimes 1.$$

Thus we obtain from (1.5) a well known theorem of Hilton:

The rational homotopy group $\pi_*(\Omega W)\otimes\mathbb{Q}$ of a wedge of spheres W is the free Lie algebra generated by $s^{-1}\tilde{H}_*(W)\otimes\mathbb{Q}$.

Furthermore we obtain from (1.6) an *isomorphism of Milgram* in [9]

$$H_*(\Omega\Omega W) = H_*(\underline{\Omega}(T(s^{-1}\tilde{H}_*W), \underline{\Delta})).$$

(1.8) EXAMPLE: Let P_N be the real projective N-space. The truncated spaces

$$P_{R,N} = P_N/P_{R-1} = e^0 \bigcup e^R \bigcup e^{R+1} \bigcup \dots \bigcup e^N$$

are CW-complexes with exactly one cell e^n in each dimension $R \leq n \leq N$. We obtain P_∞ as the realization of the geometric bar construction $\underline{B}(\mathbb{Z}_2)$ which is a simplicial set. P_N is its N-dimensional skeleton and the cell e^n is given by the single non degenerate element $x_n = (1, \dots, 1) \in (\mathbb{Z}_2)^n$. Thus $C_*P_{R,N}$ is a free chain complex generated by $x_0 = 1$ and $x_R, \dots, x_N$ with degree $|x_n| = n$. The boundary is

$$dx_n = \sum_{i=0}^{n} (-1)^i d_i^* x_n = (1 + (-1)^n)x_{n-1}, \; n > R,$$

since $d_i^* x_n = \mu_i(x_n)$ is degenerate for $0 < i < n$. By use of (1.1) the diagonal on $C_*P_{R,N}$ is

$$\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1 + \sum_{i=R}^{N-R} x_i \otimes x_{n-i}.$$

Let $y_n = s^{-1}x_n$ be the desuspension of the element x_n and let

$$T_{R,N} = T(y_R, \dots, y_N), \; R \geq 2,$$

be the free ring generated by $y_R, \dots, y_N$. Thus $T_{R,N} = \underline{\Omega}(C_*P_{R,N}, \Delta)$ is the underlying algebra of the cobar construction for $P_{R,N}$, see (0.2). The differential is given on generators by

$$dy_n = -(1 + (-1)^n)y_{n-1} + \sum_{i=R}^{n-R} (-1)^i y_i y_{n-i}.$$

By use of (1.3) we even have a diagonal

$$\underline{\Delta} : T_{R,N} \longrightarrow T_{R,N} \otimes T_{R,N}$$

which is an algebra homomorphism defined on generators by

$$\underline{\Delta}(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{\substack{i_1+\cdots+i_j+k=n+j \\ i_1,\ldots,i_j \text{ odd} \geq R \\ k \geq R,\, k \geq j \geq 1}} \binom{k}{j} y_{i_1} \ldots y_{i_j} \otimes y_k$$

$\binom{k}{j}$ denotes the binomial coefficient.

(1.9) THEOREM: *$\underline{\Delta}$ is a chain map which induces the diagonal D_* on $H_*(T_{R,N}, d) \cong H_*(\Omega P_{R,N})$. For $R \geq 3$ $\underline{\Delta}$ provides $(T_{R,N}, d)$ with a coalgebra structure and we have an isomorphism*

$$H_*(\underline{\Omega}(T_{R,N}, \underline{\Delta})) \cong H_*(\Omega\Omega P_{R,N})$$

of algebras.

This seems to be the first example in literature computing the homology of a double loop space $\Omega^2 X$ where X is no double suspension.

Since our construction is adapted to the cell structure of $P_{R,N}$ we can identify the Hopf maps. Let

$$\tau_i : \pi_N(P_{R,N}) \cong \pi_{N-i}(\Omega^i P_{R,N}) \longrightarrow H_{N-i}(\Omega^i P_{R,N})$$

be the composition of the adjunction isomorphism and the Hurewicz map and let $h_N : S^N \to P_N \to P_{R,N}$ be the Hopf map, that is the attaching map of the $(N+1)$-cell in $P_{R,N+1}$. We derive from (1.6). *The homology classes $\tau_i(h_N)$, $i = 0, 1, 2$, are represented by cycles as follows:*

$$\begin{array}{llll} \tau_0(h_N) \text{ by } & dx_{N+1} & \text{in} & C_* P_{R,N} \\ \tau_1(h_N) \text{ by } & dy_{N+1} & \text{in} & (T_{R,N}, d), \\ \tau_2(h_N) \text{ by } & ds^{-1}y_{N+1} & \text{in} & \underline{\Omega}(T_{R,N}, \underline{\Delta}). \end{array}$$

Clearly similar calculations are available for all discrete abelian groups H instead of $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

PROOF: For convenience of the reader we recall the classifying space construction $\underline{B}$. For a topological monoid H the simplicial space

$$\underline{B}H : \Delta^* \longrightarrow Top$$

maps $\Delta(n)$ to the n-fold product H^n and is defined on generating morphisms d_i, s_i in Δ^* by

$$d_i^* = \begin{cases} pr_1 : H^n \longrightarrow H^{n-1}, & i = 0 \\ \mu_i : \quad\quad\quad\quad\quad\quad , & i = 1, \ldots, n-1 \\ pr_n : \quad\quad\quad\quad\quad\quad , & i = n \end{cases}$$

$$s_i^* = j_{i+1} : H^{n-1} \to H^n, \qquad i = 0, \ldots, n-1$$

where pr_i is the projection omitting the i-th coordinate and j_i is the inclusion filling in $*$ as the i-th coordinate of the tuple. μ_i is given by the multiplication μ on H, that is

$$\mu_i = 1 \times \mu \times 1 : H^{i-1} \times H^2 \times H^{n-i-1} \to H^{i-1} \times H \times H^{n-i-1}$$

(As usual $d_i : \Delta(n-1) \to \Delta(n)$ is the injective map with image $\Delta(n) - \{i\}$ and $s_i : \Delta(n) \to \Delta(n-1)$ is the surjective map with $s_i(i) = s_i(i+1)$). (If H is path-connected and well-pointed it is well known that the realization $|\underline{B}H|$ is a classifying space for H. This is a 'pointed' variant of the original Dold-Lashof result, see [5], [8].) We now consider $\underline{B}(\mathbb{Z}_2)$.

For $\sigma = x_n = (1, \ldots, 1) \in \mathbb{Z}_2^n$ and $b = \{b_1 < \cdots < b_r\} \subset \overline{n-1}$ we have by definition of d_i^*

$$\sigma(0, b, n) = (b_1, b_2 - b_1, \ldots, b_r - b_{r-1}, n - b_r) \in (\mathbb{Z}/2\mathbb{Z})^{r+1}.$$

This element is non degenerate only if all cooordinates are odd and in this case $\sigma(0, b, n) = x_{r+1}$. Furthermore we have $\sigma(b_i, \ldots, b_{i+1}) = x_{b_i+1-b_i}$. Now let $i_1' = b_1$, $i_s' = b_s - b_{s-1}$ for $s = 2, \ldots, r$ and $i_{r+1}' = n - b_r$. Thus all i_s' are odd and clearly $i_1' + \cdots + i_{r+1}' = n$. The shuffle sign is $\epsilon_{a,b} = 1$ and thus we obtain the above formula for $\underline{\Delta}(y_n)$ from (1.3).

□

§2. A multiplication for the bar construction

In this section coalgebras and algebras are negative and algebras are also connected. For a graded R-module V let $V^{\#} = \mathrm{Hom}(V, R)$ be its dual with $(V^{\#})_{-n} = \mathrm{Hom}(V_n, R)$. We have the canonical map

$$\psi : V^{\#} \otimes W^{\#} \to (V \otimes W)^{\#} \quad \text{with} \quad \psi(\xi \otimes \eta)(x \otimes y) = \xi(x) \cdot \eta(y).$$

Let X be a simplicial set with $X_0 = *$. We can dualize the results of &1 as follows. The diagonal (1.1) induces the multiplication on the cochains $C^*X = (C_*X)^{\#}$

$$\mu = \Delta^{\#}\psi : C^*X \otimes C^*X \longrightarrow C^*X \tag{2.1}$$

which provides C^*X with an algebra structure. Its homology is the cohomology ring of $|X|$.

Now assume that $|X|$ has trivial 1-skeleton. For the bar construction on (C^*X, μ) Eilenberg and Moore (compare [17] and [15]) obtained the following result which is dual to (1.2): There is a natural isomorphism

$$\phi^* : H^*(\underline{B}C^*X) \cong H^*(\Omega|X|) \tag{2.2}$$

of cohomology groups so that for the loop addition map m on $\Omega|X|$ the diagram

$$\begin{array}{ccc} H^*(\underline{B}C^*X) & \xrightarrow{\Delta^*} & H^*(\underline{B}C^*X \otimes \underline{B}C^*X) \\ \wr\| \ \phi^* & & \wr\| \ (\phi \otimes \phi)^* \\ H^*(\Omega|X|) & \xrightarrow{m^*} & H^*(\Omega|X| \times \Omega|X|) \end{array} \tag{2.3}$$

commutes. Thus with coefficients in a field, ϕ^* is an isomorphism of coalgebras. In (2.3) we suppressed an Eilenberg Zilber map from notation.

We now determine the cup product ring structure by introducing a multiplication on $\underline{B}C^*X$. ψ above yields mappings $\psi : (V^{\#})^{\otimes n} \to (V^{\otimes n})^{\#}$ and

$$\psi : \underline{B}C^*X \longrightarrow (\underline{\Omega}C_*X)^{\#}. \tag{2.4}$$

ψ is compatible with the differentials of §0 and induces isomorphisms in homology. Consider the commutative diagram

(2.5)
$$
\begin{tikzcd}
s\bar{C}^*X \arrow[d, "p^1"] \arrow[r, equal] & (s^{-1}\tilde{C}_*X)^{\#} \\
\underline{B}C^*X \arrow[r, "\psi"] \arrow[d, shift right, "\Delta"'] & (\Omega C_*X)^{\#} \arrow[u, "i^{1\#}"'] \arrow[d, shift right, "\mu^{\#}"'] \\
\underline{B}C^*X \otimes \underline{B}C^*X \arrow[u, shift right, "\underline{\mu}"'] \arrow[r, "\psi"] \arrow[dr, "\psi\otimes\psi"'] & (\underline{\Omega}C_*X \otimes \underline{\Omega}C_*X)^{\#} \arrow[u, shift right, "\underline{\Delta}^{\#}"'] \\
& (\underline{\Omega}C_*X)^{\#} \otimes (\underline{\Omega}C_*X)^{\#} \arrow[u, "\psi"']
\end{tikzcd}
$$

with p^1 and i^1 as in (0.3) and (0.2) and with $\underline{\psi}$ defined in the same way as ψ in (2.4). ψ and $\underline{\psi}$ induce isomorphisms in homology. One can check that $\mu^{\#}\psi = \underline{\psi}\Delta$, so that the commutativity of (2.3) follows. We define the multiplication

(2.6)
$$\underline{\mu} : \underline{B}C^*X \otimes \underline{B}C^*X \longrightarrow \underline{B}C^*X$$

to be the unique coalgebra map with component $p^1\underline{\mu} = i^{1\#}\underline{\Delta}^{\#}\underline{\psi}$. (We make use of the convention that the tensor product $C \otimes C'$ of coalgebras is a coalgebra by means of the diagonal $(1_C \otimes T \otimes 1_{C'})(\Delta \otimes \Delta')$ where T is the switching homomorphism.) We see that $\underline{\Delta}^{\#}\underline{\psi} = \psi\underline{\mu}$. Therefore we get the following result dual to (1.4) from the proof of (1.4).

(2.7) THEOREM: *Using the isomorphism* (2.2) *of Eilenberg-Moore the diagram*

$$
\begin{tikzcd}
H^*(\underline{B}C^*X) \otimes H^*(\underline{B}C^*X) \arrow[r, phantom, "\cong"] \arrow[d, "x"] & H^*(\Omega|X|) \otimes H^*(\Omega|X|) \arrow[dd, "\cup"] \\
H^*(\underline{B}C^*X \otimes \underline{B}C^*X) \arrow[d, "\underline{\mu}^*"] & \\
H^*(\underline{B}C^*X) \arrow[r, phantom, "\cong"] & H^*(\Omega|X|)
\end{tikzcd}
$$

commutes, where $x([\xi]\otimes[\eta]) = [\xi \otimes \eta]$ *and where* $\cup$ *is the cup product.*

Applying the bar construction again we get dually to (1.6).

(2.8) THEOREM: *If X has trivial 2-skeleton we have a natural isomorphism of cohomology groups*

$$H^*(\underline{BB}C^*X) \cong H^*(\Omega\Omega|X|)$$

As in (2.3) the loop addition on $H^*(\Omega\Omega|X|)$ is given by the diagonal on $\underline{BB}C^*X$. Thus with coefficients in a field, this is an isomorphism of coalgebras.

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(Oblatum 26-VI-1980 & 12-XI-1980)

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