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**ON SINGULAR COMPLEX SURFACES WITH VANISHING
GEOMETRIC GENUS, AND PARARATIONAL SINGULARITIES***

Lawrence Brenton

Let X be a normal compact two-dimensional complex analytic space with (necessarily isolated) singular points x_1, \dots, x_r . We want to examine the relationship between global properties of X and local properties of the germs $(x_i, \mathcal{O}_{X, x_i})$. A standard technique for doing this is to resolve the singularities and try to lift the problem under consideration to the non-singular model, where the powerful machinery of the theory of compact complex manifolds can be brought into play. In particular this idea has proved quite successful in studying the classical analytic invariants $q = \dim H^1(X, \mathcal{O}_X)$, the irregularity, and $p_g = \dim H^2(X, \mathcal{O}_X)$, the geometric genus. (Cf. Artin [2], Laufer [22] for local versions of these ideas.)

Let $\pi: \tilde{X} \rightarrow X$ be a resolution of the singularities of X with exceptional set $E = \pi^{-1}(\{x_1, \dots, x_r\})$. Then the Leray spectral sequence shows that for $\chi = 1 - q + p_g$ the Euler characteristic, $\chi(\mathcal{O}_X) - \chi(\mathcal{O}_{\tilde{X}}) = \sum_{i=1}^r \dim(R^1 \pi_* \mathcal{O}_{\tilde{X}, x_i})$, for $R^1 \pi_* \mathcal{O}_{\tilde{X}}$ the first derived sheaf of $\mathcal{O}_{\tilde{X}}$ via the map π . That is to say, in passing from X to \tilde{X} , q goes up, p_g goes down, and the total change in χ is completely determined by the analytic structure of the germs of the singular points. However, whether in a particular example it is q or p_g (or both) that changes depends not only on the singularities but on global properties of X . (To see this, let \tilde{X} be the space obtained from the complex projective plane \mathbb{P}^2 by blowing up 10 points on a non-singular cubic curve Γ , and let X be the singular space obtained from \tilde{X} by collapsing the proper

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transform $\tilde{\Gamma}$ to Γ of a point. Let \tilde{Y} be the compact surface obtained by adjoining the infinity section to the total space of the normal line bundle on $\tilde{\Gamma}$ to the embedding of $\tilde{\Gamma}$ in \tilde{X} , and let Y be the space resulting from blowing down the zero section of \tilde{Y} . Then X and Y are biholomorphic in a neighborhood of the singular point ([23], Theorem 6.2), but $q(X) = q(\tilde{X}) = 0$, $p_g(X) = 1 \neq p_g(\tilde{X}) = 0$, while $p_g(Y) = p_g(\tilde{Y}) = 0$ but $q(Y) = 0 \neq q(\tilde{Y}) = 1$. The singular spaces X and Y have an important topological difference, too. Namely, on Y the intersection (cup product) pairing $H^2(Y, \mathbb{R}) \times H^2(Y, \mathbb{R}) \rightarrow \mathbb{R}$ is non-singular (as is the case with non-singular spaces), while on X this pairing has null eigenspace of dimension 2.)

In this paper we want to study normal compact analytic surfaces with vanishing geometric genus. For singular surfaces this is a very strong condition having surprising local consequences as well as global ones. As will be seen such surfaces are very “manifold-like” in their global properties and can have only very special kinds of singularities. Below a *surface* is any reduced, irreducible two-dimensional complex space, with or without singularities.

1. DEFINITION: A normal surface singularity $x \in X$ is ν^{th} -order rational, $\nu = 0, 1, 2, \dots$, if there is a normal (good) resolution $\pi: \tilde{X} \rightarrow X$ of the singularity of X at x for which the sheaf $R^1\pi_*\mathcal{J}_E^\nu$ vanishes at x , for $E = \pi^{-1}(x)$ the exceptional curve of the resolution and for $\mathcal{J}_E \subset \mathcal{O}_{\tilde{X}}$ the ideal sheaf of germs vanishing on E . If $x \in X$ is first-order rational and if the associated dual intersection graph of the curve E is acyclic, then $x \in X$ is called *pararational*.

REMARKS: (1) First-order rational singularities were introduced in [8] where I called them “analytically rational” points to contrast them with the so-called “topologically rational” points – those with $H^1(E, \mathbb{R}) = 0$ – and expressly to spare the reader terms like “pararational”, “pseudorational”, “almost rational”, etc. The point of this terminology was that there is a natural split exact sequence

$$0 \rightarrow (R^1\pi_*\mathcal{J}_E)_x \rightarrow (R^1\pi_*\mathcal{O}_{\tilde{X}})_x \rightarrow H^1(E, \mathcal{O}_E) \rightarrow 0$$

which breaks the stalk $(R^1\pi_*\mathcal{O}_{\tilde{X}})_x$ into an “analytic” component $(R^1\pi_*\mathcal{J}_E)_x$ and a component $H^1(E, \mathcal{O}_E)$ which is determined by the topology of E . Thus *rational* ($(R^1\pi_*\mathcal{O}_{\tilde{X}})_x = 0$) is equivalent to analytically rational and topologically rational together. Later it became clear to me that “analytically rational” was not good terminology, and in the first version of the present paper I called first-order rational points with acyclic graphs “quasi-rational” – only to discover that

Abhyankar [1] had already coined the term *quasirational* to refer to those singularities whose exceptional curves have only rational components (a larger class than “topologically rational”). In this language, rational is Abhyankar’s quasirational plus the present pararational. The term “ ν^{th} -order rational” is intended to suggest the ν^{th} -order infinitesimal neighborhood $(E, \mathcal{O}_{\tilde{X}}/\mathcal{I}_E^\nu)$ of the embedding of E in \tilde{X} .

(2) 0^{th} -order rational is the same as rational and implies 1^{st} -order rational, but in general there are no other implications between ν^{th} -order and μ^{th} -order rationality for various ν and μ . For example, among rational double points only those of type A_k (the singularities $x^{k+1} + y^2 + z^2 = 0$) are second-order rational, while the minimally elliptic singularity $x^3 + y^3 + z^3 = 0$ is ν^{th} -order rational $\forall \nu \geq 1$, but not rational. A singular point is *super-rational* if it is ν^{th} -order rational $\forall \nu$.

(3) For $\nu = 0$ and 1 the vanishing of $R^1\pi_*\mathcal{F}_{\pi^{-1}(x)}^\nu$ is independent of the resolution π , but this is not true for general ν . Indeed, if $\pi: \tilde{X} \rightarrow X$ is any normal resolution and if $\hat{\pi}: \hat{X} \rightarrow X$ is obtained from π by blowing up a point of intersection of two of the components of the exceptional curve of π , then $\dim(R^1\pi_*\mathcal{F}_{\pi^{-1}(x)}^\nu)_x > 0$ for $\nu \geq 2$. On the other hand, for fixed ν if this group vanishes for any normal resolution then it vanishes for the unique minimal normal resolution. All of these ideas carry over immediately to higher dimensions and are especially useful in the study of 3-dimensional isolated singular points.

In the present paper, one of the main results (Theorem 10 below) is that a normal algebraic surface with $p_g = 0$ can have only pararational singularities. If we assume in addition that X is *Gorenstein* – that is, that the canonical line bundle K_{X_0} on the regular points X_0 of X is trivial in a neighborhood of each singular point – then much finer results are possible. Indeed, under some further global conditions it is shown (Theorems 14 and 15 below) that only rational double points and minimally elliptic singularities of type EI (Laufer’s classification [25]) can occur. A summary of results of this kind classifying singular Gorenstein surfaces with $p_g = 0$ and their possible singularities is given in the chart at the end of section III below.

I. Pararational singularities

Let X be a normal Stein surface with exactly one singular point x . Let $\pi: \tilde{X} \rightarrow X$ be the minimal normal resolution with exceptional set $E = \pi^{-1}(x) = \cup_{i=1}^s E_i$, E_i irreducible. That is, each E_i is a non-singular curve, E_i meets E_j (if at all) transversally in a single point, $E_i \cap E_j \cap E_k = \emptyset$ for distinct indices i, j , and k , and if $\pi': X' \rightarrow X$ is any other

resolution whose exceptional set has these properties then π' factors through π . Denote by $\mathcal{J}_E \subset \mathcal{O}_{\tilde{X}}$ the (reduced) ideal sheaf of E in \tilde{X} . In the unpublished notes [7], first-order rational points (those for which the derived sheaf $R^1\pi_*\mathcal{J}_E$ identically vanishes) were studied, with these conclusions among others:

2. LEMMA: *Let $\pi: \tilde{X} \rightarrow X$ be the minimal normal resolution of a normal singular point $x \in X$ with exceptional curve $E = \cup_{i=1}^s E_i$. For each i put $n_i = \sum_{j \neq i} E_j \cdot E_i =$ the number of curves E_j meeting E_i , $i \neq j$, and put $m_i = \sum_{j < i} E_j \cdot E_i =$ the number of curves E_j meeting E_i with $j < i$. Denote by g_i and E_i^2 respectively the genus of E_i and the self-intersection number of E_i on \tilde{X} , and suppose that $\forall i$ we have the estimate*

$$-E_i^2 \geq \max\{n_i, n_i + m_i + 2g_i - 1\}.$$

Then $(R^1\pi_*\mathcal{J}_E)_x = 0$.

3. COROLLARY: *For $\pi: \tilde{X} \rightarrow X$ as above, if E has no cycles in its graph and if*

$$(1) \quad -E_i^2 \geq n_i + 2g_i$$

for all i ; or if for some index i_0 ,

$$(2) \quad g_{i_0} > 0 \text{ and } -E_{i_0}^2 \geq n_{i_0} - 2g_{i_0} - 1,$$

while (1) holds $\forall i \neq i_0$; then $x \in X$ is pararational.

Corollary 3 follows from the Lemma by noting that E acyclic implies that the E_i can be ordered in such a way as to make the integers m_i of the Lemma all equal to 1, except the first, which is equal to zero. The proof of the Lemma follows the Grauert-Laufer technique of considering successive products of powers of the ideal sheaves of the E_i . Since 2 and 3 will not be needed explicitly in the sequel the proofs will not be given, but we will need the following converse, whose proof will illustrate the technique.

4. LEMMA: *Let $x \in X$ be a pararational two-dimensional singular point and let $\pi: \tilde{X} \rightarrow X$ be a normal resolution with exceptional set $E = \cup_{i=1}^s E_i$. Then for E_i^2 , g_i , n_i the self-intersection of E_i , genus of E_i ,*

and number of components $E_j \neq E_i$ meeting E_i , for all i we have

$$(*) \quad -E_i^2 \geq n_i + g_i - 1.$$

PROOF: Let \mathcal{I}_{E_i} denote the locally principal, locally prime ideal sheaf of germs of functions vanishing on E_i . Then if $\mathcal{I}_E = \bigotimes_{i=1}^s \mathcal{I}_{E_i}$ is the ideal sheaf of E we have the exact sequence of sheaves $0 \rightarrow \mathcal{I}_{E_i} \mathcal{I}_E \rightarrow \mathcal{I}_E \rightarrow \mathcal{I}_E / \mathcal{I}_{E_i} \mathcal{I}_E \rightarrow 0$. Taking cohomology on \tilde{X} , this gives the exact sequence

$$\dots \rightarrow H^1(\tilde{X}, \mathcal{I}_E) \rightarrow H^1(\tilde{X}, \mathcal{I}_E / \mathcal{I}_{E_i} \mathcal{I}_E) \rightarrow H^2(\tilde{X}, \mathcal{I}_{E_i} \mathcal{I}_E) \rightarrow \dots.$$

Without loss of generality we may take X Stein and contractible onto x and with x as its only singular point. Then $H^2(\tilde{X}, \mathcal{F}) = 0$ for any coherent analytic sheaf \mathcal{F} , and we conclude that $H^1(\tilde{X}, \mathcal{I}_E) \rightarrow H^1(\tilde{X}, \mathcal{I}_E / \mathcal{I}_{E_i} \mathcal{I}_E)$ is surjective. But also X Stein and contractible implies that $H^1(\tilde{X}, \mathcal{I}_E) \cong (R^1 \pi_* \mathcal{I}_E)_x$, which by the condition x pararational is zero. Therefore $H^1(\tilde{X}, \mathcal{I}_E / \mathcal{I}_{E_i} \mathcal{I}_E) = 0$.

Now $\mathcal{I}_E / \mathcal{I}_{E_i} \mathcal{I}_E$ is supported on the curve E_i , and on E_i is the sheaf of germs of sections of the line bundle $L = \bigotimes_{j=1}^s [E_j]^{-1}|_{E_i}$, for $[E_j]$ the bundle on \tilde{X} of the divisor of E_j . This line bundle has Chern class (degree) equal to $-\sum_{j=1}^s E_j \cdot E_i = -E_i^2 - n_i$. By Riemann-Roch on E_i ,

$$\begin{aligned} \chi(L) &= \dim H^0(E_i, L) - \dim H^1(E_i, L) = (\text{degree of } L) + 1 - g_i \\ &= -E_i^2 - n_i - g_i + 1. \end{aligned}$$

Since $H^1(E_i, L) = H^1(\tilde{X}, \mathcal{I}_E / \mathcal{I}_{E_i} \mathcal{I}_E) = 0$ we conclude that

$$-E_i^2 - n_i - g_i + 1 = \dim H^0(E_i, L) \geq 0, \text{ or}$$

$$(*) \quad -E_i^2 \geq n_i + g_i - 1,$$

5. COROLLARY: Let x be a pararational Gorenstein point of a surface X . Suppose that x is not a rational double point and that the minimal resolution $\pi : \tilde{X} \rightarrow X$ is normal. Denote by $E = \cup_{i=1}^s E_i$ the exceptional curve of π , and in a neighborhood of E put $K_{\tilde{X}} = \bigotimes_{i=1}^s [E_i]^{-k_i}$ for some integers k_i . Then for g_i the genus of E_i we have the inequality

$$(**) \quad \sum_{i=1}^s (g_i - 1)(k_i - 3) \leq 2(s - 1).$$

PROOF: Since π is the minimal resolution, $k_i \geq 0 \forall i$. Indeed, since x is not a rational double point, each $k_i \geq 1$, for the adjunction formula

$$(***) \quad 2g_i - 2 = K_{\tilde{X}} \cdot E_i + E_i^2 = -\sum_{j=1}^s k_j E_j \cdot E_i + E_i^2$$

shows that if $k_i = 0$ then $g_i = 0$, $E_i^2 = -2$, and the same is true for every E_j meeting E_i . Since E is connected, then, $K_{\tilde{X}}$ is trivial in a neighborhood of E , which property characterizes rational double points.

Put $n_i = \sum_{j \neq i} E_j \cdot E_i$ as in Lemma 4. Then (***) can be written as

$$\begin{aligned} 2g_i - 2 + n_i &= -\sum_{j=1}^s (k_j - 1)(E_j \cdot E_i) \\ &= -\sum_{j \neq i} (k_j - 1)(E_j \cdot E_i) + (k_i - 1)(-E_i^2). \end{aligned}$$

Applying Lemma 4, since $k_i \geq 1 \forall i$,

$$2g_i - 2 + n_i \geq \sum_{j \neq i} (k_j - 1)(E_j \cdot E_i) + (k_i - 1)(n_i + g_i - 1)$$

for each i . Summing,

$$\begin{aligned} 2 \sum_{i=1}^s g_i - 2s + \sum_{i=1}^s n_i &\geq -\sum_{i=1}^s \sum_{j \neq i} (k_j - 1)(E_j \cdot E_i) + \sum_{i=1}^s (k_i - 1)(n_i + g_i - 1) \\ &= -\sum_{j=1}^s (k_j - 1) \sum_{i \neq j} (E_j \cdot E_i) + \sum_{i=1}^s (k_i - 1)n_i + \sum_{i=1}^s (k_i - 1)(g_i - 1) \\ &= -\sum_{j=1}^s (k_j - 1)n_j + \sum_{i=1}^s (k_i - 1)n_i + \sum_{i=1}^s (k_i - 1)(g_i - 1) \\ &= \sum_{i=1}^s (k_i - 1)(g_i - 1). \end{aligned}$$

By the condition of no cycles in the graph of E , $\sum_{i=1}^s n_i = 2(s - 1)$, so we have

$$2 \sum_{i=1}^s g_i - 2 \geq \sum_{i=1}^s (k_i - 1)(g_i - 1),$$

which is equivalent to (**).

REMARK: It will be noted that only topological properties of the canonical bundle $K_{\tilde{X}}$ were used in the proof, nor did the fact that the k_i were *integers* (instead of just rational numbers) play a role. But if X is any contractible Stein surface and $\pi: \tilde{X} \rightarrow X$ is a resolution, then the first Chern class $c_1 = c(K_{\tilde{X}}^{-1})$ satisfies $c_1 = \sum_{i=1}^s k_i c([E_i]) \in H^2(\tilde{X}, \mathbb{Q}) \cong H^2(E, \mathbb{Q})$, for the k_i some rational numbers. The formula (**) remains valid in this case for any pararational singularity for which $k_i \geq 1 \forall i$ when π is the minimal normal resolution.

6. DEFINITION (Wagreich [27], Laufer [25]): A normal two dimensional singular point $x \in X$ is *elliptic* if $\chi(Z) = 0$ for Z the fundamental cycle of the minimal resolution. (If $\pi: \tilde{X} \rightarrow X$ is the minimal resolution, with $\pi^{-1}(x) = E = \cup_{i=1}^s E_i$, then the *fundamental cycle* $Z = \sum_{i=1}^s a_i E_i$ is the smallest positive cycle supported on E with $Z \cdot E_i \leq 0 \forall i$.) $x \in X$ is *minimally elliptic* if $\chi(Z) = 0$ and $\chi(Z') > 0$ for all positive cycles $Z' < Z$. A minimally elliptic singularity is of *type El* if E consists of a single non-singular elliptic curve, and is of *type No* (for “node”) if E is an irreducible rational curve with a node (ordinary double point) or a collection of non-singular rational curves forming one big cycle ([25], section 5). In the present paper singularities of *type No* will appear later in connection with non-algebraic surfaces with $p_g = 0$ – see chart below.

7. LEMMA: If $x \in X$ is *pararational Gorenstein*, and if X admits a normal resolution $\pi: \tilde{X} \rightarrow X$ of the singularity of X at x whose exceptional set E is irreducible, then for g the genus of E and k satisfying $K_{\tilde{X}} = [E]^{-k}$ as in Corollary 5, either

- (a) $g = 0, k = 0$, and x is the rational double point A_1 , or
- (b) $g = 1, k = 1$, and x is a *minimally elliptic singularity of type El*.

PROOF: Again by adjunction

$$2g - 2 = (k - 1)(-E^2),$$

whence $g = 0 \Leftrightarrow k = 0$ and $E^2 = -2$, and $g = 1 \Leftrightarrow k = 1$. These are respectively the cases (a) and (b). We must show that nothing else can occur, namely that $g > 1$ is impossible.

Suppose $g > 1$. By Corollary 5 $(g - 1)(k - 3) \leq 0$, so k must be 2 or 3. If $k = 2$, then arguing as in the proof of Lemma 4 and using Serre duality,

$$\begin{aligned}
 0 &= H^1(\tilde{X}, \mathcal{F}_E/\mathcal{F}_E^2) \cong H^1(E, [E]^{-1}|_E) \\
 &\cong H^0(E, \mathcal{K}_E \otimes [E]|_E) \\
 &\cong H^0(E, [E]^{-k+2}|_E) \\
 &\cong H^0(E, \mathcal{O}_E) \\
 &\cong \mathbb{C}
 \end{aligned}$$

a contradiction. Similarly if $k = 3$, $0 = H^1(E, [E]^{-1}|_E) = H^0(E, [E]^{-1}|_E)$ (Serre duality and adjunction). Then from the sequence

$$\dots \rightarrow H^0(E, [E]^{-1}|_E) \rightarrow H^1(\tilde{X}, \mathcal{F}_E^2) \rightarrow H^1(\tilde{X}, \mathcal{F}_E) \rightarrow \dots$$

we have $H^1(\tilde{X}, \mathcal{F}_E^2) = 0$. But then from

$$H^1(\tilde{X}, \mathcal{F}_E^3) \rightarrow H^1(\tilde{X}, \mathcal{F}_E^2) \rightarrow H^1(E, [E]^{-2}|_E) \rightarrow 0$$

we conclude that $H^1(E, [E]^{-2}|_E) = H^0(E, \mathcal{O}_E) = 0$, an absurdity. This completes the proof.

II. Singular surfaces with $p_g = 0$

Pararational singularities are interesting objects in their own right and deserve, I think, further study (for instance, Corollary 5 for Gorenstein singularities can be improved by the technique of Lemma 7). But now we want to turn to global considerations.

We will rely heavily on the relations between global topological and analytic invariants of a singular surface X and its non-singular model \tilde{X} . If X is a normal surface and if $\pi : \tilde{X} \rightarrow X$ is a normal resolution of singularities with exceptional curve E , then π induces an exact commuting diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^1(X, \mathcal{O}_X) & \xrightarrow{\tilde{\pi}^1} & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \xrightarrow{\varphi} & H^0(X, R^1\pi_*\mathcal{O}_{\tilde{X}}) \\
 & & \uparrow j^1 & & \uparrow \tilde{j}^1 & & \uparrow \Psi \\
 (8) & \rightarrow & H^1(X, \mathbb{R}) & \xrightarrow{\pi^1} & H^1(\tilde{X}, \mathbb{R}) & \xrightarrow{i^1} & H^1(E, \mathbb{R}) \\
 & & & & & & \\
 & & \rightarrow & H^2(X, \mathcal{O}_X) & \xrightarrow{\tilde{\pi}^2} & H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \rightarrow 0 \\
 & & & \uparrow j^2 & & \uparrow \tilde{j}^2 & \\
 & \xrightarrow{\delta^1} & H^2(X, \mathbb{R}) & \xrightarrow{\pi^2} & H^2(\tilde{X}, \mathbb{R}) & \xrightarrow{i^2} & H^2(E, \mathbb{R}) \\
 & & & & & & \\
 & & \xrightarrow{\delta^2} & H^3(X, \mathbb{R}) & \xrightarrow{\pi^3} & H^3(\tilde{X}, \mathbb{R}) & \rightarrow 0
 \end{array}$$

where the top row is the Leray spectral sequence and the vertical arrows are induced by the inclusions of \mathbb{R} into \mathcal{O}_X and $\mathcal{O}_{\tilde{X}}$ (see [8], Lemmas 1 and 2, where the properties of this diagram are worked out in great detail). \tilde{j}_1 , hence also j_1 , is injective. Since E is exceptional its components E_i span a negative definite (with respect to the intersection pairing) subspace \mathcal{E} of $H^2(\tilde{X}, \mathbb{R})$ and i^2 maps \mathcal{E} isomorphically onto $H^2(E, \mathbb{R})$. Thus i^2 is surjective and π^3 is an isomorphism. π^2 preserves positive eigenspaces, while δ^1 maps $H^1(E, \mathbb{R})$ onto the null space of $H^2(X, \mathbb{R})$.

9. LEMMA: *Let X be a normal (possibly) singular complete abstract algebraic surface over \mathbb{C} . Suppose that X admits a resolution of singularities $\pi : \tilde{X} \rightarrow X$ which preserves the geometric genus. Then*

- (a) X is a projective variety,
- (b) each singular point of X is pararational, and
- (c) the intersection pairing on X is non-singular with positive eigenspace of dimension $2p_g + 1$.

PROOF: Since p_g is a birational invariant of non-singular surfaces we may assume that π is the minimal normal resolution. \tilde{X} , being a non-singular algebraic surface, is projective. That π preserves projectivity (in both directions) in the presence of the condition $p_g(\tilde{X}) = p_g(X)$ is shown in [8] (Corollary 5 and Proposition 7 and their proofs). Thus X is projective.

For parts (b) and (c) we chase the diagram (8). By hypothesis $\tilde{\pi}^2$ is an isomorphism, so φ is surjective. Since \tilde{X} is algebraic, \tilde{j}^1 is an isomorphism (of real vector spaces). Thus Ψ is surjective and we conclude that

$$(1) \quad h^0(R^1\pi_*\mathcal{O}_{\tilde{X}}) \leq \frac{1}{2}b_1(E).$$

(Here b_i denotes the i^{th} Betti number and $h^i(\mathcal{F})$ the (complex) dimension of $H^i(\mathcal{F})$ for \mathcal{F} a coherent analytic sheaf.) On the other hand for \mathcal{I}_E the ideal sheaf of E the exact sequence

$$0 \rightarrow H^0(X, R^1\pi_*\mathcal{I}_E) \rightarrow H^0(X, R^1\pi_*\mathcal{O}_{\tilde{X}}) \rightarrow H^1(E, \mathcal{O}_E) \rightarrow 0$$

shows that

$$(2) \quad h^0(R^1\pi_*\mathcal{O}_{\tilde{X}}) = h^0(R^1\pi_*\mathcal{I}_E) + h^1(\mathcal{O}_E).$$

Since $h^1(\mathcal{O}_E) = \frac{1}{2}b_1(E) + \frac{1}{2}g(E)$ for $g(E)$ the number of independent

cycles in the graph of E , (1) and (2) imply $h^0(R^1\pi_*\mathcal{J}_E) = g(E) = 0$ (i.e., the singularities are pararational; this proves (b)), and Ψ is an isomorphism. But then i^1 is surjective and the nilspace of $H^2(X, \mathbb{R}) = \text{im}(\delta^1) = 0$, as claimed in (c). The last assertion in (c) follows from the fact that π^2 preserves the positive eigenspace and that (c) is valid for any non-singular algebraic surface ([20], I, Theorem 3).

10. THEOREM: *Let X be a normal compact complex analytic surface with $p_g = 0$. Then for $\pi: \tilde{X} \rightarrow X$ a resolution with exceptional curve E , $R^1\pi_*\mathcal{J}_E \equiv 0$ on X .*

Indeed, either

(a) X is a projective variety with only pararational singular points and with intersection pairing non-singular with 1-dimensional positive eigenspace, or

(b) X is bimeromorphically equivalent to a non-singular surface of type VII_0 (Kodaira's classification [20]—i.e., to a non-algebraic surface with $q = b_1 = 1$), and either

(i) all singularities of X are rational and $q(X) = b_1(X) = 1$, or

(ii) X is regular ($q(X) = b_1(X) = 0$) and all singularities of X are rational except one, that one exception being elliptic.

In (b) the cup product pairing is either negative definite (\Leftrightarrow all exceptional curves E_i are rational) or with a 1-dimensional null eigenspace and no positive eigenspace (\Leftrightarrow one of the E_i 's is elliptic).

REMARK: Singular surfaces of type (b)(i) can always be constructed by the following trivial device: start with a non-singular surface \tilde{X} having whatever properties you like, perform some iterated monoidal transformations, thus introducing some rational curves with various negative self intersections, then blow down some of these. The resulting singular surface X will have only rational singularities and will have essentially the same global properties as \tilde{X} .

The most interesting examples of type (b)(ii) are the singular surfaces of Inoue-Hirzebruch ([19]; cf. [18], [15], [16]). These spaces are obtained by compactifying quotients mod certain groups Γ of automorphisms of $\mathbb{H} \times \mathbb{C}$, \mathbb{H} the half-plane, with two singular points, then identifying the points by an involution. Such a surface X has $q = b_1 = 0$, $p_g = b_2 = 0$, and its singular point is elliptic of nodal type (type No). Alternatively, for spaces of our type (b)(ii) with $b_2 > 0$, resolve just one of the two singular points of the compactification \hat{X} of $(\mathbb{H} \times \mathbb{C})/\Gamma$. (\hat{X} itself, however, has $p_g = 1$.)

Further examples of (b)(ii) are obtained from non-singular elliptic

Hopf surfaces by blowing up points on a general fibre E of an elliptic fibration, then blowing down the resulting negatively embedded elliptic curve \tilde{E} , or a star consisting of \tilde{E} and some of the rational curves born of the blowings up.

PROOF OF THEOREM 10: Let $\pi: \tilde{X} \rightarrow X$ be a resolution. Then $p_g(\tilde{X}) = 0$, so the classification of surfaces shows that either \tilde{X} is algebraic or else $q(\tilde{X}) = b_1(\tilde{X}) = 1$, and \tilde{X} is a surface of type VII₀. In the first case X is also algebraic and Lemma 9 applies. In the second we have

$$\sum_{i=1}^r \dim(R^1\pi_*\mathcal{O}_{\tilde{X}})_{x_i} = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_{\tilde{X}}) = 1 - q(X),$$

the sum taken over the singular points x_1, \dots, x_r of X . The possibilities are ((b)(i)) $q(X) = 1$ and all the singularities are rational, or $q(X) = 0$ and we are in case (b)(ii). (A singular point x with $(R^1\pi_*\mathcal{O}_{\tilde{X}})_x = \mathbb{C}$ is necessarily elliptic ([25], Theorem 4.1).)

For the vanishing of $R^1\pi_*\mathcal{F}_E$ and the assertions about the cup product, in the non-algebraic case the diagram (8) reduces to

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(X, \mathcal{O}_X) & \rightarrow & \mathbb{C} & \rightarrow & H^0(X, R^1\pi_*\mathcal{O}_{\tilde{X}}) \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & H^1(X, \mathbb{R}) & \rightarrow & \mathbb{R} & \rightarrow & H^1(E, \mathbb{R}) \rightarrow \text{Nil}(H^2(X, \mathbb{R})) \rightarrow 0. \end{array}$$

If the singularities are rational, $b_1(E) = h^0(R^1\pi_*\mathcal{O}_{\tilde{X}}) = 0$ and the diagram shows that $b^0(X) = 0$, for b^0 the dimension of the nilspace. In the elliptic case,

$$(\ddagger) \quad h^0(R^1\pi_*\mathcal{O}_{\tilde{X}}) = h^0(R^1\pi_*\mathcal{F}_E) + h^1(\mathcal{O}_E) = 1,$$

and by the diagram

$$(\ddagger\ddagger) \quad b_1(E) = 1 + b^0(X).$$

Since $b_1(E) > 0$, also $h^1(\mathcal{O}_E) > 0$, whence $h^0(R^1\pi_*\mathcal{F}_E) = \sum_{i=1}^r \dim(R^1\pi_*\mathcal{F}_E)_{x_i} = 0$ as claimed. Indeed, (\ddagger) and $(\ddagger\ddagger)$ show that $h^1(\mathcal{O}_E) = 1$, so there are two possibilities:

(α) One of the E_i is elliptic, all the rest are rational, and the graph of E is acyclic. Then $b_1(E) = 2$ and by $(\ddagger\ddagger)$ $b^0(X) = 1$. Or

(β) E_i is rational $\forall i$ and the graph of E contains one cycle. Then $b_1(E) = 1$ and $b^0(X) = 1$. The desired conclusions now follow from the

facts that non-singular surfaces of type VII_0 have negative definite intersection pairing (\tilde{X} is not necessarily minimal, so this remark is not vacuous even for spaces derived from Hopf surfaces and Inoue's surfaces S_M , $S_N^{(\pm)}$ ([18]), whose minimal models have $b_2 = 0$), and that b^+ is preserved by resolutions of singularities.

REMARK: To anticipate the needs of the next section we observe here that if each singular point is Gorenstein then the rational points are all double points, while the one elliptic point occurring in case (b) (ii) of Corollary 10 is necessarily minimally elliptic of type El or of type No ([25], Theorems 3.5, 3.10, and 4.3 and its proof).

III. Gorenstein surfaces

In this section we want to make fuller use of the classification of non-singular surfaces and in particular the notions of plurigena and canonical dimension. We review briefly some facts about Gorenstein surfaces. A Noetherian commutative ring R is *Gorenstein* if it has finite injective dimension. If $(x, \mathcal{O}_{X,x})$ is the germ of an n -dimensional analytic space, then the ring $\mathcal{O}_{X,x}$ is Gorenstein if and only if it is Cohen-Macaulay and the canonical module $\Omega_{\mathcal{O}_{X,x}} = \text{Ext}^{N-n}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{C}^N,x})$ is free of rank one, for $X \subset \mathbb{C}^N$ a local embedding. An analytic space (X, \mathcal{O}_X) is *Gorenstein* if $\mathcal{O}_{X,x}$ is a Gorenstein ring $\forall x \in X$. If X is Gorenstein then evidently the canonical sheaf Ω_X whose stalk at x is $\Omega_{\mathcal{O}_{X,x}}$ is the sheaf of germs of sections of a holomorphic line bundle K_X , the *canonical line bundle of X* . K_X restricted to the regular points X_0 of X is the usual bundle of holomorphic n -forms on the complex manifold X_0 . ([14]; [17], section 3).

Let X be a two-dimensional Gorenstein analytic space (a *Gorenstein surface*). If $\pi: \tilde{X} \rightarrow X$ is a resolution of singularities with exceptional curve $E = \cup_{i=1}^s E_i$, then the canonical bundle $K_{\tilde{X}}$ on \tilde{X} satisfies $K_{\tilde{X}} = \pi^* K_X \otimes (\bigotimes_{i=1}^s [E_i]^{-k_i})$ for some integers k_i (= order of the pole on E_i of a meromorphic 2-form defined in a neighborhood of E and without zeros or poles except on E). $k_i \geq 0$ unless E_i is a non-singular rational curve with self-intersection -1 (an "exceptional curve of the first kind"); if E contains no such components then $k_i = 0 \Leftrightarrow E_i$ is a non-singular rational curve with self-intersection -2 , and so is each analytic component E_j of the connected component of E containing E_i . If E contains no exceptional components of the first kind and if $k_i = 1$ for some i , then $g_i > 0$ implies that E_i is a non-singular elliptic

curve, while if E_i contains a singular point then E_i is a rational curve with one simple node or one simple cusp; in each of these cases E_i is an isolated component of C . (These observations follow, as in the proof of Corollary 5 above, from the adjunction formula $2g_i - 2 + 2\delta(E_i) = -\sum_{j \neq i} k_j E_j \cdot E_i - (k_i - 1)E_i^2$, where $\delta(E_i)$ is the “number of nodes and cusps”, counted with appropriate multiplicities, etc., on E_i .)

If X is Gorenstein we may define as usual the m^{th} pluri-genus $P_m(X) = \dim H^0(X, K_X^m)$ and the canonical dimension $\kappa(X) =$ order of (polynomial) growth of P_m as a function of m . Since $P_m(X) \geq P_m(\tilde{X}) \forall m$, $\kappa(X) \geq \kappa(\tilde{X})$.

11. DEFINITIONS: Let X be a normal compact complex surface. Then

(a) X is *minimal* if $b_2(X)$ is smallest among all surfaces bimeromorphically equivalent to X .

(b) X is *essentially non-singular* if it has only rational double points as singularities.

(c) X is called a *projective ruled cone* if it is the space derived from a non-singular projective ruled surface, with no singular fibres, by blowing down the base curve. (A non-singular surface \tilde{X} is *ruled* if there is a proper regular map $\rho: \tilde{X} \rightarrow \Gamma$ onto a non-singular algebraic curve Γ , whose general fibre is a non-singular rational curve. If ρ has no singular fibres then \tilde{X} is the total space of a \mathbb{P}^1 -bundle on Γ and as such contains a zero-section, the base curve $B \cong \Gamma$, which we may take to have non-positive self-intersection, and an “infinity-section” B_∞ , with $B_\infty^2 = -B^2 \geq 0$.)

(d) X is *quasi-rational* if it is (abstract) algebraic and Gorenstein and if $q(X) = p_g(X) = 0$ and $P_m(X) = 0 \forall m = 1, 2, \dots$

REMARK: The point of definition (b) is that rational double points make no contribution at all to many global properties of surfaces, while to others their contribution is completely straightforward and easy to deal with. Thus the theory of complex manifolds of dimension 2 can easily be expanded to take in the “essentially non-singular” surfaces. For example

12. PROPOSITION: Let X be an essentially non-singular compact surface. Denote by b_i , b^+ , b^- , b^0 , and e respectively the i^{th} Betti number, the dimension of the positive, negative, and null eigenspace of the cup product pairing, and the topological Euler number. Denote by K_X the canonical bundle and by $\mu(X)$ the sum of the Milnor numbers of the singular points x_i —i.e., of the number of vertices k_i of the

Dynkin diagrams $A_k, D_k,$ or E_k associated to the x_i (see [13], [3], for the graphs A_k, D_k, E_k ; [24], [26] for μ). Then

(a) (Poincaré duality, Hirzebruch index theorem, etc.) $b_i(X) = b_{4-i}(X) \forall i, b^0(X) = 0,$

$$b^+(X) = \left\{ \begin{array}{c} 2p_g(X) + 1 \\ \text{or} \\ 2p_g(X) \end{array} \right\} \Leftrightarrow b_1(X) = \left\{ \begin{array}{c} 2q(X) \\ \text{or} \\ 2q(X) - 1 \end{array} \right\} \Leftrightarrow b_1(X) \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{or} \\ \text{odd} \end{array} \right\},$$

and $b^+(X) - b^-(X) = \frac{1}{3}(K_X^2 - 2e(X) + \mu(X)).$

(b) (Kodaira’s algebraicity criteria.) If X admits two algebraically independent meromorphic functions, or if X admits a holomorphic line bundle with positive self-intersection (in particular if $K_X^2 > 0$), or if $p_g(X) = 0$ and $b_1(X) \neq 1$, then X is projective algebraic.

(c) (Serre duality and Hirzebruch Riemann-Roch.) For L a holomorphic line bundle on $X, H^i(X, \mathcal{O}_X(L)) \cong H^{2-i}(X, \mathcal{O}_X(K_X \otimes L^{-1})) \forall i,$ and

$$\chi(\mathcal{O}_X(L)) = \frac{1}{2}L \cdot (K_X \otimes L^{-1}) + \frac{1}{12}(K_X^2 + e(X) + \mu(X)) \text{ ([17, [24]).}$$

(d) If $\pi: \tilde{X} \rightarrow X$ is the minimal resolution of singularities then $b_i(\tilde{X}) = b_i(X)$ for $i \neq 2, b^+(\tilde{X}) = b^+(X), b^-(\tilde{X}) = b^-(X) + \mu(X), p_g(\tilde{X}) = p_g(X), q(\tilde{X}) = q(X), K_{\tilde{X}} = \pi^*K_X, H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*L)) \cong H^i(X, \mathcal{O}_X(L)) \forall$ line bundles L on X, X is projective $\Leftrightarrow \tilde{X}$ is projective ([3], Theorem 2.3; [8], Corollary 11), etc.

See also Burns and Wahl [11] for the contribution of rational double points to deformations. The relations (d) provide the proofs of (a)–(c).

With respect to definition (d) we note that by Theorem 10 above a quasi-rational surface has only pararational singularities. Furthermore, since $P_m(\tilde{X}) \leq P_m(X) = 0$ for \tilde{X} a non-singular model of a quasi-rational surface X , it follows from the classification of non-singular surfaces that every such X is birationally equivalent to a ruled surface over a curve Γ of genus $g = \frac{1}{2}b_3(X)$ ([18], Proposition 16.) In particular,

13. OBSERVATION: A quasi-rational surface X is rational (birationally equivalent to the complex projective plane \mathbb{P}^2) $\Leftrightarrow b_3(X) = 0 \Leftrightarrow X$ is essentially non-singular.

PROOF. For $\pi: \tilde{X} \rightarrow X$ a resolution, $\frac{1}{2}b_3(X) = \frac{1}{2}b_3(\tilde{X}) = q(\tilde{X}) = \chi(\mathcal{O}_{\tilde{X}}) - \chi(\mathcal{O}_X) = \sum \dim(R^1\pi_*\mathcal{O}_{\tilde{X}})_x,$ the sum taken over the singular

points x_1, \dots, x_r of X . Now a ruled surface \tilde{X} is rational $\Leftrightarrow q(\tilde{X}) = 0$, so X , being birationally equivalent to \tilde{X} , is rational $\Leftrightarrow b_3(X) = \sum_{i=1}^r \dim(R^1 \pi_* \mathcal{O}_{\tilde{X}})_{x_i} = 0$, i.e., $\Leftrightarrow b_3(X) = 0$ and each x_i is rational. But among rational singularities only the classical double points A_k , D_k , and E_k are Gorenstein ([25], [12]). This completes the proof.

14. THEOREM (*Classification of minimal quasi-rational surfaces*): *Let X be a minimal quasi-rational surface. Then either*

- (a) *X is essentially non-singular rational, or*
- (b) *X is a projective ruled cone over an elliptic curve.*

Indeed, in the first case either $X = \mathbb{P}^2$, $X =$ the singular quadric hypersurface $Q_0^2 = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3 = \mathbb{P}^3$, or X is obtained from \mathbb{P}^2 by the successive application of some number s , $3 \leq s \leq 8$, of monoidal transformations, followed by the blowing down of precisely s non-singular rational curves, each with self-intersection -2 (cf. [9], Proposition 4).

PROOF: In [10] are determined all Gorenstein surfaces with negative canonical bundle. They are all of type (a) or (b) above. Of these only the spaces described by the last statement of the theorem have $b_2 = 1$. Thus it suffices to show that $b_2(X) = 1$ and K_X is negative. But as remarked above $P_m(X) = 0 \forall m \Rightarrow X$ is birationally equivalent to a projective ruled surface and so has (an infinite family of) ruled cones X' in its birational equivalence class. Since $b_2(X') = 1$ for X' a ruled cone, minimality of X guarantees $b_2(X) = 1$.

Now by Lemma 9 X is projective, so X admits a positive line bundle L . Thus, positivity of line bundles being a topological property, $b_2(X) = 1 \Rightarrow$ every holomorphic line bundle on X is either positive, negative, or topologically torsion, while $q(X) = 0$ implies that a topologically torsion bundle is analytically torsion. Since no positive power of K_X admits a section, K_X cannot be positive or torsion. Thus K_X must be negative, and we are done.

It remains to consider those Gorenstein surfaces whose non-singular models are not ruled. This turns out to be quite easy.

15. THEOREM: *Let X be a complete algebraic Gorenstein surface with $p_g = 0$, and let $\pi: \tilde{X} \rightarrow X$ be a resolution of singularities. Suppose that \tilde{X} is not ruled. Then either*

- (a) *X is essentially non-singular, or*
 - (b) *X has exactly one minimally elliptic singularity of type E_1 , and every other singular point is a rational double point.*
- Case (b) does not occur if \tilde{X} is of general type.

Chart for the classification of complex Gorenstein surfaces with $p_g = 0$, by third Betti number

	Regular ($q = 0$)	Irregular ($q > 0$)
$b_3 = 0$ $\Leftrightarrow X$ is regular and essentially non-singular	X is projective algebraic and essentially non-singular and admits the same classification as non-singular surfaces, namely either (a) X is rational ($\Leftrightarrow P_m(X) = 0 \forall m$), (b) X is an Enriques surface ($\Leftrightarrow K_X \neq 0$ but $K_X \otimes K_X = 0$), (c) X is properly elliptic ($\Leftrightarrow \kappa(X) = 1$), or (d) X is of general type ($\Leftrightarrow \kappa(X) = 2$).	Does not occur
$b_3 = 1$ $\Leftrightarrow X$ is non-algebraic \Leftrightarrow the transcendence degree of the field of meromorphic functions on $X < 2$.	X is derived from a non-singular surface of type VII_0 and has precisely one non-rational point, a minimally elliptic point of type EI ($\Leftrightarrow b^0(X) = 1$) or N_0 ($\Leftrightarrow b^0(X) = 0$).	X is essentially non-singular of type VII_0

<p>$b_3 = 2$</p>	<div style="border: 1px solid black; padding: 5px; margin-bottom: 10px;"> <p>X is projective algebraic</p> </div> <p>X has precisely one non-rational point, a minimally elliptic point of type $E1$, and the non-singular model of X is either properly elliptic, hyperelliptic, or ruled over an elliptic curve.</p>	<p>X is essentially non-singular and its non-singular model is either properly elliptic $(\Leftrightarrow \kappa(X) = 1)$, hyperelliptic $(\Leftrightarrow \kappa(X) = 0)$, or ruled over an elliptic curve $(\Leftrightarrow \kappa(X) = -1)$. ($\kappa(X) = 2$ does not occur.)</p>
<p>$b_3 > 2$ $(\Leftrightarrow b_3$ is even).</p>	<div style="border: 1px solid black; padding: 5px; margin-bottom: 10px;"> <p>X is projective algebraic and birationally equivalent to a projective ruled surface over a curve of genus $g = \frac{1}{2}b_3$; X has only pararational singularities, and if $\pi: \tilde{X} \rightarrow X$ is a resolution with exceptional curve $E = \cup_{i=1}^s E_i$, then:</p> </div> <p>$\sum_{i=1}^s g(E_i) = \frac{1}{2}b_3$</p>	<p>$\sum_{i=1}^s g(E_i) < \frac{1}{2}b_3$.</p> <p>$X$ is essentially non-singular \Leftrightarrow $\sum_{i=1}^s g(E_i) = 0$.</p>

PROOF: By the classification of non-singular surfaces $\chi(\tilde{X}) \geq 0$ if \tilde{X} is not ruled ([20], [6]). Since $p_g(\tilde{X}) = p_g(X) = 0$, the only possibilities are $q(\tilde{X}) = 0$ or 1. Thus $q(\tilde{X}) - q(X) = \sum_{i=1}^r \dim(R^1\pi_*\mathcal{O}_X)_{x_i} \leq 1$, so every singular point of X is rational except at most one, and that one exception x_0 , if it occurs, has $R^1\pi_*\mathcal{O}_{X,x_0} = \mathbb{C}$. As before Gorenstein implies that the rational points are double points while x_0 is minimally elliptic. Checking Laufer's lists [25] of minimally elliptic singularities it is immediate that only those of type *El* are pararational. Thus (a) and (b) are the only possibilities.

If \tilde{X} is of general type, then $\chi(\mathcal{O}_{\tilde{X}}) \geq 1$, so $p_g(\tilde{X}) = 0 \Rightarrow q(\tilde{X}) = q(X) = 0$ and we are in case (a), Q.E.D.

REMARK: Examples of such surfaces are, for (a) simply take a surface \tilde{X} of general type with $p_g = 0$ and blow down all curves which are simultaneously base curves for all pluri-canonical systems $|K_{\tilde{X}}^m|$. The resulting surface will be essentially non-singular with $p_g = 0$ and will still be of general type – in fact X , but not in general \tilde{X} , will admit pluri-canonical projective embeddings (Bombieri [5], Bombieri and Husemoller [6], part V). For (b) the easiest construction is to take a non-singular elliptic surface with $p_g = 0$, blow up a point on a non-singular fibre F of an elliptic fibration, then blow down the proper transform \tilde{F} .

With the above results in hand the accompanying chart is constructed merely by comparing our findings above with the classification theory for non-singular surfaces ([20], [6]).

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