COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 43, nº 2 (1981), p. 239-252
<http://www.numdam.org/item?id=CM_1981__43_2_239_0>

COMPOSITIO MATHEMATICA, Vol. 43, Fasc. 2, 1981, pag. 239–252

DERIVATIONS OF THE LIE ALGEBRAS OF ANALYTIC VECTOR FIELDS

Janusz Grabowski

0. Introduction

A linear operator D of a Lie algebra L into L such that

$$D([X, Y]) = [D(X), Y] + [X, D(Y)]$$

for all $X, Y \in L$ is termed a derivation of L.

The set $\mathrm{Der}(L)$ of all derivations of L is a Lie subalgebra of the Lie algebra $\mathrm{End}(L)$ of all linear mappings of L.

For each $X \in L$, we denote by ad_X the mapping $Y \mapsto [X, Y]$ of L into L. The mapping $X \mapsto \mathrm{ad}_X$ is a homomorphism of L into $\mathrm{Der}(L)$.

The derivations of the form ad_X are called the inner derivations of L. The set ad_L of all inner derivations of L is a Lie ideal of $\mathrm{Der}(L)$. Dimensions of ad_L in $\mathrm{Der}(L)$ give us an idea of how much non-commutative is the Lie algebra L. For example, if L is commutative, then $\mathrm{ad}_L = \{0\}$ and $\mathrm{Der}(L) = \mathrm{End}(L)$. On the other hand, for finite-dimensional semi-simple Lie algebras we have $\mathrm{Der}(L) = \mathrm{ad}_L$.

The notion of semi-simplicity is well understood and useful in the finite-dimensional case. It seems to us that the class of the Lie algebras of vector fields is an interesting class of infinite-dimensional Lie algebras which have semi-simple-like properties.

Derivations of such Lie algebras were examined by Avez, Lichnerowicz, Diaz-Miranda [1], Morimoto [7] and Takens [9]. For example, it was shown by F. Takens that the derivations of the Lie algebra ${}^{\infty}\mathcal{U}$ of all C^{∞} vector fields on a smooth manifold are inner.

The same has been proved by T. Morimoto for the following classes of classical Lie algebras of vector fields on smooth manifolds

0010–437X/8105039–14$00.20/0

with additional structures:

(i) the Lie algebras of vector fields of constant divergence

(ii) the Lie algebras of vector fields preserving a hamiltonian structure up to constant factors

(iii) the Lie algebras of vector fields preserving a contact structure.

The aim of this paper is to consider an analytic version of such theorems, i.e. to prove the following

THEOREM: *Derivations of the Lie algebras of all holomorphic vector fields on Stein spaces are inner.*

Note that a local version of the above result follows from J. Heinze [6].

In view of the fact that real-analytic manifolds are real-analytic submanifolds of Stein spaces (Grauert [4]) we derive from the above theorem the similar fact for the case of the real-analytic vector fields:

THEOREM: *Derivations of the Lie algebras of all real-analytic vector fields on real-analytic manifolds are inner.*

Because the first cohomology group $H^1(L, L)$ of a Lie algebra L with adjoint representation is equal to $\mathrm{Der}(L)/\mathrm{ad}_L$, the first cohomology groups $H^1(L, L) = 0$ for L being the Lie algebras of vector fields from the above theorems.

By using the Stein spaces in the real-analytic case we overcome several technical difficulties which appear because of nonexistence of partition of unity to localize whole arguments.

It is well known that derivations of the algebra $C^\infty(N)$ of all smooth functions on a smooth manifold N are exactly the smooth vector fields.

We prove

THEOREM: *Derivations of the algebra of holomorphic functions on Stein spaces are holomorphic vector fields.*

Grauert's theorem allows us to prove also the same in $\mathbb{R}$-analytic case:

THEOREM: *Derivations of the algebra $C^\omega(N)$ of all real analytic functions on a real-analytic manifold N are the real-analytic vector fields on N.*

The Lie algebras of vector fields have been examined in [3]. It is proved there that automorphisms of such Lie algebras are "inner", i.e. they are generated by diffeomorphisms (bianalytic or biholomorphic mappings) of the underlying manifold.

The result of this paper demonstrate that, like in the finite-dimensional case, also for the Lie algebras of vector fields there is a close correspondence between their Lie algebras of derivations and their groups of automorphisms.

It is interesting if the presented results about derivations can be obtained via an explicit form of the above mentioned correspondence. In this paper they are proved independently.

1. Notation and preliminary remarks

Let $\mathscr{A}$ be the associative algebra of all holomorphic functions on a Stein space M of complex dimension r and let $\mathscr{U}$ be the complex Lie algebra of all holomorphic vector fields on M, i.e. the vector fields which in local coordinates $z_1, \ldots, z_r$ have the form $\Sigma_{i=1}^{r} f_i(\partial/\partial z_i)$, where f_i are holomorphic functions of variables $z_1, \ldots, z_r$.

Recall that $\mathscr{U}$ can be regarded as a Lie algebra of derivations of $\mathscr{A}$ and that $\mathscr{U}$ is a module over the ring $\mathscr{A}$ so that for all $f, g \in \mathscr{A}$ and $X, Y \in \mathscr{U}$ we have

$$[fX, gY] = (fX(g))Y - (gY(f))X + fg[X, Y].$$

We will use the following notation:

1. For $p \in M$, let I_p be the ideal of $\mathscr{A}$ of all functions vanishing at p.

2. Let I_p^n denote the ideal of $\mathscr{A}$ generated by

$$\{f_0 \ldots f_n : f_i \in I_p, i = 0, \ldots, n\}.$$

Note that $I_p^o = I_p$ and $I_p^m \subset I_p^n$ if $m \geq n$.

3. For natural n and $p \in M$, let

$$\mathscr{U}_p^n = \{X \in \mathscr{U} : j_p^n(X) = 0\},$$

where $j_p^n(X)$ denotes the n-th jet of X at p, and let $\mathscr{U}_p^k = \mathscr{U}$ for k being negative integers.

Note that $\mathscr{U}_p^n \subset \mathscr{U}_p^m$ if $n \geq m$ and

$$[\mathcal{U}_p^n, \mathcal{U}_p^m] \subset \mathcal{U}_p^{n+m} \tag{1.1}$$

for integer n and m.

Thus $\{\mathcal{U}_p^n\}_{n \in Z}$ is a filtration of $\mathcal{U}$.

Observe also that $\mathcal{U}_p^n$ are submodules of the module $\mathcal{U}$ over $\mathcal{A}$.

4. For an ideal J of $\mathcal{A}$, let $J\mathcal{U}$ denote the submodule of $\mathcal{U}$ generated by $\{fX : f \in J \text{ and } X \in \mathcal{U}\}$.

5. Let

(i) $\mathcal{O}$ be the analytic sheaf over M of germs of holomorphic functions on M,

(ii) $\mathcal{F}$ be the analytic sheaf over M of germs of holomorphic vector fields on M,

(iii) $\mathcal{S}_p^n$ be the analytic sheaf over M of germs of holomorphic vector fields which n-th jets at p vanish.

The algebras considered are the spaces of sections of respective sheaves: $\mathcal{U} = \Gamma(M, \mathcal{F})$, $\mathcal{U}_p^n = \Gamma(M, \mathcal{S}_p^n)$ and $\mathcal{A} = \Gamma(M, \mathcal{O})$.

Note also that all the sheaves are coherent.

6. Let $j_p(\mathcal{U}) = \underset{k}{\overset{\lim}{\longleftarrow}} \mathcal{U}/\mathcal{U}_p^k$ and denote by j_p the canonical projection from $\mathcal{U}$ to $j_p(\mathcal{U})$.

$j_p(\mathcal{U})$ has the natural Lie algebra structure and a filtration $\{j_p(\mathcal{U}_p^n)\}_{n \in Z}$ which used as a system of fundamental neighborhoods of $j_p(\mathcal{U})$ makes it a topological Lie algebra called the formal algebra of $\mathcal{U}$ at p.

The topology is separated and complete.

Note 1.1. For each $p \in M$ the Lie algebra $j_p(\mathcal{U})$ is isomorphic to the complex Lie algebra $\mathcal{L}$ of all formal vector fields in r-variables.

Proof: By Theorem A of Cartan $\Gamma(M, \mathcal{F}) = \mathcal{U}$ generates each stalk $\mathcal{F}_p$ as a $\mathcal{O}_p$ module.

Let $z_1, \ldots, z_r \in \mathcal{A}$ be local coordinates in a neighborhood of p with $z_i(p) = 0$ (there are local coordinates on Stein spaces which are global defined functions).

For a given $Y = \Sigma_{i=1}^r (\Sigma_\alpha a_{i,\alpha} z^\alpha)\partial_i$ (we denote as usual $\alpha = (\alpha_1, \ldots, \alpha_r)$, α_i natural numbers, $|\alpha| = \Sigma_{k=1}^r \alpha_k$ and $z^\alpha = z_1^{\alpha_1} \ldots z_r^{\alpha_r}$) choose $X_1, \ldots, X_n \in \mathcal{U}$ and $\hat{f}_k = \Sigma_\alpha b_{k,\alpha} z^\alpha \in \mathcal{O}_p$ such that $j_p^m(\Sigma_{k=1}^n \hat{f}_k X_k) = j_p^m(Y)$. Putting $Y_m = \Sigma_{k=1}^n (\Sigma_{|\alpha| \le m} b_{k,\alpha} z^\alpha) X_k$ we have $Y_m \in \mathcal{U}$ and $Y_m \overset{m}{\to} Y$ in the topology in $j_p(\mathcal{U})$. □

Corollary 1.2: *$\mathcal{U}_n^p = \{X \in \mathcal{U}_p^{n-1}: [X, Y] \in \mathcal{U}_p^{n-1}$ for all $Y \in \mathcal{U}\}$ for each $p \in M$ and integer n.*

To prove this it suffices to observe that the elements of $\mathcal{U}$ span the tangent space $T_p(M)$.

Note 1.3. For each $p \in M$ and natural n we have

$$\mathcal{U}_p^n = I_p^n \mathcal{U}.$$

PROOF: Let $z_1, \ldots, z_r \in \mathcal{A}$ be local coordinates in a neighborhood of p. If $X \in I_p^n \mathcal{U}$ then $X(z_i)$ – the local coordinates of X-belong to I_p^n. Thus $j_p^n(X) = 0$ and we get

$$I_p^n \mathcal{U} \subset \mathcal{U}_p^n. \tag{1.2}$$

There are $g_1, \ldots, g_m \in \mathcal{A}$ such that $(g_1, \ldots, g_m): M \to \mathbb{C}^m$ is an imbedding of M (R. Narasimham [8]) and $g_i(p) = 0$.

Put $Q = \{(i_1, \ldots, i_m): i_k \text{ natural and } \sum_{k=1}^m i_k = n+1\}$.

For $j = (i_1, \ldots, i_m) \in Q$, denote $g_1^{i_1} \ldots g_m^{i_m}$ by g^j.

Consider a sheaf homomorphism $\tau: \mathcal{F}^l \to \mathcal{S}_p^n$ (where $l = \text{card}(Q)$ and $\mathcal{F}^l$ is the product of l sheaves $\mathcal{F}$) defined on the stalks $\mathcal{F}_q^l$ in the following way:

$$(\hat{X}_{j,q})_{j \in Q} \mapsto \sum_{j \in Q} \hat{g}_q^j \hat{X}_{j,q},$$

where $\hat{X}_{j,q} \in \mathcal{F}_q$ and $\hat{g}_q^j$ denote the germs of g^j at q. Let us see that τ is an epimorphism.

In fact, if $q \neq p$, then there exists k such that $g_k(q) \neq 0$. Hence $\hat{g}_q^j$ is invertible for $j = (i_1, \ldots, i_m)$ with $i_k = n+1$ and $i_u = \dot{0}$ for $u \neq k$. This implies that τ is onto at q.

For the point p, there are $g_{i_1}, \ldots, g_{i_r}$ such that they are local coordinates in a neighborhood of p, since $(g_1, \ldots, g_m): M \to \mathbb{C}^m$ is regular at each point.

If $\hat{Y}_p \in \mathcal{S}_{p,p}^n$, then $\hat{Y}_p$ has the form

$$\hat{Y}_p = \sum_{k=1}^r \hat{a}_k \frac{\partial}{\partial g_{i_k}}$$

in local coordinates $g_{i_1}, \ldots, g_{i_r}$, and all germs $\hat{a}_k$ vanish at p with all their derivatives of degree $\leq n$. Then, $\hat{a}_k$ have the form

$$\hat{a}_k = \sum_{j \in Q} \hat{g}_p^j \hat{h}_{j,k}$$

for some germs $\hat{h}_{j,k}$ of holomorphic functions. Hence

$$\hat{Y}_p = \sum_{j\in Q} \hat{g}^j_p \left(\sum_{k=1}^{r} \hat{h}_{j,k} \frac{\partial}{\partial g_{i_k}}\right).$$

To see that τ is "onto" at p, it suffices to put

$$\hat{X}_{j,p} = \sum_{k=1}^{r} \hat{h}_{j,k} \frac{\partial}{\partial g_{i_k}}.$$

Let $\mathcal{T}$ be the sheaf $\operatorname{Ker}(\tau)$. Since $\mathcal{F}^1, \mathcal{S}^n_p$ are coherent analytic sheaves, $\mathcal{T}$ is a coherent analytic sheaf over M as the kernel of an epimorphism between two coherent analytic sheaves over M. From the exact sequence of sheaf homomorphism

$$0 \to \mathcal{T} \xrightarrow{i} \mathcal{F}^l \xrightarrow{\tau} \mathcal{S}^n_p \to 0$$

we obtain the exact sequence of the groups of cohomology with coefficients in the sheaves:

$$0 \to H^0(M, \mathcal{T}) \xrightarrow{i} H^0(M, \mathcal{F}^l) \xrightarrow{\tau} H^0(M, \mathcal{S}^n_p) \xrightarrow{\delta} H^1(M, \mathcal{T}) \to \dots$$

By the Theorem B of Cartan $H^1(M, \mathcal{T}) = 0$ and since the 0-groups of cohomology are the spaces of all global sections of the sheaves, we get the exact sequence:

$$0 \to \Gamma(M, \mathcal{T}) \xrightarrow{i} \Gamma(M, \mathcal{F}^l) \xrightarrow{\tau} \Gamma(M, \mathcal{S}^n_p) \to 0.$$

Hence $\tau : \Gamma(M, \mathcal{F}^l) \to \Gamma(M, \mathcal{S}^n_p)$ is "onto" and for every $Y \in \mathcal{U}^n_p$ there are $X_1, \dots, X_l \in \mathcal{U}$ such that $Y = \Sigma_{j\in Q}\, g^j X_j$. Since $g^j \in I^n_p$, the note follows by this and (1.2). □

Let $I_{p,n}$ be the ideal of $\mathcal{A}$ of all functions with vanishing n-th jets at $p \in M$.

Note 1.4. For each $p \in M$ and each natural n we have

$$I_{p,n} = I^n_p.$$

The proof is almost parallel to the proof of Note 1.3 and we omit it.

Note only that it suffices to consider a sheaf homomorphism between sheaves of germs of functions instead of sheaves of germs of vector fields.

2. Global approach

Observe that if D if an inner derivation of $\mathcal{U}$ then by (1.1) $D(X) \in \mathcal{U}_p^0$ for each $X \in \mathcal{U}_p^1$.

LEMMA 2.1: *If D is a derivation of $\mathcal{U}$ then $D(\mathcal{U}_p^4) \subset \mathcal{U}_p^0$ for each point $p \in M$.*

PROOF: Put $\Omega_p = \{X \in \mathcal{U}_p^1 : D(fX) \in \mathcal{U}_p^0 \text{ for all } f \in \mathcal{A}\}$.
From the equality

$$2g[fX, X] = [gfX, X] + [fX, gX],$$

which holds true for all $g, f \in \mathcal{A}$ and $X \in \mathcal{U}$, we get

$$-2D(gX(f)X) = [D(gfX), X] + [gfX, D(X)] + [D(fX), gX] + [fX, D(gX)].$$

If $X \in \mathcal{U}_p^1$, then by (1.1) the right side term belongs to $\mathcal{U}_p^0$ and thus $X(f)X \in \Omega_p$ for all $f \in \mathcal{A}$.
Denote by $\omega_p(X)$ the ideal of $\mathcal{A}$ equal to $\{g \in \mathcal{A} : gX \in \Omega_p\}$.
Then

$$X(f) \in \omega_p(X) \text{ for all } f \in \mathcal{A} \text{ and } X \in \mathcal{U}_p^1. \tag{2.1}$$

Observe that Ω_p is a Lie ideal of $\mathcal{U}_p^0$.
To see that, consider the equality

$$-D(Y(f)X) + D(f[X, Y]) = D([fX, Y]) = [D(fX), Y] + [fX, D(Y)].$$

If $X \in \Omega_p$, $f \in \mathcal{A}$, $Y \in \mathcal{U}_p^0$, then $D(Y(f)X) \in \mathcal{U}_p^0$, $[D(fX), Y] \in [\mathcal{U}_p^0, \mathcal{U}_p^0] \subset \mathcal{U}_p^0$ and $[fX, D(Y)] \in [\mathcal{U}_p^1, \mathcal{U}] \subset \mathcal{U}_p^0$.
Hence $D(f[X, Y]) \in \mathcal{U}_p^0$ for each $f \in \mathcal{A}$, i.e.

$$[\Omega_p, \mathcal{U}_p^0] \subset \Omega_p. \tag{2.2}$$

Now assume that $X \in \Omega_p$, $Y \in \mathcal{U}_p^0$ and $f \in \mathcal{A}$. We have

$$[X, fY] = X(f)Y + f[X, Y],$$

and since $[X, fY]$, $f[X, Y] \in \Omega_p$ by (2.2), we conclude that $X(f) \in \omega_p(Y)$. Because if $h \in \omega_p(gY)$ then $hg \in \omega_p(Y)$, we get

$$I_p\Omega_p(\mathscr{A}) \subset \bigcap_{Y \in \mathscr{U}} \omega_p(Y), \tag{2.3}$$

where $I_p\Omega_p(\mathscr{A})$ is an ideal of $\mathscr{A}$ generated by

$$\{fX(g) \colon f \in I_p,\ X \in \Omega_p \text{ and } g \in \mathscr{A}\}.$$

It is easy to see from (2.1) that $h^2X(f)X \in \Omega_p$ for all $h \in I_p^1$, $f \in \mathscr{A}$ and $X \in \mathscr{U}$. Since $2hg = (h+g)^2 - h^2 - g^2$, also $hgX(f)X \in \Omega_p$ for all $h, g \in I_p^1$, $f \in \mathscr{A}$ and $X \in \mathscr{U}$.

Thus, by (2.3),

$$I_p^4X(f)X(g) \subset \bigcap_{Y \in \mathscr{U}} \omega_p(Y) \tag{2.4}$$

for all $X \in \mathscr{U}$ and $f, g \in \mathscr{A}$.

The algebra $\mathscr{A}$ has the property that if $f_1, \ldots, f_m \in \mathscr{A}$ have no common zeros, then there exist $g_1, \ldots, g_m \in \mathscr{A}$ such that $\Sigma_{i=1}^m f_ig_i = \mathbb{1}$ (see e.g. [5] p. 244).

We will have shown in the next lemma that there exist $X_1, \ldots, X_m \in \mathscr{U}$ and $f_1, \ldots, f_m \in \mathscr{A}$ such that $X_i(f_i)$ have no common zeros. Then also $(X_i(f_i))^2$ have no common zeros and there exist $g_1, \ldots, g_m \in \mathscr{A}$ such that

$$\sum_{i=1}^m g_i(X_i(f_i))^2 = \mathbb{1}.$$

Let $h \in I_p^4$. Then, by (2.4),

$$\bigcap_{Y \in \mathscr{U}} \omega_p(Y) \ni \sum_{i=1}^m (hg_i)(X_i(f_i))^2 = h\left(\sum_{i=1}^m g_i(X_i(f_i))^2\right) = h,$$

which means that $I_p^4 \subset \bigcap_{Y \in \mathscr{U}} \omega_p(Y)$.

Because by Note 1.3 $I_p^4\mathscr{U} = \mathscr{U}_p^4$, the lemma is proved. □

LEMMA 2.2: *There exist $f_1, \ldots, f_m \in \mathscr{A}$ and $X_1, \ldots, X_m \in \mathscr{U}$ such that $\Sigma_{i=1}^m X_i(f_i) = 1$.*

We used this fact in [3] but the proof in that paper is false. We will prove this here using results of the theory of coherent analytic sheaves on Stein spaces as in the proof of Note 1.3.

PROOF: There is a family of functions $f_1, \ldots, f_m \in \mathcal{A}$ such that the rank of the mapping

$$(f_1, \ldots, f_m): M \to \mathbb{C}^m$$

is ≥ 1 at each point $p \in M$. (Such a mapping could even be regular at each point as in the proof of Note 1.3.)

It means that for each point $p \in M$ there is $i \in \{1, \ldots, m\}$ and a germ $\hat{X}_p$ of a holomorphic vector field at p such that

$$(\hat{X}_p(\hat{f}_{i,p}))(p) \neq 0, \tag{2.5}$$

for $\hat{f}_{i,p}$ being the germ of f_i at p.

Consider a sheaf homomorphism $\rho: \mathcal{F}^m \to \mathcal{O}$ defined on the stalk over p by

$$(\hat{X}_{1,p}, \ldots, \hat{X}_{m,p}) \mapsto \sum_{i=1}^{m} \hat{X}_{i,p}(\hat{f}_{i,p}).$$

Since by (2.5) there is $\hat{X}_{i,p}$ such that $\hat{X}_{i,p}(\hat{f}_{i,p})$ is invertible, ρ is an epimorphism.

Denote $\operatorname{Ker}(\rho)$ by $\mathcal{T}$ which is an analytic coherent sheaf over M. We have the exact sequence of sheaf homomorphisms:

$$0 \to \mathcal{T} \xrightarrow{i} \mathcal{F}^m \xrightarrow{\rho} \mathcal{O} \to 0$$

which generate the exact sequence

$$0 \to \Gamma(M, \mathcal{T}) \xrightarrow{i} \Gamma(M, \mathcal{F}^m) \xrightarrow{\rho} \Gamma(M, \mathcal{O}) \to 0.$$

Because the mapping $\rho: \Gamma(M, \mathcal{F}^m) \to \Gamma(M, \mathcal{O}) = \mathcal{A}$ is "onto", there are $X_1, \ldots, X_m \in \mathcal{U}$ such that $\Sigma_{i=1}^m X_i(f_i) = 1$. □

3. The main result

LEMMA 3.1: *If D is a derivation of $\mathcal{U}$ then for any point $p \in M$ there is induced an unique continuous derivation D_p of the formal algebra*

$j_p(\mathcal{U})$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{D} & \mathcal{U} \\ {\scriptstyle j_p}\downarrow & & \downarrow{\scriptstyle j_p} \\ j_p(\mathcal{U}) & \xrightarrow{D_p} & j_p(\mathcal{U}) \end{array}$$

PROOF: We need to show that for each integer n there is an integer k such that $D(\mathcal{U}_p^k) \subset \mathcal{U}_p^n$.

Let us put $k = n + 4$ and use induction.

If $n = 0$ then $D(\mathcal{U}_p^4) \subset \mathcal{U}_p^0$ by Lemma 2.1. Assume $D(\mathcal{U}_p^{i+4}) \subset \mathcal{U}_p^i$ for $i < n$. Then, for each $X \in \mathcal{U}_p^{n+4}$ and all $Y \in \mathcal{U}$ we have $[X, Y] \in \mathcal{U}_p^{n+3}$ and $[X, D(Y)] \in \mathcal{U}_p^{n+3}$. Hence $D([X, Y]) \in \mathcal{U}_p^{n-1}$ by the inductive assumption and we get

$$[D(X), Y] = D([X, Y]) - [X, D(Y)] \in \mathcal{U}_p^{n-1}$$

for all $Y \in \mathcal{U}$. Thus $D(X) \in \mathcal{U}_p^n$ by Corollary 1.2. □

THEOREM 3.2: *Derivations of the Lie algebra of all holomorphic vector fields on a Stein space are inner.*

PROOF: For a given derivation D we get by above lemma the induced derivations D_p of $j_p(\mathcal{U})$. By T. Morimoto [7] (or by J. Heinze [6]) D_p are inner as derivations of the Lie algebras of formal vector fields (see the Note 1.1).

Let $D_p = \mathrm{ad}_{Y_p}$ where $Y_p \in j_p(\mathcal{U})$.

Let $z_1, \ldots, z_r$ be local coordinates from $\mathcal{A}$ in a neighborhood U of a chosen $p \in M$ and $z_i(p) = 0$.

If $X \in \mathcal{U}$ then the coefficients x_i of $j_p(X) = \Sigma_{i=1}^r x_i \partial_i$ and the coefficients of $j_p(D(X))$ are formal power series in variables $z_1, \ldots, z_r$ which converge on U.

Because $j_p(D(X)) = [Y_p, j_p(X)]$, it is not hard to verify that the coefficients of Y_p also converge on U.

In fact, if $Y_p = \Sigma_{i=1}^r y_i \partial_i$ then by $D_p(z_k X) = y_k j_p(X) + z_k D_p(X)$ the formal power series $y_k x_i$ converge on U for $i = 1, \ldots, r$.

Choosing X such that $j_p^0(x_i) \neq 0$ we can divide $y_k x_i$ by x_i and conclude that y_k converges on U.

Hence Y_p defines a holomorphic vector field Y_U on U such that $[Y_U, X_{|U}] = D(X)_{|U}$. Obviously, such Y_U is on U exactly one. Covering M with such coordinate neighborhoods, we get in this way a global defined $Y \in \mathcal{U}$ satisfying

$$D(X) = \mathrm{ad}_Y(X) \text{ for all } X \in \mathcal{U}^{\cdot}. \quad \square$$

4. The real-analytic case

Now, let ${}^{\omega}\mathcal{A}$ be the algebra of all real-analytic functions on a real-analytic paracompact manifold N of dimension r, and let ${}^{\omega}\mathcal{U}$ be the Lie algebra of all real-analytic vector fields on N.

We are going to sketch a proof of the following

THEOREM 4.1: *Derivations of ${}^{\omega}\mathcal{U}$ are inner.*

PROOF: The proof is almost parallel to that in the complex case.

It is known (F. Bruhat and H. Whitney [2]) that N can be regarded as a real-analytic submanifold of a complex manifold $\mathcal{M}$ such that $\mathcal{M}$ has a covering with local coordinate patches $(\mathcal{U}_\alpha; {}^{\alpha}z_1, \ldots, {}^{\alpha}z_r)$ satisfying

$$N \cap U_\alpha = \{{}^{\alpha}z \in U_\alpha : {}^{\alpha}z_k = \text{real}, k = 1, \ldots, r\}.$$

If $f \in {}^{\omega}\mathcal{A}$, then we can extend f to a holomorphic function $\hat{f}$ on a neighborhood U of N in $\mathcal{M}$ with the same coefficients of its power series in the points of N.

By theorem of Grauert [4] there is a neighborhood V of N contained in U such that V is a Stein space.

By ${}^{\omega}I_{p,n}$ denote the ideal of all real-analytic functions on N which vanish at $p \in N$ with all their derivatives of degree $\leq n$. If $f \in {}^{\omega}I_{p,n}$, then $\hat{f} \in I_{p,n}$ and by Note 1.4 $\hat{f} \in I_p^n$, i.e. there are $g_1^j, \ldots, g_n^j$ holomorphic on V and vanishing at p such that

$$\hat{f} = \sum_{j=1}^{m} \prod_{k=1}^{n} g_k^j.$$

The functions $\mathrm{Re}(g_k^j)$ and $\mathrm{Im}(g_k^j)$ are real analytic and

$$f = \hat{f}|N = \mathrm{Re}\left(\sum_{j=1}^{m} \prod_{k=1}^{n} (\mathrm{Re}(g_k^j) + i\,\mathrm{Im}(g_k^j))\right)_{|N}.$$

Hence $f \in {}^{\omega}I_p^n$, i.e.

$$(4.1) \qquad {}^{\omega}I_{p,n} = {}^{\omega}I_p^n.$$

By Grauert [4] N can be imbedded in $\mathbb{R}^m$ for some m:

$$(f_1, \ldots, f_m): N \to \mathbb{R}^m,$$

where $f_1, \ldots, f_m \in {}^{\omega}\mathcal{A}$.

Let ∂_i $(i = 1, \ldots, m)$ be the coordinate vector fields in $\mathbb{R}^m$.

Having the natural scalar product in $\mathbb{R}^m$ (which is a real-analytic operation) we can obtain (as in [3], p. 17) vector fields $X_i \in {}^{\omega}\mathcal{U}$ $(i = 1, \ldots, m)$ such that $X_{i,p}$ is the image of $\partial_{i,p}$ under the orthogonal projection of $\mathbb{R}^m$ onto the tangent space $T_p(N)$ at each $p \in N$.

Considering $Y \in {}^{\omega}\mathcal{U}$ as a vector field on a submanifold in $\mathbb{R}^m$ we have

$$Y = \sum_{k=1}^{m} Y(f_k)\partial_k.$$

Therefore

$$(4.2) \qquad Y = \sum_{k=1}^{m} Y(f_k)X_k.$$

If $Y \in {}^{\omega}\mathcal{U}_p^n$ then $Y(f_k) \in {}^{\omega}I_{p,n}$ and by (4.1) and (4.2) $Y \in {}^{\omega}I_p^{n\omega}\mathcal{U}$, that proves the Note 1.3 for the real-analytic case.

To prove the analog of Note 1.1 observe that the germs of $X_1, \ldots, X_m$ as above generate the module of germs of real-analytic vector fields over the algebra of germs of real-analytic functions at each point $p \in N$.

Corollary 1.2, Lemma 2.1, and Lemma 3.1 follow as well without any changes in their proofs, and Lemma 2.2 is proved for the real-analytic case in Proposition 3.5 of [3].

Then, the proof of Theorem 4.1 is the same reasoning as in the proof of Theorem 3.2. □

5. Derivations of the algebras of functions

Because by Note 1.4 $I_{p,1} = I_p^1$, for X being a derivation of $\mathcal{A}$ (i.e. a linear operator of $\mathcal{A}$ into $\mathcal{A}$ satisfying the equality $X(fg) = X(f)g +$

$fX(g)$ for all $f, g \in \mathscr{A}$) we have

$$X(I_{p,1}) = X(I_p^1) \subset I_p. \tag{5.1}$$

Let $z_1, \ldots, z_r \in \mathscr{A}$ be local coordinates of M in a neighborhood of p and let $f \in \mathscr{A}$. Then

$$f - f(p) - \sum_{i=1}^{r} \frac{\partial f}{\partial z_i}(p) z_i \in I_{p,1}$$

and hence by (5.1)

$$(X(f))(p) = \sum_{i=1}^{r} \frac{\partial f}{\partial z_i}(p)(X(z_i))(p).$$

Putting

$$\hat{X}_p = \sum_{i=1}^{r} (X(z_i))(p) \frac{\partial}{\partial z_i} \in T_p(M)$$

we get a vector field $\hat{X}$ on M such that $\hat{X}(f)^{\cdot} = X(f)$ for all $f \in \mathscr{A}$.

Because $X(z_i) \in \mathscr{A}$ it is obvious that $\hat{X}$ is a holomorphic vector field.

The real-analytic case follows as well if we use (4.1) instead of Note 1.4. Thus we obtain

THEOREM 5.1.: *Derivations of the algebra of all holomorphic* (*real-analytic*) *functions on a Stein space* (*real-analytic manifold*) *M are exactly the holomorphic* (*real-analytic*) *vector fields on M.*

Note that in [3] $\mathscr{U}$ and ${}^{\omega}\mathscr{U}$ are regarded as Lie subalgebras of $\mathrm{Der}(\mathscr{A})$ and $\mathrm{Der}({}^{\omega}\mathscr{A})$ which are modules over $\mathscr{A}$ and ${}^{\omega}\mathscr{A}$ and it suffices to obtain the results about automorphisms and ideals.

Thus we know that $\mathscr{U}$ and ${}^{\omega}\mathscr{U}$ are not proper Lie subalgebras, i.e. $\mathscr{U} = \mathrm{Der}(\mathscr{A})$ and ${}^{\omega}\mathscr{U} = \mathrm{Der}({}^{\omega}\mathscr{A})$.

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(Oblatum 28-IV-1980 & 9-IX-1980)

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