COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 43, nº 2 (1981), p. 181-215

<http://www.numdam.org/item?id=CM_1981__43_2_181_0>

COMPOSITIO MATHEMATICA, Vol. 43, Fasc. 2, 1981, pag. 181–215

Printed in the Netherlands

COMPACT FIBERINGS OF HOMOGENEOUS SPACES. I

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Given a compact Lie group G and a closed subgroup H, the quotient space G/H is a highly symmetric smooth manifold. Furthermore, if K is a closed subgroup strictly between H and G, then one can fiber G/H over G/K with fiber K/H (compare [46]). Results due to W. Browder, D. Gottlieb, and several others show that under suitable conditions a converse is either true or very nearly so [2, 11]; namely, all smooth fiberings of G/H look much like those given by homogeneous spaces from Lie theory. This paper is a further attempt to discover the extent to which homogeneous fiberings of G/H can be used to describe all "nice" fiberings (e.g., smooth fiber bundles). We concentrate on homogeneous spaces G/H where no intermediate subgroup exists, in which case the aim is to show there are generally no "nice" fiberings.

One motivation for this study is a result from [2, 12], which states that even-dimensional projective spaces over the reals, complex numbers, quaternions, or Cayley numbers cannot be fibered nicely. This may be viewed as a generalization of the fact that there are no closed subgroups of the group $CL(2n+1)$ – where CL denotes O, U, or Sp – strictly between this group and $CL(2n) \times CL(1)$; the latter may be derived from the work of Borel and de Siebenthal, for example [7]. Thus the conjecture that G/H cannot be fibered if there are no intermediate subgroups suggests itself immediately. The principal results of this paper give evidence in favor of this conjecture, at least after it has been reformulated to exclude counterexamples that are more or less predictable from Lie theory itself (see Section 2). Specifically, we prove that the following homogeneous spaces cannot be fibered nicely:

(i) The quotient of a compact Lie group G by the normalizer of a

0010–437X/81050181-35$00.20/0

maximal torus, provided the latter is a maximal closed subgroup (Theorem 3.1).

(ii) Grassmann manifolds of 2-planes in complex or quaternionic n-space, $n \geq 5$ (Theorem 4.19).

(iii) Odd-dimensional quaternionic projective spaces except for the quaternionic projective line (Theorem 5.1).

(iv) Grassmann manifolds of 2-planes in real n-space for all but possibly a very sparse set of integers n (Theorem 6.13).

There are numerous other examples too, but they will be postponed to a later paper.

Here is a more specific description of this paper's contents: The first section discusses the homotopy-theoretic analog of a smooth fiber bundle (a *compact fibering* in Gottlieb's terminology [23]) and its relation to several more familiar notions. In the second section we formulate the conjecture on fibering homogeneous spaces precisely, and in the third section we determine the fiberability of a compact Lie group mod the normalizer of its maximal torus. The fourth section demonstrates the nonfiberability of complex and quaternionic Grassmannians of 2-planes, while the fifth section discusses odd dimensional quaternionic projective spaces. Our results on fibering real Grassmannians of 2-planes appear in Section 6. Finally, Section 7 gives examples of fiberable homogeneous spaces G/H with H maximal when either G is noncompact (but still semisimple!) or H does not have maximum rank, with speculations about alternate conjectures in such situations.

Acknowledgments

The motivation for this paper came from work of D.H. Gottlieb [2, 12, 23]. I am very grateful to him for his willingness to discuss various aspects of this problem and for several key suggestions and questions (which directly motivated Theorem 3.1 in particular). Results of S. Halperin on rational fibrations and minimal models [25, 26] were a direct stimulus for me to test Conjecture 2.3 on Grassmannians of 2-planes; I appreciate very much his comments on my efforts, some of which led directly to the present proof of Theorem 4.3. In this connection I would also like to thank Larry Smith for sharing with me his private notes on calculating signatures of Grassmann manifolds. Comments by J.F. Adams, J.C. Becker, D. Davis, W.-C. Hsiang, H. Glover, and W. Neumann have also been valuable, often setting me straight in places where I had first said something completely stupid. Surely others deserve mention too, but I regret that I cannot recall all the names. Finally, it is a pleasure to acknowledge that this work was supported in part by Grants GP-19530A2, MPS74-03609, MCS76-08794, MCS78-02913, and MCS78-02913A1, from the National Science Foundation.

1. Compact fiberings

Following Gottlieb [23], we define a *compact fibering* of a finite complex X to be a Hurewicz fibration $F \to E \to B$ with E homotopy equivalent to X and F, B both homotopy equivalent to finite complexes. Of course, the most obvious examples are smooth fiber bundles with F and B compact smooth manifolds with boundary (if both boundaries are nonempty, then E has corners). In fact, if E is a compact finite-dimensional ANR (hence homotopic to a finite complex X by results of J. West [50]), then *every* fiber bundle with total space E gives rise to a compact fibering of X:

PROPOSITION 1.1: *Let $F \to E \to B$ be a locally trivial fiber bundle with (say) B compact T_2 (see* [38] *for remarks about T_2). Then F and B both have the homotopy types of finite complexes, and $F \to E \to B$ is exact* [45] *(hence is a compact fibering).*

PROOF: It follows from [45, p. 96] that $F \to E \to B$ is a Serre fibration, and exactness follows if we know both F and B have the homotopy types of complexes. Because of this and the result of West, it suffices to prove that F and B are both compact (already known) finite-dimensional ANR's. This is easy for F, which is closed in E and also is a cartesian factor of some open set in E (the latter automatically being an ANR). Furthermore, an argument of C.B. de Lyra [38] yields this also for B; the only changes needed in de Lyra's argument are to replace S^{2n+1} with E (all he uses is local contractibility) and to replace the dimension-theoretic assertion with the weak statement that P, Q compact metrizable and $\dim P \times Q$ finite imply $\dim P$ finite. ■

REMARK: The following result of A. Edmonds [15] sheds still further light on topological fiber bundles with total space a manifold: Suppose $F \to E \to B$ is a fiber bundle with E and F compact manifolds, and assume $\operatorname{codim} F \geq 5$. Then there is a fiber bundle $F \to E \to B'$ with B' a manifold; in fact, the latter is induced by a map

$f: B' \to B$ and the canonical map $E = f^*E \to E$ is homotopic to the identity. As noted in [15], there are many examples where B itself is not a manifold.

While discussing the recognition of compact fiberings, we mention another basic fact:

THEOREM 1.2: *Suppose that E and B are closed* CAT *manifolds,* CAT = DIFF *or* TOP, *and $f: E \to B$ is a* CAT *submersion* [41, 44]. *Then f is a* CAT *fiber bundle.* ■

This is due to Ehresmann for DIFF [16] and follows from more general results of Siebenmann for TOP [44, pp. 150–151]. It is of course commonplace to contrast this with the corresponding results for submersions with noncompact unbounded domains [41].

There are several other ways in which compact fiberings are strongly related to the more common notion of a fiber bundle. One is given by the fiber smoothing theorems of A. Casson [12], which state that compact fiberings are equivalent to smooth bundles with fibers that are interiors of compact manifolds with boundary, and one can get compact manifolds with boundary if one is willing to multiply the fiber by a suitable torus. A second link is given by a remarkable formula first recognized by F. Quinn [42]. Following Bredon [9], we shall use the term *Poincaré-Wall complex* to denote a finite complex satisfying Poincaré duality with arbitrary local coefficients as in [49] (these include all closed manifolds).

THEOREM 1.3: (Quinn's Formula, Special Case). *Let $F \to E \to B$ be a fibration with all three spaces homotopy equivalent to finite complexes F', E', B'. Then E' is a Poincaré-Wall complex if and only if F' and B' are. Furthermore, the formal dimension of E is the sum of the formal dimensions of B and F.*

Several proofs exist; see [24] for one in print (also compare [54]). Although the dimension formula is not stated explicitly, it follows immediately from most if not all proofs of Quinn's Formula. ■

Gottlieb has defined a finite complex X to be *prime* if the only compact fiberings of X have either contractible fibers or contractible bases. For connected complexes he has also considered a weaker notion we shall call *connectedwise prime* complexes, for which one considers only compact fiberings with connected fibers. Given such definitions, the following question is an obvious one:

(1.4) *Which finite (resp., connected finite) complexes are prime (resp., connectedwise prime)?*

One's initial intuition is that "almost all" finite complexes are prime, but – even ignoring the question of formulating it precisely – this question seems well beyond current skills in general. Our purpose here is to look at some special cases where primeness can be compared to basic mathematical patterns in other contexts; we describe this more clearly in Section 2. We shall close this section with the simplest possible example of a prime complex:

PROPOSITION 1.5: *A one point space is prime.*

PROOF: If $F \to E \to B$ is a compact fibering with E contractible, then Quinn's Formula implies B is a zero-dimensional Poincaré-Wall complex. In addition, B is connected. But Wall has shown that all such complexes are contractible [49]. (This can also be shown directly in other ways.) ∎

This result overlaps (but does not contain) classical theorems on the nonfiberability of $\mathbb{R}^n$ as a fiber bundle with compact fiber (see [47, pp. 300–301] for a summary of the literature). Although the proof of Proposition 1.5 is certainly quite trivial, the result does illustrate the significance of Quinn's formula and Wall's result [49] for the study of compact fiberings of closed manifolds.

2. Homogeneous spaces

We have already stated that the following result is our starting point:

THEOREM 2.1: *Let $\mathbb{A}$ denote the reals complex numbers, quaternions, or Cayley numbers. Then there are no Hurewicz fibrations $F \to E \to B$ with E homotopy equivalent to the equivalent to the projective space $\mathbb{A}P^{2n}$ ($n = 1$ only for the Cayley numbers) and B, F homotopy equivalent to noncontractible finite complexes.* ∎

Apparently at least some version of this result was known to A. Borel many years ago. Partial results first appeared in [2], and the result in full generality appears in [12].

Since $\mathbb{A}P^{2n}$ is the homogeneous space of a compact Lie group, by the remarks in the introduction we know that 2.1 generalizes known results about $CL(2n)\times CL(1)$ being a maximal closed subgroup of $CL(2n+1)$ for $CL = O$, U, or Sp (see Routine Exercise 7.3). Since the Cayley projective plane is the homogeneous space F_4/Spin_9 [5], this also generalizes the fact that Spin_9 is a maximal closed subgroup of F_4. However, it is too much to ask that G/H never fiber if H is a maximal closed subgroup of G. The easiest counterexamples are the odd-dimensional projective spaces $\mathbb{C}P^{2n+1} = U(2n+2)/U(2n+1)\times U(1) = SU(2n+2)/\rho U(2n+1)$, where $\rho\colon U(a)\to U(a)\times U(1)$ takes A into $(A, \det A^{-1})$. By [7] we know $\rho U(2n+1)$ is a maximal closed *connected* subgroup of $SU(2n+2)$, and routine matrix computations imply it is its own normalizer (7.3). However, one has the well-known fiber bundles

$$S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{K}P^n \tag{2.2}$$

arising from the free linear S^3 action on S^{4n+3}. There is, however, a simple group-theoretic explanation for this: The subgroup $Sp(n+1)$ acts transitively on $\mathbb{C}P^{2n+1}$, so that $Sp(n+1)/\rho U(2n+1)\cap Sp(n+1) = \mathbb{C}P^{2n+1}$, but $\rho U(2n+1)\cap Sp(n+1) \cong Sp(n)\times S^1$ is *not* a maximal closed subgroup of $Sp(n+1)$ because $Sp(n)\times Sp(1)$ contains it. One can avoid this by excluding all G/H where some closed proper subgroup $\Gamma \subsetneq G$ acts transitively by translation; in standard terminology, we shall consider only homogeneous spaces for which G acts *irreducibly transitively* on G/H. With this in mind, we may state the fibering problem as follows:

CONJECTURE 2.3: *Let G be a connected compact Lie group, and let H be a closed subgroup of G such that*
(i) *H is a maximal closed subgroup,*
(ii) *G acts irreducibly transitively on G/H,*
(iii) *H contains a maximal torus of G.*
Then G/H is prime.

The third condition is needed to eliminate some counterexamples given in Section 7, but it also serves (i) to let us use the work of Borel and de Siebenthal on the classification of subgroups having maximum rank, (ii) to make the following simple observation a useful tool:

PRODUCT FORMULA 2.4 (compare [2]): *If $F\to E\to B$ is a compact fibering, then the Euler characteristics satisfy $\chi(E) = \chi(B)\chi(F)$.* ■

Rather than discuss other tools that are necessary, we shall proceed to our verifications of Conjecture 2.3 in special cases.

3. Reduced flag manifolds

For certain geometrical reasons the homogeneous space $U(n)/T^n$ is often called a *complex flag manifold*, and this has carried over to the name *(generalized) flag manifold* for arbitrary homogeneous spaces G/T with G connected compact Lie and T a maximal torus. If we let $N(T)$ denote the normalizer of T in G, then $G/T \to G/N(T)$ is a finite covering obtained from a free action of the Weyl group $W(G) = N(T)/T$ on G/T, and the base has Euler characteristic one. For these reasons we call $G/N(T)$ a *reduced flag manifold.* Such manifolds are our first test cases here; I wish to thank D. Gottlieb for suggesting them as potentially simple examples.

Without loss of generality we may assume G is simple; otherwise the splitting of some finite covering as $G_1 \times G_2$ and the corresponding splitting for the normalizer implies $N(T)$ is not a maximal closed subgroup.

THEOREM 3.1: *Let G be a simple, connected, compact Lie group. Then the following are equivalent*:
(i) *$N(T)$ is a maximal closed subgroup.*
(ii) $H_1(G/N(T); \mathbb{Z}) \cong \mathbb{Z}_2$
(iii) *$G/N(T)$ is prime.*

PROOF: We shall prove the above theorem in the sequence (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(i), and (i)$\Rightarrow$(ii). Of course (iii)$\Rightarrow$(i) is trivial by the observations already made, and therefore we shall not consider it further.

The proof of (ii)$\Rightarrow$(iii) begins with two basic observations:

(3.2) *$G/N(T)$ is rationally acyclic.*

(3.3) *If $F \to E \to B$ is a compact fibering with E homotopic to $G/N(T)$, then the Euler characteristics of F and B are* 1.

PROOF OF (3.2): The integral cohomology of G/T is torsion free, concentrated in even dimensions, and of total rank $|N(T)/T| =$ index of covering $G/T \to G/N(T)$. Therefore a transfer argument implies that $H^*(G/N(T); \mathbb{Q})$ is a direct summand of $H^*(G/T; \mathbb{Q})$ with Euler characteristic 1. This happens only if $G/N(T)$ is rationally acyclic. ■

PROOF OF (3.3): By 2.4 we know that $1 = \chi(E) = \chi(F)\chi(B)$, and hence $\chi(F) = \chi(B) = \pm 1$. On the other hand, as noted in [2, 12] the projection map $H^*(B; \mathbb{Q}) \to H^*(G/N(T); \mathbb{Q})$ is then a monomorphism. Since $G/N(T)$ is rationally acyclic, the same must be true of B, which implies $\chi(F) = \chi(B) = 1$. ■

These have the following simple consequence:

(3.4) *Under the assumptions of* 3.3, *the fiber F is arcwise connected.*

PROOF OF (3.4): Since $E = G/N(T)$ is arcwise connected, the exact sequence of the fibration ends with $\to \pi_1(B) \to \pi_0(F) \to \pi_0(E) = \{pt.\}$. It follows that each component of F is homotopy equivalent to the other, so that $\chi(F) = n\chi(F_0)$, where F_0 is an arbitrary component and n is the number of components. Since $\chi(F) = 1$, this means n must also equal 1. ■

We now remark that Quinn's Formula is a powerful statement that allows us to handle compact fiberings with the same ease as smooth fiber bundles. For example, this allows us to form first Stiefel–Whitney classes and oriented double coverings; specifically, the first Stiefel–Whitney class arises from the homomorphism $\pi_1 \to \mathbb{Z}_2 = \{\pm 1\}$ induced by the deck transformation action of π_1 on a twisted orientation class in the universal covering (see [49]).

With these facts at our disposal, we now prove (ii) $\Rightarrow$ (iii). Suppose that we have a nondegenerate compact fibering $F \to E \to B$ with E homotopy equivalent to $G/N(T)$ and $H_1(G/N(T)) = \mathbb{Z}_2$ (Note: The class of all G where (ii) applies is nonempty because $\pi_1(SU_n/N(T))$ is the symmetric group on n letters and H_1 is the abelianization of π_1). Since G is simple, it is clear that $\dim E = \dim G/N(T) > 0$. But a connected Poincaré–Wall complex of formal dimension zero is contractible, so in our situation the numbers $b = \dim B$ and $f = \dim F$ are both positive (both are connected).

Since B and E are rationally acyclic and have positive dimension, they are both nonorientable. Furthermore, the Poincaré–Hurewicz theorem for π_1 and the connectedness of F show that $H_1(E)$ maps onto $H_1(B)$. Applying the assumption $H_1(E) \cong \mathbb{Z}_2$, we see that the projection $H_1(E) \to H_1(B)$ must be an isomorphism. In particular, the pullback of the oriented double covering $\hat{B} \to B$ to E must be the oriented double covering $\hat{E} \to E$. It follows that the fiber inclusion $F \to E$ lifts to $\hat{E}$ and the sequence $F \to \hat{E} \to \hat{B}$ is again a compact fibering.

Since $\hat{E}$ has a universal covering space homotopy equivalent to G/T, it follows that $H^*(\hat{E}, \mathbb{Q})$ injects into $H^*(G/T; \mathbb{Q})$, which is concentrated in even dimensions. Also, the Euler characteristic of $\hat{E}$ is 2 because it is a double covering of $E \simeq G/N(T)$. Therefore $\hat{E}$ must be a rational cohomology sphere of algebraic dimension $b+f = \dim G/N(T)$ ($=\dim E$, etc.).

On the other hand, the fiber space transfer [3, 12] associated to $F \to E \to B$ shows that the projection induces a monomorphism $H^*(\hat{B}; \mathbb{Q}) \to H^*(E; \mathbb{Q})$. But $H^b(\hat{B}; \mathbb{Q}) = \mathbb{Q}$ by orientability and $H^b(\hat{E}; \mathbb{Q}) = 0$ by the previous paragraph combine to give us a contradiction. In particular, the hypothetical compact fibering $F \to E \to B$ cannot exist. ■

We now come to the *Proof that* (i) *implies* (ii): We shall prove the contrapositive; namely, if $H_1(G/N(T)) \not\cong \mathbb{Z}_2$ then there is a closed subgroup K strictly between $N(T)$ and G. To do this, we must use the classification theory for simple compact Lie groups to determine the possibilities for $H_1(G/N(T))$.

(3.5) *Let G be a connected compact Lie group. Then $H_1(G/N(T)) \cong \mathbb{Z}_2$ if G is of type A_n, D_n, E_6, E_7, or E_8, and $H_1(G/N(T)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if G is of type B_n, C_n, F_4, or G_2.*

PROOF: The group $H_1(G/N(T))$ is merely the abelianized Weyl group $W^{\#}(G)$, and this may be computed quite easily using the Coxeter presentation (a very accessible treatment appears in [4]). It follows from the specifics of the Coxeter presentation that $W^{\#}(G)$ is generated by symbols $v_1, \ldots, v_r$ corresponding to the vertices of the Dynkin diagram of G, with relations $2v_i = 0$ and $c_{ij}(v_i + v_j) = 0$, where the c_{ij} are also expressible via the Dynkin diagram; namely, $c_{ij} = 3$ if a single line joins v_i to v_j and c_{ij} is even otherwise. It follows that $H_1 = W^{\#}(G) = \mathbb{Z}_2$ if the Dynkin diagram of G has only single lines (the A, D, and E cases), while $H_1 = W^{\#}(G) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in the remaining cases (the B, C, F, and G cases). ■

In each of the cases B_n, C_n, F_4, and G_2 we shall exhibit an explicit subgroup strictly between $N(T)$ and G. Consider first the B_n case, where we may take $G = SO_{2n+1}$, $n \geq 2$. In this case $N(T)$ is isomorphic to the wreath product $\Sigma_n \int O_2$; the latter is being embedded in $SO_{2n+1} \subseteq O_{2n+1}$ by the map $j \oplus \det(j)$: $\Sigma_n \int O_2 \to O_{2n} \times O_1 \subseteq O_{2n+1}$, where j: $\Sigma_n \int O_2 \to O_{2n}$ is inclusion and "det" refers to the determinant map. This embedding extends to an embedding

$id \oplus \det(id)$: $O_{2n} \to O_{2n} \times O_1 \subseteq O_{2n+1}$ which also factors through SO_{2n+1}, and therefore the subgroup O_{2n} is strictly between $N(T)$ and SO_{2n+1}. Next consider the C_n case, where $G = Sp_n$. If $N(S^1 : S^3)$ denotes the normalizer of S^1 in S^3, then $N(T)$ is isomorphic to the wreath product $\Sigma_n \int N(S^1 : S^3)$. But this group is contained in the larger wreath product $\Sigma_n \int S^3 \subseteq Sp_n$. This disposes of the classical cases.

As one would expect, the exceptional cases must be treated more specifically. Since the case G_2 is easier, we shall do it first. Consider the transitive linear action of G_2 on S^6 with isotropy subgroup SU_3. Since G_2 acts transitively and linearly on S^6, it also acts transitively on $\mathbb{R}P^6$, and the isotropy group of this action must be a two component extension DSU_3 of SU_3. It follows that $\chi(G/DSU_3) = 1$; we claim that DSU_3 contains the normalizer of T. To see this, let $N'(T)$ denote the normalizer of T in DSU_3. The maximal torus theorem implies that $\chi(G/N(T)) = 1$ even if G is disconnected ($N(T)$ meets every component of G [30]), and from this it follows that the order of $N'(T)/T$ has order $\chi(DSU_3/T) = \chi(G_2/T) = \text{order } N(T)/T$. Since $N'(T)/T$ is a subgroup of $N(T)/T$, this implies they are equal, from which $N(T) = N'(T) \subseteq DSU_3$ follows.

For the case of F_4, considerably more is required. We use the standard realization of F_4 as the automorphism group of the exceptional Jordan algebra $\mathbb{M}_3^8$, whose elements are Hermitian 3×3 matrices over the Cayley numbers [32]. Let H be the subgroup of F_4 that leaves the 3-dimensional subspace of diagonal matrices pointwise fixed. Then H is of type D_4 (compare [32, 33]); moreover, the representation of H on $\mathbb{M}_3^8$ splits as a 3-dimensional trivial representation plus three mutually inequivalent 8-dimensional irreducible representations (see [32, p. 144] and [33, p. 18] for the Lie algebra version of this statement). It follows from this and the work of [31, 34] that H must be isomorphic to the simply connected group Spin_8. Define a group $N\ \mathrm{Spin}_8$ to be the subgroup of automorphisms that leave the diagonal matrix subspace *setwise* fixed, the action of *each element corresponding to a fixed permutation* of coordinates on $\mathbb{R}^3$. This group is strictly larger than Spin_8, for it contains a copy of Σ_3 acting on M_3^8 by permuting the rows and columns. In fact, $N\ \mathrm{Spin}_8$ is a semidirect product of Spin_8 with Σ_3. Elementary computations show that $\chi(F_4/\mathrm{Spin}_8) = 6$, and consequently $\chi(F_4/N\ \mathrm{Spin}_8) = 1$. It now follows as in the case $G = G_2$ that $N\ \mathrm{Spin}_8$ contains $N(T)$. ■

(3.6) Final Remarks: The argument for (ii) $\Rightarrow$ (iii) actually shows more: *If M^m is a closed manifold that is rationally acyclic, nonorientable, and has $H_1(M^m) = \mathbb{Z}_2$, then M^m is prime.* The homogeneous spaces G/K constructed above – with K strictly between $N(T)$ and G – all have this property. Of course, if G has type B_n, then $G/K = \mathbb{R}P^{2n}$, while if G has type G_2, then $G/K = \mathbb{R}P^6$. However, *in the other cases* (C_n, F_4) *this yields new examples of prime homogeneous spaces.* ■

4. Manifolds of complex and quaternionic two planes

In many respects, Grassmann manifolds give the dominant examples of homogeneous spaces G/H with H a maximal subgroup of maximum rank. For instance, suppose we consider the following alternate to Problem 2.4.

PROBLEM 4.1: *Let G be a connected compact simple Lie group, and suppose that H is a connected subgroup of maximum rank. Furthermore, assume that no closed* CONNECTED *subgroup lies strictly between H and G, and G acts irreducibly transitively on G/H. Is G/H connectedwise prime?*

REMARKS: If G is only semisimple, then H splits into factors corresponding to the simple factors of G [7], and thus the question above has maximum generality. On the other hand if $G = SU_4$ and $H = U_2 \times U_2 \cap SU_4$, then H is a maximal connected subgroup whose normalizer is generated by H and the matrix switching the first and last two coordinates in $\mathbb{C}^4$.

The pairs (G, H) satisfying the conditions except irreducible transitivity are exactly those listed by Borel and de Siebenthal in a table at the end of [7]. Specifically, their work shows that G/H must be a Grassmann manifold of quaternionic, complex, or *oriented* real k-planes (with not *both* the dimension and codimension odd in the latter case), a Hermitian symmetric space of the form Sp_n/U_n, a homogeneous space $F_n = SO_{2n}/U_n$, or one of about 15 exceptional cases such as G_2/SO_4 or $F_4/\mathrm{Spin}_9 = \mathbb{C}ayP^2$. It is well known that F_n fibers over S^{2n-2} with fiber F_{2n-1}, and this fibration corresponds to the fact that the subgroup SO_{2n-1} acts transitively on F_n by translation. Thus Grassmann manifolds, the much smaller family Sp_n/U_n, and a finite list of remaining cases are the only objects that arise in connected fibering problem 4.1.

Of course, projective spaces are special cases of Grassmann manifolds, and accordingly one is led directly to ask if the nonfiberability results of [2, 12] can be pushed just a little further – say to the remaining projective spaces and to manifolds of two-planes. Since the complex and quaternionic two-plane manifolds are the easiest to study, we shall deal with them in the present section.

The following consequence of work done by D. Gottlieb [20, 21] is very useful for our purposes:

LEMMA 4.2: *Let X be a finite complex, and let $F \xrightarrow{i} E \to B$ be a compact fibering of X. Assume that the Euler characteristic of X is nonzero. Then $i_*: \pi_1(F, *) \to \pi_1(E, *)$ is a monomorphism. In particular, if F is connected and X is* 1*-connected, then F and B are also* 1*-connected.*

PROOF: Since $O \neq \chi(X) = \chi(B)\chi(F)$ by 2.4, we know that $\chi(F)$ is also nonzero. By [20, 21, 22], the latter implies that the boundary map $\pi_2(B) \to \pi_1(F)$ is trivial, and the assertion about i_* follows immediately from the homotopy exact sequence of a fibration. ■

Given the relative ease with which one can handle simply connected spaces in general, the final statement of Lemma 4.2 is quite helpful in studying fiberability problems such as the following one:

THEOREM 4.3: *If $n \geq 4$, the Grassmann manifolds $G_{n,2}(\mathbb{F})$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{K}$, is connectedwise prime.*

Note. If $n = 3$ the Grassmann manifold is a projective plane, and if $n = 2$ it is a point.

PROOF: Only finitely many of the rational homotopy groups of a 1-connected homogeneous space are nonzero, and accordingly such spaces can be studied using results of S. Halperin [25, 26]. Specifically, a minimal model for the rational homotopy type of $G_{n,2}(F)$ has polynomial generators in dimensions d and $2d$ (where $d = \dim_R \mathbb{F}$), exterior generators in dimensions $nd - 1$ and $nd - d - 1$, and differentials on the exterior generators corresponding to the two polynomial relations in $H^*(G_{n,2}(\mathbb{F}); \mathbb{Q})$.

Suppose we are given a compact fibering of $G_{n,2}(\mathbb{F})$ with connected fiber. The work of Halperin on rational fibrations imposes some very strong conditions [26]. For example, F and B (which are 1-connected by 4.2) both have finite rational homotopy, the Serre spectral sequence for $F \to E \to B$ collapses over the rationals, and if $\mathbb{P}(X)$ denotes the rational Poincaré polynomial of a finite complex X, we have

$$\mathbb{P}(B) = \frac{(1-t^{dn})(1-t^{d(n-1)})}{(1-t^{dr})(1-t^{ds})}$$

(4.4)

$$\mathbb{P}(F) = \frac{(1-t^{dr})(1-t^{ds})}{(1-t^{d})(1-t^{2d})}.$$

It follows that

$$\dim \mathbb{F} = d(r+s-3). \tag{4.5}$$

The dimension of $G_{n,2}(\mathbb{F})$ is $d(2n-4)$, which is divisible by 4 because $d=2$ or 4. Consequently we may consider the signature of this manifold, at least up to sign. Since everything is 1-connected, an argument due to Chern, Hirzebruch, and Serre [13] implies that

$$\operatorname{sgn}(E) = \operatorname{sgn}(B)\operatorname{sgn}(F), \tag{4.6}$$

where $E = G_{n,2}(\mathbb{F})$. The latter may be given as follows:

(4.7) *The signature of $G_{n,2}(\mathbb{F})$ is equal to $\left[\frac{n}{2}\right]$ (brackets denoting the greatest integer function), which is also $\dim_{\mathbb{Q}} H^{d(n-2)}(G_{n,2}(\mathbb{F}); \mathbb{Q})$. Consequently, the cup product form in the middle dimensions is (positive or negative) definite.*

Remarks on (4.7): There are several different methods for proving 4.7. For example, the Schubert calculus implies that the dual Schubert classes $[j, n-j-1]^*$ form an orthonormal basis of the middle dimensional cohomology (to do this, combine some results from [29] – specifically, Theorem II on p. 352, Theorem I on p. 327, and Theorem II on p. 331). In the *complex* case one can use Hodge theory, first noting that all cohomology has type (k, k) and then using the Hodge signature theorem (compare [28]); the quaternionic case then follows by analogy. Finally, one can do this directly from the cohomology of the Grassmann manifold and the flag manifold $U_n/U_{n-2} \times U_1 \times U_1$ which fibers over it; this approach has been carried out by L. Smith, who also mentioned to me that a general signature formula for Grassmann manifolds was known to R. Stong. ■

In particular, (4.7) implies that the signatures of $G_{n,2}(\mathbb{F})$, B, and F are all nonzero; this has another obvious consequence:

(4.8) *The dimensions of F and B are both divisible by four.*

A further consequence of (4.8) is the following:

(4.9) *One of the numbers r, s in* (4.4) *is even, and the other is odd.*

PROOF OF (4.9): Let $u = t^d$, so that $\mathbb{P}(F)$ is in fact a polynomial in u. Because of this, we know that $(1+u)$ divides the product $(1-u^r)(1-u^s)$, which can only happen if at least one of r, s is even. On the other hand, $\dim F = d(r+s-3)$ must be divisible by 4 if $\mathbb{F} = \mathbb{C}$ to ensure $\operatorname{sgn} F \neq 0$, and likewise it must be divisible by 8 if $\mathbb{F} = \mathbb{K}$ (if $\dim F \equiv 4 \bmod 8$ in the quaternionic case, the middle dimension has no rational cohomology). In any case $r+s-3$ is even, and therefore not both r and s are even. ■

The balance of the proof of Theorem 4.3 involves a close scrutiny of the conditions imposed on the middle dimensional rational cohomology of $G_{n,2}(\mathbb{F})$ by the compact fibering. This begins with some conclusions about the cohomology of F and B near *their* middle dimensions.

(4.10) *The group $H^{dk}(F; \mathbb{Q})$ is nonzero if $0 \leq k \leq r+s-3$.*

This follows immediately by inspection of the Poincaré polynomial. ■

(4.11) *Let M be defined by $2dM = \dim B$ (hence $M = n - \frac{1}{2}(r+s+1)$). Then $H^{dj}(B; \mathbb{Q}) = 0$ if $0 < |j - M| \leq (r+s-3)/2$.*

If (4.11) were false, then the collapse of the Serre spectral sequence for $F \to E \to B$, (4.10), Poincaré duality, and the multiplicative properties of the Serre spectral sequence would yield a nonzero subspace of the middle dimensional cohomology of $G_{n,2}(\mathbb{F})$ that would be self orthogonal under cup product. ■

Of course, the rational cohomology of B is fully given by (4.4), and the objective now is to prove that (4.4) and (4.11) are inconsistent. Here is the first step:

(4.12) *Suppose that $j \leq n-2$. Then $H^{dj}(B) \neq 0$ if and only if $j = xr + ys$, where x, $y \geq 0$ are integers.*

REMARK: It follows that $x < (n-1)/r$ and $y < (n-1)/s$, but these are not really important.

PROOF OF 4.12:

Case A. Suppose $r, s \leq n-2$. Then the minimal model gives a $(dn-d-1)$-connected map

$$B \otimes \mathbb{Q} \rightarrow K(\mathbb{Q}, dr) \times K(\mathbb{Q}, ds),$$

which in turn gives the Poincaré polynomial of B through dimension $dn-d-2$. Inspection of the polynomial shows the assertion of (4.12) to be true.

Case B. At least one of $r, s \geq n-1$; if both are, then B is rationally acyclic and therefore contractible (being a Poincaré complex), and we are done. So let us assume $r \geq n-1 > s$. Then one has a $(dn-d-1)$-connected map $B \otimes \mathbb{Q} \rightarrow K(\mathbb{Q}, ds)$, and using this one can again verify the given assertion. ■

CONCLUSION OF PROOF OF THEOREM 4.3: We let M be given as in (4.11). If $\dim F = 0$, then it follows that F is contractible (satisfying Poincaré duality) and we are done. Thus in any case we may assume $\dim F \geq 4$, and, since all rational cohomology lies in dimensions divisible by d, in the quaternionic case we may assume $\dim F \geq 8$. In other words, we may assume $\dim F \geq 2d$. It follows that $\dim B \leq d(2n-6)$, and accordingly $M \leq n-3$. Thus (4.12) applies with $j = M$, so that $M = xr + ys$ with $x, y \geq 0$.

Case A. M can be written as a positive multiple of r *or* a positive multiple of s, *but not both*, and $M \neq xr + ys$ *with both* x *and* y *positive*. Inspection of the Poincaré polynomial of B implies that $H^{dM}(B; \mathbb{Q})$ is one-dimensional, and therefore the signature of B is ± 1. It follows that the middle dimensional cohomology of F must have dimension at least $\operatorname{sgn} G_{n,2}(F) = \left[\frac{n}{2}\right]$. But the middle dimension for F is $d(r+s-3)$. Look at $\mathbb{P}(F)$ to estimate its coefficient in that dimension. An upper estimate is given by

$$\dim H^{d(r+s+-3)/2}(F; \mathbb{Q}) \leq 1 + \left[\frac{r+s-3}{4}\right]. \tag{4.13}$$

By the reasoning as before, $\dim B$ is at least $4d$; it follows that

$$\frac{r+s-3}{4} \leq \frac{n}{2} - 3, \tag{4.14}$$

which implies the greatest integers satisfy

$$\left[\frac{r+s-3}{4}\right] < \left[\frac{n}{2}\right] - 2. \tag{4.15}$$

Therefore $\operatorname{sgn} F < \operatorname{sgn} G_{n,2}(F)$, which contradicts $\operatorname{sgn} B = \pm 1$ and (4.6). This eliminates Case A.

Case B. Either $M = xr + ys$ *with both* $x, y > 0$ *or* $M = xr = ys$ *with both* $x, y > 0$. It follows from (4.12) that both $H^{d(M-r)}(B;\mathbb{Q})$ and $H^{d(M-s)}(B;\mathbb{Q})$ are nonzero. Therefore (4.11) implies

$$\begin{aligned} 2r &> r+s-3 \\ 2s &> r+s-3, \end{aligned} \tag{4.16}$$

which in turn imply $|s-r| < 3$. But $s-r$ is an odd integer by (4.9), and accordingly $|s-r| = 1$.

For the sake of convenience let us assume $r = s+1$ and $M = xr + ys$ with $y > 0$. Then $M - 1 = (x+1)r + (y-1)s$ where $x+1, y-1 \geq 0$, and accordingly

$$H^{d(M-1)}(B;\mathbb{Q}) \neq 0 \tag{4.17}$$

by (4.12). According to (4.11), this can only happen if

$$\frac{r+s-3}{2} < 1, \tag{4.18}$$

which translates to $5 > r+s$, or $3 \geq r+s$ (which is odd). Since r and s, are positive integers with one even and the other odd, this implies $r = 1$ and $s = 2$. Hence (4.4) implies that $\mathbb{P}(F) = 1$, from which it follows that F is contractible. Therefore we have shown that $G_{n,2}(\mathbb{F})$ is connectedwise prime. ■

Remark: For $n = 4$ this was proved first in [26]; for $n = 8$ this was first proved in a letter from S. Halperin to the author.

Once one knows that $G_{n,2}(\mathbb{F})$ is connectedwise prime, it is not too difficult to determine its compact fiberability more or less completely.

Theorem 4.19: *If* $n \geq 5$, *then* $G_{n,2}(\mathbb{F})$ *is prime, where* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{K}$. *If* $n = 4$, *let* T *be the free involution on* $G_{n,2}(\mathbb{F})$ *sending a 2-plane* $W \subseteq \mathbb{F}^4$

into its orthogonal complement (with respect to the usual dot product $y \cdot y = \Sigma x_j \bar{y}_j$). Then $G_{n,2}(\mathbb{F})/T$ is prime.

REMARK: The normalizer N of $CL(2) \times CL(2)$ in $CL(4)$ – where $CL = U$ or Sp – is a two component group, and T represents the action of $N/CL(2)^2 = \mathbb{Z}_2$ on $G_{n,2}$.

PROOF: The first step is to show that if $G_{n,2}(\mathbb{F})$ admits a non-degenerate compact fibering (i.e., neither B nor F is contractible), then this fibering is equivalent to a finite covering on a finite complex homotopic to $G_{n,2}(\mathbb{F})$.

Suppose that $F \to E \to B$ is a compact fibering of $G_{n,2}(\mathbb{F})$. Then the exact sequence of homotopy implies that $\pi_1(B)$ is in 1-1 correspondence with the necessarily finite set $\pi_0(F)$. Thus if $\tilde{B}$ denotes the (finite!) universal covering of B, we also have a related compact fibering

$$F_0 \to E \to \tilde{B}. \tag{4.20}$$

By Theorem 4.3 either F_0 or $\tilde{B}$ is contractible. If $\tilde{B}$ is contractible, then we have a finite complex which is contractible and has a free action of the finite group $\pi_1(B)$. By P.A. Smith theory, this happens only if $\pi_1(B)$ is trivial [6, 8]; but then B is contractible. On the other hand, if F_0 is contractible, then E is homotopy equivalent to $\tilde{B}$, and we have a finite complex K with a free action of $\pi_1(B)$ that is homotopy equivalent to E or equivalently $G_{n,2}(\mathbb{F})$.

Therefore, to prove $G_{n,2}(\mathbb{F})$ is prime if $n \geq 5$, it is only necessary to check that every periodic self-map of a finite complex K homotopic to $G_{n,2}$ $(\mathbb{F})$ has a fixed point. The most fundamental tool for such investigations is the Lefschetz Fixed Point Theorem [45], and the natural idea is to determine how much can be said on that basis. This problem has been studied extensively by H. Glover and his coworkers [10, 19, 40], and their results give us all we need. Specifically, the results of [19, 40] establish the fixed point property if $n \geq 6$, while results of S. Brewster determine cohomological self maps determined by self-equivalences more generally [10]. The only possible zero Lefschetz number occurs if $n = 4$, in which case we indeed have the free involution sending W to its orthogonal complement. On the other hand, it turns out that this involution provides the only possible endomorphism from a periodic self-map aside from the identity [10].

To conclude the argument for $n = 4$, suppose $F \to E \to B$ is a compact fibering of $G_{n,2}(\mathbb{F})/T$. Then the composite $\tilde{E} \to E \to B$ (where

$\tilde{E}$ = universal 2-fold covering) is a compact fibration, and it follows that the fiber F' is either
(i) a double cover of F,
(ii) half of F – i.e., F is homotopic to $F' \amalg F'$.
By what we already know, F' is contractible and $\tilde{E} \to B$ corresponds to a finite covering if $F \to E \to B$ is nondegenerate. Smith theory now shows that (i) is impossible, so that $E \to B$ is also homotopically a covering space, with half as many sheets as $\tilde{E} \to B$. The results on Lefschetz numbers now imply that $\tilde{E} \to B$ must be a double covering homotopically; thus $E \to B$ must be a homotopy equivalence, and F must be contractible. Therefore $G_{4,2}(\mathbb{F})/T$ is prime as claimed. ∎

5. Quaternionic projective spaces

In this section we shall answer the fiberability question for those projective spaces not covered in [2, 12]. The method of proof will also be useful in later sections, and for this purpose we shall reformulate our calculations in a highly abstract way (Proposition 5.4).

THEOREM 5.1: *The quaternionic projective spaces $\mathbb{K}P^{2n+1}$ are prime for $n \geq 1$.*

PROOF: We begin by proving they are *connectedwise prime.* In this connection the rational theory of Halperin tells us that for any compact fibering $F \to E \xrightarrow{\pi} B$ of $\mathbb{K}P^{2n+1}$ with connected fiber we have

$$P(B) = \frac{(1 - t^{8n+4})}{(1 - t^{4d})}. \tag{5.2}$$

It follows that $\chi(F) = d$ and d divides $2n + 2$. To ensure the fibration is nondegenerate, assume that $d \neq 2n + 2$.

The map $\pi^*: H^*(B; \mathbb{Z}_p) \to H^*(\mathbb{K}P^{2n+1}; \mathbb{Z}_p)$ is a monomorphism for all p not dividing d (compare [3]). Therefore, by the torsion freeness of $H^*(\mathbb{K}P^m; \mathbb{Z})$ and (5.2) we know that $H^*(B; \mathbb{Z}_p)$ is the subalgebra generated by x^d, where $x \in H^4\,(\mathbb{K}P^{2n+1}; \mathbb{Z}_p)$ is a generator.

Let P^1 be the first Steenrod power operation for the prime p; then $P^1 x^d = 2dx^{d+[(p-1)/2]}$ [48], and by the naturality of P^1 we see that the odd primes can be put into three distinct categories:

CATEGORY I. *The odd primes p dividing $d = \chi(F)$.*

CATEGORY II. *The odd primes p such that* $d + \frac{1}{2}(p-1) \geq 2n+2$.
CATEGORY III. *The odd primes p such that d divides* $(p-1)/2$.

To verify this trichotomy, suppose p is not in the first two categories. Then P^1x^d is nonzero. But x^d generates the image of π^* multiplicatively, so that $x^{d+[(p-1)/2]} = Kx^{md}$ for some K, m. From this it is immediate that p falls into Category III.

Since d is a proper divisor of $2(n+1)$, it follows that $d \leq n+1$. Using this it is easy to verify that $p \leq d$ for Category I and $p > 2d$ for Categories II and III. Hence no odd prime p satisfies $d < p < 2d$. On the other hand, the following result is a standard fact in number theory, and a proof may be found in (say) [27] or [37]:

BERTRAND'S HYPOTHESIS: *If* $d > 1$, *there is a prime p so that* $d < p < 2d$; *if* $d > 3$, *then* $2d$ *is replaceable by* $2d-2$ [27, p. 373].

The only possibility is that $d = 1$. Hence $\mathbb{K}P^{2n+1}$ is always connectedwise prime. If $F \to E \to B$ is a compact fibering of $\mathbb{K}P^{2n+1}$ we may now proceed as in Section 4, considering the compact fibering $F_0 \to E \to \tilde{B}$, where F_0 is a component of F (all are homotopy equivalent to each other) and $\tilde{B}$ is the (finite) universal covering of B. As before, if $\tilde{B}$ is contractible then Smith theory implies $B = \tilde{B}$ and the fibering is degenerate, while if F_0 is contractible then the fibering is equivalent to a finite covering $E \to B$ with B a finite complex and E homotopic to $\mathbb{K}P^{2n+1}$. However, it is well known that $\mathbb{K}P^{2n+1}$ has the fixed point property if $n \geq 1$ (see [8] or [17]), and the same Lefschetz number argument works for any finite complex of the same homotopy type. Therefore $\mathbb{K}P^{2n+1}$ must be prime if $n \geq 1$. ■

Since the algebra in this proof is so simple and appears again when one tries to fiber $G_{2n+1,2}(\mathbb{R})$, we shall recast the proof in an abstract setting. Recall that the Becker–Gottlieb transfer $\tau: B^+ \to E^+$ of a compact fibering $F \to E \xrightarrow{\pi} B$ is really an S-map, and hence it commutes with all stable cohomology operations. In particular, it follows that if h^* is a cohomology theory in which $\chi(F)$ is a unit, then the map π^* induced by projection is split monic *over the algebra* $\mathcal{A}(h^*)$ *of stable cohomology operations* (compare [3]). For such cohomology theories the map $\tau^*\pi^*$ is an automorphism; therefore we can define an idempotent

$$\epsilon: h^*(E) \to h^*(E)$$

by the formula $\epsilon = \pi^*(\tau^*\pi^*)^{-1}\tau^*$. The basic properties of transfer then imply

(5.2) *The map* ϵ *is an* $\mathscr{A}(h^*)$*-module map.*

(5.3) *The map* ϵ *is an* $h^*(B)$*-module map; specifically,* $\epsilon(x \cdot \pi^*(y)) = \epsilon(x) \cdot \pi^*(y)$ *for* $y \in h^*(B)$.

Using this, we can abstractify the proof of Theorem 5.1 as follows:

PROPOSITION 5.4: *Let* $B^* \subseteq H^*(\mathbb{K}P^m; \mathbb{Z}[\frac{1}{2}])$ *be a subalgebra whose rank divides* $m+1$, *and let* ϵ *be an idempotent operator on* $B^* \otimes R$, *where* R *denotes the integers with* 2 *and* $d = m + 1/rkB^*$ *inverted. Assume that* ϵ *satisfies* (5.3) *for singular cohomology with coefficients in* R *and all its localizations and finite quotients, and assume that* $\epsilon \otimes \mathbb{Z}_p$ *satisfies* (5.2) *for ordinary cohomology over all primes* p *in* R. *Then* $B^* \otimes R$ *is either zero or all of* $H^*(\mathbb{K}P^m; R)$.

PROOF: The only substantial point that needs checking is that $B^* \otimes R$ is generated by x^d; given this, everything proceeds as in 5.1. But suppose B^{4e} is the first nonzero group in positive degree. Then the idempotent ϵ implies that $B^{4e} \otimes R = H^{4e} \otimes R$, so that $x^e \in B^{4e} \otimes R$. It follows that $\epsilon(x^j) = 0$ if $1 \leq j < e$ and $\epsilon(x^e) = x^e$. Repeated application of (5.3) shows that ϵ is nonzero precisely in dimensions divisible by e. Suppose we write $m + 1 = qe - r$, where $0 \leq r < e$. Then $rkB^* = q$ is immediate. But $qd = m + 1$ by assumption, and therefore $r = 0$ and $d = e$ must hold. ∎

REMARK 5.5: D. Gottlieb has suggested that an alternate proof of 5.1 is possible using the following diagram, whose rows and columns are all compact fiberings up to homotopy:

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d] & S^3 \arrow[d] & \\
Y \arrow[r] \arrow[d] & S^{8n+7} \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & B \arrow[d] \\
F \arrow[r] & \mathbb{K}P^{2n+1} \arrow[r] & B
\end{tikzcd}$$

If F is connected, then so is Y and Browder's results [11] imply that $Y \simeq S^1$, S^3, or S^7. With further work one can proceed in this fashion; in fact, Y is homotopic to S^7, $n \leq 2$, and F is homotopic to S^4. (In [11] there is a slight misstatement about spaces with cohomology a truncated polynomial algebra on an 8-dimensional class; actually, the Adem relations for $p = 3$ show that the third power must be zero). From there the necessary calculations are almost trivial. Of course, Browder's proof uses 2-primary information very heavily, and accordingly this alternate method is not applicable to the problems in Section 6.

6. Manifolds of real two-planes

We now turn to the remaining cases involving Grassmann manifolds of 2-planes – namely, those over the real numbers. Actually, there are two related but distinct questions in this case. The first is whether the Grassmann manifold $G_{n,2}(\mathbb{R})$ is prime, and the second is whether its double covering $G^+_{n,2}(\mathbb{R})$ – the manifold of *oriented* 2-planes in $\mathbb{R}^n$ – is connectedwise prime. As in the previous sections, an answer to the second question will lead to an answer to the first. Unfortunately, we have not been able to answer the second question completely for odd values of n, but we can say that S^2 is the only possible connected fiber in any case (see Theorem 6.11). From this we are able to show that $G_{n,2}(\mathbb{R})$ is always prime if $n+1$ is not a power of two. Actually, one *can* fiber $G_{7,2}(\mathbb{R})$ over G_2/SO_4 with fiber $\mathbb{R}P^2$; this follows from the fact that G_2 acts transitively on $V_{7,2}(\mathbb{R})$ via the standard representation on $\mathbb{R}^7$ as automorphisms of pure Cayley numbers (see Example 6.14). Thus any study of $G_{n,2}(\mathbb{R})$ for $n+1 = 2^r, r \geq 4$, must necessarily be more delicate than ours here (some further comments appear in Remark 6.19).

We begin with $G^+_{2n,2}(\mathbb{R})$. A clear, thorough discussion of this manifold's cohomology and its well-known identification with the non-singular complex hyperquadric in $\mathbb{C}P^{2n}$ have been given by H.F. Lai [35]. In particular, its integral cohomology is generated by $c \in H^2$ and $\omega \in H^{2n}$ with relations $2\omega c = c^{n+1}$ and $\omega^2 = (1+(-1)^n)c^n\omega$.

THEOREM 6.1: *The manifold* $G^+_{2n+2,2}(\mathbb{R})$ *is connectedwise prime,* $n \geq 2$.[1]

PROOF: The first step is to see how far we can get by rational homotopy methods as in Section 4. The Poincaré series for the base and fiber must take the forms

$$\mathbb{P}(F) = \frac{(1-t^{2a})(1-t^{2b})}{(1-t^2)(1-t^{2n})}, \tag{6.2}$$

$$\mathbb{P}(B) = \frac{(1-t^{4n})(1-t^{2n+2})}{(1-t^{2a})(1-t^{2b})}. \tag{6.3}$$

If we make the change of variables $u = t^2$ and note that $\mathbb{P}(F), \mathbb{P}(B)$ are both polynomials, it is immediate that n divides either a or b, the

[1] The proof given here is incomplete if $n = 3, 5$. Corrections for these cases will appear shortly.

number a divides either $2n$ or $n+1$, and similarly for b. Thus if we assume that n divides a, it follows that a must divide $2n$; hence either $a = n$ or $a = 2n$ must hold. We shall call these two possibilities Case I and Case II respectively.

The following observation is extremely helpful in understanding (6.2) and (6.3):

(6.4) *The number b divides* $n+1$. In Case II this is immediate from (6.3) and the linear factorizations of all polynomials over the complex numbers.

Case I is not quite so easy, for one must exclude the possibility that b divides $2n$; however, this can be done by a similar but more delicate argument.

Elimination of Case II. We know that $\mathbb{P}(B) = 1 + u^b + \cdots + u^{n+1-b}$, where $u = t^2$. In particular, either $b = n+1$ or else $\dim B \leq 2n$ and $H^*(B; \mathbb{Q})$ contains the rationalization of c^b. The case $b = n+1$ implies that $\dim B = 0$ and hence B is contractible. But in the latter case $0 \neq c^{n+1}$ is a power of c^b because b divides $n+1$, contradicting the fact that $H^*(B; \mathbb{Q})$ is a subalgebra of $H^*(G_{2n+2,2}(\mathbb{R}); \mathbb{Q})$. Thus Case II is impossible and Case I must apply.

Elimination of Case I, $b \neq n+1$. In this case

$$\mathbb{P}(B) = (1 + u^n)\left(\frac{1 - u^{n+1}}{1 - u^b}\right).$$

It follows that $H^{2j}(B; \mathbb{Q}) = 0$ for all j strictly between n and $n+b$. But again we know that $H^{2(n+1)} \neq 0$ as in Case II, which means b must equal 1. Hence $\dim F = 0$ and the fibering is again degenerate.

Elimination of Case I, $b = n+1$. In this case B is a simply connected cohomology $2n$-sphere over all coefficients such that $n + 1 = \chi(F)$ is invertible. If n is even, then the signature of $G^+_{2n+2,2}(\mathbb{R})$ is 2 (use Lai's calculations or the Hodge signature theorem), and the Chern–Hirzebruch–Serre Theorem shows that no fibering over such a B (with zero signature) exists.

If n is odd, more work is needed. First of all we observe that for an odd prime p the Steenrod operations satisfy

(6.5) $P^1(\omega) = 2P^1(\omega) = P^1(c^n)$, provided $2(p-1) \leq 2n - 2$. To see this, use the identity $2\omega c = c^{n+1}$ plus the Cartan formula. Now let p

be an odd prime such that

$$\frac{n+1}{2} < p < n+1. \tag{6.6}$$

This exists by Bertrand's Hypothesis since n is odd; it follows that p does not divide $n+1 = \chi(F)$. Let

$$\alpha \in H^{2n}(G^+_{2n+2,2}(\mathbb{R}); \mathbb{Z}_p)$$

be a generator for the image of $H^{2n}(B; \mathbb{Z}_p)$ (recall the fibration projection induces a strongly split monomorphism). Of course $P^1\alpha$ must be zero, which means that α must be a multiple of $2\omega - c^n$ by (6.5). On the other hand, from the Serre spectral sequence we can conclude that $H^*(G^+_{2n+2,2}(\mathbb{R}); \mathbb{Z}_p)$ is a free $H^*(B; \mathbb{Z}_p)$-module on $1, c, \ldots, c^n$. Since $2\omega - c^n$ annihilates c in particular, this gives a contradiction. ∎

REMARK 6.6: Since $G^+_{4,2}(\mathbb{R}) = SO_4/SO_2 \times SO_2 = S^3 \times S^3/S^1 \times S^1 = S^2 \times S^2$, the restriction $n \geq 2$ is appropriate.

Although $G^+_{m,2}(\mathbb{R})$ is not prime because it double covers $G_{m,2}(\mathbb{R})$, it is still important to study all cyclic covering spaces over finite complexes with total space homotopic to $G^+_{m,2}(\mathbb{R})$. Here is the basic application of the Lefschetz theorem we need:

THEOREM 6.7: *Let T be a free periodic map on a finite complex homotopic to $G^+_{m,2}(R)$, where $m \geq 5$. Then T is an involution, and the induced map in rational cohomology coincides with that induced by the involution that reverses orientations of 2-planes.*

PROOF: Since the particulars are very routine, we shall simply summarize the steps. First of all, in dimension 2 we must have $T^* = -I$ to ensure periodicity and a zero Lefschetz number. In particular, T must be an involution.

We now consider the case where $m = 2n+2$ is even; the other case will be very easy once we make some standard observations. Observe that $n \geq 2$. The only ambiguity in the cohomological action of T^* lies in dimension $2n$. Here, however, we know that T^* must preserve the cup product form and have trace $(-1)^n - 1$. It follows that the only algebraic possibility is

$$T^*c^n = (-1)^n c^n, \tag{6.8}$$

(6.9) $$T^*\omega = (1+(-1)^n)c^n.$$

To dispose of the case $G^+_{2n+1,2}(\mathbb{R})$, we note the following elementary fact:

(6.10) *Away from the prime 2, the space $G^+_{2n+1,2}(\mathbb{R})$ has the same cohomology as $\mathbb{C}P^{2n-1}$.*

One way of seeing this is to notice that $G^+_{2n+1,2}(\mathbb{R}) = V_{2n+1,2}(\mathbb{R})/S^1 = SO_2$ and the Stiefel manifold $V_{2n+1,2}(\mathbb{R})$ is equivalent to S^{4n-1} away from 2 (compare [43]; E. Friedlander has given extensive generalizations of this).

Once we know (6.10), it is immediate that $T^*c = -c$ must occur for a zero Lefschetz number, and the balance of the argument follows familiar lines. ■

Observation (6.10) is quite useful in studying the fiberability of $G^+_{2n+1,2}(\mathbb{R})$ further. Unfortunately, our results are significantly less complete than for $G^+_{2n+2,2}(\mathbb{R})$.

THEOREM 6.11: *Suppose that $F \to E \to B$ is a nondegenerate compact fibering of $G^+_{2n+2,2}(\mathbb{R})$ with connected fiber. Then F is homotopic to S^2.*

REMARK: If $E = G^+_{2n+2,2}(\mathbb{R})$ and the fibering is actually a bundle, then (i) the fiber is actually S^2 because it is a two-dimensional manifold factor and two-dimensional manifold factors are manifolds [52] (ii) the subgroup O_3 is a deformation retract of the full homeomorphism group of S^2 (Kneser's Theorem; compare [18]).

PROOF OF THEOREM 6.11: Let $A^* \subseteq H^*(G^+_{2n+1,2}(\mathbb{R}); \mathbb{Z}[\frac{1}{2}])$ be generated by the class $c^2 \in H^4$: we use the identification with $H^*(\mathbb{C}P^{2n-1})$ by (6.10). Denote the intersection of A^* with the image of $H^*(B; \mathbb{Z}[\frac{1}{2}])$ by X^*. It follows from (6.10) that X^* satisfies the conditions of Proposition 5.4, and consequently $X^* = A^*$ must hold. Therefore we know that B must have dimension at least $4n-4 = \dim G^+_{2n+1,2}(\mathbb{R}) - 2$. Hence $\dim F \leq 2$. But $\chi(F)$ is nonzero and F is simply connected, which mean that F is either contractible or S^2. ■

REMARK 6.12: It *is* possible to prove that *$G^+_{2n+1,2}(\mathbb{R})$ is connected-wise prime for infinitely many n – for example, whenever $n \equiv 5(6)$*. The methods are similar to others in this paper, and we shall not burden the reader with the rather unenlightening specifics.

Despite the limited nature of 6.11, we can settle the fiberability question for $G_{m,2}(\mathbb{R})$ except in a relatively sparse set of cases.

THEOREM 6.13: *The manifold $G_{m,2}(\mathbb{R})$ is prime if $m \geq 5$ and $m+1$ is not a power of two.*

EXAMPLE 6.14: The restriction $m+1 \neq 2^N$ is not totally removable, for $G_{7,2}(\mathbb{R})$ *is not prime.* If G_2 acts on $\mathbb{R}^7$ via automorphisms of pure Cayley numbers, then G_2 inherits a transitive action on $V_{7,2}(\mathbb{R})$. Using this, one can fiber $G_{7,2}(R)$ over G_2/SO_4 with fiber RP^2; similarly, the oriented Grassman manifold fibers with fiber S^2. Of course, this does not contradict 2.3. It seems plausible that $m=7$ is the only exception. However, as noted in Remark 6.19, more sophisticated techniques – perhaps tied to real K-theory – will be needed, even if $m = 15$ or 31.

PROOF: As before, the essential part of the proof is to show that $G_{m,2}(\mathbb{R})$ is connectedwise prime. Suppose that $F \to E \to B$ is a compact fibering of $G_{m,2}(\mathbb{R})$, and let $\tilde{F} \to \tilde{E} \to \tilde{B}$ be the induced maps of universal coverings; the latter is a compact fibering of $G^+_{m,2}(R)$. It follows that the latter is nondegenerate only if m is odd and $\tilde{F}$ is homotopic to S^2. There are now two possibilities. One is that F is homotopic to $\mathbb{R}P^2$ and B is simply connected, and the other is that $\tilde{B} \to B$ is a double covering and F is homotopic to S^2.

The latter possibility may be eliminated as follows: Consider the subalgebras of $H^*(G_{2n+1,2}(\mathbb{R}); \mathbb{Z}[2^{-1}]) \cong H^*(CP^{2n-1}; \mathbb{Z}[2^{-1}])$ generated by $H^*(B)$, $H^*(\tilde{B})$, and $H^*(G_{n,2}(\mathbb{R}))$, all with $\mathbb{Z}[2^{-1}]$ coefficients. By 5.4 the subalgebra generated by $H^*(\tilde{B})$ consists of all classes in dimensions divisible by 4, while 6.7 and a transfer argument show the same is true for $H^*(G_{2n+1,2}(\mathbb{R})) \cong H^*(\mathbb{K}P^{n-1})$. If we apply Proposition 5.4 to the image of $H^*(B)$, it follows that the latter is all of $H^*(G_{2n+1,2}(\mathbb{R}))$. Hence $\chi(B) = \chi(G_{2n+1,2}(\mathbb{R}))$, which implies that $\chi(F) = 1$; using this one can deduce that F is homotopic to $\mathbb{R}P^2$.

Since $\chi(\mathbb{R}P^2) = 1$, one can use the Becker–Gottlieb transfer and its consequences (5.2) and (5.3) for $\mathbb{Z}_2$ cohomology. It follows that the mod 2 Serre spectral sequence collapses, the fiber being totally nonhomologous to zero. Using (5.2) and (5.3) systematically one finds that $H^2(B; \mathbb{Z}_2)$ and $H^3(B; \mathbb{Z}_2)$ are one dimensional. If we write

$$H^*(G_{2n+1,2}(\mathbb{R}); \mathbb{Z}_2) = \mathbb{Z}_2[a_1, a_2]/(r_{2n-1}, r_{2n})$$

with deg $u_j = j$, then the respective classes in $H^*(B; \mathbb{Z}_2)$ are

$$x = a_1^2 + a_2$$

$$y = Sq^1 x = a_1 a_2.$$

Furthermore, through dimension $2n-2$ we have that x and y satisfy no relations, and in all dimensions they generate $H^*(B; \mathbb{Z}_2)$ algebraically.

For the sake of simplicity we eliminate the case $2n+1=5$ now. Since the Serre spectral sequence collapses, over $\mathbb{Z}_2$ we have

$$H^*(G_{2n+1,2}(\mathbb{R})) = H^*(\mathbb{R}P^2) \otimes H^*(B)$$

as graded $\mathbb{Z}_2$ vector spaces. Since $\dim H^*(\mathbb{R}P^2) = 3$ and $\dim H^*(G_{5,2}(\mathbb{R})) = 10$, this is impossible.

We wish to deduce information about B and $H^*(B; \mathbb{Z}_2)$ until we obtain a contradiction. The previous calculations show that $H^*(G_{2n+1,2}(\mathbb{R}))$ – all coefficients in this context are $\mathbb{Z}_2$ – is a free $H^*(B)$-module over 1, a_1, and a_1^2, with multiplicative relation $a_1^3 = a_1 x + y$. From this it is routine to calculate the Wu classes of E in terms of the Wu classes of B. The formula is

$$V(G_{2n+1,2}(\mathbb{R})) = \pi^* V(B)(1 + a_1 + (1 + \tilde{S}q + \tilde{S}q\tilde{S}q + \cdots)x) \tag{6.15}$$

where $\tilde{S}qz = Sq^1 z + Sq^2 z + \cdots$ (one uses the free $H^*(B)$ module structure plus the Cartan formula with the definition $V(M)\alpha \cap [M] = Sq\alpha \cap [M]$). Since $W = SqV$, the above formula tells us that

$$w(G_{2n+1,2}) = \pi^* w(B)(1 + a_1 + a_2). \tag{6.16}$$

However, by results of K.Y. Lam [36] we know

$$W(G_{m,2}(\mathbb{R})) = \frac{(1 + a_1 + a_2)^m}{(1 + a_1^2)}, \tag{6.17}$$

and combining this we deduce in our situation, with $m = 2n+1$, that

$$\pi^*\omega(B) = \frac{(1 + a_1^2 + a_2^2)^n}{(1 + a_1^2)}. \tag{6.18}$$

The coefficients of a_1^{2r} and a_2^{2s} in the above are

$$1 + n + \cdots \binom{n}{r-1} + \binom{n}{r} \text{ and } \binom{n}{s} \text{ respectively.}$$

Suppose we assume $r = 2$. Then we get $\pi^* w_2(B) = (n+1)a_1^2$; but $\pi^* w_2(B)$ is also a multiple of $x = a_1^2 + a_2$, and this can happen only if $n + 1 = 0$ (2). *Hence n must be odd.*

If we set $r = 2^t$ and $s = 2^{t-1}$ where $2^t \leq n - 1$, then we obtain further restrictions on n in the same way. For the standard basis of $H^{2t+1}(B)$ obtained from monomials in x and y has an obvious property: The only monomial requiring either $a_1^{2^{t+1}}$ or $a_2^{2^t}$ in its expansion is $x^{2^t} = a_1^{2^{t+1}} + a_2^{2^t}$. Therefore the coefficients of $a_1^{2^{t+1}}$ and $a_2^{2^t}$ in expression (6.18) must agree. For instance, if $t = 1$ this yields $\binom{n}{2} \equiv 1$ (2), which combined with $n \equiv 1$ (2) yields $n \equiv 3$ (4). Proceeding by induction on t, we discover that $n - 1 \geq 2^t$ always implies $n \equiv -1(2^{t+1})$. Putting these together, we see that $n + 1$ must be a power of two. This shows that $G_{m,2}(\mathbb{R})$ *is connectedwise prime unless $m + 1$ is a power of two.*

To prove that $G_{m,2}(\mathbb{R})$ is *prime* under these circumstances, assume that $F \to E \to B$ is a nondegenerate compact fibering. Then there is a finite covering $\hat{B} \to B$ such that $F_0 \to E \to \hat{B}$ is also a compact fibering, with F_0 being one of the (homotopically equivalent) components of F. By what we know, either F_0 or $\hat{B}$ is contractible, and the usual argument shows that the only genuine possibility is that F_0 is contractible and $G_{m,2}(\mathbb{R})$ is homotopically a finite covering of B. But then the universal covering of B is homotopy equivalent to $G^+_{m,2}(\mathbb{R})$, and by 6.7 we know the latter is at most a twofold covering space. It follows that the covering up to homotopy $G_{m,2}(\mathbb{R}) \to B$ must be homotopically one-sheeted; i.e., a homotopy equivalence. Thus F is contractible, contradicting the assumed nondegeneracy of $F \to E \to B$. ■

REMARK 6.19: Since the above argument is rather unmotivated, here is an explanation of the underlying causes. The mod 2 cohomology of $G_{m,2}(\mathbb{R})$ has two generators a_i $(i = 1, 2)$ in dimension i, and two relations r_{2_n}, $r_{2n+1} = Sq^1 r_{2n}$. If we have a fibering $\mathbb{R}P^2 \to E \to B$ with E homotopic to $G_{2n+1,2}(\mathbb{R})$, then it follows that r_{2n} and r_{2n+1} must be polynomials in $\pi^* x = a_1^2 + a_2$ and $\pi^* y = a_1 a_2$; furthermore, if this is true, then no contradiction can be derived from mod 2 cohomology. Our manipulations with Wu and Stiefel–Whitney classes are just a short way of proving that r_{2n} is not a polynomial in $\pi^* x$ and $\pi^* y$. If $2n + 1 = 7$, the existence of a nondegenerate compact fibering of $G_{7,2}(R)$ shows that r_6 *must* be a polynomial in $\pi^* x$ and $\pi^* y$; in fact, it is $\pi^*(x^3 + y^2)$. Further computations for $n = 7$ and $n = 15$ show that r_{14} and r_{30} are also polynomials in $\pi^* x$ and $\pi^* y$, and therefore it is clear that our methods break down in those instances. (For example, $r_{14} = \pi^*(x^7 + x^4 y^2 + x y^4)$).

7. Some examples

In this final section we include some simple examples to show that the conditions in 2.3 regarding compactness of G and maximal rank of H are necessary to avoid counterexamples. We conclude with possible alternate conjectures if G is (semi)simple and G/H is compact.

Since the closed subgroups of SO_3 are so well understood [4, 51], we begin by considering the maximal proper subgroups not of maximum rank. These are the orientation preserving symmetry groups of the regular icosahedron (dually, dodecahedron) and cube (dually, octahedron); the tetrahedron group lies inside the cube group (consider the convex hull of the vertices $(\pm 1, \pm 1, 1)$ and $(\pm 1, \pm 1, -1)$ in $[-1, 1]^3$). We shall call these maximal finite groups the *maximal Platonic groups.*

PROPOSITION 7.1: *Let $H \subseteq SO_3$ be a maximal Platonic group. Then SO_3/H is connectedwise prime but not prime.*

PROOF: Let $\hat{H} \subseteq S^3$ be the inverse image of H under the covering homomorphism $S^3 \to SO_3$, so that $S^3/\hat{H} = SO_3/H$. But it is immediate from the classification of free linear representations on S^3 that the free $\hat{H}$ action induced on S^3 extends to a free $\hat{H} \times \mathbb{Z}_r$ action where r is prime to the order of H (compare [39, 51]). Thus SO_3/H is not prime.

On the other hand, suppose $F \to E \to B$ is a compact fibering of SO_3 with F *connected.* Then exactly one of B, F is one-dimensional, and by Wall's classification of one-dimensional Poincaré–Wall complexes [49] we know that exactly one of B, F must be homotopic to S^1. Consider the exact homotopy sequence:

$$O \to \pi_2(B) \xrightarrow{\partial} \pi_1(F) \to \underset{[\cong H].}{\pi_1(SO_3/H)} \to \pi_1(B) \to 1$$

Clearly $\pi_1(B)$ is finite, and hence B is two dimensional with universal cover $\tilde{B} \simeq S^2$ and $\pi_1(B)$ at most $\mathbb{Z}_2$. Therefore $F \simeq S^1$ by our previous remarks. It follows that H has a composition series with a factor $\mathbb{Z}/\text{Image } \partial = \mathbb{Z}_q$ and $\pi_1(B) \subseteq \mathbb{Z}_2$. Since H has a unique subgroup of order two, it follows that H must contain an element of order at least $|H|/2$. But this is patently false for the maximal Platonic groups. ■

Although G/H is connectedwise prime in the previous example, the next one (suggested by J.C. Becker) shows that G/H need not be so even if H is maximal.

PROPOSITION 7.2: *Let $n \geq 3$ and let $0 < k \leq n$ be odd. Then the Grassmann manifolds $G^+_{2n,k}(\mathbb{R})$ and $G_{2n,k}(\mathbb{R})$ admit free differentiable circle actions and consequently are not prime. However, $SO_k \times SO_{2n-k}$ is a maximal connected closed subgroup of SO_{2n}, and $O_k \times O_{2n-k}$ is a maximal closed subgroup of O_{2n}.*

PROOF: We first construct the S^1 action. Let $\rho\colon S^1 \to U_n \subseteq SO_{2n}$ be the representation with n copies of the standard representation down the diagonal. Then the action $(z, xH) \to \rho(z)xH$ has an ineffective kernel of $\mathbb{Z}_2$, and the resulting action of $S^1/\mathbb{Z}_2 = S^1$ is free (it is here that we must assume k is odd – otherwise S^1 acts trivially). However, this gives us a smooth bundle $S^1 \to X \to X/S^1$ where X is the appropriate Grassmann manifold.

The proof that $SO_k \times SO_{2n-k}$ is a maximal connected subgroup of SO_{2n} follows readily from the work of Dynkin on maximal subgroups [14], the only problem being the transition from complex to real Lie algebra. Since SO_{2n} is simple, it follows that a closed subgroup strictly between $O_k \times O_{2n-k}$ and O_{2n} must be at most a finite extension of the former. We may conclude the proof with an appeal to the following:

ROUTINE EXERCISE 7.3: *Let $CL = O$ or U. Then the normalizer of $CL(p) \times CL(q)$ in $CL(p+q)$ is $CL(p) \times CL(q)$ itself if $p \neq q$ and the wreath product $Z_2 \wr CL(p)$ if $p = q$.* ■

We now consider compact homogeneous spaces G/H with B connected simple but noncompact, H a closed subgroup, and $\chi(G/H) \neq 0$. Since all Riemannian 2-manifolds are conformally equivalent to surfaces of constant curvature [51], it follows that closed surfaces of genus ≥ 2 give a basic class of examples. Motivated by this, we summarize the basic facts about compact fiberings of such manifolds.

(7.4) *The only possible compact fiberings of a compact surface with negative Euler characteristic are finite coverings up to homotopy.* ■

This follows from Wall's classification of one-dimensional Poincaré complexes. The following elementary results contain all we need about finite coverings up to homotopy:

(7.6) *If S_h is an oriented (resp., unoriented) compact surface of genus h and g satisfies $h-1=n(g-1)$ for some integer $n \geq 1$, then S_h is an n-fold covering space of the oriented (resp., unoriented) surface S_g. Conversely, if both S_g and S_h are orientable or nonorientable, the above condition is necessary for the existence of a covering.* ■

(7.7) *Suppose X and Y are Poincaré–Wall complexes of the same dimension and X is a finite cover of Y. If X is nonorientable, then so is Y.* ■

If S_g is a Riemannian manifold of genus ≥ 2 with constant curvature, then its universal cover is the hyperbolic plane $\mathbb{H}$, and it follows that $S_g = \mathrm{Iso}(\mathbb{H})/\Gamma$, where $\mathrm{Iso}\,\mathbb{H}$ denotes the Lie group of isometries of $\mathbb{H}$. The latter group has two components, and its identity component is isomorphic to $SL(2, \mathbb{R})$; in fact, the identity component may also be viewed as the set of all holomorphic automorphisms of $\mathbb{H}$, taking the latter as the open unit disk in $\mathbb{C}$ (the other component of $\mathrm{Iso}\,\mathbb{H}$ corresponds to antiholomorphic automorphisms). In fact, if S_g^* denotes the oriented single or double covering of S_g (depending on whether S_g is orientable), then S_g^* inherits a natural complex structure in this process.

The group $SL(2, \mathbb{R})$ in fact acts *irreducibly* transitively on S_g^*. To see this, notice that any closed proper subgroup that acted transitively would be two-dimensional and there are no two-dimensional Lie subalgebras of the Lie algebra $\mathfrak{sl}(2, R)$.

Now suppose that S_g is *nonorientable*, being expressed as $\mathrm{Iso}(\mathbb{H})/\Gamma$. It is immediate that the identity component $SL(2, \mathbb{R})$ of $\mathrm{Iso}(\mathbb{H})$ acts transitively on the connected manifold S_g, and accordingly we may write $S_g = SL(2, R)/S\Gamma$, where $S\Gamma = SL(2, R) \cap \Gamma$. As in the oriented case, the action must be irreducibly transitive. We want to know when Γ can be a maximal proper subgroup of $SL(2, R)$. Of course S_2 has Euler characteristic -1 and cannot finitely cover anything, so S_2 is prime by (7.7) and thus $S\Gamma$ must be maximal). In view of this we shall assume $g \geq 3$ henceforth; by (7.7) we know that $S\Gamma$ has finite index in any proper closed subgroup $\Delta \supseteq S\Gamma$.

LEMMA 7.8: *In the above notation, assume that S_g^* with its associated complex structure is not a finite unramified holomorphic covering space for a Riemann surface of lower genus. Then $S\Gamma$ is a maximal closed subgroup of $SL(2, \mathbb{R})$.*

PROOF: There is a closed subgroup $\Gamma_0 \subseteq S\Gamma$ of index 2 so that

$S_g^* = SL(2,\mathbb{R})/\Gamma_0$. By (7.6) we know that S_g is an n-sheeted covering of the nonorientable surface $S_h = SL(2,\mathbb{R})/\Delta$. Let $\Delta_0 \subseteq \Delta$ be the index two subgroup corresponding to the oriented surface S_h^*. Then we have an $n = |\Delta_0/\Gamma_0|$ sheeted holomorphic covering $S_g^* \to S_h^*$, and by assumption this implies $1 = n = |\Delta_0/\Gamma| = |\Delta/S\Gamma|$. Hence $\Delta = S\Gamma$ as claimed. ■

The usefulness of 7.8 for showing that $S\Gamma$ may be maximal if $g \geq 3$ depends strongly on the fact that a Riemann surface of type S_g^* usually is *not* a multiple-sheeted holomorphic covering space of anything else. I am indebted to W. Neumann for proposing this proof and to D. Drasin, A. Weitsman, and P. Sipe for help in filling in some details.

PROPOSITION 7.9: *Let S_g^* be an oriented surface of genus g. Then there are uncountable many distinct complex structures ξ on S_g^* for which (S_g^*, ξ) does not holomorphically cover a Riemann surface of lower genus.*

PROOF: (*Sketch*) Let Π_g denote the Teichmüller space of complex structures on S_g^* (see [1] for example), and let A_g denote the (countable) mapping class group of S_g^*. Then Π_g is naturally a $(3g-3)$-dimensional complex manifold [1, pp. 143–144], and A_g acts analytically and properly discontinuously on Π_g (compare [53]). If h is chosen so that $g-1 = n(h-1)$, then passage to n-sheeted coverings defines a map from Π_h to Π_g. Since $\dim \Pi_h < \dim \Pi_g$ if $h \geq 2$, it suffices by Sard's theorem to prove that the geometrically defined map $\Pi_h \to \Pi_g$ is smooth. For then $\Pi_g - \cup_h \text{Image } A_g \times \Pi_h$ is dense, where h runs over all integers with $h-1 \mid g-1$.

To see this, observe that local *real analytic* coordinates for Π_k may be given as follows: Consider the representation of

$$\pi_1(S_k^*) = \langle x_1, y_1, \ldots, x_k, y_k : \Pi[x_j, y_j] = 1\rangle \tag{7.10}$$

in $SL(2,R)$ induced by the complex structure (i.e., elements of π_1 determine elements of Hol. Aut. $(\tilde{S}_k^*) =$ Hol. Aut. $(\mathbb{H})$). Then the $6k-6$ real coordinates may be taken to come from the entries of the 2×2 unimodular matrices associated to $x_2, y_2, \ldots, x_k, y_k$ (compare [1, p. 143]). But if S_g^* is topologically an n-sheeted covering of S_h^*, then $\pi_1(S_g^*)$ is contained in $\pi_1(S_h^*)$. Thus if we let a_i, b_i correspond to x_i, y_i in (7.10) for $k = g$ and p_i, q_i to x_i, y_i for $k = h$, it follows that a_i and b_i can be expressed rationally in the p's and q's. Thus the entries of a_i

and b_i are merely rational in the entries of the p's and q's, which implies the desired smoothness assertion. ■

From this we obtain our desired examples:

THEOREM 7.11: *There exist (uncountably many nonconjugate) closed maximal subgroups $\Gamma \subseteq SL(2, \mathbb{R})$ such that $SL(2, \mathbb{R})/\Gamma$ is a surface of fixed genus $g \geq 3$. Consequently, Conjecture 2.3 is false for compact homogeneous spaces of noncompact semisimple Lie groups, even with $\chi(G/H) \neq 0$ replacing condition* (iii).

PROOF: Let ξ be any complex structure on S_g^* so that (S_g, ξ) does not holomorphically cover anything else, and let $\Gamma_0 \subseteq SL(2, \mathbb{R})$ be chosen so that $(S_g^*, \xi) = SL(2, \mathbb{R})/\Gamma_0$. By Lemma 7.9 the only possible closed subgroup between Γ_0 and $SL(2, \mathbb{R})$ is a group Γ with $|\Gamma/\Gamma_0| = 2$ and $SL(2, R)/\Gamma \cong S_g$ (unoriented). In any case either Γ_0 or the hypothetical group Γ is maximal. Verification that one gets uncountably many nonconjugate examples in this way is left to the reader. ■

Given the importance of commensurability in the study of discrete subgroups of Lie groups, the anomalies involving finite coverings are not a complete surprise. Thus some optimistic conjectures in the noncompact case may be the following:

(7.12A) *Suppose G is a semisimple Lie group and H is a maximal closed subgroup such that G acts irreducibly transitively on G/H by translation, G/H is compact, and $\chi(G/H) \neq 0$. Is G/H connectedwise prime?*

(7.12B) *Under the above conditions, are all the compact fiberings of G/H equivalent to finite coverings?*

(7.12C) *If either of these is false for G semisimple, is it nevertheless true for G simple?*

Of course, the fundamental group of G/H must be infinite if G is noncompact, and accordingly one would expect the fundamental group to be quite important in the noncompact case.

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(Oblatum 12-IX-1979 & 27-XI-1980)

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Added in proof

Please consult "Correction to the paper" to appear in Compositio Mathematica, Volumen 43, Fasciculus 3 (1981).