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CHERN INVARIANTS OF SURFACES OF GENERAL TYPE

Ulf Persson

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Introduction

Let X be a minimal surface of general type. Its two fundamental discrete invariants are given by the Chern numbers $c_1^2(X)$ and $c_2(X)$. It is well known that they characterize a surface up to a finite number of families.

The Chern numbers satisfy the arithmetic condition

$$c_1^2(X) + c_2(X) \equiv 0(12)$$

which is made explicit by Noether's formula

$$c_1^2 + c_2 = 12\chi \text{ (χ the holomorphic Euler characteristics).}$$

It will now be more convenient to use χ and c_1^2 as our basic discrete

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invariants, and they will henceforward be referred to, by slight abuse of terminology, as the Chern invariants of the surface.

The Chern invariants cannot be arbitrary. In fact they are restricted by the following inequalities:

$$c_1^2, \chi > 0 \text{ and } 2\chi - 6 \leq c_1^2 \leq 9\chi.$$

The first three are fairly elementary and known already to the Old Italians.

The last $c_1^2 \leq 9\chi$ (or equivalently $c_1^2 \leq 3c_2$) is much less so, and only very recently proved (1976). It is due to Bogomolov, Miyaoka on one hand and Yau on the other.

The main purpose of this article is to investigate to what extent the above inequalities are the only restrictions.

In other words, given x, y, suitably restricted, can we find a minimal surface of general type X, with $c_1^2(X) = y$, $\chi(X) = x$.

We present the following partial results:

THEOREM 2: *Let x, y be positive integers satisfying*

$$2x - 6 \leq y \leq 8x \text{ and } y \neq 8x - k$$

(*where $k = 2$, or k is odd and $1 \leq k \leq 15$ or $k = 19$*).

Then there exists a minimal surface of general type X, with $c_1^2(X) = y$, $\chi(X) = x$.

Furthermore X can be assumed to be a genus two fibration.

The surfaces exhibited, have been chosen in view of their relative simplicity of construction and do not reflect the "typical" surface with the given Chern invariants.

THEOREM 3: *Let x, y be positive integers satisfying*

$$2x - 6 \leq y \leq 8(x - Cx^{2/3}) \text{ where } C = 9/\sqrt[3]{12}.$$

Then there exists a simply connected minimal surface of general type X, which is a double covering of a rational surface, with $c_1^2(X) = y$, $\chi(X) = x$.

In view of the result that the intersection form on a surface is determined by χ, c_1^2; and that the intersection form determines the homotopy type of a simply connected surface if $c_1^2 \not\equiv 0(2)$ or if

$c_1^2 \equiv 0(2)$ in connection with the primitivity of K- the canonical divisor (i.e. if $K \equiv 2C$ or not for some divisor C); there is much less a degree of arbitrariness of the surfaces constructed (see [4]).

It is widely believed that there exists homotopically equivalent but yet non-diffeomorphic surfaces (a four-dimensional counterexample to Smale). Furthermore it is not held unlikely that there exists totally unrelated diffeomorphic surfaces.

No example of either phenomena has yet been found.

The next step would be to describe all surfaces as opposed to just one example with given Chern invariant (χ, c_1^2).

For very special cases, e.g. (3, 1); (4, 2) this can be done explicitly, but in general it seems hardly feasible.

Setting ones aims lower, one would hope to establish effective bounds for the number of different families, corresponding to a given pair of Chern invariants. Even this seems quite hard, and it is suggested, by the proofs of the theorems above, that those bounds may be quite high.

The subsidiary purpose of this article, is to address itself to questions, of what I would like to call, geography of surfaces of general type.

One can ask whether additional restrictions on the surfaces (e.g. of topological nature, simply connectedness, irregularity; of complex analytical nature, canonical embeddability, etc.) is reflected in the possible Chern invariants.

Or, conversely, if further restrictions on the Chern invariants, force additional properties of the surfaces.

This is not the place to give a comprehensive survey of those "geographical" questions.

Suffices it to indicate some basic features.

The line $c_1^2 = 8\chi$ does in many ways act as a watershed. At our present state of knowledge, the region $8\chi < c_1^2 \leq 9\chi$, could rightly be termed arctic. Only sporadic examples of surfaces have been found there. The notable cases being, compact quotients of the unit ball in $\mathbb{C}^2$ (Hirzebruch-Borel) whose invariants satisfy $c_1^2 = 9\chi$ (by Yau, those are in a sense the only examples on that line), certain non-degenerate fibrations (Kodaira [7]) and now very recently some examples by Miyaoka.

The major conjecture, mainly supported by the lack of evidence to the contrary*, is

* Added in proof: It has recently been announced by Holzapfel (Berlin) that simply connected surfaces exist with c_1^2/c_2 arbitrarily close to 3.

CONJECTURE: *If X is simply connected, then $c_1^2 < 8\chi$.*

In Proposition 4.5 we exhibit simply connected surfaces with high c_1^2.

It seems to be a basic feature, that the smaller c_1^2 is, the more special is the surface and hence more amendable to explicit study.

Horikawa has shown ([5]) the possibility to elucidate in detail the structure of the surfaces for which $c_1^2 = 2\chi - 6$ (or more generally $c_1^2 = 2\chi - k$, $k = 5, 4$ etc.). In particular, they turn out to be simply connected genus two fibrations.

Beauville has proved ([2]) that if $c_1^2 < 3\chi - 10$, then the surface is by necessity a double covering of a ruled surface. In particular, a canonically embeddable surface must satisfy $c_1^2 \geq 3\chi - 10$.

Reid has recently indicated ([15]) that some hold can be gotten for surfaces with $c_1^2 \leq 4\chi$.

For higher c_1^2 the situation becomes rapidly more chaotic, due to the great profusion of surfaces encountered.

Bearing in mind that surfaces fibered with genus two curves should in some sense be considered the most special surfaces of general type, we present the following

THEOREM 1′: *Let X be a surface of general type, with a pencil of genus two curves, then $c_1^2 < 7\chi$.*

The proof presented is very simple minded and rather crude, I believe a more sophisticated and careful approach would yield a much sharper bound.

On the other hand, we have

PROPOSITION 3.12′: *There exists an infinite family of surfaces of general type, with rationally based genus two fibrations and with $c_1^2 = 4\chi - 4$.*

And also an example due to Oort and Peters ([10]) of a Campedelli surface with $c_1^2 = \chi = 1$, with a genus two fibration.

For general genus two fibrations we cannot hope for more than the conjecture below, in view of Theorem 2.

CONJECTURE: *If X is a surface of general type, and with a genus two fibration, then $c_1^2 \leq 8\chi$.*

We have only been able to verify the conjecture for those irregular pencils based over elliptic or genus two curves (Theorem 1).

The methods are too weak to give any interesting corresponding bounds for higher genus hyperelliptic fibrations.

The plan of the paper will be as follows:

1. *Double coverings*

As there does not exist a standard reference for double coverings, I will briefly set down the conventions I adhere to, and state the main facts. For a fuller treatment along the same lines the reader is referred to [12].

The major part of this section is devoted to the singularities of double coverings, stemming from singularities of the branch locus. In this context we elaborate on the well-known concept of inessential sigularities of the branch locus, and the characterization of the resulting surface singularities as rational double points. All of this is of course known, but due to the centrality of the concept we find it worthwhile to give a systematic account.

We also treat in detail the simplest essential singularity, that of the so called infinitely close triple point, due to its importance in the methods of the proofs of the main theorems.

Finally we touch upon more global aspects, and compute the Chern invariants of some double coverings, to be used later.

2. *Estimates of Genus Two Fibrations.*

In this section we recall the characterization of genus two fibrations, as double coverings of ruled surfaces along sextic branch loci, whose only essential singularities can be assumed to be infinitely close triple points.

A straightforward but crude estimate of their number then yields the bound on c_1^2 spelled out in Theorem 1.

We also indicate some partial results on the opposite question, of giving bounds on c_1^2 forcing a surface to have a genus two fibration.

3. *A Family of Genus Two Fibrations.*

Here we will recall, refine, and also develop and put into context the construction presented provisionally in [13]. We hope also that the Family constructed here also will have some independent interest, as a source of various phenomena. Its function in this paper is both to supplement the constructions in the next section, and to give a simple foretaste of the techniques elaborated there.

4. *The Main Theorems.*

Here we prove Theorems 2 and 3. The strategy in both cases is quite similar. The first step is to exhibit a family of surfaces, whose invariants are truly spread out. Those constructions act as a scaffolding, a skeleton, to be fleshed out and completed. This constitutes the last step, and entails imposing singularities. This presupposes complete control of their nature and number, and presents in my view the main difficulty. Our solution consists in working with double coverings, and inducing the singularities via singularities of the branch curve. This not only reduces it to a one-dimensional problem, but also, due to that fortunate quirk of nature, the encompassing notion of inessential singularity, makes the constructions get off ground.

I am indebted to Professor D. Mumford for exciting my "geographical" interest in surfaces of general type at a talk at Avignon (1977).

I also would like to thank Professor B. Moishezon and his student K. Chakiris for many stimulating discussions and encouragement during the initial phase of this work. To them I also owe the statement and proof of the pivotal lemma 3.20.

I have also had fruitful discussions with Professor Van de Ven at the final reworking.

Finally I would like to thank Mrs. Sylvia Carleson for excellent typing.

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1. Double Coverings

The material is standard, but unfortunately there exists no standard reference. (Each author hence tends to nourish his own notations and idiosyncracies, the present author being no exception.)

A fuller treatment along the same lines is to be found in [12].

By a double covering $\pi: Y \to X$ is meant a finite map of degree two, or equivalently an involution on Y with no isolated fixpoints.

The Data of the covering is given by X a nonsingular surface, C an even branchcurve and finally a square root, i.e. a divisor B (up to linear equivalence) such that $C \in |2B|$.

Given these Data a surface Y is easily constructed, either as

1. $\mathrm{Spec}(\mathcal{O}_X \otimes \mathcal{O}_X(B))$, where a ringstructure is imposed by the embedding of $\mathcal{O}_X(-2B) = \mathcal{O}_X(-B) \otimes \mathcal{O}_X(-B)$ into $\mathcal{O}_X$ via C.

or

2. given by the equation $z^2 = f$ in the linebundle $[B]$ on X, where f are the local equations of C in $[2B]$ and z the fiber co-ordinate.

PROPOSITION: 1.1: *Y is normal iff C has no multiple components. Y is nonsingular iff C is nonsingular.*

PROOF: See [12].

In the second case we have the following.

PROPOSITION 1.2: *If Y is non-singular, then*

$$K_Y = \pi^*(K_X + B) \textit{ thus,}$$

$$c_1^2(Y) = 2(K_X + B)^2 \,(= 2(c_1^2(X) + 2g(C) - 2) - \frac{3}{2}C^2)$$

$$c_2(Y) = 2c_2(X) + 2g(C) - 2$$

$$\chi(Y) = 2\chi(X) + (g(C) - 1)/2 - C^2/8$$

and finally we note

$$(\pi^*D \cdot \pi^*D') = 2(D \cdot D'), \quad H^i(\pi^*(D)) = H^i(D) \oplus H^i(D - B)$$

PROOF: Standard. See e.g. [12].

Notice that if D is a component of the branchlocus C, then $\pi^*(D)$ is double and if D_0 denotes its reduced part, $D_0^2 = 1/2\, D^2$.

This might be the place to insert the following observation.

PROPOSITION 1.3: *If Y is non-ruled, its exceptional divisors are of two kinds:*

(a) *pullbacks of exceptional divisors of X (those always come in pairs)*

(b) *reduced pullbacks of rational components with self-intersection -2, of the branch curve C.*

PROOF: Given an exceptional divisor on Y, look at its image under the involution. This is a new exceptional divisor, which is either disjoint from the original (case a)) or coincides (case b)). Note also that in case a) the exceptional divisors are pullbacks of exceptional divisors disjoint from the branch curve.

We will now study two aspects of local nature.

First we will investigate the nature of the surface singularities of Y

stemming from singularities of the branch curve, and specifically how Proposition 1.2. has to be modified. This leads us to the very important technical concept of inessential singularities of the branch-locus. As a digression we will show how this ties up with the classical notion of rational double points. We will also elucidate in detail the simplest, and perhaps most basic essential singularity, that of an infinitely close triple point.

Secondly we will describe how singularities located on the non-singular branchlocus will change when the corresponding curves are pulled back. This will constitute another technical tool.

The isolated surface singularities of double coverings stemming from curve singularities of the branchlocus can easily be characterized locally.

PROPOSITION 1.4: *An isolated singularity P of a surface S is equivalent to a branch induced singularity iff*

> *P is a hypersurface singularity (the embedding dimension of P is three) of multiplicity two. i.e. S is given locally be an equation $F(x, y, z) = 0$, with multiplicity of F at P equal to two.*

PROOF: The only if part is immediate (cf. Proposition 1.2.). And the if part follows almost equally directly. Indeed by a change of coordinates we can write F under the form $z^2 - f(x, y)$, locally.

Thus branch induced singularities compromise the simplest case of isolated singularities. But extensive enough to include e.g. all the rational double points.

They are very accessible to explicit study, as they can be directly related to the corresponding branch curve singularity. There is indeed a simple and well-known algorithm for their resolution in terms of the resolution of the corresponding curve singularity. This algorithm hinges on the following concept (cf. also [5]).

DEFINITION 1.5: *The pair (X', C') is said to be the even resolution of the pair (X, C), where C is an even curve without multiple components on a smooth surface X, if $(X', C') = (X_\infty, C_\infty)$ where (X_k, C_k), $0 \leq k \leq \infty$ is defined inductively as follows*

i) $(X_0, C_0) = (X, C)$
ii) *given (X_k, C_k) define $(X_{k+1}, C_{k+1}) = (X_k, C_k)$ if C_k is non-singular, otherwise pick a singular point on C_k, let $\pi_p : X_{k+1} \to X_k$ be the*

blow up with exceptional divisor E_p. And let $C_{k+1} = \pi_p^*(C_k) - 2[m/2]E_p$, where m is the multiplicity of C_k at p.
(Note that E_p is a component of C_{k+1} iff m is odd).

iii) *If $(X_{k+1}, C_{k+1}) = (X_k, C_k)$, let $(X_\infty, C_\infty) = (X_k, C_k)$.*

Note. It should of course be checked that this process ultimately stabilizes–to make sense of (X_∞, C_∞). And also that (X', C') is independent of the particular choices of singular points p in ii). This is safely left to the reader.

We are now able to give the explicit resolution of any branch induced singularity. For simplicity we will assume a global setting. It is left to the reader to translate this into a local one.

DEFINITION 1.6: *Let X be a non-singular surface and C an even (possibly singular) curve with no multiple components. By the resolution $\bar{Y}$ of the double covering Y of X along C is meant the minimalization of the double covering Y' of X' along C', (X', C') being the even resolution of (X, C). (In the minimalization, only exceptional divisors in the fibers of $Y' \to Y$ are blown down).*

We are now ready to define the important technical concept of inessential singularity.

DEFINITION 1.7: *A branch curve C is said to have only inessential singularities iff $c_1^2(\bar{Y}) = c_1^2(Y)$; $\chi(\bar{Y}) = \chi(Y)$.*

Note. The right sides are defined by Proposition 1.2. by interpreting $g(C)$ as the (formal) arithmetic genus.

There is a local characterization of inessential singularities, which can then be taken as the definition in a more general setting.

PROPOSITION 1.8: *A branch curve C has only inessential singularities iff all its singularities have multiplicity less than three, and no infinitely close points have multiplicity more than two.*

PROOF: Standard, see e.g. [5]. The reader can supply his own by characterize those C for which $\pi^*(K_X + B) = \pi'^*(K_{X'} + B')$ (where the $'$ is selfexplanatory). This observation will be taken up below.

In other words, double points (with arbitarily high order of contact) and triple points for which not all three branches are tangent, are all inessential.

PROPOSITION 1.9: *The singularities stemming from inessential singularities of the branchcurve are all rational double points.*

PROOF: See e.g. [5], or notice that the characterization in Prop. 1.8. shows that the resolution does not contribute to the canonical divisor, and thus appeal to a standard characterization of rational double points. See [1].

Below we will give an additional independent proof of 1.9. using the characterization of inessential singularities of 1.8. and the explicit algorithm of even resolutions.

In order to appreciate the connection between 1.9 and 1.7. the reader should bear in mind Brieskorn's result on the simultaneous resolution of rational double points [3]. Hence in the future branch curves with only inessential singularities can be treated for all intents and purposes as smooth.

The simplest essential singularity, that of an infinitely close triple point, will play a pivotal role in our future constructions. It will be treated in detail, its elucidation involves a typical application of the algorithm of even resolutions.

DEFINITION 1.10: *A singularity is called an infinitely close triple point, iff it consists of three tangent branches no longer simultaneously tangent after one blow up.*

PROPOSITION 1.11: *If $\bar{Y}$ is the standard resolution (see 1.6) of a double covering Y, branched along a curve with one infinitely close triple point. Then*

$$c_1^2(\bar{Y}) = c_1^2(Y) - 1 \text{ and } \chi(\bar{Y}) = \chi(Y) - 1.$$

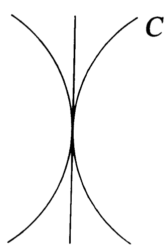
PROOF: The arguments are best described by the following tables and diagrams.

Initial situation on X:

Branchcurve C

Square root B

Canonical divisor $K + B$

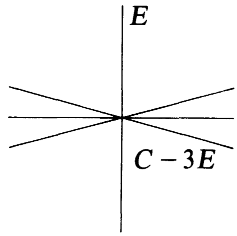


After one blow up. On X_1:

Branchcurve $C_1 = (C - 3E) + E$

Square root $\quad B - E$

Canonical divisor $K + B$

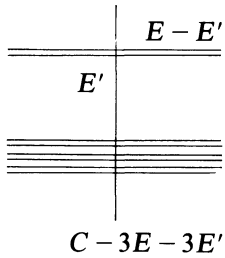


After two blow ups. On $X_2 = X'$:

Branchcurve $\quad C' = C_2 = (C - 3E) + E - 4E'$

Square root $\quad B - E - 2E'$

Canonical divisor $\quad K + B - E'$



Now C' is smooth, the even resolution of C, furthermore $g(C') = g(C) - 7$, and $c_2(X') = c_2(X) + 2$. Thus using Prop. 1.2., $c_1^2(Y') = 2(K + B - E')^2 = c_1^2(Y) - 2$ and $c_2(Y') = c_2(Y) - 10$. By Noether's formula $\chi(Y') = \chi(Y) - 1$. But the pullback of $E - E'$ is exceptional on Y' by Prop. 1.3. To form $\bar{Y}$ it has to be blown down, and hence $c_1^2(\bar{Y}) = c_1^2(Y) - 1$; $\chi(\bar{Y}) = \chi(Y) - 1$.

Note 1.12: The minimal resolution of a singularity stemming from an infinitely close triple point consists of an elliptic curve with selfintersection -1.

Note 1.13: As the local equation for an infinitely close triple point can be given by $x^3 + y^6 = 0$, the local equation for the above surface singularity can be represented by $z^2 + x^3 + y^6 = 0$ (cf. [8] p27).

Note 1.14: There is also the concept of higher order infinitely close triple points. They will naturally appear in the future, and sometimes sloppily be referred to as infinitely close triple points.

A curve singularity is said to be an infinitely close triple point of order k if it turns into an ordinary triple point after k successive blow ups.

The generalization of Proposition 1.11. now becomes

$$c_1^2(\bar{Y}) = c_1^2(Y) - \left[\frac{k+1}{2}\right] \text{ and } \chi(\bar{Y}) = \chi(Y) - \left[\frac{k+1}{2}\right].$$

Thus we can think of an infinitely close triple point of order k as $[(k+1)/2]$ ordinary (i.e. of order 1) infinitely close triple points (infinitely close).

In general we have the notion of so called specialization vector of a singular branchcurve.

DEFINITION 1.15: *To a singular branchcurve C is associated a so called specialization vector $(-a,-b)$(cf. [12]) defined by $\chi(\bar{Y}) = \chi(Y) - a$, $c_1^2(\bar{Y}) = c_1^2(Y) - b$, where we recall that $\bar{Y}$ denotes the standard desingularization of Y.*

The determination of specialization vectors can be done purely mechanically (cf. proof of Prop. 1.11.) and hence they will be presented in the future when needed without explicit proof.

The following observation is helpful. (We are conserving the terminology of Definition 1.6.)

OBSERVATION 1.16: *Let Y_k be the (possibly singular) double covering of X_k along C_k, and $n = [m/2]$, then*

$$c_1^2(Y_{k+1}) = c_1^2(Y_k) - 2(n-1)^2$$

$$\chi(Y_{k+1}) = \chi(Y_k) - \frac{n(n-1)}{2}.$$

Note 1.17: To compute the invariants of Y_∞, we only have to keep track of the multiplicities that occur in the even resolution. Finally to compute the invariants of $\bar{Y}$, we have to look for exceptional divisiors in the resolution, which is slightly more subtle.

We are now going to list in a table, all the possible inessential

TABLE 1.18

Equation of curve singularity	graphic representation	"name"	surface singularity	
x^2+y^{2n}	$(1, 1\to 2\to \ldots 2)$ (n arrows)	tacnodal of order $n-1$	A_{2n-1}	$(n \geq 1)$
x^2+y^{2n+1}	$(2\to 2\to \quad 2)$ (n arrows)	cuspidal of order n	A_{2n}	
x^2y+y^{2n+3}	$(1, 1\to 2\to \quad 2\to 3)$ ($n+1$ arrows)		D_{2n+4}	$(n \geq 0)$
x^2y+y^{2n+4}	$(2\to 2\to \quad 2\to 3)$ ($n+1$ arrows)		D_{2n+5}	
x^3+y^4			E_6	
x^3+xy^3			E_7	
x^3+y^5			E_8	

singularities of a branch curve on the left side, and on the right side their characterization as isolated surface singularities.

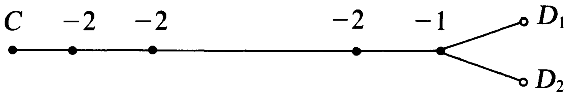
This is straightforward combinatorics, and is a nice exercise of the above methods and proves Proposition 1.9. (cf. also Proposition 2.8. in [14]).

To cover our second aspect of local nature of double coverings we present the following two elementary but useful observations,

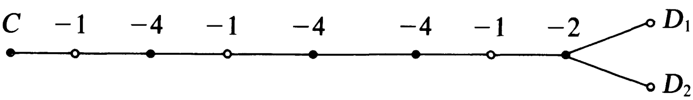
LEMMA 1.19: *Let D_1 and D_2 be two curve branches transversal at a point p to the branchcurve C of a double covering $\pi: Y \to X$.*

Assume that D_1 and D_2 have contact of order k at p (i.e. k blow ups are needed to separate them). Then the pullbacks π^*D_1 and π^*D_2 have contact of order $2k$ at $\pi^*(p)$.

PROOF: Separate D_1 and D_2 by k blow ups of X.



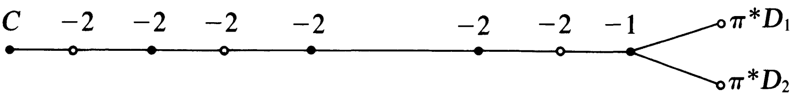
(k-dots)



($2k$-dots)

The exceptional divisors are all part of the branch locus because of evenness. Hence k further blow ups are necessary for the even desingularization of the branch locus.

By taking the double covering of the last configuration, we end up with



($2k$ exceptional divisors)

and we are done.

We will mainly be interested in the special case of $k = 1$. I.e. transversal intersections on the branchlocus become tangential upstairs. The most important application will be

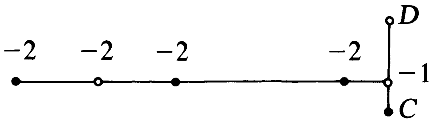
COROLLARY 1.20: *An ordinary triple point located on the branch locus becomes an infinitely close triple point when pulled up.*

LEMMA 1.21: *Let D be a curve branch with contact of order (a) $2k$ (b) $2k+1$ with the branch curve C.*

Then the pullback π^*D becomes (a) a tacnode of order $k-1$ $(1, 1 \to 2 \to \ldots 2)$ (k arrows) and with contact of order $2k$ with C or (b) a cuspidal of order k $(2 \to \ldots 2)$ (k arrows) and with contact of order $2k+1$ with C.

PROOF: The following sequence of diagrams should be self-explanatory by now.

Step I (Separation)

(a) 

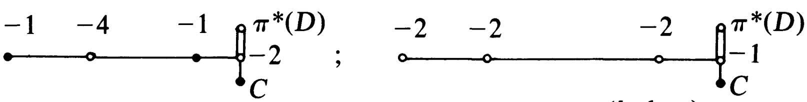
(k black "−2" dots)

(b) 

(k + 1 black "−2" dots)

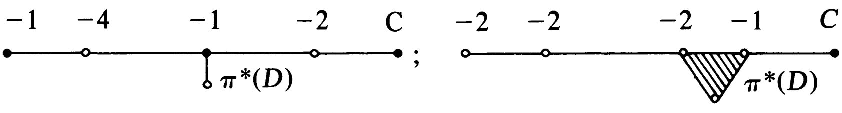
Step II (Pullback and reduction)

(a)



(k dots)

(b)



($k+1$ dots)

As before the main case of interest will be $k = 1$.

COROLLARY 1.22: *A branch tangential to the branchcurve becomes an ordinary double point with branches transversal to the ramification curve.*

A branch flexed to the branchcurve becomes a cusp, tangent to the ramification curve.

Finally we will consider some global aspects. Namely the computation of Chern numbers of some simple but basic double coverings, by straightforward application of Proposition 1.2. These calculations will be used in the final and main section of this paper.

EXAMPLE 1.23: As the minimal rationally ruled surfaces will play an important role, it might be appropriate to review them quickly and to establish some notations.

They are classified by the positive integers, and to each N corresponds the so called Hirzebruch surface of type N, which we will denote by $\mathbb{F}_N$.

$\mathbb{F}_N$ can be realized as the completion of the linebundle $\mathcal{O}_{\mathbb{P}^1}(N)$ on $\mathbb{P}^1$, by adding a section at infinity.

The Picard group of $\mathbb{F}_N$ is freely generated by a section S, characterized by $S^2 = N$, and a fiber F.

By $|aS + bF|_N$ we will denote the linear system $PH^0(\mathbb{F}_N, aS + bF)$ (If no confusion can arise, the subscript N is suppressed). Note that in order for this to be non-empty, then $a, b \geq 0$ with one important exception.

Indeed if $N > 0$, there exists a unique section $S_\infty \equiv S - NF$ on $\mathbb{F}_N$. This will be referred to as the minimal section, or sometimes the infinity section, because to present $\mathbb{F}_N$ as a completed linebundle is equivalent to give two disjoint sections. One of those has to be S_∞ and the other an element $S_0 \in |S|_N^0$. (Where $|S|_N^0$ denotes the Zariski open subset of irreducible (or equivalent smooth) sections). (Observe that the reducible sections of $|S|_N$ will have S_∞ as transversal component and N fibers. Thus they are parametrized by the sections of $\mathcal{O}_{\mathbb{P}^1}(\mathrm{N})$).

Given S_0, S_∞ disjoint and z a section of $H^0(\mathbb{F}_N, S)$ defining S_0, then $\mathbb{F}_N - S_\infty$ is a line-bundle with z as fiber coordinate.

We can now define an involution on $\mathbb{F}_N$ via $z \to -z$. This will have as ramification locus $S_0 \cup S_\infty$, and its quotient will be $\mathbb{F}_{2N}$. Thus we can exhibit $\mathbb{F}_N$ as a double covering of $\mathbb{F}_{2N}$ branched at two disjoint sections.

This simple observation will be important for future constructions.

Finally we observe that the natural double covering $\pi: \mathbb{F}_N \to \mathbb{F}_{2N}$ induces a map $\pi^*: \mathrm{Pic}\ \mathbb{F}_{2N} \to \mathrm{Pic}\ \mathbb{F}_N$ and via the identification $\mathbb{Z}^2 \simeq \mathrm{Pic}\ \mathbb{F}_N((a, b) \to aS + bF)$ π^* defines the endomorphism $(a, b) \to (2a, b)$.

In particular the pullback of a section of $\mathbb{F}_{2N}$ becomes a bisection of $\mathbb{F}_N$.

EXAMPLE 1.24: The classification theory for irrational minimal ruled surfaces gets progressively complicated. Let it be sufficient to mention the following facts.

Crude invariants for a minimal ruled surface R are given by (1) the genus q of the basecurve (q turns out to be also the irregularity of R).

(2) the minimal positive selfintersection N of a section.

Such surfaces will be denoted by $\mathbb{F}_N^{q+1}$.

For $q > 0$, the Picard group becomes unwieldy, but its discrete part–the Neron-Severi group (divisors modulo numerical equivalence) is still freely generated by a section S, $S^2 = N$ and a fiber F.

We are now ready to compute the Chern invariants of some double coverings.

PROPOSITION 1.25: *Let R be as above, with $S^2 = N$ and $q(R) = q$, and let C be an even branch curve of type $2mS + 2nF$ (or $(2n, 2m)$ for short.) Let Y be the double cover of R along C, then*:

$$c_1^2(Y) = 2m(m-2)N + 4(m-2)(n+2q-2)$$

$$\chi(Y) = (m(m-1)/2)N + (m-1)(n+q-1) + 1 - q.$$

PROOF: As $K = -2S + (N + 2q - 2)F$ and $(K + C)C = 2g - 2$ and $c_2(R) = 4 - 4q$, this becomes a straightforward application of Proposition 1.2.

COROLLARY 1.26: *If Y is branched along a sextic ($m = 3$) branch-curve, then we have the following invariants*:

$$c_1^2(Y) = 2N + 4(n + 2q - 2)$$

$$\chi(Y) = 3N + 2(n + q - 1) + 1 - q.$$

EXAMPLE 1.27: We start with $\mathbb{P}^1 \times \mathbb{P}^1 (= \mathbb{F}_0)$ and two curves C_1 and C_2 of type $(2a, 2b)$ and $(2c, 2d)$ respectively.

Let Y_1 be the double covering $\pi_1 : Y_1 \to \mathbb{F}_0$ branched at C_1, and let $Y = Y_2$ be the double covering $\pi_2 : Y_2 \to Y_1$ branched at $\pi_1^*(C_2)$.

PROPOSITION 1.28: *The invariants of Y are given by*

$$c_1^2(Y) = 8(A-2)(B-2)$$

$$\chi(Y) = (A-2)(B-2) + \phi(a, b)$$

where $A = a + c$, $B = b + d$, $\phi(a, b) = (A - a)(B - b) + ab$.

PROOF: The canonical divisor K on $\mathbb{F}_0$ is given by $(-2, -2)$ then apply Proposition 1.2.

EXAMPLE 1.27: We replace $\mathbb{F}_0$ by $\mathbb{F}_N$ obtaining a surface $Y_{(N)}$:

PROPOSITION 1.28: *The invariants of $Y_{(N)}$ are given by*

$$c_1^2(Y_{(N)}) = 4(A-2)(AN+2(B-2))$$
$$\chi(Y_{(N)}) = (A-2)(AN+B-2)+\phi(a,b)+(A-ac)N.$$

EXAMPLE 1.29: We once again start with $\mathbb{F}_0$ and now n curves C_k of type $(2a_k, 2b_k)$. Define inductively Y_k as the double covering $\pi_k: Y_k \to Y_{k-1}$ branched at $\pi_{k-1}^*(C_k)$ where $Y_0 = F_0$ and $\pi_0 =$ identity.

PROPOSITION 1.30: *The invariants of $Y = Y_n$ are given by*

$$c_1^2(Y) = 2^{n+1}(A-2)(B-2)$$
$$\chi(Y) = 2^{n-2}(A-2)(B-2)+2^{n-2}\Phi$$

where $A = \Sigma a_k$, $B = \Sigma b_k$ and $\Phi = \Sigma a_k b_k$.

PROOF: Clear by induction and Proposition 1.2.

2. Estimates of Genus Two Fibrations

The following should be clear.

PROPOSITION 2.1: *Let Y be a hyperelliptic fibration. Then Y is the double covering of a ruled surface.*

COROLLARY 2.2: *Let Y be a genus two fibration. Then Y is the double covering of a ruled surface along a sextic branch locus.*

The ruled surface is uniquely determined by the fibration, but not by the surface itself, which can have many hyperelliptic fibrations.

The typical example is constructed as follows,

EXAMPLE 2.3: Let E be an elliptic curve, and consider the trivial ruled surface $E \times \mathbb{P}^1$. Choose a curve C of type $6S + 2F$, where S is a horizontal section ($S^2 = 0$) and F a vertical (rational) fiber. C could simply be the union of vertical and horizontal fibers, and as such having only inessential singularities. Now let Y be the resolution of $E \times \mathbb{P}^1$ branched along C. Note that Y has two different genus two fibrations, the second one arising from the horizontal sections. (A double cover of an elliptic curve branched at two points is a genus two curve), and determines a rational ruled surface.

Thus the question of distinguishing surfaces arisen as double coverings of ruled surfaces is a hard and perhaps subtler one than might at first been thought.

On the other hand, in view of the canonical algorithm of resolving singularities of double coverings, as expounded in the previous section, a surface is uniquely determined by a choice of minimal model and the push down of the branch locus. Now for fixed fibrations there is a canonical choice of minimal model minimizing the singularities of the pushdown of the branchcurve.

In particular we have,

PROPOSITION 2.4: *If Y is a genus two fibration, then Y is the resolution of a double covering of a minimal ruled surface branched along a sextic branch curve whose only essential singularities are infinitely close triple points.*

PROOF: Consult [6]. (Note that the infinitely close triple points could be of arbitrary order, cf. note 1.14:)

REMARK: The keytool in the proof is the use of so called elementary transformations of a ruled surface. These enable us to reduce the multiplicities of a branchcurve singularity to m for a branchcurve of type $(*, 2m)$. (See e.g. [9], [11]) As we will explicitly use this notion in the future it might be convenient to define it explicitly.

DEFINITION 2.5: *By an elementary transformation at a point p on a ruled surface R, is meant the blow up of p, and the blow down of the exceptional divisor consisting of the proper transform of the fiber through p. (This is of course ambiguous in the one case of* $\mathbb{F}_0$, but the *context will make clear what fibration is chosen).*

We will need the following.

PROPOSITION 2.6: *Let Y be a relatively minimal genus two fibration of general type.*

Then Y has at most one exceptional (necessarily transversal) divisor, and if Y is non-minimal then $c_1^2(Y) = 0$.

The key fact is the following inequality, which is the direct analogue of the Cauchy–Schwarz inequality.

INEQUALITY 2.7: (Hodge). *Let X be a surface with $K^2 > 0$ and let C be a curve, then*

$$|(K \cdot C)| \geq \sqrt{(K^2)}\sqrt{(C^2)}$$

with equality iff $C = \dfrac{(K \cdot C)}{(K^2)} K$.

PROOF: Observe $K((K^2)C - (K \cdot C)K) = 0$, and then exploit the Hodge Index theorem.

PROOF OF PROPOSITION 2.6: Let $X = \bar{Y}$($\bar{Y}$ the minimal model of Y) and let C be the image of a fiber of the genus two fibration of Y. Note that if E is a transversal exceptional divisor then $EC = 1$ (cf. proposition 3.7.)

Assuming $Y \neq \bar{Y}$ we have $C^2 \geq 1$. By the adjunction formula $KC + C^2 = 2$. As $\bar{Y}$ is of general type and C is non rational $KC > 0$. Thus $KC = C^2 = 1$.

The Hodge inequality now gives $1 = KC \geqslant \sqrt{(K^2)}$, from which follows $K^2 = 1$ and we are done.

REMARK 2.6′: The well-known example of a double covering of $\mathbb{F}_2$ branched along $5S + (S - 2F)$, gives a non-minimal genus two fibration with $K^2 = 0$ and exactly one exceptional divisor. (This is the one example for which $4K$ fails to be birational and it is also uniquely determined by $(\chi, c_1^2) = (3, 1)$, cf. Introduction).

We are now ready to state the main theorem of this section.

THEOREM 1: *Let Y be a genus two fibration over a basecurve with genus at most two. Then we have the following estimates for c_1^2.*

a) $c_1^2(Y) < 7\chi(Y)$, *rational or elliptic base curve.*

b) $c_1^2(Y) \leq 8\chi(Y)$, *genus two base curve.*

PROOF: We need the following elementary lemma.

LEMMA 2.8: *Let $C = 6S + (a + b)F$ be a branchcurve in a ruled surface of type $\mathbb{F}_N^{g+1}$ (see Ex. 1.24.) with b fibral components. Assume that k is the number of infinitely close triple points. (An infinitely close triple point of order m, is then counted with multiplicity $[(m + 1)/2]$).*

Then we have the following estimate.

$$k \leq \frac{5}{2}N + \frac{5}{6}a + \frac{2}{3}b$$

PROOF: Write $C = C_0 + bF$, where C_0 has no fibral components. The infinitely close triple points of C are of two kinds.

(i) bona fide infinitely close triple points on C_0

(ii) vertical tacnodes of C_0 (turned into infinitely close triple points of C by the addition of a fibral component).

Note that the resolution of an infinitely close triple point of order m on C_0 reduces its arithmetic genus with $3(m+1)$. For a tacnode of order m we get a corresponding reduction of $2(m+1)$.

So let there be k_1 points of type (i) and k_2 points of type (ii) counted with the appropriate multiplicity.

We thus get the following estimate.

$$(*) \qquad (K + C_0)C_0 - 12k_1 - 4k_2 \geq 2(2q-2)$$

(where the left side gives $2g'-2$, where g' is the arithmetic genus of C_0 after the resolution of the points of type (i) and (ii) and the right side gives a lower bound for $2g-2$, where g is the arithmetic genus of the desingularization of C_0, which at "worst" can be the disjoint union of six sections).

As $K = -2S + (N + 2q - 2)F$ we can rewrite $(*)$ as

$$12k_1 + 4k_2 \leq 30N + 10a.$$

As $k_2 \leq b$ we obtain

$$12(k_1 + k_2) \leq 20N + 10a + 8b$$

which clearly is the estimate of Lemma 2.6.

PROOF OF THEOREM: By Corollary 2.2. Y is the double covering of a ruled surface along a sextic branchlocus. By Proposition 2.4. we can "descend" to a minimal ruled surface, such that the pushdown C of the branchcurve has only infinitely close triple points as essential singularities.

Let C be as in the previous lemma, and let k be its number of infinitely close triple points (counted appropriately).

Using Proposition 1.11. (with note 1.14.) and Corollary 1.26. the invariants of Y are given by

$$c_1^2(Y) = 6N + 2(a+b) + 8(q-1) - k.$$

$$\chi(Y) = 3N + (a+b) + (q-1) - k.$$

Introducing $\omega = \frac{5}{2}N + \frac{5}{6}a + \frac{2}{3}b - k$

$$\theta = \frac{1}{2}N + \frac{1}{6}a + \frac{1}{3}b + \omega \quad (= \chi(Y) + (1-q))$$

we can write

$$\frac{c_1^2}{\chi} = \frac{8(q-1) + 7\theta - b - 6\omega}{(q-1) + \theta} \tag{$*$}$$

by Lemma 2.8. $\omega \geq 0$. If $q \leq 2$, and Y is of general type, then

$$\theta = \chi(Y) + (1-q) \geq \chi(Y) - 1 \geq 0.$$

If $q = 0, 1$ $(*)$ shows immediately $c_1^2 < 7\chi$ and if $q = 2$ we get $c_1^2 \geq 8\chi$.

Note 2.9: If $q > 2$, we cannot a priori conclude $\theta \geq 0$.
The cone of effective divisors in the Neron-Severi group can be quite complicated for high q, and in particular a can be very negative. But it is conceivable that for a sharper estimate of ω, one could push through the arguments for higher q.

Note 2.10: If $q = 2$, equality $c_1^2 = 8\chi$ can occur. It occurs exactly when (in the terminology of the proof) $\theta = \omega = b = 0$, examples of which are

EXAMPLE 2.10.1. Let D be a curve of genus two, and let C be six disjoint sections of $D \times \mathbb{P}^1$. The double cover then becomes the trivial fibered surface $D \times D'$ where D' is another genus two curve (a double covering of $\mathbb{P}^1$ branched at $C \cdot F$).

EXAMPLE 2.10.2. Let once again D be a curve of genus two, and let $d \not\equiv 0$ be a divisor on D such that $6d \equiv 0$. Let R be the completion of the linebundle $\mathcal{O}_D(d)$, and let C be defined as the union of all the sixth-roots of unity. C will then be irreducible (two disjoint, three disjoint components) if neither $2d$, nor $3d$ are trivial ($3d \equiv 0$, $2d \equiv 0$) and define unramified sextic (triple, double) coverings onto D.

The corresponding double covering will have isomorphic fibers, but will not be a direct product like in example 2.10.1. This phenomena occurs because of monodromy. If we pullback the fibration via an irreducible component of C, we will obtain a trivial fibration.

It would no doubt be possible to classify all cases for which $\theta = \omega = b = 0$, and hence all genus two fibrations over a genus two curve with $c_1^2 = 8\chi$ (using the classification of ruled surfaces of irregularity two, e.g. in the spirit of [9]).

Note 2.11. As observed above, the estimate of ω is very naive and hence rather crude. Thus for genus two fibrations over rational or elliptic curves, it seems possible to sharpen significantly, the sectors wherein they live. Results in the opposite direction will be presented in the next section.

One may observe the following corollary of Theorem 1.

COROLLARY 2.11: *Let Y be a surface of general type, and assume $c_1^2(Y) \geq 7\chi(Y)$, and let C be a curve of genus two inside Y, then*

(i) $C^2 \leq 0$ (ii) *C does not move.*

PROOF: (i) follows directly from the circle of ideas around the Hodge inequality (2.9.) and (ii) needs no comment.

The above corollary fits into the picture, that the more "general" a surface is, the less likely there is to find curves of small genera.

The same method as above also yields

PROPOSITION 2.12: *Let Y be a hyperelliptic fibration, fibered with curves of genus g over a curve of genus q, then*

$$c_1^2(Y) \geq 4\frac{g-1}{g}\chi(Y) + 4\frac{(g+1)(g-1)}{g}(q-1).$$

PROOF: By Proposition 2.1. Y is the resolution of the double covering of a ruled surface of genus q, along a branchcurve intersecting each fiber in $2m$ points, where $m = g + 1$.

If the branchcurve has no essential singularities, then Proposition 1.25. gives equality in the above proposition.

In general we can assume that the singularities of the branchcurve have at most multiplicity m. (cf. Proposition 2.4. and Definition 2.5.)

Recalling the notion of specialization vector (Definition 1.15.) we can state

LEMMA 2.13. *The specialization vector of a singularity of multi-*

plicity k, has a slope bounded by

$$\text{(i) } 4\frac{k-2}{k} \textit{ if } k \textit{ is even} \quad \text{(ii) } 4\frac{k-3}{k-1} \textit{ if } k \textit{ is odd.}$$

PROOF: Immediate from observation 1.16. Note that the presence of exceptional divisors would only improve the estimates.

We observe that the bounds given form an increasing sequence. Thus we have universally the bound $4[(m-2)/m]$ (m even or not). Now as clearly $4[(m-2)/m] < 4[(m-2)/(m-1)] = 4[(g-1)/g]$, we are done.

The significance of Proposition 2.12. is due to the following result of Beauville.

THEOREM: *If $c_1^2(Y) < 3\chi(Y) - 10$, then Y is a double covering of a ruled surface (i.e. a hyperelliptic fibration).*

PROOF: see [2] (or [12]).

PROPOSITION 2.14: *If X is a surface of general type, irregularly fibered, and $c_1^2(X) < (16/6)\chi$, then the fibration is a genus two fibration.*

PROOF: Immediate from the above theorem and proposition 2.12.

Unfortunately we cannot strengthen Proposition 2.12. so as to prove Proposition 2.14. also for linearly fibered surfaces. In fact such endeavours are doomed to fail, as is suggested by the possibility of blowing up the base points of any linear system of curves of any genus on any surface.

Instead one can, following M. Reid, consider the canonical images as embedded ruled surfaces in the appropriate $\mathbb{P}^N$. The inequalities between c_1^2, χ allows a straightforward translation into inequalities concerning N and the degree d of the image. By projective algebraic methods one can obtain bounds for d in terms of N, forcing the image to be ruled by bona fide lines. As the canonical image of any higher hyperelliptic curve never can be a line, we will be done.

We will not pursue these methods further, however, as they lie outside the scope and limited intentions of the present section.

3. A Family of Genus Two Fibrations

We will now present a simple construction of sextic branchcurves of $\mathbb{F}_N$ with many infinitely close triple points.

This will be applied to the exhibition of a family of genus two fibrations, to complement later constructions.

The ideas and techniques used in this section also constitute a foretaste of their elabouration in the next (the essence of this material has already appeared in provisional form in [13]). The main point is contained in the following construction:

PROPOSITION 3.1: *Given N, a and k, with $0 \leq k \leq 2N + 2[2a/3]$, then there exists on $\mathbb{F}_N$ a curve $C_{N,k,a}$ equivalent to $6S + 2aF$ (where S is the section with $S^2 = N$, and F the fiber (see Ex. 1.23.)) with exactly k infinitely close triple points and no other essential singularities.*

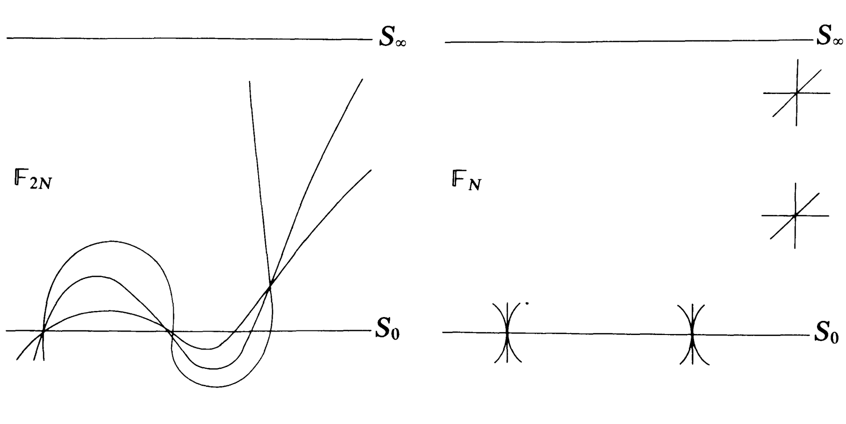
PROOF: Choose two disjoint sections $S_0 \in |S|$ and $S_\infty \in |S - NF|$. They define an involution on $\mathbb{F}_N$, and in fact presents $\mathbb{F}_N$ as a double covering of $\mathbb{F}_{2N}$ (cf. Ex. 1.23.).

Let $b \leq [2a/3]$. Choose points $q_1, q_2, \ldots q_{2N+b}$ on S_0 and $q_{2N+b+1}, \ldots q_{2N+2b}$ on S_∞, with no two points on the same fiber.

For any subset of k points $q'_1, q'_2, \ldots q'_k$ of the q_i's it is now possible to find three distinct (irreducible) sections S_1, S_2, S_3 (distinct from S_0) in $|S + bF|$ on $\mathbb{F}_{2N}$, such that their only common intersection points on $S_0 \cup S_\infty$ are $q'_1, q'_2, \ldots q'_k$. (If $b = 0$, we can think of the sections as given by polynomials of degree $2N$ and with only k common (preassigned) zeroes).

Now the pullbacks $\pi^*S_i (i = 1, 2, 3)$ are bisections of $\mathbb{F}_N$ and with common vertical tangents at $\pi^*(q'_i)$ (Lemma 1.19. and Corollary 1.20.).

Thus $C_{N,k,a} = \Sigma_i \pi^* S_i + (2a - 3b)F$ (where the fibers have been chosen generic) will satisfy the requirements.



Note 3.2: To appreciate the power of the preceeding construction, we might observe that there are twelve conditions to specify an infinitely close triple point with given tangent direction. Furthermore $\dim|6S+2aF|_N = 21N + 14a + 6$.

In view of the observations below, the above particular construction may not be as arbitrary as it first appears.

LEMMA 3.3: *Given $C \in |2S|$, C irreducible. Then there is $S_0 \in |S|$, such that C is invariant under the involution defined by S_0.*

In other words C is a pullback via $\pi : \mathbb{F}_N \to \mathbb{F}_{2N}$ for a suitable section. As such all its vertical tangents lie on a section, in fact on the appropriate S_0. Hence the following corollary.

COROLLARY 3.4: *Given $C \in |2S|$ all its vertical tangents lie on a section $S \in |S|$.*

PROOF: As observed each $S_0 \in |S|_N$ defines a map $\pi : \mathbb{F}_N \to \mathbb{F}_{2N}$. Pullbacks of sections of $|S|_{2N}$ become bisections of $|2S|_N$. Those bisections have vertical tangents at their intersections with S_0, which is hence recaptured.

Finally the observation, based on Riemann-Roch, that $\dim|2S|_N = 3N+2$, $\dim|S|_N = N+1$, and $\dim|S|_{2N} = 2N+1$ is highly suggestive.

The argument can be made precise, but becomes then somewhat involved. For completeness and its intrinsic geometric interest it will be given, but due to its length and being somewhat peripheral, it will be relegated to Appendix B.

We are thus authorized to claim and guess respectively.

COROLLARY 3.5: *The construction of Proposition* 3.1. *is the best possible using unions of bisections.*

Note that in [10] the authors exhibit on $\mathbb{F}_1$ a slightly more efficient construction, while not restricting themselves to bisections. But anyway I venture

CONJECTURE: *The construction is the best possible for vertical infinitely close triple points.*

The above construction has the following applications to the construction of surfaces.

In fact construct surfaces accordingly.

DEFINITION 3.6: *Let $X_{N,k,a}$ be the resolution of the double covering of $\mathbb{F}_N$ along $C_{N,k,a}$ (in the sense of Definition* 1.6.).

We will mainly be interested in those $X_{N,k,a}$ which are minimal and of general type.

PROPOSITION 3.7: *$X_{N,k,a}$ is minimal under the following circumstances*

$N \geq 2$ *(always)*

$N = 1, a \geq 1$

$N = 0, a = 2$ *and* $k = 0$; *or* $a = 3$ *and* $k \leq 3$; *or* $a \geq 4$.

PROOF: In order to check minimality, we will restrict ourselves to surfaces with effective canonical divisor. As is well-known exceptional divisors then show up as fixed components of the canonical system.

The canonical divisor of $X_{N,k,a}$ is given by the pullback of

$$S + (N + a - 2)F - \sum_{i \leq k} E_i + \sum_{i \leq k} (E_i - E_i') = K + B \quad (2B = C_{N,k,a})$$

(where the E_i are the exceptional divisors stemming from the blow up of the q_i' and E_i' the blow ups of the infinitely close points to the q_i', (in the notation of Proposition 3.1)) and the subsequent blow down of the exceptional divisors arising from pullbacks of $\Sigma_{i \leq k}(E_i - E_i')$ (cf. Proposition 1.11.)

This is an immediate consequence of Proposition 1.2. From the same we infer that all effective canonical divisors of $X_{N,k,a}$ are invariant under the involution, as $H^0(K) = 0$ for rational surfaces.

Thus to find effective divisors in the canonical system on $X_{N,k,a}$ is equivalent to find effective divisors in $|K + B|$ where $2B = C_{N,k,a}$.

Let $b = [2a/3]$ as in Proposition 3.1. Recall from the construction, that if $k \leq 2N + b$, all the q_i''s can be chosen to lie on S_0.

Thus $S - \Sigma_{i=1}^{2N+b} E_i$ is always effective.

To take care of the case $k > 2N + b$, we have to add fibral components $N - E_i$, with $i > 2N + b$, corresponding to points q_i' necessarily on S_∞ (there is at most "b" such points).

In other words $N - 2 + a$ gives the number of additional points on S_∞ that can be "taken care of".

One easily checks $a \geq b, a \geq b - 1$ if $a \geq 1, a \geq b - 2$ if $a \geq 4$. Thus effective canonical divisors can be found in case

$$N \geq 2 \quad \text{always } (N-2+a \geq b)$$
$$N = 1 \quad a \geq 1 \quad (a-1 \geq b)$$
$$N = 0 \quad a = 2\ k \leq 1;\ a = 3\ k \leq 3;\ a \geq 4.$$

We also note that in addition to the above obvious cases, we can also add

$$N = 1 \quad a = k = 0 \quad (\text{as } S - F \text{ is effective containing } S_\infty).$$

It is not hard, albeit slightly cumbersome, to prove that these are the only cases leading to effective divisors in the system $K + B$, and hence to $p_g > 0$.

The candidates for exceptional divisors on $X_{N,k,a}$ are the pullbacks of

(a) the fibral components $F - E_i$ $(i > 2N + b)$

(b) the transversal component $S - \Sigma_{i \leq k_0} E_i\ k_0 \leq 2N + b$.

The former have self intersection zero (in fact they are part of the induced fibration) and are hence automaticallly dismissed.

In the latter case, the self intersection is given by $2(N - k_0) + k_0 = 2N - k_0$, and it is ramified at $6N + 2a - 2k_0$ points.

Thus we get two cases

i) $6N + 2a - 2k_0 = 0;\quad 2N - k_0 = -2$

ii) $6N + 2a - 2k_0 = 2; 2N - k_0 = -1.$

The first case leads to $k_0 = 2N + 2$, $2a = 4 - 2N \leq 4$. As $a \geq 3$ by the first equality, we get a contradiction.

In the second case we get $k_0 = 2N + 1$, $2a = 4 - 2N$, with the only possibility $N = 0$, $a = 2$ and hence $k_0 = 1$.

To check the additional case ($N = 1$, $a = k = 0$) we observe that $S - F$ is exceptional and disjoint from the branch curve $C_{1,0,0}$. Thus upstairs we obtain two exceptional divisors.

This completes the proof of the proposition.

PROPOSITION 3.8: *The Chern invariants of $X_{N,k,a}$ are given by*

$$c_1^2(X_{N,k,a}) = 6N + 4a - 8 - k$$
$$\chi(X_{N,k,a}) = 3N + 2a - 1 - k$$

PROOF: Clear from Corollary 1.26. and Proposition 1.11.

OBSERVATION 3.9: *By looking at N, k, a for which $\chi(X_{N,k,a}) \geq 2$ we get sufficient conditions for the surfaces $X_{N,k,a}$ to have effective canonical divisors. The list incidentally coincides with the one given in the proof of Proposition* 3.7.

Note 3.10: It might be instructive to give the list of the remaining cases, discarded in Proposition 3.7. and see how they fit into the general classification of surfaces.

N	k	a	c_1^2	χ	p_g	q	comments
1	2	0	-4	0	0	1	elliptic, ruled
1	1	0	-3	1	0	0	rational
1	0	0	-2	2	1	0	K-3 blown up twice
0	4	3	0	1	0	0	rational
0	2	2	-2	1	0	0	rational
0	1	2	-1	2	1	0	K-3 blow up once
0	0	1	-4	1	0	0	rational
0	0	0	-8	-1	0	2	$C \times \mathbb{P}^1$, $g(C) = 2$ (minimal)

The quickest way is simply to look at the pluricanonical divisors, and use classification of surfaces.

An entertaining task could be to find rulings directly. E.g. in case $(1,2,0)$, a ruling is obtained by looking at the pullback of the linear system $|2S - E_1 - E_2 - E_1' - E_2'|_1$. In other words at the bisections of type $2S$ with vertical tangents at q_1 and q_2 in the terminology of Proposition 3.1. One can then easily check all the degenerate fibers and get in that way an independent calculation of C_1^2. A challenge would be to find elliptic sections of the ruling explicitly!

Note 3.11: Not all the minimal surfaces described in Proposition 3.7. are of general type. The exceptions are listed below.

N	k	a	c_1^2	χ	p_g	q	comments
2	4	0	0	1	1	1	elliptic
1	2	1	0	2	1	0	elliptic
0	0	2	0	3	2	0	elliptic.

As to the surfaces of general type, in the families $X_{N,k,a}$, we have

PROPOSITION 3.12: *Given x, y positive integers, satisfying* $2x - 6 \leq$

$y \leq 4x-4, y \neq 4x-5$, *then there exists a minimal surface* $X_{N,k,a}$ $(a=0, 1, 2)$ *with* $c_1^2(X_{N,k,a}) = y$; $\chi(X_{N,k,a}) = x$.

PROOF: Consider for each $a = 0, 1, 2$ the linear maps given by

$$\begin{cases} y = 6N + 4a - 8 - k \\ x = 3N + 2a - 1 - k. \end{cases} \quad \text{(cf. Proposition 3.8.)}$$

The image points are characterized by $y - x \equiv 2a - 1(3)$. Furthermore restrict them to the sectors given by $0 \leq k \leq 2N + 2b$, $N \geq 0$ where $b = 0$ $a = 0, 1$; $b = 1$ $a = 2$ (cf. Proposition 3.1.) Those will be mapped onto sectors bounded below by $y \geq 2x - 6$ and above by $y \leq 4x - 4$, $(4x - 8, 4x - 6)$ for $a = 0$ $(1, 2)$.

Alternatively we can recapture N and k from x, y by $N = y - x + 7 - (2a/3)$, $k = y - 2x + 6$. The above bounds on x, y ensures $0 \leq \mathrm{k} \leq 2\mathrm{N} + \mathrm{b}$.

Note 3.13: The "lower line" alluded to in the proof of the previous proposition, is obtained by putting $k = 0$ and corresponds to $y = 2x - 6$. This is the well-known Horikawa line. This is also a lower limit for all surfaces of general type (cf. Introduction). And the surfaces corresponding to its invariants have been exhaustively classified by Horikawa (see [5]).

Note 3.14: By allowing all a no new invariants are obtained.
In fact

$$c_1^2(X_{N,k,a+3}) = c_1^2(X_{N+2,k,a})$$

$$\chi(X_{N,k,a+3}) = \chi(X_{N+2,k,a})$$

In particular we see that a given pair in the sector described in Proposition 3.12. can be obtained in many different ways as the Chern invariants of a surface $X_{N,k,a}$.

One can easily estimate the number of different ways $d(x, y)$ a pair of Chern invariants (x, y) can be covered in the above manner.

It is easy to compute $d(x, y)$, using the alternative observation of 3.12. and the equalities above. In fact

$$d(x, y) = 1 + \left[\frac{N}{2}\right] = \left[\frac{y + x + 13 - 2a}{6}\right]$$

with $a = 0, 1, 2$ chosen appropriately.

It would be very interesting to know to what extent those ways correspond to different families of surfaces, i.e. of different deformation type.

Note 3.15: The explicit construction of the branchcurves $C_{N,k,a}$ in Proposition 3.1. allows a variety of specializations.

The fundamental ones are the following.

i) the acquisition of a new infinitely close triple point.

$$X_{N,k,a} \to X_{N,k+1,a} \text{ (provided } k \leq 2N + b - 1)$$

ii) the coalescence of two infinitely close triple points lying on S_0

$$X_{N,k,a} \to X_{N-1,k-2,a} \ (k \leq 2)$$

ii)′ the coalescence of two infinitely close triple points lying on S_∞

$$X_{N,k,a} \to X_{N+1,k-2,a-3} (a \geq 3, k \geq 2)$$

iii) the coalescence of a fibral component with an infinitely close triple point on S_0

$$X_{N,k,a} \to X_{N-1,k,a+1} (a \geq 1, 1 \leq k \leq 2N + 2(b-1))$$

iii)′ the coalescence of a fibral component with an infinitely close triple point on S_∞

$$X_{N,k,a} \to X_{N+1,k,a-2} (a \geq 2, 1 \leq k \leq 2N + 2(b-1))$$

The verifications of the surface degenerations are a straightforward application of Lemma 1.19, 1.20. and Proposition 2.4.

One has a natural partial order on the $X_{N,k,a}$ by means of specialization. This partial order is made explicit by the above.

E.g. $X_{N',k',a'}$ is a specialization of $X_{N,k,0}$ iff $a = 0$, $N' \leq N$, $k' \geq k + N' - N$.

PROPOSITION 3.16: *We have*

$$q(X_{N,k,a}) = 0 \textit{ unless } a = 0, k = 2N$$

$$q(X_{N,2N,0}) = 1, N > 0 \ (q(X_{0,0,0}) = 2).$$

PROOF: As $H^1(X_{N,k,a}, 0) = H^1(\mathbb{F}_N, 0) \oplus H^1(\mathbb{F}_N, -B)$ (with $2B \equiv C_{N,k,a}$) by Leray spectral sequence (see e.g. [12]).

It suffices to compute $H^1(-B)$ as $H^1(\mathbb{F}_N, 0) = 0$.

By Serre duality $H^1(-B) = H^1(K+B)$, the computation of the latter is intimately connected with the computation of $H^0(K+B)$ (which incidentally is equal to $p_g(X_{N,k,a})$).

Now recall $K+B = S + (N-2+a)F - \Sigma_{i=1}^k E_i$; (cf. proof of Proposition 3.7.)

Denote by $h_k^0(N,a) = \dim H^0(K+B)\,(= p_g(X_{N,k,a}))$ and by $h_k^1(N,a) = \dim H^1(K+B)\ (= q(X_{N,k,a}))$.

The crucial observation is that $h_k^0 - 1 \le h_{k+1}^0 \le h_k^0$ which follows because the requirement that curves should go through an assigned point imposes at most one condition.

By Riemann-Roch it is immediate that the two different cases are related to the behaviour of h_k^1. In fact

$$(1)\qquad \begin{aligned} h_{k+1}^0 &= h_k^0 - 1 \text{ iff } h_{k+1}^1 = h_k^1 \\ h_{k+1}^0 &= h_k^0 \text{ iff } h_{k+1}^1 = h_k^1 + 1. \end{aligned}$$

Those are easy to compute when $k=0$ (we are then working on minimal $\mathbb{F}_N$). In fact

$$(2)\qquad \begin{aligned} &h_0^1(N,a) = 0,\ h_0^0(N,a) = 3N+2a-2 \quad (N,a) \neq (0,0) \\ &h_0^1(0,0) = 2,\ h_0^0(0,0) = 0 \end{aligned}$$

On the other extreme if $k \ge 2N-1+a$, S_0 becomes a fixed component of any effective divisor in $K+B$, and the only moving part will consist of fibers. This makes the computation of h_k^0 exceedingly simple.

In fact

$$h_{2N-1}^0(N,0) = h_{2N}^0(N,0) = N-1 \quad N > 0$$

and

$$h_{2N+2b}^0(N,a) = N-2+2(a-b) \quad a \ge 1.$$

We conclude

$$h_{2N-1}^0(N,0) + 2N - 1 = h_0^0(N,0)$$

and

$$h^0_{2N+2b}(N, a) + 2N + 2b = h^0_0(N, a).$$

This implies by (1)

$$\text{(3)} \quad \begin{aligned} &h^1_k(N, 0) = 0, 0 \le k \le 2N - 1, h^0_{2N}(N, 0) = 1 \quad N > 0 \\ &h^1_k(N, a) = 0, 0 \le k \le 2N + 2b \quad a \ge 1 \end{aligned}$$

(3) (together with (2)) then proves the proposition.

Note 3.17: In particular we note that although $X_{N,2N,0}$ and $X_{N-2,2N,3}$ have the same Chern numbers, they correspond to altogether different families.

One should extract one interesting observation from the proof above. Namely, in its terminology S_0 being a fixed component of the linear system $K + B$, means that the canonical divisor of the double covering is composite with a pencil.

COROLLARY 3.18: *If $k \ge 2N - 1 + a$ and $0 \le a \le 3$, then the canonical map of the surface $X_{N,k,a}$ is composite with a pencil.*

Note that the condition $a \le 3$ is necessary, as according to our construction we can only impose $2N + b$ infinitely close triple points on S_0. Thus $b \ge a - 1$.

This has as a consequence (cf. conjecture R.5 in [15])

PROPOSITION 3.19: *If x,y are positive integers satisfying $4x - 10 \le y \le 4x - 4$, $y \ne 4x - 5$.*

Then there exists a genus two fibration X, whose canonical mapping is composite with a pencil (of genus two) and such that $c_1^2(X) = y$, $\chi(X) = x$.

After this digression, we return to our main concern. As pointed out in the introduction it is of some interest to find out which of the $X_{N,k,a}$ are simply connected.

The following topological lemma will enable us to give sufficient conditions.

LEMMA 3.20: *Let X be a real four dimensional manifold with a* *fibration $\pi: X \to B$ onto a real two dimensional simplyconnected manifold, with path connected fibers.*

Assume that there are no multiple fibers and that there exists at least one simply connected fiber.

Then X is simply connected.

PROOF: See Appendix A.

PROPOSITION 3.21: *We have*

$$\pi_1(X_{N,k,a}) = 0 \text{ if } a \neq 0(3)$$
$$\text{or } a \equiv 0(3) \; a \geq 3 \; k \leq 2N + \frac{4a}{3} - 2.$$

PROOF: In the above cases the branchcurves $C_{N,k,a}$ can be chosen to have fibral components. If $a \not\equiv 0(3)$ this is always the case, if $a \equiv 0(3)$ $a \geq 3$ and $k \leq 2N + (4a/3) - 2$, then there exist $C_{N,k,a-1}$, and $C_{N,k,a-1} + 2F \equiv C_{N,k,a}$, where the fibers are chosen generically.

The genus two fibration of $X_{N,k,a}$ will hence have at least one rational fiber.

Note 3.22: It would be interesting to compute the fundamental group in the remaining cases, i.e. for $X_{N,k,0}$ $k \leq 2N - 1$ and $X_{N,2N+4a/3,a}$, $X_{N,2N+(4a/3)-1,a}$ for $a \equiv 0(3)$ $a \geq 3$.

Some of those cases are already known by classification theory for surfaces, in fact all the examples in note 3.10. which are regular ($q = 0$) are also simply connected.

We are now ready to state the main proposition of this section.

PROPOSITION 3.23: *Given x,y with $2x - 6 \leq y \leq 4x - 8$, then there exists a (minimal) simply connected surface X of general type, such that $c_1^2(X) = y$, $\chi(X) = x$.*

Furthermore X can be chosen to be a genus two fibration.

PROOF: This is now an immediate consequence of Propositions 3.12. and 3.21.

4. The Main Theorems

The strategy of the proofs of the main theorems boils down to the following.

First there is the exhibition of a family of surfaces whose invariants are truly spread out, and reasonably densly so. In fact one has the notion of density of a spread, a notion which can be made very precise in a specified context. (cf. Lemmas 4.13 and 4.2.1)

The second step consists in imposing singularities of specified nature and number upon the surfaces. Their desingularizations will provide new invariants, and for technical purposes it is important to have simple relations between the imposed invariants and the original ones. We are, as has been mentioned in the introduction, working with double coverings and with singularities induced from the branchcurves.

We will exclusively restrict ourselves to infinitely close triple points, where the relations are exceedingly simple (recall Proposition 1.11.)

Now we are forced to consider the notion of content of a surface, or of a pair of invariants. It essentially gives the range of the number of singularities that can controllably be imposed. In its context it can be made quite precise (cf. Lemmas 4.1.6. (table 4.1.9$^{(\prime)}$ and 4.2.3.)

The basic problem is now to find sufficiently large regions in the plane of invariants, where the contents of the invariants of the initial family "matches up" with its density.

To be more specific let us concentrate on the proof of Theorem 3. (The proof of Theorem 2 is sufficiently simple minded for us to disregard the articulation above in its elucidation).

We are now constructing surfaces by taking repeated double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$. The corresponding spread of invariants is then easy to compute (Propositions 1.28. and 1.30.)

We take great care to ensure that the surfaces constructed comply with the conditions set forth in Lemma 3.20. This will then also be true for the imposed surfaces, ensuring that all the surfaces constructed are simply connected. For brevity this is left to be implicit in the all-over presentation, but dealt with explicitly in the proof of Lemma 4.2.3.

We now establish regions, in fact sectors, in the plane of invariants, where the densities match up with the contents. Specifically we consider potential invariants (x,y) such that there is an integer t such that $(x+t, y+t)$ is a pair of invariants of a surface (in the initial family) allowing exactly t infinitely close triple points to be imposed. Proposition 1.11. now shows that (x,y) is a pair of invariants for some surface.

Finally we exploit the method of repeated double coverings to give examples of simply connected surfaces with high c_1^2.

We are now ready to proceed with the proofs.

THEOREM 2: *Let x, y be integers satisfying*

$$2x - 6 \leq y \leq 8x$$

and $y \neq 8x - k$ (where for even k, $k = 2$ and for odd $1 \leq k \leq 15$, $k = 19$).

Then there exists a minimal surface X of general type, such that $c_1^2(X) = y$, $\chi(X) = x$.

Furthermore X can be assumed to be a genus two fibration.

We will first select an appropriate family of surfaces, whose Chern invariants are densely spread out.

DEFINITION 4.1.1: *By a surface of type $X_{a,c}^{N,i}$ is meant a double covering of a ruled surface over a curve of genus $a-1$, along a branchcurve of type $6S + 2(2c + i)F$ where S is a section with selfintersection N, and F a fiber.*

The surfaces above have the following Chern invariants.

LEMMA 4.1.2:

$$c_1^2(X_{a,c}^{N,i}) = 8(A-2) + 6N + 4i = 8a - 16 + 2(4c + 3N + 2i)$$

$$\chi(X_{a,c}^{N,i}) = (A-2) + 3c + 3N + 2i = a - 2 + (4c + 3N + 2i)$$

where $A = a + c$. (Note that $X_{a,c}^{0,0}$ is simply example 1.27. specialized to $b = 0$, $d = 3$).

Note also that it is sufficient to let $N, i = 0, 1$. This will tacitly be assumed in the future.

The Chern invariants of the surfaces $X_{a,c}^{N,i}$, form a subset S of the integral lattice.

More precisely

LEMMA 4.1.3: *The subset S is the intersection of the sector $2x - 6 \leq y \leq 8x$ with a sublattice (of the integral lattice) with coarea six.*

The following reformulation turns out to be convenient.

LEMMA 4.1.4: *The integers x, y are the Chern invariants of a* *surface of type* $X^{N,i}$ *iff* $y = 8x - 6p$ $(p \geq 0, p \neq 1)$ *and* $y \geq 2x - 6$, *where*

$$p \equiv 0(4)\ i = 0\ N = 0\ p \equiv 3(4)\ i = 0\ N = 1$$

$$p \equiv 2(4)\ i = 1\ N = 0\ p \equiv 1(4)\ i = 1\ N = 1.$$

PROOF: $c_1^2 = 8\chi - 6(4c + 3N + 2i)$ (from Lemma 4.1.2), $8\chi - 6p \geq 2\chi - 6$ is equivalent to $\chi \geq p - 1$, but by Lemma 4.1.2. $\chi = a - 2 + p$, and any $a \geq 1$ is used.

In order to fill up the gaps, we will have to impose essential singularities on the surfaces $X^{N,i}$. As those are given by double coverings, we will obtain those by imposing given numbers of infinitely close triple points, and no other essential singularities on the branchcurves.

Let us call a surface $X^{N,i}$ of type (p) if its Chern invariants satisfy $c_1^2 = 8\chi - 6p$.

LEMMA 4.1.5: *Let X be a surface of type* (p), *and assume that we on X impose k infinitely close triple points and no other essential singularities. Then the Chern invariants of the corresponding desingularized surface* $\bar{X}$ *satisfy*

$$c_1^2 = 8\chi - 6p + 7k.$$

The strategy will now simply be, given p what numbers of infinitely close triple points can be imposed.

We remind the reader of Proposition 2.4. which shows that it is pointless to impose other kinds of essential singularities.

In particular if we can prove the following, the bulk of Theorem 2 will follow immediately.

LEMMA 4.1.6: *There is a* p_0 *such that if* $p \geq p_0$, *then for any k,* $1 \leq k \leq 5$ *we can impose on a surface of type* (p) *exactly k infinitely close triple points and no other essential singularities.*

The above lemma is an immediate corollary of the following universal construction.

PROPOSITION 4.1.7: *On a ruled surface of type* $\mathbb{F}_N^a$ *(see Example 1.24.) with* $N = 0, 1$; *there exists for every k, such that* $0 \leq k \leq 2c' + N$, *a curve of type* $(6, 6c')$ *(i.e.* $6S + 6c'F$*) with exactly 2k infinitely close triple points and no other essential singularities.*

We can also find for every k, such that $1 \leq k \leq 2c' + 1 - N$ a curve of type $(6, 6c' + (-1)^N 3)$ with exactly $2k - 1$ infinitely close triple points (and no other essential singularities).

Before proving the above proposition, let us show how it implies Lemma 4.1.6.

Let $c \geq 4$, then $2(2c + i) \geq 6c' + 3$, with $c' \geq 2$. Thus we can find branchcurves of type $(6, 2(2c + i))$ on $\mathbb{F}_N^a$ with exactly k infinitely close triple points, for $0 \leq k \leq 5$, by using Proposition 4.1.7. and if necessary, adding generic fibers. Now if $p \geq 4 \cdot 4 + 3 \cdot 1 + 2 = 21$, then necessarily $c \geq 4$, and we are done.

The bound $p_0 = 21$ given above is of course rather crude, with more care it can be sharpened. It suffices to give the following preliminary version of the theorem.

THEOREM 2′: *Let x, y be integers satisfying*

$$2x - 6 \leq y \leq 8x - 126$$

Then there exists a (minimal) surface X of general type, such that $c_1^2(X) = y$, $\chi(X) = x$.

The rest of this subsection is hence devoted to trying to fill in as many of the finite number of missing lines.

This boils down to the rather pedantic task of trying to represent low numbers on the form $6p - 7k$, where the allowable k depend on p.

For that purpose, we will squeeze out more precise information out of Proposition 4.1.7, present it in tabular form. And supplement the universal construction by a few ad hoc constructions.

Ideally we would have liked to fill in all the missing lines, but this becomes progressively harder the smaller we want $6p - 7k$, until it becomes a challenge we were unable to meet. Hopefully this will be done in the future.

But first we will present the delayed proof of Proposition 4.1.7, which will follow by the lemma below.

LEMMA 4.1.8: *There exists for every k; $0 \leq k \leq 2c' + N$, a pencil of sections of type $(1, c')$ on $\mathbb{F}_N$, such that exactly k of its basepoints (which can all be assumed simple) lie on the union of two given disjoint sections S_0 and S_∞.*

PROOF: Choose an irreducible section S_1 of type $(1, c')$ intersecting S_0 and S_∞ transversally.

S_1 intersects S_0 in $c'+N$ distinct points $p_1, p_2, \dots p_{c'+N}$ and it intersects S_∞ in c' distinct points $q_1, q_2, \dots q_{c'}$.

Clearly no two points lie on the same fiber.

Construct another (reducible) section S_2 of type $(1, c')$ as follows.

i) If $k \le c'$, choose a section S of type $(1, 0)$ not passing through any of the points $p_1, p_2, \dots$ Add fibers passing through k of the points $q_1, q_2, \dots$ and in addition add $c'-k$ generic fibers.

ii) If $k \ge c'$, start out with $S_\infty(=(1, -N))$, add $(k-c')$ fibers passing through $k-c'$ of the points $p_1, p_2, \dots$ and $2c'+N-k$ generic fibers.

The pencil generated by $S_1 \& S_2$ will have the required properties.

We are now ready to construct the curves in Proposition 4.1.7.

For this purpose we consider the following sequence of double coverings

$$\mathbb{F}_N^a \xrightarrow{\pi_1} \mathbb{F}_{2N}^a \xrightarrow{\pi_2} \mathbb{F}_N$$

The branchcurve of π^2 is given by 2a judiciously chosen fibers of $\mathbb{F}_N$.

The branchcurve of π_1 is given by the pullback of two disjoint sections of $\mathbb{F}_N$, denoted by S_0, S_∞ below (cf. Example 1.23.)

On $\mathbb{F}_N$ choose three distinct generic curves C_1, C_2, C_3 in a pencil with exactly k base points on $S_0 \cup S_\infty$. (This is possible by Lemma 4.1.8.)

Choose the branch locus of π_2 to be disjoint from the basepoints. On $\mathbb{F}_{2N}^a$ we have three sections $\pi_2{}^*C_i$ $(i = 1, 2, 3)$ of type $(1, 2c')$ meeting in $2k$ points lying on the branch locus of π_1.

In the final pullback $\pi_1{}^*\pi_2{}^*C_i$ are bisections of type $(2, 2c')$ and their union has exactly $2k$ infinitely close triple points (cf. Corollary 1.20.)

To construct an odd number of infinitely close triple points, we perform an elementary transformation (Definition 2.5.) at a base point of the pencil lying outside the branch locus of π_1 (this is possible exactly when k is short of its maximal possible value), by our construction there will be another triple point on the same fiber, now turned exceptional, which will turn into an infinitely close triple point.

In case $N=0$, an elementary transformation has the effect of changing a curve of type $(2, 2c')$ into $(2, 2c'-1)$, and in case $N=1$, it changes $(2, 2c')$ into $(2, 2c'+1)$.

In our applications below we want the curves to be even, that is easily achieved by adding the appropriate number of generic fibers. It is now easy to establish the following tables:

TABLE 4.1.9. *Even specializations*

Characteristics of branch curve			# of infinitely close triple points	
$c \geq 3$	$p \geq 12$	$p \equiv 0(4)$	$2k$	$0 \leq k \leq 4$
$c \geq 3$	$p \geq 14$	$p \equiv 2(4)$	$2k$	$0 \leq k \leq 4$
$c \geq 1$	$p \geq 9$	$p \equiv 1(4)$	$2k$	$0 \leq k \leq 3$
$c \geq 2$	$p \geq 11$	$p \equiv 3(4)$	$2k$	$0 \leq k \leq 3$
$c \geq 2$	$p \geq 8$	$p \equiv 0(4)$	$2k$	$0 \leq k \leq 2$
$c \geq 1$	$p \geq 6$	$p \equiv 2(4)$	$2k$	$0 \leq k \leq 2$
$c \geq 0$	$p \geq 5$	$p \equiv 1(4)$	$2k$	$0 \leq k \leq 1$
$c \geq 0$	$p \geq 3$	$p \equiv 3(4)$	$2k$	$0 \leq k \leq 1$

TABLE 4.1.9′. *Odd specializations*

Characteristics of branch curve			# of infinitely close triple points	
$c \geq 2$	$p \geq 13$	$p \equiv 1(4)$	$2k-1$	$1 \leq k \leq 4$
$c \geq 3$	$p \geq 15$	$p \equiv 3(4)$	$2k-1$	$1 \leq k \leq 4$
$c \geq 3$	$p \geq 12$	$p \equiv 0(4)$	$2k-1$	$1 \leq k \leq 3$
$c \geq 2$	$p \geq 10$	$p \equiv 2(4)$	$2k-1$	$1 \leq k \leq 3$
$c \geq 1$	$p \geq 9$	$p \equiv 1(4)$	$2k-1$	$1 \leq k \leq 2$
$c \geq 1$	$p \geq 7$	$p \equiv 3(4)$	$2k-1$	$1 \leq k \leq 2$
$c \geq 1$	$p \geq 4$	$p \equiv 0(4)$	1	
$c \geq 1$	$p \geq 6$	$p \equiv 2(4)$	1	

We can first observe that from the tables above it is clear that if $p \geq 10$ any number up to five of infinitely close triple points can be imposed. Thus any line $y = 8x - k$ is covered for $k \geq 60$.

A more careful study reveals that we can cover all the lines with the exception of for even k, $k = 2, 6, 10$ and 14 and odd k, $1 \leq k \leq 15$, $k = 19, 23, 27$ and 31.

To sharpen our result we have to resort to ad-hoc constructions. The following supplement the universal construction of Proposition 4.1.7.

LEMMA 4.1.10: *On a ruled surface of type F_N^a we can find a curve of type $(6, 2(2c+i))$ with exactly k infinitely close triple points and no* *other essential singularities. In the following cases*

	N	i	c	k	remark
(a)	1	0	1	4	
(b)	0	0	1	2	$a \geq 2$
(a′)	1	0	2	5	
(b′)	0	0	2	3	
(c′)	1	1	0	1	

PROOF: As in the proof of Proposition 4.1.7, we are making use of the sequence of double coverings

$$\mathbb{F}_N^a \xrightarrow{\pi_1} \mathbb{F}_{2N}^a \xrightarrow{\pi_2} \mathbb{F}_N$$

Start with $\mathbb{F}_0$. Consider two generic sections of type $(1, 2)$ and $(1, 1)$ respectively. They intersect in three points p_1, p_2 and p_3. Choose a third section C_3 through the points p_i, $i = 1, 2, 3$. Now we can choose S_0 and S_∞ such that $p_1 \in S_0$ and $p_2 \in S$ but $p_3 \notin S_0, S$.

(a) Let the branchlocus of π_2 pass through p_1 and p_3 but *not* through p_2.

(b) Let the branchlocus of π_2 pass through all points p_i, $i = 1, 2, 3$ (note $a \geq 2$).

In case (a) the total pullback of the sections C_i has one six-tuple point (see Corollary 1.22.) and four infinitely close triple points stemming from p_2 and p_3. Perform an elementary transformation at the six-tuple point and we have the first case of Lemma 4.1.10.

In case (b) the total pullback of the sections C_i has two six-tuple points stemming from p_1 and p_2 and two infinitely close triple points stemming from p_3. Perform two elementary transformations at the two six-tuple points respectively. And we have the second case of Lemma 4.1.10.

Finally let the branchlocus of π_2 be (a′) totally disjoint from p_i, (b′) containing p_1 but not p_2 and p_3, and (c′) passing through p_1 and p_2 but not p_3.

In all three cases there will be two ordinary triple points stemming from p_3. Perform an elementary transformation at one of them, turning the other into an infinitely close triple point.

In case (b′) there will be a six-tuple point as well, above p_1, perform an additional elementary transformation at that point.

In case (c′) there will be two six-tuple points above p_1 and p_2. Perform two elementary transformations at those points.

Note 4.1.11: It is now trivial to check the values of $6p - 7k$ in the five examples above. They turn out to be 14, 10, 31, 27, and 23 respectively.

Note 4.1.12: It seems hard to press those results further while restricting oneself to sextic branchcurves. The natural step would be to consider higher degree branchcurves, giving rise to higher genus fibrations. But even then results do no follow easily.

LEMMA 4.1.13: *On a ruled surface of type* $\mathbb{F}_1^a(a \geq 3)$ *one can find a curve of type* $(8,2)$ *with* 6 *infinitely close triple points and no other essential singularities.*

Note 4.1.14: It is straightforward to compute the Chern invariants of a double covering Y of a ruled (type $\mathbb{F}_1^a$) along a branchcurve of type $(8,2)$. In fact Proposition 1.25. yields:

$$c_1^2(Y) = 16a - 8, \chi(Y) = 2a + 5.$$

Thus $c_1^2 = 8\chi - 48$. The imposition of 6 infinitely close triple points gives

$$c_1^2(Y) = 16a - 14, \quad \chi(Y) = 2a - 1,$$

which covers the line $y = 8\chi - 6$!

PROOF OF LEMMA 4.1.13: As usual we consider the sequence of double coverings

$$\mathbb{F}_N^a \xrightarrow{\pi_1} \mathbb{F}_{2N}^a \xrightarrow{\pi_2} \mathbb{F}_N$$

This time with $N = 0$. On $\mathbb{F}_0$ we choose generically four points p_1, p_2, p_3 and p_4, conditioned to p_1, p_4 lying on a horizontal section.

Through p_1 and p_2 we choose a (unique) section C_1, of type $(1,1)$, and similarly we choose C_2 of type $(1,1)$ through p_3 and p_4.

C_1 and C_2 will intersect at two points q_1 and q_2.

Consider all sections of type $(1,2)$ passing through the points p_1, p_2, p_3 and p_4. They form a linear pencil.

In the pencil choose D_1 to pass through q_1 and D_2 to pass through q_2.

Now let the branchcurve of π_2 consist of fibers, six of which (it is here we need $a \geq 3$) pass through q_1, q_2, p_1, p_2, p_3 and p_4.

As the branchcurve of π_1, we consider the pullbacks $\pi_2^*(S_0) \cup$

$\pi_2^*(S_\infty)$ of two disjoint sections S_0, S_∞ on $\mathbb{F}_0$, chosen such that $q_1 \in S_\infty$.

Let $\pi = \pi_1\pi_2$, and $C = C_1 \cup C_2 \cup D_1 \cup D_2$.

It is clear that $\pi^*(C)$ is a curve of type $(8, 12)$ on $\mathbb{F}_0^a$, having three six-tuple points above q_1, p_1 and p_4 respectively, and also having two infinitely close triple points above each point p_2, p_3 and q_2.

We now perform elementary transformations at the six tuple points of $\pi^*(C)$. This defines a rational map: e: $\mathbb{F}_0^a \to \mathbb{F}_1^a$.

The proper image $\bar{e}_*\pi^*(S_\infty) = \bar{S}_\infty$, becomes a section with self-intersection -1.

It is elementary to check the intersections $(\bar{S}_\infty \cdot \bar{e}_*\pi^*(C_i)) = (\bar{S}_\infty \cdot \bar{e}_*\pi^*(D_1)) = 0$ and $(\bar{S}_\infty \cdot \bar{e}_*\pi^*(D_2)) = 2$.

Thus $(\bar{S}_\infty \cdot \bar{e}_*\pi^*(C)) = 2$ and $\bar{e}_*\pi^*(C)$ is of type $(8, 2)$ with six infinitely close triple points and no other essential singularities.

Note 4.1.15: Variations of the above construction do not, unfortunately, lead to anything new. E.g. if we in the proof of Lemma 4.1.13. demand that p_2 and p_3 as well lie on a horizontal section, choosing that instead as our S_0; we will, after the appropriate elementary transformations, end up with a curve of type $(8, 4)$ in $\mathbb{F}_0^a$ $(a \geq 3)$ with four infinitely close triple points. The corresponding double coverings will have their Chern invariants on the line $y = 8x - 4$, which, however, was covered earlier.

Note also that our constructions are fairly efficient in producing an even number of infinitely close triple points. But far less so in exhibiting an odd number.

Note 4.1.16: For completeness we should observe that all the double coverings considered above are minimal. If $a > 1$ this is obvious, because then our surfaces are fibered over non-rational curves, and there is no need to worry about the transversal exceptional divisors. If $a = 1$, then we are simply considering subcases of Proposition 3.7.

Note 4.1.17: The observant reader has noticed that on the line $8x - 6$, we have missed (due to the condition $a \geq 3$ in Lemma 4.1.3.) the two cases $(1, 2)$ and $(3, 18)$.

The first is a wellknown invariant e.g. the Campedelli double plane. The second is left to the same observant reader.

Let us now turn to the more involved proof of the more interesting result.

THEOREM 3: *Let x and y be positive integers satisfying*

$$4x - 8 \leq y \leq 8(x - Cx^{2/3}) \text{ where } C = 9/\sqrt[3]{12}.$$

Then there exists a simply-connected minimal surface of general type X, with $c_1^2(X) = y$ and $\chi(X) = x$.

Furthermore X can be assumed to be a double covering of a rational surface.

PROOF: To get our skeleton of surfaces, we will consider the family of Example 1.27. specialized to $b = 1$. The intermediate surfaces (depending on the parameter a) will then be rational, and the intermediate branchcurve will turn into $4d$-sections. As we have great freedom in varying the branchcurve, we can assume it to contain fibral components. By Lemma 3.20. the corresponding double coverings will be simply connected.

The invariants of the surfaces will according to Proposition 1.28. be given by

$$(*) \qquad \begin{aligned} c_1^2 &= 8(A-2)(d-1) \\ \chi &= (A-2)(d-1) + A + c(d-1) \end{aligned}$$

where we recall $A = a + c$ and $0 < c < A$.

It is now convenient to consider a parameter space (a, c, d) on one hand, and the invariant space (x, y) $(=(\chi, c_1^2))$ on the other.

For each fixed d, $(*)$ defines a linear mapping L_d from the (a, c) plane into the (x, y) plane. (It takes e.g. slanted lines $a + c = A$ onto horizontal lines $y = 8(A-2)(d-1)$).

The image of the integral (a, c)-lattice will be a translated sublattice of the integral (x, y)-lattice.

From $(*)$ one readily sees that (x, y) is in the image iff it satisfies $y \equiv 0(8(d-1))$, $x \equiv A(d-1)$ where $A - 2 = (y/8(d-1))$.

Thus the coarea of the sublattice is equal to $8(d-1)^2$.

In fact we have the following more precise statement.

LEMMA 4.2.1. *Given (x, y) then there exists an integer t, which can be assumed to satisfy $0 \leq t < 8(d-1)^2$, such that $(x + t, y + t)$ lies in the image of the integral (a, c)-lattice under the linear map L_d.*

PROOF: Put $t = 8(d-1)(x - y - 2) - y + 8(d-1)^2 m$!

Now we are only interested in the sector given by $a, c > 0$. Because those are the parameter values giving rise to surfaces. The corresponding values of the invariants lie in a sector described below.

LEMMA 4.2.: *A given (x, y) is in the image under L_d of the first quadrant of the (a, c)-lattice, iff it is in the image of the (a, c)-lattice and satisfies the inequalities below.*

$$8\frac{d-1}{2d-1}(x-(d+1)) \leq y \leq 8\frac{d-1}{d}(x-(d+1)).$$

PROOF: The left and right lines, giving the lower and upper bound respectively, are just the images of the lines $a = 1$ and $c = 1$, under L_d.

The strategy is now clear. In order to fill out the sectors, we will have to impose given numbers of infinitely close triple points. If sufficient numbers can be constructed we are assured of success due to Lemma 4.2.1.

However, not all parameter values will give rise to surfaces allowing a sufficient number of essential singularities to be imposed.

More precisely,

LEMMA 4.2.3: *For any k, $0 \leq k \leq 8(d-1)^2$, we can impose exactly k infinitely close triple points and no other essential singularities on the final branchcurve, provided*

$$c \geq 9(d-1)+2 \quad 3a \geq 8(d-1)^2.$$

In order not to disrupt the general flow of the argument, we will postpone its proof.

The conditions of the previous lemma define displaced quadrants Q_d in the (a, c) plane; parametrizing so to speak sufficiently effective surfaces. We will also introduce the shifted quadrant $\bar{Q}_d$ by $c \geq 9(d-1)+2$ $3a \geq 8(d-1)^2+3(6d-2)$. Let us denote their images under L_d by S_d and $\bar{S}_d$ respectively. They form sectors in the (x, y) plane of invariants.

We observe

LEMMA 4.2.4: *If $(x, y) \in \bar{S}_d$, then $(x+t, y+t) \in S_d$, provided $0 \leq t \leq 8(d-1)^2$.*

We can now conclude by 4.2.1. and 4.2.3.

PROPOSITION 4.2.5: *If an integral lattice point* $(x, y) \in \bar{S}_d$, *then there is a simply connected surface* X *(which can be assumed to be a double covering of a rational surface) with* $c_1^2(X) = y$, $\chi(X) = x$.

What is left, modulo the proof of 4.2.3. is to describe the region $S = \cup_d \bar{S}_d$ in the (x, y) plane.

LEMMA 4.2.6: *The region* S *contains the region defined by*

$$4x - 8 \leq y \leq 8(x - Cx^{2/3}) \text{ with } C = 9/\sqrt[3]{12}.$$

PROOF: Let $l_d(x)$ denote the linear function whose graph is the image under L_d of the line $c = 9(d-1)^2$, similarly let $m_d(x)$ be associated to $3a = 8(d-1)^2 + 3(6d-2)$.

It is clear that l_d and m_d define the upper and lower bounds respectively of the sector $\bar{S}_d$.

The most interesting aspect is to compute an effective lower bound on $s(x) = \sup_d l_d(x)$.

A straightforward computation yields

$$y = 8\frac{d-1}{d}(x - 9(d-1)^2 - 2d) = l_d(x).$$

Note that the slopes $8(d - 1/d)$ increase as d increases. Furthermore if x_d satisfies $l_d(x_d) = l_{d+1}(x_d)$, then

$$x_d = 18d^3 + 2d^2 - 16d + 9, \text{ which forms an increasing sequence.}$$

Thus $s(x)$ is a monotonically increasing piecewise linear function.

We want a bound of type $s(x) \geq A(x - Cx^t)$. $(A > 0)$. The right hand function is convex for $0 < t < 1$, thus it is sufficient to check the inequality for the values x_d.

Now

$$y_d = l_d(x_d) = 8(18d^3 - 25d^2 + 7d).$$

Thus

$$y_d \geq 8(x_d - 27d^2) \geq 8(x_d - Cx_d^{2/3})$$

with

$$C = 9/\sqrt[3]{12}.$$

Finally we compute

$$y = 8\frac{d-1}{2d-1}(x - (2 - \frac{8}{3}(d-1)^2 - 6d + 2)(d-1) - 2) = m_d(x).$$

Tedious calculations show that the vertex of the sector $\bar{S}_{d+1}$ is contained in $\bar{S}_d$ ($d \geq 3$). As the vertex of $\bar{S}_3$ lies below $4x - 8$, and the slopes of l_3 and m_3 are strictly bigger and strictly less than four respectively, we are done.

Now to the proof of Lemma 4.2.3. It will be worthwhile to isolate the following elementary observation.

Observation 4.2.7: *Let S be a set of N points on* F_0, *generic in the sense* that each fiber (regardless of the fibration) contains at most two points of S. If L is a complete linear system of bisections, with* $\dim|L| \geq N$, *then the generic element of L passing through the points of S, contains no fibral components.*

Proof: The key point is that there are three conditions for a linear system of bisections to contain a given fiber as a fixed component.

Corollary 4.2.9: *Let S be a set of N points on* F_0, *generic in the sense** that each fiber contains at most one point of S. If L is a complete linear system of bisections, with* $\dim|L| \geq 2N$, *then one can find for each subset S′ of S, an element of L, with no fibral components, and passing through the points of S′ but avoiding all points of* $S - S'$.

Proof: Construct a set T generic in the sense* and disjoint from $S - S'$, by adding to S' two points from each fiber passing through $S - S'$. The cardinality of T is then at most $2N$, an application of Observation 4.2.7. exhibits an element of L, with no fibral components, passing through T. Such an element cannot pass through any points of $S - S'$, as it can only intersect each fiber twice.

Note that the dimension requirement for L in the corollary can no doubt be weakened for "generic" generic points. We prefer, however, to play safe, and it is sufficient for our purposes.

Now to the proof of Lemma 4.2.3.

Proof: The main branchcurve on the intermediate surface will be constructed as the pullback of three distinct non singular curves in a pencil on $\mathbb{F}_0$ of type (c', d') plus the appropriate number of generic horizontal and vertical fibers to make up a curve of type $(2c, 2d)$.

We will choose c' and d' such that $2c-2 \leq 3c'+1 \leq 2c$ and $2d-2 \leq 3d' \leq 2d$.

The three curves will be assumed to intersect transversally (i.e. having distinct base points) and no two on the same fiber.

If $c \geq 9(d-1)+2$ the number of intersections will be $(c', d')^2 = 2c'd' \geq (8/9)(c-3/2)(d-1) \geq 8(d-1)^2$.

Now by counting constants and applying Corollary 4.2. we see that if $3a \geq 8(d-1)^2$, we can find an initial branchcurve, without fibral components, on $\mathbb{F}_0$ of type $(2a, 2)$ passing through exactly k of the basepoints. By Corollary 1.20. we have produced a branchcurve on the intermediate rational surface, with the required essential singularities.

Notice also that by our carefulness, the final double covering will satisfy the requirements of Lemma 3.20.

Finally, for completeness we should indicate why the surfaces constructed are minimal.

LEMMA 4.2.10: *On each of the surfaces contructed above one can find a canonical divisor with no rational components.*

COROLLARY 4.2.10: *All the surfaces constructed are minimal.*

PROOF: Choose a curve of type (c', d') of $\mathbb{F}_0$, in the terminology of the proof of Lemma 4.2.3. passing through all the push downs of the infinitely close triple points upstairs (on the intermediate double coverings). Now add the required number of vertical and horizontal fibers to add up to $(a+c-2, d-1)$, choosing the fibers generically. It is clear that the pullback will indeed be the canonical divisor (cf. Proposition 1.2.), and the check that it has no rational components is straightforward.

The corollary follows from the well known fact (cf. Proposition 3.7.) that all exceptional divisors show up as fixed components of the canonical divisor.

REMARK 4.3: In the above case we have not used the full force of our spread. Rather than looking at each d seperately, we could have considered them simultaneously. This would ostensibly have increased the "density" of the spread of invariants, and hence lessened the demands on "content", which of course would have allowed wider sectors to be covered.

But to actually show that the densities do increase, one is lead to considering certain number theoretical lemmas, which I have been unable to deal with. The type or results needed is indicated below.

In the interval $I_d = [d, d^2]$ consider the union $A^{(d)}$ of all additive subgroups $A_k = \{nk: n \in \mathbb{Z}\}$ for $d \le k \le 2d$.

A value $x \in I_d \setminus A^{(d)}$ is said to be a gap value. And we say we have a gap of length l, if there is l consecutive numbers $x, x+1, \dots x+l-1 \in I_d \setminus A^{(d)}$.

We are now interested in asymptotic bounds of l in terms of d.

CONJECTURE: *If l is the length of a gap, then*

$$l \le d^{\varepsilon}, \text{ for arbitrarily small } \varepsilon, \text{ provided } d \text{ large enough.}$$

If the conjecture is true, we can replace the exponent 2/3 in Theorem 3 with $(1/2) + \varepsilon$.

We are lead to believe that Theorem 3 cannot be significantly sharpened, by the methods presented here. There might of course exist totally different approaches.

If we are not primarily interested in filling out sectors of invariants, but just to exhibit simply connected surfaces with high c_1^2, we may state

PROPOSITION 4.4: *There exists simply connected surfaces X, with $c_1^2(X) = y$, $\chi(X) = x$, such that $y \ge 8(x - 2x^{1/2})$ for arbitrarily large x.*

PROOF: We simply put $c = 1$, $a = d$ in our construction for Theorem 3, letting the final branchcurve have two vertical components. By Lemma 3.20. the resulting surfaces will be simply connected and their Chern invariants will be given by $c_1^2 = 8(a-1)^2$, $\chi = a^2 + 1$, and we are done!

It would be interesting to know whether the exponent 1/2 that comes up, merely reflects a deficiency in the method of constructing surfaces or if it is charged with deeper significance.

Appendix A

We are now going to give the proof of the purely topological Lemma 3.20. I am indebted to Prof. Moishezon for supplying the arguments.

Recall the situation of a map $f\colon\ V \to \mathbb{P}^1$, where V is a 4-$\dim_{\mathbb{R}}$ manifold. Let $a_1, \dots .\ a_n$ denote the critical values of f, i.e. corresponding to the singular fibers $f^{-1}(a_i)$.

Assume (a) none of the singular fibers is multiple.

(b) at least one fiber is simply connected.

The claim is now that V is simply connected.

PROOF: Let $S' = \mathbb{P}^1 \setminus \cup_{i=1}^{n}\{a_i\}$, $V' = f^{-1}(S')$, $f' = f_{|V'}\colon V' \to S'$. We first observe that we have a natural surjection $\pi_1(V') \xrightarrow{\gamma} \pi_1(V)$ (Any loop in V can be removed from the singular fibers, those having real codimension two).

Now let F denote a generic fiber in V. F being connected we obtain the following exact sequence.

$$(*) \qquad \pi_1(F) \xrightarrow{\alpha} \pi_1(V') \xrightarrow{s} \pi_1(S') \to 1.$$

Let $c_1, \dots c_k$ be generators of $\pi_1(F)$ and $d_1, \dots d_n$ be generators of $\pi_1(S')$ corresponding to small loops around the critical points a_i.

Now let $\bar{d}_i$ be elements in $\pi_1(V')$ such that $s\bar{d}_i = d_i$. Because of surjectivity of γ, we note that $\gamma(\bar{d}_i)$, $\gamma\alpha(c_i)$ generate $\pi_1(V)$.

Let D_i be disks with boundary $\partial D_i = d_i$. We can choose loops representing $\bar{d}_i$, such as to be boundaries ∂D_i of local sections of f over D_i. (No multiple fibers are assumed to exist). This shows $\gamma\bar{d}_i = 0$.

Finally we observe that $f^{-1}(D_1)$ is homotopically equivalent to $f^{-1}(a_1)$ (a deformation retract), as the latter is assumed to be simply connected we have $\pi_1(f^{-1}(0, 1)) = 0$. Now the generic fiber F in $(*)$ can be assumed to be $f^{-1}(t)$ where t is a generic point in D_1 (i.e. $t \neq a_1$). By

$$\begin{tikzcd}
\pi_1(F) \arrow[r, "\alpha"] \arrow[d] & \pi_1(V') \arrow[d, "\gamma"] \\
\pi_1(0) \arrow[r] \arrow[d, equal] & \pi_1(V) \\
0 &
\end{tikzcd}$$

we observe $\gamma\alpha(c_i) = 0$. Hence $\pi_1(V) = 0$.

In order to be able to apply Lemma 3.20. in Proposition 3.21, Theorem 3 and Proposition 4.4, we have to check the nature of the critical fibers. Specifically we have to show that fibers have reduced components, and that the fibers arising from our constructions with fibral components are indeed simply connected.

The latter follows from the following lemma.

LEMMA: *Let $f: X \to \mathbb{P}^1$ be a rational fibration, and let C be a branchcurve with a fibral component $F = f^{-1}(a)$. The corresponding (desingularized) double covering Y will then get an induced fibration over $\mathbb{P}^1$.*

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ X \arrow[r, "f"'] & \mathbb{P}^1 \end{tikzcd}$$

We then have $g^{-1}(a) = 2\bar{F} + \Sigma \bar{E}_i$, where $\bar{F}, \bar{E}_i$ are rational curves and $\bar{F} \cdot \bar{E}_i = p_i$ and $\Sigma \pi_* p_i = C \cdot F$.
In particular, $g^{-1}(a)$ is simply connected.

PROOF: Consider

$$\begin{tikzcd} Y \arrow[d, "\pi'"'] \arrow[dr, "g"] & \\ X' \arrow[r, "f'"] \arrow[d, "p"'] & \mathbb{P}^1 \\ X \arrow[ur, "f"'] & \end{tikzcd}$$

where X' is the even desingularization of X, cf. Definition 1.5.

If $C = C_0 + F$, we can represent C' (locally along F) as $C' = (C_0 - \Sigma E_i) + (F - \Sigma E_i)$, where E_i are the exceptional divisors corresponding to the points $C_0 \cdot F$ $(= C \cdot F)$.

Thus $f'^{-1}(a) = (F - \Sigma E_i) + \Sigma E_i$, where the first component F' is part of the branchlocus C'.

Letting $\bar{F} = \pi'^{-1}(F')$ and $\bar{E}_i = \pi'^{-1}(E_i)$ we obtain $g^{-1}(a) = 2\bar{F} + \Sigma \bar{E}_i$.

Finally to show that the remaining critical fibers of the fibrations considered do have reduced components it is sufficient to check the following

LEMMA: *Let $f: X \to D$ be a fibration, and assume that $f^{-1}(a)$ contains a reduced component not part of the branchlocus C. Then the fiber $g^{-1}(a)$ of the corresponding double cover Y with its induced fibration, also contains a reduced component.*

PROOF: Conserve the notation of the proof of the previous lemma. If F denotes a reduced component of $f^{-1}(a)$ not part of C, then $p^{-1}(F)$ will be a reduced component of $f'^{-1}(a)$ not part of C'. Thus $\pi^{-1}(F)$ will be a reduced component of $g^{-1}(a)$.

We finally observe that in the relevant cases (3.2.1, 3.4.4) we construct double coverings of rationally fibered surfaces, all of whose fibers have but reduced components.

The above lemmas together with the proof of 3.20. then shows that the surfaces so constructed are indeed simply connected.

Appendix B

We are now going to make the suggested arguments following Lemma 3.3. more precise.

In order not to be too lengthy we will not give all the details, but because of the beautiful geometry involved we will make occasional digressions.

The plan of the proof is as follows,

We will exhibit a rational map

$$\Phi : P \times |S|_{2N} \to |2S|_N$$

where P is some compact (although mildly singular) parameter space, which will parametrize all double coverings $\Pi : \mathbb{F}_N \to \mathbb{F}_{2N}$ (cf. ex. 1.23.)

We will then show:

(1) There is a Zariski open subset P_0 of P such that

(a) $\Phi : P_0 \times |S|_{2N} \to |2S|_N$ is given by the pullback of sections of $\mathbb{F}_{2N}$.

(b) $(\mathrm{Im}\,\Phi(P \setminus P_0) \times |S|_{2N}) \cap |2S|_N^0 \subseteq (\mathrm{Im}\,\Phi(P_0)) \cap |2S|_N^0$ where $|2S|_N^0$ denotes the non-singular bisections of type $2S$.

(2) If z is a generic element of the image of Φ,

$$\dim \Phi^{-1}(z) + \dim |2S|_N = \dim(P \times |S|_{2N})$$

Now (2) (which is essentially the suggestive counting of constants) shows that Φ is surjective. The crucial fact (1b) shows

$$\mathrm{Im}\,\Phi(P_0 \times |S|_{2N}) \supseteq |2S|_N^0$$

and finally (1a) completes the arguments for the validity of Lemma 3.3.

Proof: We can clearly assume $N > 0$ ($N = 0$ there being nothing to prove).

There are now unique projections $p_N : \mathbb{F}_N \to \mathbb{P}^1$, those have "canonical" sections $s_N : \mathbb{P}^1 \to \mathbb{F}_N$ given by the minimal sections S_∞. (cf. ex. 1.23.)

We can now form the fiberproduct

$$\mathbb{F}_{N \times \mathbb{P}^1} \mathbb{F}_{2N} = \{(x, y) \in \mathbb{F}_N \times \mathbb{F}_{2N} | p_N(x) = p_{2N}(y)\}.$$

The space P will be defined as follows.

Let $p = (\psi, \zeta) \in H^0(\mathbb{F}_N, X) \times H^0(\mathbb{F}_{2N}, S)$ define a $\mathbb{C}^*$ action by $tp = (t\psi, t^2\zeta)$, and quotient out to get P. P will clearly be proper.

We will now show how P parametrizes all the involutive maps $\pi : \mathbb{F}_N \to \mathbb{F}_{2N}$.

First we define a subvariety $\Pi \subseteq P \times \mathbb{F}_{N \times \mathbb{P}^1} \mathbb{F}_{2N}$. Choose two non-zero sections $\psi_0 \in H^0(\mathbb{F}_N, S - NF)$ and $\zeta_0 \in H^0(\mathbb{F}_{2N}, S - 2NF)$ fixed from now on.

(*) Define $\Pi = \{(\psi, \zeta; x, y) | \zeta_0(y)\psi^2(x) = \zeta(y)\psi_0^2(x)\}$.

A moments thought shows that this is indeed well-defined.

Now let P_0 be the open subset of P, consisting of (ψ, ζ) with ψ, ζ defining irreducible sections.

If $p \in P_0$, $\Pi_p \subseteq \mathbb{F}_{N \times \mathbb{P}^1} \mathbb{F}_{2N}$ defines an involutive map $\Pi_p : \mathbb{F}_N \to \mathbb{F}_{2N}$. The branch locus of $\mathbb{F}_{2N}$ is given by the minimal section *and* the section $(\zeta = 0)$; similarly the ramification locus on $\mathbb{F}_N$ is given by its minimal section *and* the section $(\psi = 0)$.

The subset P_0 is the first part of a natural stratification $P = P_0 \cup P_1 \cup P_2 \cup P_{12}$ of disjoint locally closed sets, where in addition to P_0

$$P_1 = \{(\psi, \zeta) | \ \psi \text{ reducible, } \zeta \text{ irreducible}\}$$

$$P_2 = \{(\psi, \zeta) | \ \psi \text{ irreducible, } \zeta \text{ reducible}\}$$

$$P_{12} = \{(\psi, \zeta) | \ \psi, \zeta \text{ reducible}\}.$$

In our analysis which follows P_{12} will be subjected to a further, finer stratification.

The subvariety Π will be unnecessarily big and we will consider $\bar{\Pi}$ – the transversal irreducible component of Π. (Note that $\bar{\Pi} = \Pi$ above $P - P_{12}$).

Using $\bar{\Pi}$ we can exhibit the rational map

$$\Phi : P \times |S|_{2N} \to |2S|_N$$

it will be defined as follows.

(**) $\Phi(p, s) = \{x \in \mathbb{F}_N | (x, y) \in \bar{\Pi}_p \text{ and } y \in s\}$.

It will become clear that Φ is indeed a rational map and that its image is indeed contained in $|2S|_N$.

First, however, we observe that if $p \in P_0$, then $\Phi(p, s) = \Pi_p^*(s)$ which shows (1a).

The proof of (1), which incidentally verifies the claim above, will follow from a detailed study of $\bar{\Pi}$ on P_1, P_2 and P_{12} respectively.

It is convenient to make the preliminary observation, (cf. ex. 1.23.) that if $\phi \in H^0(\mathbb{F}_N, S)$ defines a reducible section, we can write $\phi = \gamma\phi_0$, with $\phi_0 \in H^0(\mathbb{F}_N, S - NF)$ and $\gamma \in H^0(\mathbb{F}_N, NF) \cong H^0(\mathbb{P}^1, N)$.

If ϕ_0 is fixed, this sets up a $1-1$ correspondence between reducible sections and sections of the appropriate line bundle on $\mathbb{P}^1$ (the geometric interpretation, independent of the choice of ϕ_0, in terms of divisors on the minimal section is obvious).

(A) $p \in P_1$ $\quad \Pi_p = \bar{\Pi}_p$

$\bar{\Pi}_p$ reduces into a sum $2V + W$, where V projects onto the minimal section of $\mathbb{F}_N$ and W projects onto a section of type S on $\mathbb{F}_{2N}$.

(B) $p \in P_2$ $\quad \Pi_p = \bar{\Pi}_p$

Π_p reduces into a sum $V + W$, where V projects onto a bisection of type $2S$ on $\mathbb{F}_N$ and W projects onto the minimal section of $\mathbb{F}_{2N}$

The projection of V onto $\mathbb{F}_N$ is defined by a section of type $\psi^2 - s\psi_0^2$. In its projection onto the minimal section, it is ramified exactly when $s = 0$, along points given by its intersection with the section $(\psi = 0)$. (cf. Corollary 3.4.) [Conversely a bisection whose vertical tangents lie on a section $(\psi = 0)$ can be defined by a section as above].

(C) Now ψ, ζ being both reducible, we can write $\psi = \alpha\psi_0$, $\zeta = s\zeta_0$ with $\alpha \in H^0(\mathbb{P}^1, N)$, $s \in H^0(\mathbb{P}^1, 2N)$.

i) Consider the open subset $P_{12}^0 \subseteq P_{12}$ where $\alpha^2 \neq s$. Then $\Pi_p = \bar{\Pi}_p = V + 2W + \Sigma n_i E_i$, where V, W project onto the minimal sections of $\mathbb{F}_{2N}$ and $\mathbb{F}_N$ respectively.
And $E_i = \mathbb{P}^1 \times \mathbb{P}^1$ projects onto p_i on the common minimal section. The divisor $\Sigma n_i p_i$ is the zero divisor of the section $\alpha^2 - s$ of $\mathcal{O}_{\mathbb{P}^1}(2N)$.

ii) Consider $\bar{P}_{12} = P_{12} \setminus P_{12}^0$ where $s = \alpha^2$.
In order to study $\bar{\Pi}_p$ we will consider the normal directions to $\bar{P}_{12}$ in P.

Thus pick $\psi^1 \in H^0(\mathbb{F}_N, S)$; $\zeta^1 \in H^0(\mathbb{F}_{2N}, S)$ and consider the segments $p(\epsilon) = (\psi(\epsilon), \zeta(\epsilon))$ where $\psi(\epsilon) = \alpha\psi_0 + \epsilon\psi^1$; $\zeta(\epsilon) = \alpha^2\zeta_0 + \epsilon\zeta^1$.

We are now going to compute $\lim_{\epsilon\to 0}\Pi_{p(\epsilon)} = \Pi_{p(0)}$ which after straightforward manipulation comes out accordingly.

$$\bar{\Pi}_{p(0)} = \{(x, y) \in \mathbb{F}_{N \times_{\mathbb{P}^1}} \mathbb{F}_{2N} \,|\, \psi_0(2\alpha\zeta_0\psi^1 - \zeta^1\psi_0) = 0\}.$$

(a) Assume neither ψ^1, ζ^1 reduces.

$$\bar{\Pi}_{p(0)} = V + W$$

where V projects onto the minimal section of $\mathbb{F}_N$ and W sets up a birational correspondence between $\mathbb{F}_N$ and $\mathbb{F}_{2N}$.

W projects naturally onto the common minimal section, which is a section of W as well.

W cannot be minimal ruled its fibers become reducible exactly when α vanishes. In fact W is non-singular and sets up the birational morphism of N elementary transformations along the divisor of α.

(b′) Assume $\psi' = s\psi_0$, ζ' irreducible.

$$\bar{\Pi}_{p(0)} = 2V + W$$

where V projects onto the minimal section of $\mathbb{F}_N$, and W projects onto the section given by $\zeta - 2\alpha s\zeta_0$.

(b″) Assume $\zeta' = \gamma\zeta_0$, ψ' irreducible.

$$\bar{\Pi}_{p(0)} = V_1 + V_2 + W$$

where V_i projects onto the minimal section of $\mathbb{F}_{iN}$, $i = 1, 2$, and W projects onto the section given by $2\alpha\psi' - \gamma\psi_0$ on $\mathbb{F}_N$.

(c) Assume $\psi' = s\psi_0$, $\zeta' = \gamma\zeta_0$.

$$\bar{\Pi}_{p(0)} = 2V + W + \sum n_i E_i$$

with V, W self-explanatory by now and $E_i = \mathbb{P}^1 \times \mathbb{P}^1$ projecting onto p_i, $\Sigma n_i p_i = \mathrm{div}(2\alpha s - \gamma)$.

Finally if $2\alpha s = \gamma$, then the corresponding $p(\epsilon)$ is not normal to $\bar{P}_{12}$!
It is now easy to check the behaviour of Φ. It has already been done on P_0, so let us now do it systematically.

(A) If s is a section of type $|S|_{2N}$, then its image under Φ will be $2S_\infty + 2NF$, where the $2N$ fibers come from the intersection of s with the projection of W.

In case s coincide with the projection of W, Φ is not well defined via (**). But it is straightforward to compute the images, which are all of the above type.

(B) If s is a section of type $|S|_{2N}$, then Φ maps it onto the projection of V, which is a pullbacked bisection of $\mathbb{F}_N$. (As shown earlier its vertical tangents lie on a section, which defines the involution).

(C) The images of a section of type $|S|_{2N}$ become respectively

i) $2S_\infty + 2NF$, (the $2N$ fibers defined by $\operatorname{div}(\alpha^2 - s)$)
ii) (a) $S_\infty + (S + NF)$
(b)′ $2S_\infty + (2NF)$
(b)″ $S_\infty + (S + NF)$
(c) $2S_\infty + (2NF)$

In particular they are all reducible, and this list proves (1b).

Finally to prove (2) we note first

$$\dim P = \dim H^0(\mathbb{F}_N, S) + \dim H^0(\mathbb{F}_{2N}, S) - 1.$$

Furthermore if z is a generic element of the image of Φ, $z \in |2S|_N$, it determines uniquely a section $s \in |S|_N$ passing through its $2N$ ramification points with respect to its projection. Thus $\dim \Phi^{-1}(z) = \dim H^0(\mathbb{F}_{2N}, S)$ (because the involution on $\mathbb{F}_N$ is determined by s, leaving as the only free parameter ζ cf def (*)).

By Riemann-Roch

$$\begin{aligned} \dim H^0(\mathbb{F}_N, S) &= N + 2 \\ \dim H^0(\mathbb{F}_{2N}, S) &= 2N + 2 \\ \dim H^0(\mathbb{F}_N, 2S) &= 3N + 3. \end{aligned}$$

A simple calculation then gives (2).

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