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ON THE LIMITING DISTRIBUTION OF NON NEGATIVE ADDITIVE FUNCTIONS

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Let A be a set of positive integers and let for $N \geq 2$

$$A(N) = \sum_{n \in A, n \leq N} 1.$$

Let N_0 be the smallest element of A .

Let f be an arithmetic function and define for $N \geq N_0$,

$$F_{N,A}(t) = \frac{1}{A(N)} \sum_{\substack{n \leq N, n \in A \\ f(n) \leq t}} 1.$$

When A is fixed, or $A = N^*$, the set of positive integers, we write simply $F_N(t)$.

We say that f has a limiting distribution of the set A if there exists a non-decreasing function F satisfying $\lim_{t \rightarrow -\infty} F(t) = 0$, $\lim_{t \rightarrow +\infty} F(t) = 1$ and, at every continuity point, t , of F , $F_N(t)$ tends, when $N \rightarrow +\infty$, to $F(t)$.

Erdős and Wintner [2] showed that an additive function f (i.e. $f(m \cdot n) = f(m) + f(n)$ for every coprime m and n) has a limiting distribution on the set of all integers if and only if the following series converge

$$\sum_{|f(p)| \leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| \leq 1} \frac{|f(p)|^2}{p}, \quad \sum_{|f(p)| > 1} \frac{1}{p}.$$

Kátai [3] proved that if these series converge, then the additive function f has a limiting distribution on the set $\{p + 1 \mid p \text{ is prime}\}$.

Elliott [1] proved that if $f(p) \geq 0$ for every prime p and $f(p^r) = f(p)$ if $r \geq 1$ and if f has a limiting distribution on the set $\{p + 1\}$ these series converge. We shall be concerned in the following with non negative additive functions (i.e. $f(p^r) \geq 0$ for every prime power p^r).

THEOREM: *Let A be a set of positive integers satisfying*

(i) *for every $d \in \mathbb{N}^*$,*

$$\lim_{N \rightarrow \infty} \frac{1}{A(N)} \sum_{\substack{d|n, n \leq N \\ n \in A}} 1$$

exists and is equal to $\omega(d)/d$.

(ii) *ω is a multiplicative function satisfying*

$$\sum_p \sum_{r \geq 2} \frac{\omega(p^r)}{p^r} < \infty.$$

Let f be a non negative additive function satisfying

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)\omega(p)}{p} + \sum_{1 < f(p)} \frac{\omega(p)}{p} = +\infty$$

then f has not a limiting distribution on the set A and more precisely

$$\lim_{N \rightarrow +\infty} F_N(t) = 0 \quad \text{for every } t.$$

COROLLARY 1: *If $A = \{p + 1\}$ and if f is a non negative additive function satisfying*

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)}{p} = +\infty \quad \text{or} \quad \sum_{1 < f(p)} \frac{1}{p} = \infty$$

then f has not a limiting distribution on A and $\lim_{N \rightarrow \infty} F_N(t) = 0$ for every t .

This corollary contains Elliott's result.

COROLLARY 2: *Let A be a set of positive integers such that*

$$\liminf \frac{A(N)}{N} > 0.$$

If f is a non negative additive function and if

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)}{p} = +\infty \quad \text{or} \quad \sum_{1 < f(p)} \frac{1}{p} = +\infty$$

then f has not a limiting distribution on A and $\lim_{N \rightarrow \infty} F_{N,A}(t) = 0$.

PROOF OF THE COROLLARIES: Corollary 1 is immediate by remarking that for every

$$d \in \mathbb{N}^*, \frac{1}{A(N)} \sum_{\substack{p^+ | \leq N \\ d | p^+}} 1 \text{ tends to } \frac{1}{\varphi(d)} \text{ that is } \omega(d) = \frac{d}{\varphi(d)}$$

where φ is Euler's function.

For the proof of Corollary 2, let

$$F_{N,A}(t) = \frac{1}{A(N)} \sum_{\substack{n \leq N, n \in A \\ f(n) \leq t}} 1$$

and

$$F_N(t) = \frac{1}{N} \sum_{\substack{n \leq N \\ f(n) \leq t}} 1.$$

From the theorem with $A = \mathbb{N}^*$ we see that $\lim_{N \rightarrow \infty} F_N(t) = 0$. As

$$F_{N,A}(t) \leq \frac{N}{A(N)} F_N(t)$$

we get $\lim F_{N,A}(t) = 0$ for every t .

PROOF OF THE THEOREM: We first remark that

$$\sum_p \frac{\omega(p)(1 - e^{-f(p)})}{p} = +\infty.$$

This is easily deduced from the following inequalities:

$$(1 - 1/e)t \leq 1 - e^{-t} \leq t \quad \text{if } 0 \leq t \leq 1$$

$$(1 - 1/e) \leq 1 - e^{-t} \leq 1 \quad \text{if } t > 1$$

and the hypohese on f .

For $y \geq 2$, define the non negative additive function f_y by

$$f_y(p^r) = \begin{cases} f(p^r) & \text{if } p^r \leq y \\ 0 & \text{if } p^r > y \end{cases} \text{ for every prime power } p^r.$$

Let $g_y = e^{-f_y} * \mu$ where μ is the Möbius function. Clearly g_y is multiplicative, $g_y(p^r) = e^{-f_y(p^r)} - e^{-f_y(p^{r-1})}$ which shows that $|g_y(p^r)| \leq 2$ and that $g_y(n) = 0$ except on a finite set of integers S_y , say.

Let $\Pi_y = \sum \frac{g_y(n)}{n} \omega(n)$. Then one sees easily that

$$\Pi_y = \prod_{p \leq y} \left(1 - \frac{(1 - e^{-f(p)})\omega(p)}{p} + \sum_{r \geq 2} \frac{g_y(p^r)\omega(p^r)}{p^r} \right).$$

As $\sum_p \frac{(1 - e^{-f(p)})\omega(p)}{p} = +\infty$ and $\sum_p \sum_{r \geq 2} \frac{\omega(p^r)}{p^r} < \infty$ we get $\lim_{y \rightarrow \infty} \Pi_y = 0$.

Now, as $e^{-f_y(n)} = \sum_{d|n} g_y(d)$ we have

$$\frac{1}{A(N)} \sum_{\substack{n \leq N \\ n \in A}} e^{-f_y(n)} = \sum_{d \in S_y} g_y(d) \frac{1}{A(N)} \sum_{\substack{n \leq N \\ n \in A, d|n}} 1$$

and so

$$\lim_{N \rightarrow \infty} \frac{1}{A(N)} \sum_{\substack{n \leq N \\ n \in A}} e^{-f_y(n)} = \sum \frac{g_y(d)\omega(d)}{d} = \Pi_y.$$

Remarking that

$$e^{-f(n)} = \prod_{p^r \parallel n} e^{-f(p^r)} \leq \prod_{\substack{p^r \parallel n \\ p^r \leq y}} e^{-f(p^r)} = e^{-f_y(n)}$$

we obtain

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{A(N)} \sum_{\substack{n \leq N \\ n \in A}} e^{-f(n)} \leq \Pi_y$$

and by taking the limit when $y \rightarrow \infty$ we get

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{A(N)} \sum_{\substack{n \leq N \\ n \in A}} e^{-f(n)} = 0.$$

As

$$\frac{1}{A(N)} \sum_{\substack{n \leq N \\ n \in A}} e^{-f(n)} = \int_0^\infty e^{-x} dF_N(x)$$

and for every t

$$\int_0^{\infty} e^{-x} dF_N(x) \geq e^{-t} F_N(t)$$

we obtain

$$\overline{\lim}_{N \rightarrow \infty} F_N(t) = 0$$

and so the theorem.

REFERENCES

- [1] P.D.T.A. ELLIOTT: On the limiting distribution of $f(p+1)$ for non negative additive functions. *Acta Arithmetica XXV* (1974) 259–264.
- [2] P. ERDÖS and A. WINTNER: Additive arithmetical functions and statistical independence. *Amer. J. Math.* 61 (1939) 713–721.
- [3] I. KÁTAI: On the distribution of arithmetical functions on the set of primes plus one. *Compositio Math.* 19 (1968) 278–289.

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