# COMPOSITIO MATHEMATICA

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## Singularities of nets of quadrics

# SINGULARITIES OF NETS OF QUADRICS

## C.T.C. Wall

In the paper [8] I enumerated the different types of net (linear system of freedom 2) of quadrics in (complex, projective) 3-space. The present paper grew out of an appendix to that one in which I listed, for each type, the way the 8 base points coincide in groups. This result was needed for a forthcoming study of affine cubic functions in $\mathbb{C}^3$. The results came out in a particularly neat and satisfactory form, which suggested the existence of a general mechanism.

Some hints at such a mechanism are given in this paper. In Chapter 1 we first show that for a general linear system of quadrics, the singularities of the variety $V$ defined by the family are 'essentially' the same as those of the intersection $B$ of the quadrics of the system. Thus study of the base locus $B$ is reduced to the (easier) study of $V$. Singularities of $V$ occur only on singular quadrics, so we project to the discriminant variety $\Delta$ of the family. Our main theme is the relationship between singularities of $\Delta$ and of $V$; particularly in the case of nets, and isolated singularities.

The results are somewhat complicated, but exhibit some striking numerical relationships (see particularly (3.4), (3.5)) which again suggest deeper mechanisms at work.

The plan of the paper is as follows:

## Chapter 1 Singularities of linear systems

### 1.1 The singularity sets

We consider the general linear system of quadrics given by the vanishing of

$$F(\lambda, x) \equiv \sum_{i=0}^{r} \sum_{j,k=0}^{n} a_{jk}^{i} \lambda_i x_j x_k = x^t \left( \sum_{i=0}^{r} \lambda_i A_i \right) x,$$

where the matrices $A_i = (a_{jk}^i)$ are symmetric. Define the *total variety*

$$V = \{(\lambda, x) : F(\lambda, x) = 0\},$$

the *variety of base points*

$$B = \{x : \forall \lambda, F(\lambda, x) = 0\},$$

and the *discriminant*

$$\Delta = \{\lambda : \det\left( \sum_{i=0}^{r} \lambda_i A_i = 0 \right).$$

Write also for any $\lambda$, $Q_\lambda$ for the quadric $\{x : F(\lambda, x) = 0\}$, and $Q_i$ for the quadric $x^t A_i x = 0$.

For any variety $X$, we write $S(X)$ for the variety of its singular points: we first consider this as a point set. Thus $S(Q_\lambda)$ is the vertex of the quadric $Q_\lambda$. We are mainly concerned with the interrelations between $S(\Delta)$, $S(V)$ and $S(B)$.

LEMMA 1.1: $(\lambda, x) \in S(V)$ *if and only if* $x \in S(Q_\lambda) \cap B$.

Note that $S(Q_\lambda)$ is non-empty only for $\lambda \in \Delta$.

PROOF: If $(\lambda, x) \in S(V)$, then since each $\partial F / \partial \lambda_i$ vanishes at $(\lambda, x)$, $x$ belongs to each $Q_i$, i.e. $x \in B$. Since each $\partial F / \partial x_i$ vanishes, $\Sigma(\lambda_i A_i)x$ vanishes identically, i.e. $x$ is in the vertex $S(Q_\lambda)$. These arguments are reversible.

LEMMA 1.2: $x \in S(B)$ *if and only if there exists* $\lambda \in \Delta$ *with* $x \in S(Q_\lambda)$.

PROOF: $B$ is defined by equations $x^t A_i x = 0$ for $0 \leq i \leq r$ i.e. as the intersection of the $Q_i$. We have a singular point if and only if the tangent planes $x^t A_i y = 0$ do not neet transversely, i.e. satisfy some linear relation

$$0 \equiv \sum \lambda_i x^t A_i y.$$

But this is equivalent to having $x \in S(Q_\lambda)$, where $\lambda = \sum \lambda_i e_i$.

Combining these lemmas, we see that the image of $S(V)$ under the projection on $C^{n+1}$ defined by $x$ is precisely $S(B)$. This projection often gives a bijection $S(V) \to S(B)$ (at least when we regard $\lambda$ as belonging to projective space $P_r = P(C^{r+1})$). Indeed given $x \in B$, we again consider the tangent planes $x^t A_i y = 0$. In general these are independent, spanning an $(r+1)$-dimensional vector space. We have $x \in S(B)$ when they are dependent: call $x$ *tame* if they span an $r$-dimensional space. In this case there is a unique linear relation (up to scalar multiples), hence a unique $\lambda \in P_r$ with $x \in S(Q_\lambda)$ and so $(\lambda, x) \in S(V)$.

From now on we consider $x$, $\lambda$ as points in projective spaces $P_n$, $P_r$. Then $S(V) \to S(B)$ is bijective if all points in $S(B)$ are tame.

## 1.2 The relation of $S(V)$ to $S(B)$

We now study the types of singularity presented. Say that two isolated hypersurface singularities are of *the same type* if either they

are analytically equivalent or they can be reduced by analytic equivalence to hypersurfaces defined by two functions

$$f(z_1, \ldots, z_k) = 0 \text{ in } \mathbb{C}^k$$

$$f(z_1, \ldots, z_k) + \sum_1^t z_{k+i}^2 = 0 \text{ in } \mathbb{C}^{k+t}.$$

Here of course, $\Sigma z_{k+i}^2$ could be replaced by any nonsingular quadratic form.

PROPOSITION 1.3: *Let* $(\boldsymbol{\lambda}, x) \in S(V)$ *where* $x$ *is tame in* $S(B)$. *Then the two singular points have the same type.*

Although $B$ is not a hypersurface in $P_n$, we have seen that it is contained in the (locally) transverse intersection of $r$ quadrics: a manifold in which it has codimension 1.

PROOF: Choose coordinates such that the given point has $\boldsymbol{\lambda} = e_0$ (in $\mathbb{C}^{r+1}$), $x = e_0$ (in $\mathbb{C}^{n+1}$) and $Q_i$ (for $0 \le i \le r$) has tangent plane $x_i = 0$ at $e_0$. Then in affine coordinates $(x_0 = 1)$

$$F(\boldsymbol{\lambda}, x) = \sum_{i=1}^r 2\lambda_i x_i + \sum_{i=0}^r \sum_{j,k=1}^n a_{jk}^i \lambda_i x_j x_k$$

$$= \lambda_0 f_0(x) + \sum_{i=1}^r \lambda_i (2x_i + f_i(x)),$$

where each $f_i(x)$ $(0 \le i \le r)$ has order at least 2—this is the only property used in the sequel. In this situation one can apply Thom's splitting lemma, but we will proceed directly.

We write $\lambda_0 = 1$ in affine coordinates, and take new local coordinates

$$x_i' = x_i + \tfrac{1}{2} f_i(x) \quad (1 \le i \le r) \qquad x_j' = x_j \quad (r < j \le n)$$

and write $f_0(x) = \phi(x')$. Then

$$F(\boldsymbol{\lambda}, x) = \phi(x') + \sum_{i=1}^r 2\lambda_i x_i'.$$

Recall that $B$ is the hypersurface in the manifold $0 = x_1' = \cdots = x_r'$

defined by $0 = f_0(x) = \phi(x')$. Setting

$$\phi(x') = \psi(x'_{r+1}, \ldots, x'_n) + \sum_{i=1}^{r} x'_i \phi_i(x')$$

we see that $B$ is given by $\psi = 0$. On the other hand, $V$ is given by the vanishing of

$$F(\lambda, x) = \psi(x'_{r+1}, \ldots, x'_n) + \sum_{i=1}^{r} x'_i(2\lambda_i + \phi_i(x')).$$

The further coordinate change $\lambda'_i = \lambda_i + \frac{1}{2}\phi_i(x')$ $(1 \le i \le r)$ reduces this to the standard form

$$\psi(x'_{r+1}, \ldots, x'_n) + \sum_{i=1}^{r} 2x'_i \lambda'_i$$

showing that the singularities are indeed of the same type.

## 1.3 The relation of $S(V)$ to $S(\Delta)$: simple case

We now turn to the study of the other projection, $S(V) \to \Delta$. The results here are more complex, and will occupy us for the rest of the paper. We introduce them with the simplest general result that seems to hold.

THEOREM 1.4: *Suppose $\lambda_0 \in S(\Delta)$, and $Q_{\lambda_0}$ has corank 1 with vertex $v_0$. Then the singular points $\lambda_0$ of $\Delta$, $(\lambda_0, v_0)$ of $V$ have the same type.*

PROOF: Again choose coordinates with $\lambda_0$ at $e_0$ and $Q_0$ as $\sum_1^n x_i^2$, so that $v_0 = (1, 0, \ldots, 0)$ also. Near $(\lambda_0, v_0)$ we use affine coordinates, setting $\lambda_0 = x_0 = 1$. Then

$$F(\lambda, x) = \sum_1^n x_i^2 + \sum_{k=1}^{r} \left( \sum_{i,j=1}^{n} a_{ij}^k \lambda_k x_i x_j + 2 \sum_{i=1}^{n} a_{0i}^k \lambda_k x_i \right)$$

as by Proposition 3, $x \in B$ so $a_{00}^k = 0$.

As function of $z = (x_1, \ldots, x_n)$, this is the sum of a quadratic form with matrix $C$, say, where

$$(C)_{ij} = \delta_{ij} + \sum_{k=1}^{r} \lambda_k a_{ij}^k \quad 1 \le i, j \le n$$

and a linear form corresponding to the vector $b$, where

$$b_i = \sum_{k=1}^{r} \lambda_k a_{0i}^k:$$

$$F(\lambda, z) = 2b^t z + z^t C z.$$

For small $\lambda$, $b$ is small and $C$ is approximately the identity. For each $\lambda$, move the origin to the centre of $Q_\lambda$, which is given by

$$0 = b^t + z^t C, \quad \text{so is at } z = -C^{-1} b.$$

Setting $y = z + C^{-1} b$ is clearly a local coordinate change, and we find after reduction

$$F(\lambda, x) = y^t C y - b^t C^{-1} b.$$

Again this suggests the splitting theorem. As $C$ is close to $I$ we can find a symmetric square root $R$ which depends analytically on $\lambda$; the further coordinate change $w = Ry$ reduces $F$ to $w^t w = b^t C^{-1} b$.

   On the other hand, $\Delta$ is defined by the vanishing of a determinant which is, in the above notation,

$$\begin{vmatrix} 0 & b^t \\ b & C \end{vmatrix} = -(\det C) b^t C^{-1} b$$

as we see on postmultiplying the matrix by $\begin{pmatrix} 1 & 0 \\ -C^{-1}b & I \end{pmatrix}$.

Since $\det C \neq 0$ for $\lambda$ near 0, we could have used instead the equation $0 = -b^t C^{-1} b$. Hence the singular points are indeed of the same type.

## Chapter 2  Reduction of the problem

### 2.1 Preliminary remarks

   If $\lambda \in S(\Delta)$ and the corank of $Q_\lambda$ exceeds 1, there is still a relationship between the types of corresponding singularities, but other considerations also play a role.

   We shall be interested only in the case when $(\lambda, x)$ is an isolated singularity of $V$. Indeed, since the relationship is between $\lambda \in S(\Delta)$ and set of all the $(\lambda, x) \in S(V)$, we want each of these to be isolated. Thus $\lambda$ must give an isolated singularity of $\Delta$ and $S(V) \cap B$ must be

finite; conversely, these conditions clearly suffice (and hence imply that the corresponding points $x \in S(B)$ are all tame).

The nature of the singular point $\lambda$ of $\Delta$ does not suffice by itself to determine the types of all the corresponding singular points of $V$. In our paper [8] we saw that, at least in the case of nets, the essential further 'singularity-theoretic' information was given by what we there called the base points of the adjugate system. We now recall and re-analyse this notion.

Our linear system is given by

$$x^t \left( \sum_{i=0}^{r} \lambda_i A_i \right) x = 0$$

Thus the tangential equation of $Q$ is given in dual coordinates by

$$X^t \, \mathrm{adj} \left( \sum_{i=0}^{r} \lambda_i A_i \right) X = 0.$$

We reinterpret this equation, fixing $X$, as a hypersurface in $P_r(\mathbb{C})$. As $X$ varies, we have a system of hypersurfaces which cuts a system of divisors on $\Delta$. This is the 'adjugate system'. It is not linear; its linear span is spanned by the system of functions which are the entries in the adjugate matrix. This defines a rational map from $P_p(\mathbb{C})$ to the projective space of the space of symmetric matrices, which is defined at all $\lambda$ except those for which rank $(\sum_{i=0}^{r} \lambda_i A_i) \leq n - 1$. The resulting blow-up $\Delta_B$ of $\Delta$ is also defined by blowing up the ideal generated by the entries in adj $(\sum_{i=0}^{r} \lambda_i A_i)$. The general principle (at least for nets) is that the interesting structure is determined by the equisingularity type of the pair $(\Delta, \Delta_B)$.

## 2.2 Reduction of the number of variables

We now show how the methods of Theorem 1.4 can be used to reduce the number of variables involved in the equations.

Take coordinates such that the point $\lambda$ under investigation is at $\Lambda_0 = (1, 0, \ldots, 0)$ and $Q_0$ (of corank $(k + 1)$) has equation

$$\sum_{i=k+1}^{n} x_i^2 = 0.$$

We can take affine coordinates by setting $\lambda_0 = 1$, but for now we

retain all the $x_i$. We partition all the matrices into blocks, separating the first $(k + 1)$ rows and columns from the remaining $(n - k)$; thus

$$M = A_0 + \sum_{i=1}^{r} \lambda_i A_i = \begin{pmatrix} A & B^t \\ B & C \end{pmatrix}.$$

Here $A$, $B$ and $C - I$ are homogeneous linear in the $\lambda_i$. So $C$ is invertible at, and hence near $\Lambda_0$. We use the identity

$$\begin{pmatrix} A & B^t \\ B & C \end{pmatrix} \begin{pmatrix} I & 0 \\ -C^{-1}B & I \end{pmatrix} = \begin{pmatrix} A - B^t C^{-1} B & B^t \\ 0 & C \end{pmatrix}$$

to compute the determinant and the adjugate of $M$. We find

$$\det M = \det C \det (A - B^t C^{-1} B),$$

so the equation of $\Delta$ can be written $0 = \det (A - U)$, where $U = B^t C^{-1} B$. Likewise,

$$\text{adj } M = \begin{pmatrix} \det C \text{ adj } (A - U) & -(\text{adj } (A - U)) B^t \text{ adj } C \\ -(\text{adj } C) B \text{ adj } (A - U) & \det (A - U) \text{ adj } C + (\det C) C^{-1} B \text{ adj } (A - U) B^t C^{-1} \end{pmatrix}$$

Since $\det C$ is invertible, the ideal generated (in the ring of germs at 0 of holomorphic functions in $\lambda_1, \ldots, \lambda_r$) by the entries in adj $M$ coincides with the ideal of entries in adj $(A - U)$.

To study $V$, we partition the coordinate vector $x^t = (y^t, z^t)$ correspondingly. Then

$$F(\lambda, x) = x^t M x = y^t A y + 2 y^t B^t z + z^t C z;$$

for fixed $y$ (and small $\lambda$) this is a nonsingular quadratic in $z$, with centre $x = -C^{-1} B y$. Setting $z = z' - C^{-1} B y$, we have

$$F(\lambda, x) = z''^t C z' + y^t A y - y^t B^t C^{-1} B y$$

$$= z''^t C z' + y^t (A - U) y,$$

and the singularity of this has the same type as that of $y^t (A - U) y$.

We collect these observations as

PROPOSITION 2.1: *With the notations introduced above, the singularity of $\Delta$ is given by*

$$0 = \det(A - U),$$

*those of V have the same type as those of*

$$0 = y^t(A - U)y,$$

*and $\Delta_B$ is defined by blowing up the ideal of entries in*

$$\text{adj}(A - U).$$

Thus for all three purposes the $(n + 1) \times (n + 1)$ matrix $M$ has been replaced by the $(k + 1) \times (k + 1)$ matrix $(A - U)$. This is most useful when $k$ is small; the case $k = 0$ gives Theorem 1.4 immediately and the case $k = 1$ will be explored in Chapter 3.

We conclude this section by observing that while $A$ is homogeneous linear in $\lambda_1, \ldots, \lambda_r$, all the terms in $U$ have order $\geq 2$. Indeed, the terms of degree 2 in $U$ are given by $B^t B$. The tangent cone of $\Delta$ at $\Lambda_0$ is given by the terms of lowest degree in $\det(A - U)$, i.e. by $\det A$ (provided this does not vanish identically). Note that $A$ is the matrix of the linear system of quadrics cut by the given systems on the vertex $z = 0$ of $Q_0$. This is spanned by $r$ quadrics in $\mathbb{C}^{k+1}$ (or $P_k\mathbb{C}$): if the base points are to be isolated in $P_k(\mathbb{C})$, we must have $r \geq k$. If $r = k$, $S(Q_0) \cap B$ is always non-empty, but if $r > k$, this need not be the case and there may be no singular point of $V$ projecting to $\Lambda_0$.

## 2.3 Recognition principles

In the examples to be discussed below, the singular points will all be simple in the sense of Arnol'd [1], and indeed will belong to one of the classes typified by the normal forms

$$A_m : z_1^{m+1} + \sum_{2}^{n} z_i^2 = 0 \qquad (m \geq 1)$$

$$D_m : z_1^{m-1} + z_1 z_2^2 + \sum_{3}^{n} z_i^2 = 0 \quad (m \geq 4).$$

The problem arises, for a function not in simple form, of recognising when a local analytic coordinate transformation will exist to reduce it to one of the above forms.

Some progress is easily made by inspection of the lowest degree terms in the power series expansion: suppose $f(z_1, \ldots, z_n)$ has a singular point at 0, and $f(0) = 0$. Then the lowest degree terms have

degree 2, and yield a quadratic form in $z_1, \ldots, z_n$. If (the matrix of) this has corank $k$ (i.e. rank $n - k$), then:

for $k = 0$   we have a point of type $A_1$.

for $k = 1$   we have a point of type $A_m$ (some $m \geq 2$).

for $k = 2$;   suppose $z_1, z_2$ do not occur in the form. Then we look at the cubic terms in $z_1$ and $z_2$: a binary cubic. If this has distinct factors, we have a point of type $D_4$; if a repeated factor (but not a perfect cube), of type $D_m$ for some $m \geq 5$. Otherwise we have a higher singularity.

It is less easy to determine the number $m$. The basic method is Arnol'd's normal form theorem [2] in a somewhat simplified special case. Observe that the normal forms above are weighted homogeneous, if we assign weights by

$$(A_m) \quad \mathrm{wt}(z_1) = (m + 1)^{-1}, \quad \mathrm{wt}(z_i) = \tfrac{1}{2} \ \text{ for } i > 1,$$

$$(D_m) \quad \mathrm{wt}(z_i) = (m - 1)^{-1}, \quad \mathrm{wt}(z_2) = \frac{m - 2}{2(m - 1)}, \quad \mathrm{wt}(z_i) = \tfrac{1}{2} \ \text{ for } i > 2.$$

For any weighted homogeneous polynomial (with positive weights) the recognition problem is much simpler (particularly if few variables are involved).

THEOREM 2.2: [2] *Suppose* $f(z_1, \ldots, z_n)$ *a holomorphic function with a critical point at 0. Suppose there are weights* $\mathrm{wt}(z_i) = w_i$, $0 < w_i < 1$, *such that the terms* $a_\omega z_1^{\varphi_1} \ldots z_n^{\omega_n}$ *occurring in the Taylor expansion of* $f$ *(with* $a_\omega \neq 0$*) have weights* $\Sigma \omega_i w_i \geq 1$, *and the sum* $f_0$ *of the terms of weight 1 is equivalent to one of the normal forms above, by local analytic coordinate change. Then* $f$ *is equivalent to* $f_0$.

## 2.4 The case of pencils

For pencils $(r = 1)$ it follows that for isolated singularities, $k \leq 1$. Since $\Delta$ is a set of points on a line, its 'singularities' are the points occurring with multiplicity $m + 1 \geq 2$: such a point being of type $A_m$. If $k = 0$, the corresponding singularity of $V$ also has type $A_m$, by Theorem 4. We now investigate the case $k = 1$.

It is natural here to refer to the classification of pencils described by Segre [6]. As $\Delta$ has isolated singularities, we can choose $\Lambda \notin \Delta$, i.e. $A_1$ nonsingular. The eigenvalues $\lambda$ of $A_0 A_1^{-1}$ then given the values of $\lambda = -\lambda_1/\lambda_0$ at the points $P_\lambda$ of $\Delta$ (with multiplicities). Moreover, if we

put $A_0 A_1^{-1}$ in Jordan canonical form, and the eigenvalue $\lambda$ is asso-
ciated to blocks of sizes $m_1 \geq m_2 \geq \ldots \geq m_t > 0$, we say $(m_1,$
$m_2, \ldots, m_t)$ is the corresponding term in the Segre characteristic of
the pencil. Here, $m = \Sigma_1^t m_i$. Also, by inspection $t = k + 1$.

LEMMA 2.3:
(i) *The multiplicity of $P_\lambda$ as adjugate base point is $\Sigma_2^t m_i = m'$.*
(ii) *$P_\lambda$ corresponds to isolated singularities of $V$ if and only if $m' \leq 1$.*

PROOF: For a single Jordan block of size $m$ one has the canonical
form

$$\lambda_0 \left( \sum_0^{n-1} y_i y_{n-1-i} \right) + \lambda_1 \left( \sum_0^n y_i y_{n-i} \right).$$

The dual equation has $\lambda_0^{n-1}$ as coefficient of $Y_n^2$ and $\lambda_1^{n-1}$ as coefficient
of $Y_0 Y_n$, thus has no base-points.
    Now for any matrices $P$ and $Q$,

$$\mathrm{adj}(P \oplus Q) = \det Q . \mathrm{adj}\, P \oplus \det P . \mathrm{adj}\, Q.$$

If $P$ is a block of the above type, the h.c.f. of the corresponding
entries in the adjugate matrix is thus $\det Q = \det M / \det P$. The lowest
power of $\lambda \lambda_0 + \lambda_1$ dividing this is when $\det P$ has the highest such
power, $m_1$.
    (ii) We have $t = k + 1 \leq 2$. Now $m' = 0 \Leftrightarrow t = 1$, and the singularity
is then certainly isolated.
    For the single block above, the vertex $Y_n$ of $Q_0$ is contained in $Q_1$
unless $n = 0$. Thus if $m_1 \geq m_2 \geq 2$, $Q_1$ contains a line in $S(Q_0)$ and we
have a non-isolated singularity, whereas if $t = 2$ and $m_2 = 1$ (which is
equivalent by (i) to $m' = 1$), $S(Q_0)$ is a line not wholly contained in $Q_1$,
so the intersection is finite.
    Part (ii) of the above lemma is due to Knörrer [4] as is the next
result.

PROPOSITION 2.4: *If $P_\lambda$ corresponds to $(m, 1)$ in the Segre charac-
teristic (so has type $A_m$ on $\Delta$), then the singular points of $V$ on $S(Q_\lambda)$ are
as follows:*

$m = 1$: *two points, each of type $A_1$,*

$m = 2$: *one point of type $A_3$,*

$m \geq 3$: *one point of type $D_{m+1}$.*

Observe that in the sequence of Dynkin diagrams $D_{m+1}$, it is natural to interpret $D_3$ as $A_3$ and $D_2$ as $A_1^2$. The same identification holds for the corresponding Lie algebras.

PROOF: Although the result is found in [4] and [9] we show here how it follows from the methods just developed. We use the notations of (2.2).

*Case $m = 1$* The matrix $A$ has rank 2, and we may take

$$A = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} u(\lambda) & \lambda + v(\lambda) \\ \lambda + v(\lambda) & w(\lambda) \end{pmatrix}$$

with $u$, $v$, $w$ of order $\geq 2$. Thus det $U$ has order 2 (confirming $m = 1$). The equation of $V$ is (equivalent to)

$$u(\lambda)x^2 + 2(\lambda + v(\lambda))xy + w(\lambda)y^2$$

with singular points at $x = 0$ and at $y = 0$. Taking (for example) $x = 0$, use inhomogeneous coordinates $y = 1$

$$ux^2 + 2(\lambda + v)x + w:$$

the terms of degree 2 are $2\lambda x + a\lambda^2$ (for some $a$) giving a nonsingular quatratic form, hence of type $A_1$.

*Case $m > 1$* Here $A$ has rank 1 and we take

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \lambda + u & v \\ v & w \end{pmatrix}$$

again with $u$, $v$, $w$ of order $\geq 2$. We take $\lambda + u$ as new coordinate in place of $\lambda$, and so reduce $u$ to 0. By hypothesis, $\lambda w - v^2 = \det U$ has order $(m + 1)$ in $\lambda$.

$V$ has a unique corresponding singular point, at $x = 0$, and we have the local equation

$$0 = \lambda x^2 + 2v(\lambda)x + w(\lambda)$$

$$= \lambda \left( x + \frac{v(\lambda)}{\lambda} \right)^2 + \frac{\lambda w - v^2}{\lambda}.$$

Here, take $\xi = x + (v/\lambda)$ as new coordinate in place of $x$. Now assign-

ing weights

$$\text{wt}(\lambda) = \frac{1}{m}, \qquad \text{wt}(\xi) = \frac{m-1}{2m}$$

we see the only terms of weight $\leq 1$ are

$$\lambda \xi^2 + c\lambda^m \qquad (\text{some } c \neq 0).$$

If $m \geq 3$, this is the normal form for $D_{m+1}$; if $m = 2$ we rewrite it as

$$c\left(\lambda + \frac{\xi^2}{2c}\right)^2 - \frac{\xi^4}{4c}$$

to recognise a singularity of type $A_3$.

## Chapter 3  Singularities of nets

### 3.1 General discussion

For the rest of this paper we restrict to the case of nets of quadrics ($r = 2$ above), which is the case where the relation between the system and the discriminant curve $\Delta$ seems to be closest. If $\Lambda_0$ is an isolated singularity of $\Delta$, the corank $(k+1)$ of $Q_0$ must satisfy $k \leq r = 2$. The results for $k = 0$ are already contained in Theorem 1.4; we shall concentrate mainly on the case $k = 1$.

The linear system cut on the vertex $S(Q_0)$ (a projective line) is spanned by two forms, so can be reduced to one of the normal forms

(i) $\lambda x^2 + \mu y^2$    (ii) $2\lambda xy + \mu x^2$    (iii) $2\lambda xy$    (iv) $\lambda x^2$    (v) $0$,

with respective discriminants

(i) $\lambda\mu$    (ii) $-\lambda^2$    (iii) $-\lambda^2$    (iv) $0$    (v) $0$.

The corresponding singularities of $V$ (or $B$) are the base points of this system. For case (v) we get the whole line as a non-isolated singularity: this we will not discuss further. In cases (ii), (iv) we get one point $Y$ ($x = 0$); in case (iii) two points $X$ ($y = 0$) and $Y$, and in case (i) no singularities.

Since the above discriminant is the tangent cone to $\Delta$ at $\Lambda_0$, we

have a singularity of type $A_1$ in case (i), a higher double point $A_n$ in cases (ii) and (iii) and a triple (or higher) point in the remaining cases. Using a similar discussion for the case $k = 2$ (to be given below) we see that every singular point of $\Delta$, with the sole exception of those of type $A_1$ (with $k = 1$), corresponds to singularities of $V$ and $B$. This result is due to Beauville [3], and was a major stimulus to the author in formulating the results of this paper in a general framework.

PROPOSITION 3.1: *In the above situation, $\Lambda_0$ has multiplicity 1 as adjugate base point on $\Delta \Leftrightarrow$ we have case (i) or case (ii) above.*

PROOF: By Proposition 2.1, the adjugate system is equivalent to the system of curves
$$X^t \operatorname{adj}(A - U)X = 0.$$

The multiplicity of $\Lambda_0$ as adjugate base point exceeds 1 if and only if these have a common tangent at $\Lambda_0$. But they have the same tangents as the curves $X^t \operatorname{adj} AX = 0$ where $A = \lambda A_0 + \mu A_1$ is the matrix of the above pencil. But the matrices $\operatorname{adj} A$ (as $\lambda$, $\mu$ vary) are multiples of a common matrix only in cases (iii), (iv) and (v) above.

## 3.2 Case (ii)

We have just seen that this case is characterised by having adjugate base point multiplicity 1. The corresponding singular points are related by

THEOREM 3.2: *In case (ii), for some integer $m \geq 2$, $\Lambda_0$ is a singular point of type $A_m$ on $\Delta$ and $Y$ is a singular point of type $A_{m-1}$ on $V$.*

PROOF: As above, we can normalise

$$A = \begin{pmatrix} \mu & \lambda \\ \lambda & 0 \end{pmatrix};$$

we also write

$$U = \begin{pmatrix} u & v \\ v & w \end{pmatrix}.$$

By Proposition 2.1, $\Delta$ is given (locally) by

$$w(u - \mu) = (v - \lambda)^2$$

and we may consider $V$ as given (near $Y$) by

$$(u - \mu)x^2 + 2(v - \lambda)x + w = 0$$

in affine coordinates ($y = 1$).

Moreover, $u$, $v$ and $w$ are functions of $\lambda$ and $\mu$ of order $\geq 2$. We may thus take $\lambda - v$ and $\mu - u$ as new local coordinates in the $(\lambda, \mu)$-plane at $\Lambda_0$. This has the effect of replacing the above by equations of the same form, but with $u$ and $v$ set equal to 0.

The equation of $\Delta$ is now $\lambda^2 = -\mu w$. By the preparation theorem

$$\lambda^2 + \mu w = (\lambda^2 + 2a(\mu)\lambda + b(\mu))U,$$

where $U$ does not vanish at $\Lambda_0$. Making the further substitution $\lambda' = \lambda + a(\mu)$ (which reintroduces $v$ as $-a(\mu)$) brings this to the form

$$(\lambda^2 + c(\mu))U.$$

Let $m + 1$ be the order of $c(\mu)$, so we can write

$$c(\mu) = \mu^{m+1}(c + o(1)), \quad \text{with } c \neq 0.$$

Then the above can be written as

$$\lambda^2 + c\mu^{m+1} + o(\lambda^2, \lambda\mu^{(m+1)/2}, \mu^{m+1}),$$

where the small $o$ in the remainder signifies that the monomials occurring in the power series expansion are divisible by at least one of those listed, with a quotient zero at $\Lambda_0$ (without this final clause, we would use a capital $O$). Thus if we set

$$w = w_0(\mu) + \lambda w_1(\mu) + O(\lambda^2),$$

we find

$$\mu w_0 + v^2 = \mu^{m+1}(c + o(1)),$$

$$\mu w_1 - 2v = o(\mu^{(m+1)/2}).$$

But now $V$ is given by the vanishing of

$$\mu x^2 + 2(\lambda - v)x - w$$

$$= \mu x^2 + 2(\lambda - v(\mu))x - (w_0(\mu) + \lambda w_1(\mu) + O(\lambda^2))$$

$$= (2x - w_1(\mu))(\lambda - v(\mu) + \tfrac{1}{2}\mu x + \tfrac{1}{4}\mu w_1(\mu))$$

$$+ \{\tfrac{1}{4}\mu w_1^2 - w_0 - v w_1\} + O(\lambda^2).$$

The term in braces here equals

$$-\mu^{-1}(\mu w_0 + v^2) + \frac{1}{4\mu}(\mu w_1 - 2v)^2 = \mu^m(-c + o(1))$$

by the above.

Now take $\xi = x - \frac{1}{2}w_1(\mu)$ as new coordinate in place of $x$, and assign weights to coordinates by

$$\mathrm{wt}(\mu) = \frac{1}{m}, \qquad \mathrm{wt}(\lambda) = \frac{1}{2} + \frac{1}{4m}, \qquad \mathrm{wt}(\xi) = \frac{1}{2} - \frac{1}{4m}.$$

The first term above is

$$2\xi(\lambda - v + \tfrac{1}{2}\mu\xi + \tfrac{1}{2}\mu w_1)$$

$$= 2\lambda\xi + \mu\xi^2 + \xi(\mu w_1 - 2v)$$

and the second and third terms here have weights $> 1$ since

$$\mathrm{wt}(\mu w_1 - 2v) > \frac{m+1}{2}\,\mathrm{wt}(\mu) = \frac{1}{2} + \frac{1}{2m}.$$

Thus modulo terms of weight $> 1$, the function reduces to

$$2\lambda\xi - c\mu^m \quad (c \neq 0).$$

So by the recognition theorem we have a singularity of type $A_{m-1}$.


**3.3 Case (iii)**

This is the most delicate, and most interesting case that I am able to treat fully.

THEOREM 3.3: *If $\Lambda_0$ is a singular point of type $A_m$ on $\Delta$, with multiplicity $s \geq 2$ as adjugate base point, then $V$ has two singular points on $S(Q_0)$, of respective types $A_{s-1}$ and $A_{m-s}$.*

PROOF: It follows from Proposition 3.1 and other remarks in §3.1 that the hypothesis places us in Case (iii) above. We have

$$A = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

and by Proposition 2.1 we have equations

$$V : ux^2 + 2(v - \lambda)xy + wy^2 = 0$$

$$\Delta : uw = (v - \lambda)^2$$

and the ideal to blow up is $\langle u, v - \lambda, w \rangle$.

We first choose $\lambda - v$ as new coordinate, to reduce $v$ to zero. If now $u(0, \mu)$, $w(0, \mu)$ have respective orders $s$, $t$ with $s \leq t$, then the ideal equals $\langle \lambda, \mu^s \rangle$: one adjoins $\lambda\mu^{-s}$ to blow it up, so the multiplicity as adjugate base point is $s$.

This can be checked by observing that a typical singular point $\lambda^2 = \mu^{m+1}$ of type $A_m$ becomes, under the blow-up $\rho = \lambda\mu^{-s}$, a singularity $\rho^2 = \mu^{m+1-2s}$, of type $A_{m-2s}$ (if $m > 2s$: for $m = 2s$ or $2s - 1$, the curve becomes nonsingular), conforming to the conventions of [8].

Next, we apply the preparation theorem as before. Thus we can make a change of coordinates

$$\lambda = \lambda' - v(\mu)$$

(reinstating $v$) so that

$$(\lambda - v)^2 - uw = \lambda^2 + c\mu^{m+1} + o(\lambda^2, \lambda\mu^{m+1/2}, \mu^m)$$

for some $c \neq 0$. Note that the coefficient of $\lambda$ in the original $uw$ was divisible by $\mu^{s+1}$, so $v(\mu) = O(\mu^{s+1})$ and thus $u_0(\mu) = u(0, \mu)$ continues to have order $s$. Note also that we have chosen $m \geq 2$ (in fact $\geq 3$) so that $\Delta$ has a singularity of type $A_m$ at $\Lambda_0$.

In fact, let us set

$$u = u_0(\mu) + \lambda u_1(\mu) + O(\lambda^2),$$

$$w = w_0(\mu) + \lambda w_1(\mu) + O(\lambda^2);$$

then

$$v^2 - u_0 w_0 = \mu^{m+1}(c + o(1))$$

and

$$u_0 w_1 + 2v + u_1 w_0 = o(\mu^{(m+1)/2}) = O(\mu^{(m+2)/2}).$$

To analyse the local structure of $V$ at $X$, we set $x = 1$ giving

$$wy^2 + 2(v - \lambda)y + u.$$

Here we assign weights $1/s$ to $\mu$, $\epsilon$ to $y$ and $(1-\epsilon)$ to $\lambda$ for some $\epsilon$ with $0 < \epsilon < \frac{1}{2}$. Then modulo terms of weight $>1$ we can ignore terms strictly divisible by $\mu^s$ or $\lambda y$ or divisible by $\lambda^2$, thus leaving

$$w_0 y^2 + 2(v - \lambda)y + u_0 + u_1 \lambda$$

but $v$ has order $s+1$, $u_0$ order $s$ and $w_0$ order $\geq s$, so wt $v > 1$, wt $w_0 \geq 1$ and wt$(u_0 - d\mu^s) > 1$ for some $d \neq 0$. This leaves

$$-2\lambda y + d\mu^s + u_1 \lambda$$

and as $u_1$ has order at least 1, we can choose $\epsilon = 1/2s$ and eliminate the last term. By the recognition theorem, we now have a singularity of type $A_{s-1}$.

The analysis of the singularity at $Y$ is more delicate. Here we set $y = 1$ and $x = \xi - v/u_0$ giving

$$(u_0 + u_1)(\xi - v/u_0)^2 + 2(v - \lambda)(\xi - v/u_0) + (w_0 + w_1 \lambda) + O(\lambda^2)$$

$$= -2\lambda\xi + \lambda\xi\left(u_1\xi - 2u_1\frac{v}{u_0}\right) + u_0\xi^2 + \frac{\lambda}{u_0}(u_0 w_1 + 2v + w_0 u_1)$$

$$+ \frac{\lambda u_1}{u_0^2}(v^2 - u_0 w_0) - \frac{1}{u_0}(v^2 - u_0 w_0) + O(\lambda^2)$$

$$= -2\lambda\xi + C_1 + C_2 + C_3 + C_4 + C_5 + O(\lambda^2), \text{ say.}$$

Here we assign weights

$$\text{wt}(\xi) = \tfrac{1}{2} - \epsilon, \qquad \text{wt}(\lambda) = \tfrac{1}{2} + \epsilon, \qquad \text{wt } \mu = \frac{1}{m - s + 1}$$

for $\epsilon > 0$ to be determined. The weights of the above terms are then

$$\text{wt}(C_1) > 1 \qquad \text{wt}(C_2) = 1 - 2\epsilon + \frac{1}{m - s + 1}$$

$$\text{wt}(C_3) \geq \tfrac{1}{2} + \epsilon + \frac{m + 2 - 2s}{2(m - s + 1)} \qquad \text{wt}(C_4) \geq \tfrac{1}{2} + \epsilon + \frac{m + 1 - 2s}{m - s + 1} > \text{wt}(C_3)$$

and wt$(C_5) = 1$. If we now choose

$$\epsilon = \frac{2s - 1}{4(m - s + 1)},$$

we find that $C_2$, $C_3$ and hence $C_4$ have weights $> 1$. As the leading term of $C_5$ is

$$-(d\mu^s)^{-1}(c\mu^{m+1}),$$

we have, modulo terms of higher weight

$$-2\lambda\xi - cd^{-1}\mu^{m+1-s},$$

enabling us to recognise a singular point of type $A_{m-s}$.


### 3.4 Case (iv)

In this case our discussion is far from complete. By Proposition 2.1, we have equations

$$(u - \lambda)x^2 + 2vx + w = 0 \qquad \text{for } V \text{ at } Y,$$

$$(u - \lambda)w = v^2 \qquad\qquad \text{for } \Delta \text{ at } \Lambda_0.$$

Here we must first look at the terms of degree 2 in $w$, giving a homogeneous quadratic $q$ in $\lambda$ and $\mu$. If $q$ has distinct factors then $V$ has a singularity of corank 1 and $\Delta$ one belonging to the $D$-series.

CONJECTURE 3.4: *In this case, for some $m \geq 3$, $V$ has a singularity of type $A_m$ at $Y$ and $\Delta$ one of type $D_{m+1}$ at $\Lambda_0$.*

I have verified this for $m \leq 6$ by direct calculation, but an attempted proof along the lines of (3.2) and (3.3) has so far failed to work. There is little doubt that the conjecture holds, but the more general conjecture 3.5 below is of more interest.

If $q$ has a repeated root which is not $\lambda$, $V$ and $\Delta$ both have corank 2 singularities. For $\Delta$ we again have a singularity of type $D_m$ for some $m$; the singular point $Y$ of $V$ also has some type $D_p$, on account of the term $-\lambda x^2$ of degree 3.

If $q$ is a multiple of $\lambda^2$, $\Delta$ still has a triple point (of higher type) and $V$ a corank 2 singularity, but the possibilities are now much more numerous. For example, the singularity of $V$ is a $D_m$ only if the coefficient of $\mu^2$ in $v$ does not vanish.

Finally, $q$ may vanish: then $V$ has a corank 3 singularity and $\Delta$ a quadruple point (at least).

Further enumeration of subcases should also be accompanied by an analysis of the adjugate base point behaviour.

In each of the cases so far analysed, the following may be observed to hold:

Suppose $r = 2$, $k = 1$ and $V$ has isolated singularities. Then if the singular point $P$ on $\Delta$ corresponds to singular points $Q_i$ on $V$, the Milnor numbers satisfy

$$\Sigma_i \mu(Q_i) = \mu(P) - 1.$$

In view of our reduction of the equations, we can reformulate and generalise this conjecture. Note that of the varieties $V'$, $\Delta$ given by

$$x^t(A - U)x = 0,$$

$$\det(A - U) = 0 \quad \text{(respectively)},$$

$\Delta$ is a plane curve, and $V'$ a double cover of the plane, branched along $\Delta$.

CONJECTURE 3.5: *Let $X$ be a double plane, with equation*

$$u(\lambda, \mu, \nu)x^2 + 2v(\lambda, \mu, \nu)xy + w(\lambda, \mu, \nu)y^2 = 0,$$

*branched along the curve $\Delta$ ($uw = v^2$). Let $P$ be a singular point of $\Delta$, where $u = v = w = 0$, such that the corresponding line in $X$ contains only finitely many singular points $\{Q_i\}$ of $X$. Then the Milnor numbers are related by*

$$\mu(P) = 1 + \Sigma_i \mu(Q_i).$$

Not only would this clarify the proofs above and extend the results; one would also expect the methods to deal with other related problems.

### 3.5 The case $r = k = 2$

In this case, the vertex of the quadric $Q_0$ is a (projective) plane, and the net cuts a pencil of conics (defined by $Q_1$ and $Q_2$) in this plane. The corresponding points of $S(V)$ are the intersections of these conics: we wish these to be isolated. We may thus have a singular pencil of type $(x^2, y^2)$ or one of the five types of nonsingular pencil.

We give details only in the case $n = 3$: though relatively simple, this is already interesting. Here, $Q_0$ is a repeated plane, and the multiplicity of $B$ as intersection $Q_0 \cap Q_1 \cap Q_2$ is twice that of the two conics $Q_0 \cap Q_1$, $Q_0 \cap Q_2$ (in $Q_0$) at $B$. Moreover, as $Q_0$ meets $Q_1$ transversely (necessary for $B$ to be tame), $V$ can only have singularities of type $A_n$ for some $n$. The corresponding singularity of $\Delta$ is a triple point.

THEOREM 3.6: *For $n = 3$, $r = k = 2$, the following cases occur:*

| Type of singular $P$ on $\Delta$ | $D_4$ | $D_5$ | $D_6$ | $E_6$ | $E_7$ |
|---|---|---|---|---|---|
| Segre symbol of pencil | [1, 1, 1] | [2, 1] | [(1, 1), 1] | [3] | [(2, 1)] |
| Discriminant of pencil | 1, 1, 1 | 2, 1 | $2^1$, 1 | 3 | $3^1$ |
| Multiplicities of base points | 2, 2, 2, 2 | 4, 2, 2 | 4, 4 | 6, 2 | 8 |
| Types of singular points of $V$ | $A_1^4$ | $A_3 A_1^2$ | $A_3^2$ | $A_5 A_1$ | $A_7$ |

REMARKS: The singular pencil $(x^2, y^2)$ cannot occur when $n = 3$: this would imply that $Q_0$ was the tangent plane at $B$ to each of $Q_1$, $Q_2$; hence that they do not intersect transversely there. Each type of nonsingular pencil occurs in association with just one singularity type for $P$: This is the main assertion of the Theorem. The row 'discriminant of the pencil' indicates the multiplicities of points on the discriminant as such and (the superscript) as adjagate base point. The two final rows can be read off from the type of the pencil using the remarks preceding the theorem.

We again observe a relation between Milnor numbers: here of the form $\mu(P) = \Sigma_i \mu(Q_i)$. The relation between types is yet more direct. Each $P$, as a simple singularity, corresponds not only to a Dynkin diagram (of spherical type) but also to an extended Dynkin diagram (of Euclidean type). We obtain the diagram for the $Q_i$ by deleting one vertex from this extended diagram (circled in the diagrams below).
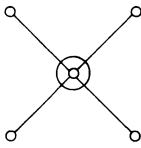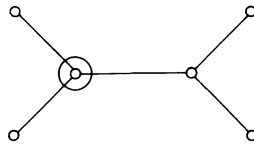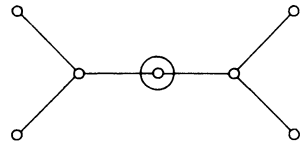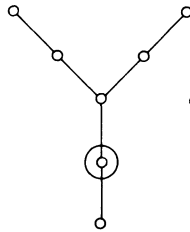
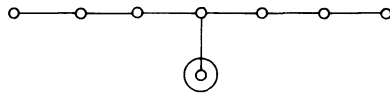$$D_4 \to A_1^4 \qquad D_5 \to A_3 A_1^2 \qquad D_6 \to A_3^2$$

$$E_6 \to A_5 A_1 \qquad E_7 \to A_7$$

PROOF OF 3.6: The tangent cone to $\Delta$ at $P$ is given by the discriminant of the pencil. If this has three distinct factors, $P$ has type $D_4$.

If the discriminant of the pencil has one repeated point and one other, $P$ has type $D_n$ (some $n$). We can take the pencil in the form

$$\mu(x_3^2 + 2ax_2x_3) + \nu(2x_2x_4),$$

where $a = 0$ if and only if the repeated point is an adjugate base point. Then the net has matrix

$$\begin{pmatrix} \lambda & \alpha & \beta & \gamma \\ \alpha & 0 & a\mu & \nu \\ \beta & a\mu & \mu & 0 \\ \gamma & \nu & 0 & 0 \end{pmatrix},$$

with determinant

$$\Delta = -\lambda\mu\nu^2 + 2\alpha\gamma\mu\nu + \beta^2\nu^2 - 2a\beta\gamma\mu\nu + a^2\gamma^2\mu^2,$$

where $\alpha$, $\beta$ and $\gamma$ are linear in $\mu$ and $\nu$, and $\gamma$ not divisible by $\nu$ (else the point $X_4$ is not tame). If $a \neq 0$ the coefficient of $\mu^4$ is nonzero, so $P$ has type $D_5$; otherwise the coefficient of $\mu^3\nu$ is nonzero (else $\nu^2$ divides $\Delta$) and $P$ has type $D_6$ (set $\lambda = 1$ and assign weights wt $\mu = 1/5$, wt $\nu = 2/5$).

If the discriminant of the pencil has a threefold point, we take the pencil as

$$\mu(x_3^2 - x_2x_4) + \nu(x_2^2 + 2ax_2x_3),$$

where again $a = 0$ if and only if we have an adjugate base point. As above, the matrix of the net is obtained by bordering by $(\lambda, \alpha, \beta, \gamma)$ where $\gamma$ is not divisible by $\mu$. The tangent cone to $\Delta$ at $P$ is $\mu^3$; the coefficient of $\nu^4$ comes from the term $a^2\gamma^2\nu^2$, so if $a \neq 0$ $P$ has type $E_6$. If $a = 0$ the coefficient of $\mu\nu^3$ comes from $-2\gamma^2\mu\nu$ and is nonzero, so here $P$ has type $E_7$.

## Chapter 4  Further remarks

### 4.1  Stability

In our previous paper [8] we defined a linear system to be *stable* if it is so in the sense of Mumford [5] for the natural action of $SL_{n+1}$. By [8, p. 232] this fails if and only if, for a suitable choice of coordinates, and $s \leq (n + 1)/2$, $a_{jk}^i = 0$ whenever $0 \leq i \leq r$, $0 \leq i \leq s$, $0 \leq j \leq n - s - 1$. The connexion with the present paper is given by

LEMMA: 4.1 *For a stable net, all base points are tame.*

PROOF: If there is a non-tame base point, $X_0$ say, then there is a line $\ell(\lambda_0 = 0$ say$)$ in $P_2$ such that for $\lambda \in \ell$, $X_0$ is on the vertex of $Q_\lambda$. Thus the polars of $X_0$ with respect to $Q_1$, $Q_2$ vanish identically; taking $x_n = 0$ as the tangent plane to $Q_0$ at $X_0$ we find that the above criterion is satisfied (with $s = 0$).

COROLLARY: *For a stable net, if $B$ has isolated singularities, so does $V$, hence also $\Delta$.*

In the case $n = 3$ we found [8, p.247] as a result of enumerations that for stable nets, dim $B > 0$ if and only if $\Delta$ has a repeated component. This observation now fits into place. For an unstable net, $\Delta$ always has a repeated component [8, p.233].

A further consequence of tameness may also be noted here: that the singularity of $B$ is equivalent to a hypersurface singularity in $\mathbb{C}^{n-r}$. In the case $n = 3$, $r = 2$, we have a subset of $\mathbb{C}$: if isolated, this is necessarily of type $A_\ell$ for some $\ell \geq 1$, and the type is determined by the multiplicity $(\ell + 1)$ as base point.

### 4.2  Nets of quadrics: remaining cases

For the case $n = 3$, $r = 2$ of nets of quadrics, an isolated singularity of $\Delta$ cannot be worse than a triple point. Double points were analysed

in (3.1)–(3.3); a triple point which is not an adjugate base point corresponds to (3.4) and indeed (since the singularities of $V$ have type $A$ by the above remark) to Conjecture 3.4. The conjecture is true in the relevant cases ($P$ of type $D_{m+1}$ on $\Delta$, a point of type $A_m$ on $V$) since $m$ can only be 3, 4 or 5 in this case (i.e. when $n = 3$). Triple points which are adjugate base points were classified in (3.5).

Other singularities are non-isolated on $\Delta$ and on $V$, but can still be isolated on $B$ (though not tame). We shall confine ourselves to the case of nets with only finitely many base points. Then there cannot be a common line, so in the classification of [8, 1.6.1] type II is excluded, and indeed all those of type I not falling under one of the cases 1i, 1ii, 2iii, 3v or 4viii of [8, 1.6.2]. These cases are classified by the table in [8, 1.6.3]. Here $S_0$ and $S_1$ define a pencil of cones with a common vertex; the types of pencil correspond to the columns in the table. For the ($xz$, $yz$) and ($x^2$, $xy$) columns, $\dim(S_0 \cap S_1) = 2$ so corresponding nets all have infinitely many base points. In all other cases we have 4 lines (which may coincide) each meeting $S_2$ in (1, 0, 0, 0).

In general these lines meet $S_2$ transversely there, and the other intersection gives a further base point. But a line (corresponding to base point $a$ of the pencil) such that $a$ belongs to the line indicated in the table will touch $S_2$ at that point. Thus multiplicities of base points are as follows

| Type of pencil | abcd | aabc | aaab | aabb | aaaa | $x^2, y^2$ |
|---|---|---|---|---|---|---|
| **Case** | | | | | | |
| 1i, 1ii | 41111 | 4211 | 431 | 422 | 44 | 44 |
| 2iii | 5111 | $(ae)_{611}(be)_{521}$ | $(ae)_{71}(be)_{53}$ | 62 | 8 | — |
| 3v | 611 | $(ab)_{71}(bc)_{62}$ | 8 | 8 | — | — |
| 4viii | — | 611 | 71 | 62 | — | 8 |

Finally one sees easily that the only non-semistable nets with finitely many base points are the cones over nets of conics without base points (types $A$, $B$, $D$, $E$). Clearly here there is a single base point, with multiplicity 8.

We shall not list the elements of the Hilbert scheme corresponding to these multiple points, but observe that although those of multiplicity

$\ell + 1 \leq 3$ must be tame, hence of type $A_\ell$, those of multiplicity $\geq 4$ in this table are neither; with the single exception of one of the two points with multiplicity 4 in the penultimate column.

## 4.3 Webs

Our more detailed results concerned only the cases $r = 1, 2$. It seems unlikely that the results of §2 can be extended to the case $r = 3$ (and à fortiori to $r > 3$). For the linear system cut on the vertex, there is only one new possibility: $(x^2, xy, y^2)$. However this case now corresponds to a singularity of type $A_1$ on the surface $\Delta$; $(x^2, y^2)$ to singularities of type $A_n(n > 1)$ and $(x^2, xy), (xy, 0)$ to singularities of corank 2. In particular, a singularity of type $A_n$ for any $n$ corresponds to no singular point of $V$, so no simple relation between Milnor numbers can persist to this case. (Perhaps one should use the Milnor number $\mu^{(1)}$ of a generic plane section).

## 4.4 Corrigendum to 'Nets of quadrics'

It seems appropriate to insert here a correction to [8] which was drawn to my attention by Professor Horrocks. In (3.1) of this paper I stated that the bundle $\xi$ over a nonsingular curve $\Delta$ corresponded to a net if and only if dim $\Gamma(\xi)$ was even. The correct condition is of course that moreover dim $\Gamma(\xi) = 0$, which is equivalent to the non-vanishing of a certain theta-constant: a condition holding in general (indeed, even this seems to be unknown for plane curves) but not always. See [7] for a fairly full discussion in modern terminology.

The conclusions I formulated will thus admit of exceptions even for nonsingular curves. Evidently the conjectures made in [8] will need corresponding modifications. I had visualised a map from a 'space of nets' to a 'space of curves' whose restriction to curves without repeated component was proper and finite-to-one (hence of constant degree). We see now that propriety breaks down.

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